

# High-precision Hadron Structure

Charge Symmetry Violation in PDFs and  
Fragmentation Functions

Whitney Armstrong  
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November 4, 2024

# Overview

- Ingredients to precision parton distributions
- Charge Symmetry and how it breaks
- Theory and Experimental Status
- Recent Measurements
- Challenges toward precision era
- New possibilities



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# Ingredients to precision parton distributions

Not a comprehensive list

- World data, including precision measurements.
- pQCD machinery and analysis framework
- Power corrections: target mass, quark mass, high-twists, etc...
- Exclusions: kinematics coverage, type (helicity independent, valence, etc...)
- **Assumptions** (stated and unstated)

It is the latter that I want to focus on first.

PDF fitters of world data nearly universally assume charge symmetry to significantly reduce the number of distributions (parameters) they need to fit.

When looking at effects approaching the level of a few percent, we need to be more careful and complete approach.

# Introduction

## What is Charge symmetry?

Charge symmetry (CS) is a specific rotation in isospin space. It is the invariance with respect to rotation of  $\pi$  about the T2 axis.

$$P_{CS} = \exp(i\pi T2)$$

$$\begin{aligned} P_{CS} |d\rangle &= |u\rangle \\ P_{CS} |u\rangle &= -|d\rangle \end{aligned}$$

### Low Energy: CS in nuclei

CS operator interchanges neutrons and protons

- CS goes back to the charge independence of N force.
- pp and nn scattering lengths are nearly the same
- $M_n \simeq M_p$
- $B(n, {}^3He) \simeq B(p, {}^3H)$  and energy levels in other mirror nuclei are equal (to 1%)
- $m({}^3He) \simeq m({}^3H)$

After electromagnetic corrections CS respected down to  $\sim 1\%$

### QCD: Quark level

- $u^p(x, Q^2) = d^n(x, Q^2)$   
 $d^p(x, Q^2) = u^n(x, Q^2)$
- Origin of CS violations:
  - Electromagnetic interaction
  - $\delta m = m_d - m_u$

Naively, one would expect CSV would be on the order of  $(m_d - m_u)/\langle M \rangle$ , where  $\langle M \rangle$  is roughly 0.5 – 1.0 GeV

→ CSV effect about 1%

# Motivation

- **Charge symmetry violation** is an important ingredient for pushing the **precision frontier in the partonic structure of the hadrons**
- Charge symmetry is often assumed in extracting PDFs from data – where the data is limited in sensitivity to CS violation
- The validity of charge symmetry is a necessary condition for many relations between structure functions and sum rules
- CSV in fragmentation functions is critical component of hadronization
- Important contribution for reducing uncertainty of  $V_{ud}$  from neutron  $\beta$  decay. (Crawford and Miller, PRC 106, 065502 (2022))

# Upper Limits on CSV

## Theoretical Limits

### Charge Symmetry Violation

$$CSV(x) = \delta d - \delta u \neq 0$$

where

$$\begin{aligned}\delta u(x) &= u^p(x) - d^n(x) \\ \delta d(x) &= d^p(x) - u^n(x)\end{aligned}$$

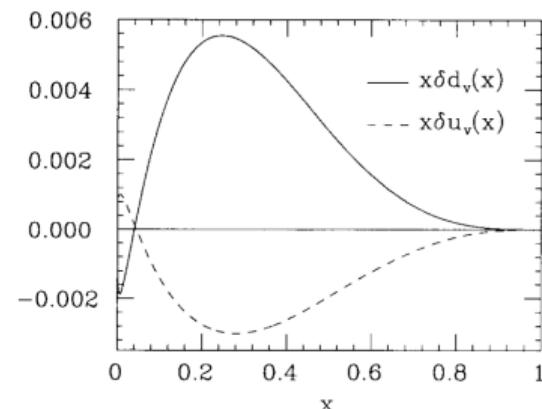
Model by Sather:

$$\delta d(x) \sim 2 - 3\%, \delta u(x) \sim 1\%$$

$$\delta d_v(x) = -\frac{\delta M}{M} \frac{d}{dx} [x d_v(x)] - \frac{\delta m}{M} \frac{d}{dx} d_v(x)$$

$$\delta u_v(x) = \frac{\delta M}{M} \left( -\frac{d}{dx} [x u_v(x)] + \frac{d}{dx} u_v(x) \right)$$

where  $\delta M = 1.3\text{MeV}$  is the n-p mass difference,  
and  $\delta m = m_d - m_u \sim 4\text{MeV}$  is the down-up  
quark mass difference. E. Sather, Phys. Lett. B274, 433  
(1992)



Model by Rodionov, Thomas and Londergan  $\delta d(x)$  could reach up to 10% at high  $x$

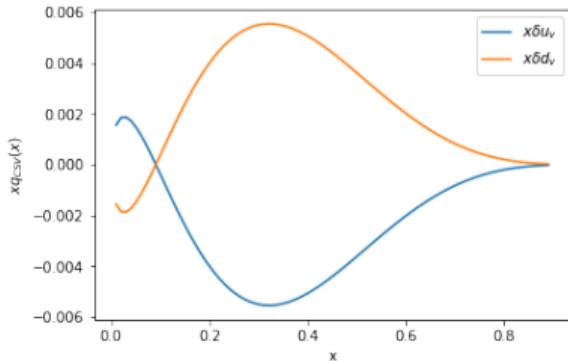
E. N. Rodionov, A. W. Thomas and J. T. Londergan, Mod. Phys. Lett. A 9, 1799 (1994)

# Upper Limits on CSV

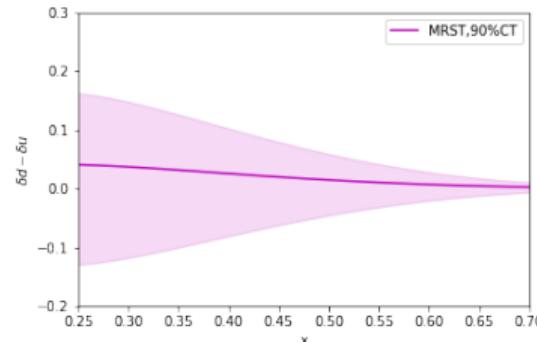
Phenomenological limits

MRST included CSV in a phenomenological evaluation of PDFs

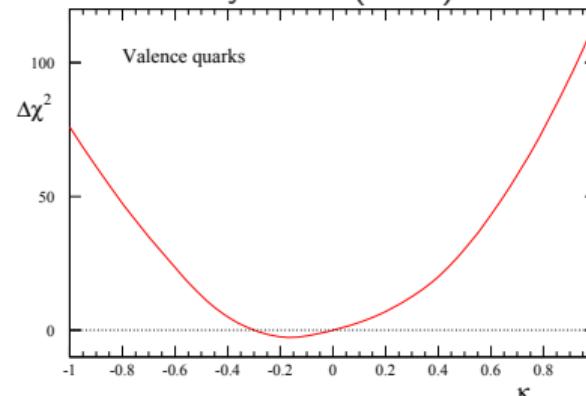
$$\delta u_v(x) = -\delta d_v(x) = \kappa f(x)$$
$$f(x) = (1-x)^4 x^{-0.5} (x - 0.0909)$$



Using the uncertainties in PDFs studied by MRST Group, CSV is constrained to less than 9%



Eur. Phys. J. 35(2004)325



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# Upper Limits on CSV

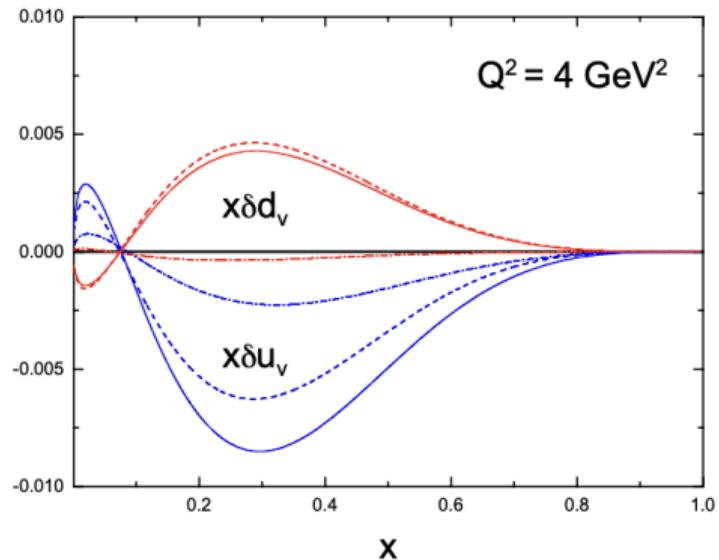
## Lattice QCD

The charge symmetry violation via lattice simulation:

$$\delta U = \int_0^1 dx x \delta u(x) = 0.0023(7)$$

$$\delta D = \int_0^1 dx x \delta d(x) = 0.0017(4)$$

The dash-dotted, dashed and solid curves represent pure QED, pure QCD and the total contributions. The results is compatible with the MRST analysis. Physics Letters B, 753:595–599



# Upper Limits on CSV

## Experimental Limits

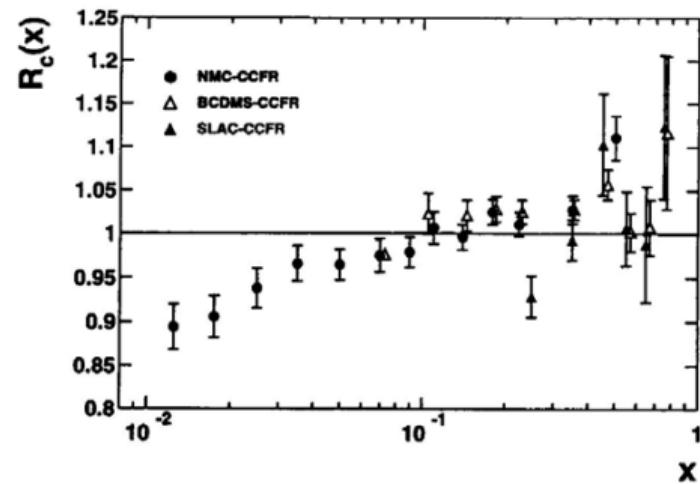
- Upper limit obtained by combining neutral and charged current data on isoscaler targets
- $F_{2\nu}$  by CCFR collaboration at FNAL (Fe data)
- $F_{2\gamma}$  by NMC collaboration using muons (D target)
- $0.1 \leq x \leq 0.4 \rightarrow 9\% \text{ upper limit for CSV effect!}$

## "Charge Ratio"

$$R_c(x) = \frac{F_2^\gamma(x) + x [s(x) + \bar{s}(x) - c(x) - \bar{c}(x)] / 6}{5\bar{F}_2^{W(x)} / 18}$$

$$\simeq 1 + \frac{3 (\delta u(x) + \delta \bar{u}(x) - \delta d(x) - \delta \bar{d}(x))}{10 \bar{Q}(x)}$$

$$\bar{Q}(x) = \sum_{u,d,s} (q(x) + \bar{q}(x))$$



Lonergan and Thomas. Prog. Part. Nucl. Phys. 41 (1998)  
49-124



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# SIDIS Formalism

## For Isoscaler Targets

### Charge Symmetry Violation

In the PDFs:

$$\delta d(x) = d^p(x) - u^n(x), \delta u(x) = u^p(x) - d^n(x). \\ CSV(x) = \delta d - \delta u$$

In Fragmentation Functions

$$\delta D(z) = \frac{D_u^{\pi^+} - D_d^{\pi^-}}{D_u^{\pi^+}}$$

Leading order methodology for iso-scaler targets (Londergan, Pang, and Thomas PRD54(1996)3154)

$$R_{meas}^D(x, z) = \frac{N^{D\pi^-}(x, z) - N^{D\pi^+}(x, z)}{N^{D\pi^+}(x, z) - N^{D\pi^-}(x, z)} = \frac{4R_Y(x, z) - 1}{1 - R_Y(x, z)} \quad (1)$$

where  $N^{D\pi^\pm}(x, z)$  is the **measured yield** of  $\pi^\pm$  electroproduction on a deuterium target,  
 $R_Y = N^{D\pi^-}/N^{D\pi^+}$  is the yield ratio, and we rely the following:

### Factorization

$$N^{Nh} \propto \sum_i e_i^2 q_i^N(x) D_i^h(z)$$

### Impulse Approximation

$$N^{D\pi^\pm}(x, z) = N^{p\pi^\pm}(x, z) + N^{n\pi^\pm}(x, z)$$

# Formalism

Leading order experimental analysis → will need higher order global analysis

Londergan, Pang and Thomas PRD54(1996)3154

$$D(z) R(x, z) + A(x) \text{CSV}(x) = B(x, z)$$

$$D(z) = \frac{1 - \Delta(z)}{1 + \Delta(z)}, \Delta(z) = \frac{D_u^{\pi^-}(z)}{D_u^{\pi^+}(z)}$$

$$R(x, z) = \frac{5}{2} + R_{meas}^D$$

$$\text{CSV}(x) = \delta d - \delta u$$

$$A(x) = \frac{-4}{3(u_v + d_v)}$$

$$B(x, z) = \frac{5}{2} + R_{sea-S}^D(x, z) + R_{sea-NS}^D(x)$$

$$R_{sea-NS}^D(x) = \frac{5(\bar{u}^p(x) + \bar{d}^p(x))}{[u_v^p(x) + d_v^p(x)]}$$

$$R_{sea-S}^D(x, z) = \frac{\Delta_s(z)[s(x) + \bar{s}(x)]/(1 + \Delta(z))}{[u_v^p(x) + d_v^p(x)]}$$

$$\Delta_s(z) = \frac{D_s^-(z) + D_s^+(z)}{D_u^+(z)}$$

$A(x)$  and  $B(x, z)$  are known and  $R(x, z)$  is measured

## CSV

Extract simultaneously  $D(z)$  and  $\text{CSV}(x)$  from each  $(Q^2, x)$  setting

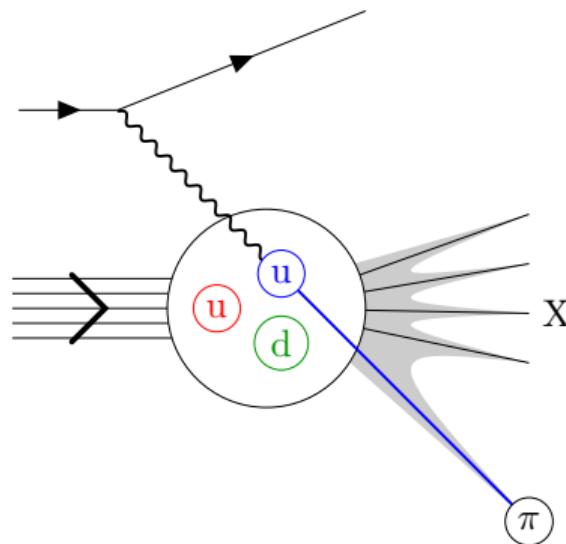
# Hall C Experiment E12-09-002

Charge Symmetry Violating Quark Distributions via Precise Measurement of  $\pi^+/\pi^-$  Ratios in Semi-inclusive Deep Inelastic Scattering.

Spokespersons: D.Dutta, D.Gaskell, K.Hafidi, W.A

Students: S.Jia H. Bhatt

- 10.6 GeV electron beam
- $\text{LH}_2$  and  $\text{LD}_2$  targets (10 cm),  
Al-dummy
- Ran in Fall 2018 and Spring 2019



# Precise measurements of charged pion ratio off deuterium

Charged pion acceptance should be independent of pion charge

For each charge setting ( $\pi^+$  and  $\pi^-$ ) keep the rates in the hadron arm the same

- Reduce the systematic error related to rate dependent effects (tracking efficiency...)
- 50  $\mu\text{A}$  (25  $\mu\text{A}$ ) beam current for negative (positive) polarity



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## HMS (electrons)

$$4.5 \leq p \leq 6.8 \text{ GeV/c}$$
$$12.5^\circ \leq \theta_e \leq 20.2^\circ$$

- $\pi$  - rate  $\simeq 10\text{s kHz}$
- Lead glass Calo: 99%  $e$  detection eff.
- 200:1  $\pi$  rejection
- Heavy gas Č @ 1 atm to further reduce  $\pi$  background

## SHMS (hadrons)

$$1.7 \leq p \leq 4.6 \text{ GeV/c}$$
$$10.7^\circ \leq \theta_\pi \leq 20^\circ$$

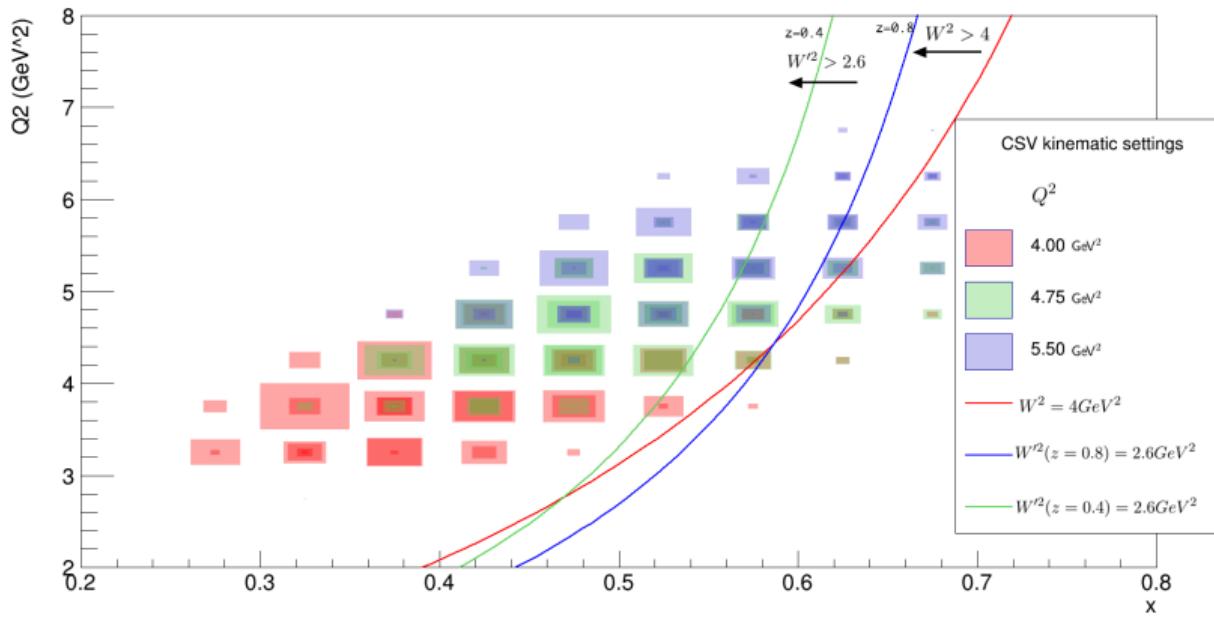
- Gas Č @ 0.96 atm  
 $p_{\text{threshold}}(\pi) = 2.65 \text{ GeV/c}$   
 $p_{\text{threshold}}(K) = 9.4 \text{ GeV/c}$
- Aerogel ( $n = 1.015$ )  
 $p_{\text{threshold}}(\pi) = 0.8 \text{ GeV/c}$   
 $p_{\text{threshold}}(K) = 2.85 \text{ GeV/c}$



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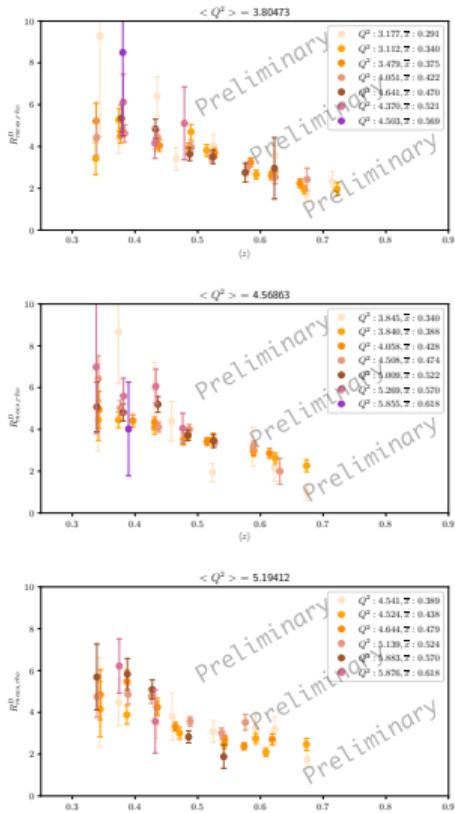
# Hall C Experiment E12-09-002

## Kinematic Coverage



$$W'^2 = M^2 + Q^2(1 - z)(1/x - 1)$$

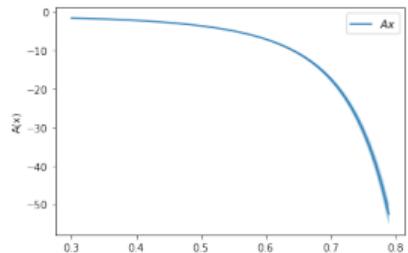
# Preliminary $R_{meas}^D$



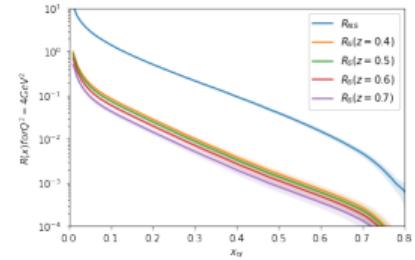
$$D(z) \left[ \frac{5}{2} + R_{meas}^D(x, z) \right] + A(x) CSV(x) = B(x, z)$$

$$\leftarrow R_{meas}^D(x, z) = \frac{4N^{D\pi^-}(x, z) - N^{D\pi^+}(x, z)}{N^{D\pi^+}(x, z) - N^{D\pi^-}(x, z)}$$

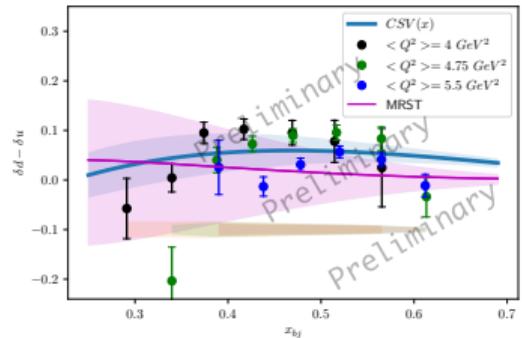
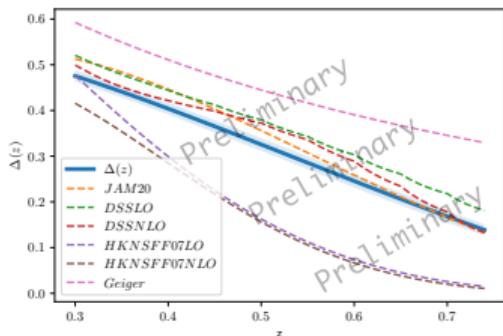
Model inputs:



$$A(x) = \frac{-4}{3(u_v + d_v)}$$



$$B(x, z) = \frac{5}{2} + R_{sea-S}^D(x, z) + R_{sea-NS}^D(x, z)$$



From Shuo Jia



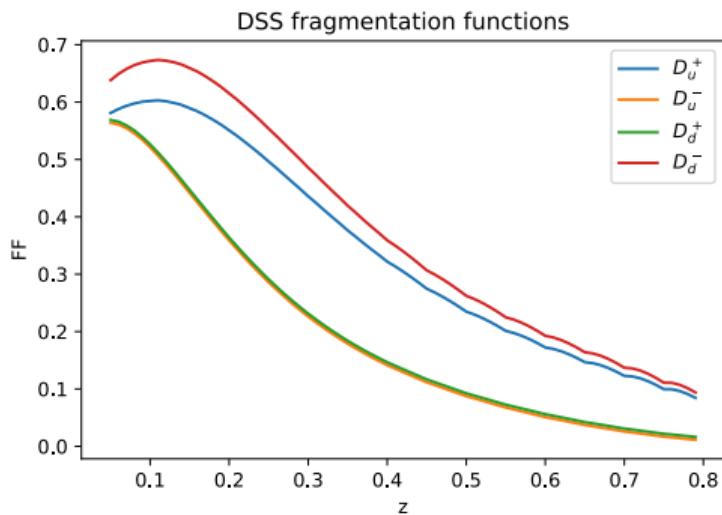
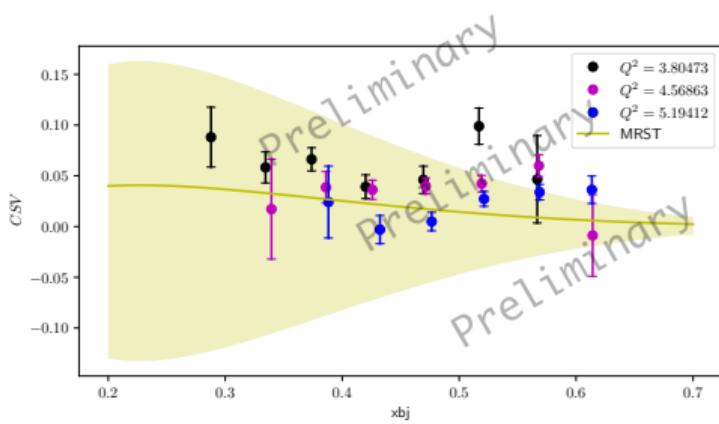
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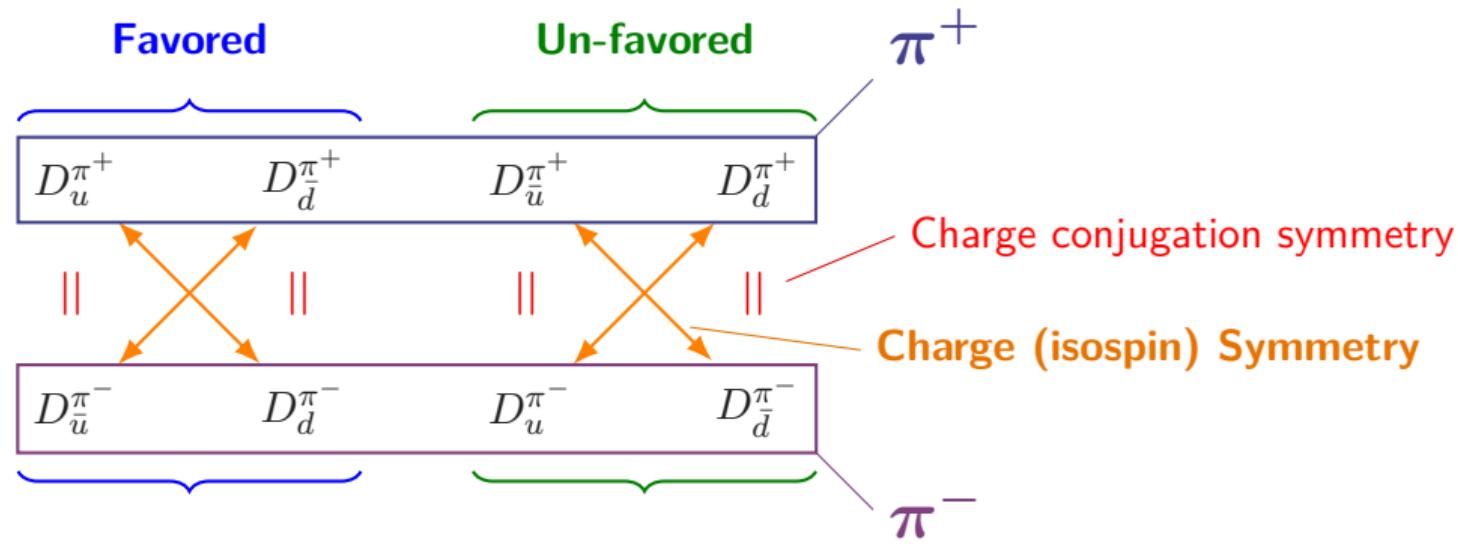
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# CSV in Parton Distribution and Fragmentation Functions



- Preliminary results hint at CSV contributions at higher  $x$ .
- CSV contributions from fragmentation have been ignored.
- Simple LO analysis insufficient for definitive statement on CSV. Need global analysis.

# Charged Pion Fragmentation Functions



$$A_{\text{quark}}^{\text{Fav}} = \frac{D_u^{\pi^+} - D_d^{\pi^-}}{D_u^{\pi^+} + D_d^{\pi^-}}$$

$$A_{\text{anti-quark}}^{\text{Fav}} = \frac{D_{\bar{u}}^{\pi^-} - D_{\bar{d}}^{\pi^+}}{D_{\bar{u}}^{\pi^-} + D_{\bar{d}}^{\pi^+}}$$

$$A_{\text{quark}}^{\text{Un-fav}} = \frac{D_u^{\pi^-} - D_d^{\pi^+}}{D_u^{\pi^-} + D_d^{\pi^+}}$$

$$A_{\text{anti-quark}}^{\text{Un-fav}} = \frac{D_{\bar{u}}^{\pi^+} - D_{\bar{d}}^{\pi^-}}{D_{\bar{u}}^{\pi^+} + D_{\bar{d}}^{\pi^-}}$$

Charge Conj. Symmetry

$$A_{\text{quark}}^{\text{Fav}} = A_{\text{anti-quark}}^{\text{Fav}}$$

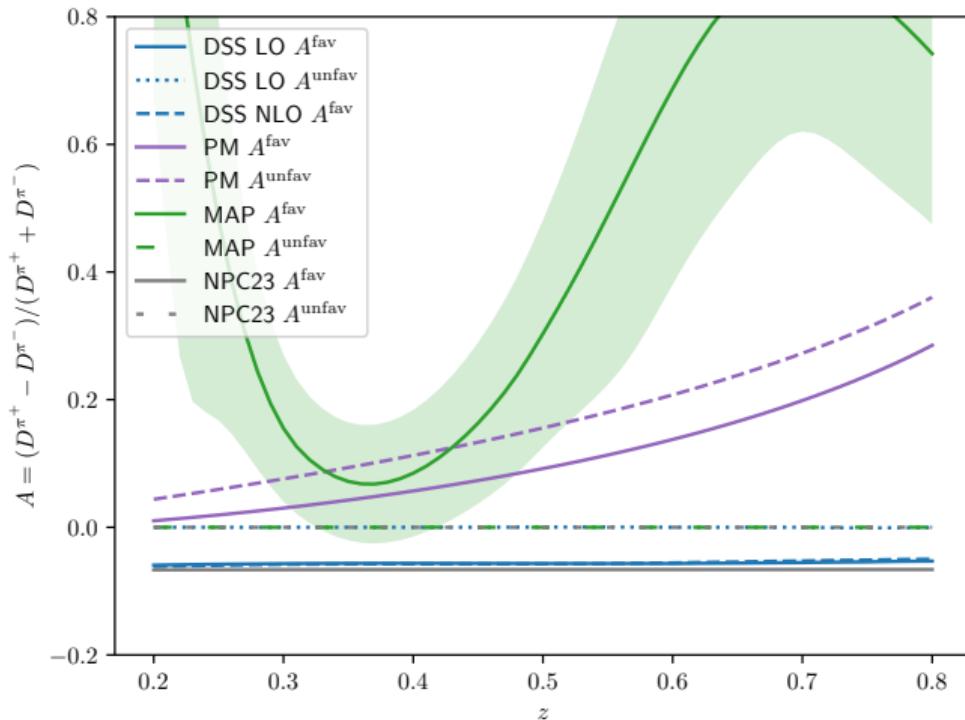
$$A_{\text{quark}}^{\text{Un-fav}} = A_{\text{anti-quark}}^{\text{Un-fav}}$$

Charge Symmetry

$$A_{\text{quark}}^{\text{Fav}} = 0$$

$$A_{\text{quark}}^{\text{Un-fav}} = 0$$

# FF Asymmetries



$$A^{\text{Fav}} = \frac{D_u^{\pi^+} - D_d^{\pi^-}}{D_u^{\pi^+} + D_d^{\pi^-}}$$
$$A^{\text{Unfav}} = \frac{D_u^{\pi^-} - D_d^{\pi^+}}{D_u^{\pi^-} + D_d^{\pi^+}}$$

- Wide variety of assumptions/fit results
- Ma and Peng only parameterization to include FF CSV in the *unfavored sectors*
- **Question for theorists: In which sector should the CSV Asymmetry be larger? Favored or Unfavored?**
- Is  $A^{\text{Fav}} > A^{\text{unfav}}$  or vice-versa? For all  $z$ ?

# Flavor Dependence of Charged Pion Fragmentation Functions

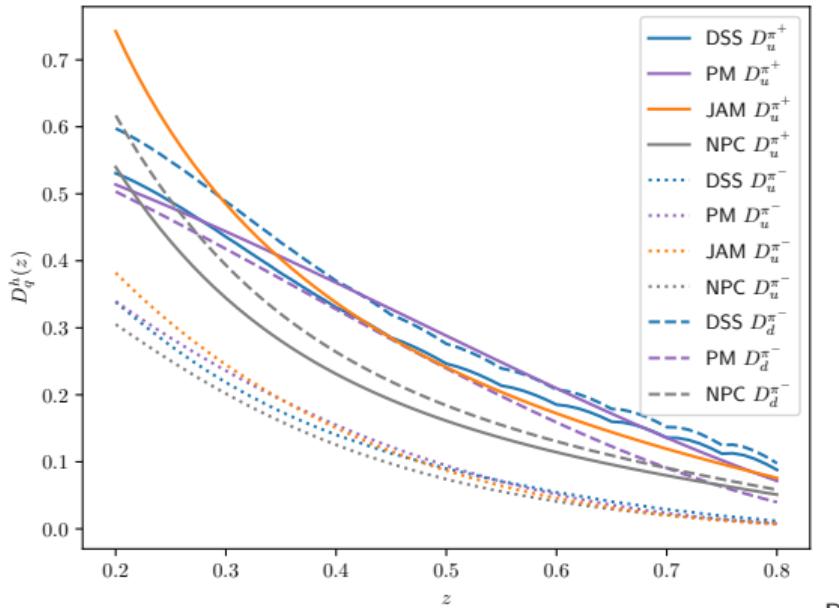
Combined analysis of CSV and Pt-SIDIS

E12-09-017 Spokespersons: H.Mkrtchyan, P.Bosted, R.Ent, E.Kinney

## Flavor Dependence of Charged Pion Fragmentation Functions

H. Bhatt,<sup>1</sup> P. Bosted,<sup>2</sup> S. Jia,<sup>3</sup> W. Armstrong,<sup>4</sup> D. Dutta,<sup>1</sup> R. Ent,<sup>5</sup> D. Gaskell,<sup>5</sup> E. Kinney,<sup>6</sup> H. Mkrtchyan,<sup>7</sup> S. Ali,<sup>8</sup> R. Ambrose,<sup>9</sup> D. Androic,<sup>10</sup> C. Ayerbe Gayoso,<sup>1</sup> A. Bandari,<sup>2</sup> V. Berdnikov,<sup>8</sup> D. Bhetuwal,<sup>1</sup> D. Biswas,<sup>11</sup> M. Boer,<sup>3</sup> E. Brash,<sup>12</sup> A. Camsonne,<sup>5</sup> M. Cardona,<sup>3</sup> J. P. Chen,<sup>5</sup> J. Chen,<sup>2</sup> M. Chen,<sup>13</sup> E. M. Christy,<sup>11</sup> S. Covrig,<sup>5</sup> S. Danagolian,<sup>14</sup> M. Dieffenthaler,<sup>5</sup> B. Duran,<sup>3</sup> M. Elaasar,<sup>15</sup> C. Elliot,<sup>16</sup> H. Fenker,<sup>5</sup> E. Fuchey,<sup>17</sup> J. O. Hansen,<sup>5</sup> F. Hauenstein,<sup>18</sup> T. Horn,<sup>8</sup> G. M. Huber,<sup>9</sup> M. K. Jones,<sup>5</sup> M. L. Kabir,<sup>1</sup> A. Karki,<sup>19</sup> S. J. D. Kay,<sup>9,20</sup> C. Keppel,<sup>5</sup> V. Kumar,<sup>9</sup> N. Leshley-Colthirst,<sup>11</sup> W. B. Li,<sup>2</sup> D. Mack,<sup>5</sup> S. Malace,<sup>5</sup> P. Markowitz,<sup>21</sup> M. McCaughan,<sup>5</sup> E. McClellan,<sup>5</sup> D. Meekins,<sup>5</sup> R. Michaels,<sup>5</sup> A. Mkrtchyan,<sup>7</sup> G. Niculescu,<sup>22</sup> I. Niculescu,<sup>22</sup> B. Pandey,<sup>11,23</sup> S. Park,<sup>24</sup> E. Pooser,<sup>5</sup> B. Sawatzky,<sup>5</sup> G. R. Smith,<sup>5</sup> H. Szumila-Vance,<sup>5,21</sup> A. S. Tadepalli,<sup>5</sup> V. Tadevosyan,<sup>7</sup> R. Trotta,<sup>8</sup> H. Voskanyan,<sup>7</sup> S. A. Wood,<sup>5</sup> Z. Ye,<sup>4,25</sup> C. Yero,<sup>21</sup> and X. Zheng<sup>13</sup>  
(for the Hall C SIDIS Collaboration)

- The favored charged pion fragmentation functions are charge symmetric within 10%.
- The unfavored fragmentation functions also consistent with charge symmetry holding but with a very large uncertainty.
- The latter is due to low statistics from down quark scattering from the neutron.



and Ma, Phys.Rev.D 107 (2023) 1, 014014

DSS Phys.Rev.D 75 (2007)

HKNS Phys.Rev.D 75 (2007) 094009

MAP Phys.Rev.D 104 (2021) 3, 034007

# Factorization

Berger's criterion:  $\Delta\eta \gtrsim 2$

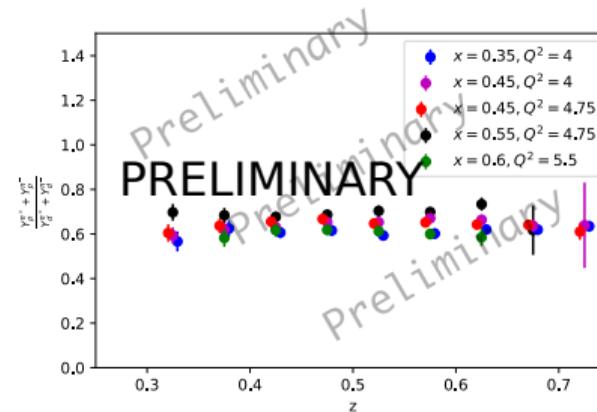
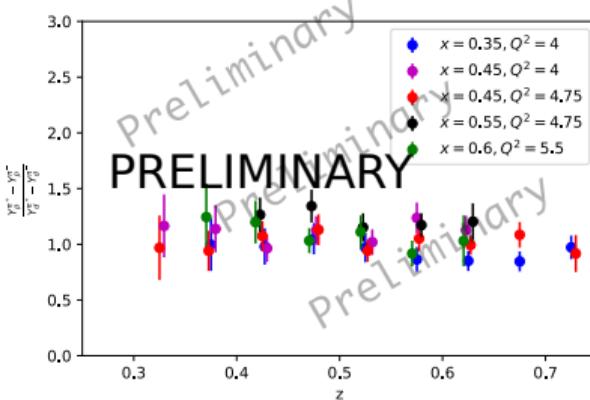
| Sets $z_{min}$ for a given $W_{max}$ (for pions) |            |        |        |
|--|------------|--------|--------|
|  | JLab 6 GeV | 11 GeV | 22 GeV |
| $z_{min}^{\pi^{in}} \rightarrow$                 | 0.29       | 0.22   | 0.16   |
| $z_{min}^R \rightarrow$                          | N/A        | 0.79   | 0.56   |
| HERMES   |            |        | 0.14   |
|  |            |        | 0.50   |

See Chapter 8 from S.J. Joosten, Ph.D. thesis, Illinois Univ., Urbana (2013).  
Mulders AIP Conf. Proc. 588 (2001) 1, 75-88

## Charge Ratio Sum and Differences

$$\frac{\sigma_p^{\pi^+} - \sigma_p^{\pi^-}}{\sigma_d^{\pi^+} - \sigma_d^{\pi^-}} = \frac{4u_v(x) - d_v(x)}{3(u_v(x) + d_v(x))} = R^-$$

$$\frac{d_v}{u_v} = \frac{4 - 3R^-}{3R^- + 1}$$



Ratios should not depend on  $z$ .

$$\frac{\sigma_p^{\pi^+} + \sigma_p^{\pi^-}}{\sigma_d^{\pi^+} + \sigma_d^{\pi^-}} = \frac{4u + 4\bar{u} + d + \bar{d}}{5(u + \bar{u} + d + \bar{d})}$$

# Factorization

Berger's criterion:  $\Delta\eta \gtrsim 2$

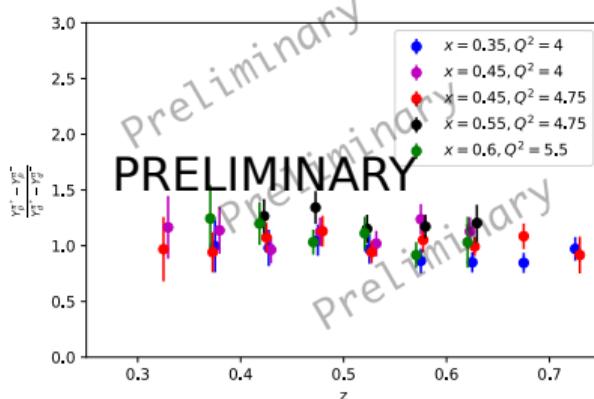
| Sets $z_{min}$ for a given $W_{max}$ (for pions) | JLab 6 GeV | 11 GeV | 22 GeV | HERMES |
|--|------------|--------|--------|--------|
| $z_{min}^\pi \rightarrow$                        | 0.29       | 0.22   | 0.16   | 0.14   |
| $z_{min}^R \rightarrow$                          | N/A        | 0.79   | 0.56   | 0.50   |

See Chapter 8 from S.J. Joosten, Ph.D. thesis, Illinois Univ., Urbana (2013).  
Mulders AIP Conf. Proc. 588 (2001) 1, 75-88

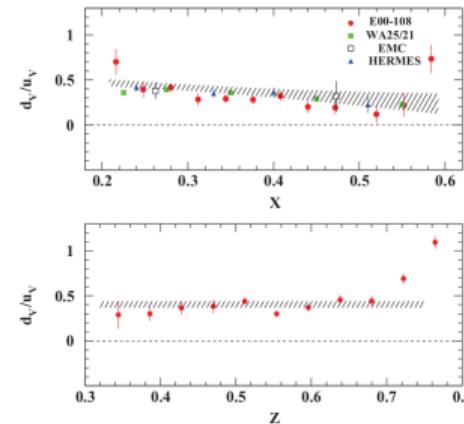
## Charge Ratio Sum and Differences

$$\frac{\sigma_p^{\pi^+} - \sigma_p^{\pi^-}}{\sigma_d^{\pi^+} - \sigma_d^{\pi^-}} = \frac{4u_v(x) - d_v(x)}{3(u_v() + d_v(x))} = R^-$$

$$\frac{d_v}{u_v} = \frac{4 - 3R^-}{3R^- + 1}$$



Ratios should not depend on  $z$ .



JLab E00-108: PRC 85, 015202 (2012)

# Future Possibilities

- CLAS12's large acceptance with positrons to make precision ratio measurements in Hall B.
  - swap clas12 magnet polarities with beam charge to keep acceptances the same between hadron and electron polarities.
  - Directly measure the  $\rho$  background → reduced systematic uncertainty.
- Measurements with SOLID – same idea but now luminosity limited by positron beam.
- Use helium isoscaler target with recoil tagging: Better access to unfavored FFs
  - detect spectator  ${}^3\text{He}$  to identify scattering on neutron.
  - use mirror process to identify scattering on proton and cut down on systematics.
  - Promising new technology: DOE Early Career Award to develop novel LHe Active target
- Explore CSV in the Polarized sector – CSV corrections to Bjorken Sum Rule (Cloët, et.al PLB 714 (2012))
  - Play same game with polarities to ensure bin acceptance the same and leverage NH3 and ND3 targets.

# Summary

- Conducted precision semi-inclusive measurements of the  $\pi^-/\pi^+$  ratio on a deuterium target
- Results for the CSV parton distribution are consistent with previous estimates
- Positrons at JLab present new opportunities to improve systematics with large acceptance detectors.



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Thank you!



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# Backups



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# Charge Symmetry in QPM

Charge-conjugation symmetry

$$D_{\bar{u}}^{\pi^\pm} = D_{\bar{u}}^{\pi^\mp}$$

Charge Symmetry

$$\begin{aligned} D_u^{\pi^+} &= D_d^{\pi^-} & D_{\bar{u}}^{\pi^+} &= D_{\bar{d}}^{\pi^-} \\ D_d^{\pi^+} &= D_u^{\pi^-} & D_{\bar{d}}^{\pi^+} &= D_{\bar{u}}^{\pi^-} \end{aligned}$$

Gottfried Sum Rule

$$\begin{aligned} S_G &= \int_0^1 dx \left[ \frac{F_2^p - F_2^n}{x} \right] \\ &= \frac{1}{3} + \frac{2}{9} \int_0^1 dx \left[ 4\bar{u}^p + \bar{d}^p - 4\bar{u}^n - \bar{d}^n \right] \\ &\stackrel{\text{CS}}{=} \frac{1}{3} + \frac{2}{3} \int_0^1 dx \left[ \bar{u}^p - \bar{d}^p \right] \end{aligned}$$

Lonergan and Thomas. Prog. Part. Nucl. Phys. 41 (1998) 49-124

Bjorken Sum Rule (Cloët, et.al PLB 714 (2012))

$$\begin{aligned} \int (g_1^p - g_1^n) dx &= \frac{G_A}{6G_V} \left[ 1 - \alpha_s(Q^2)/\pi \right] \\ &= \int \left[ \frac{\Delta u^+ - \Delta d^+}{6} + \frac{4\delta\Delta d^+ + \delta\Delta u^+}{18} \right] \end{aligned}$$