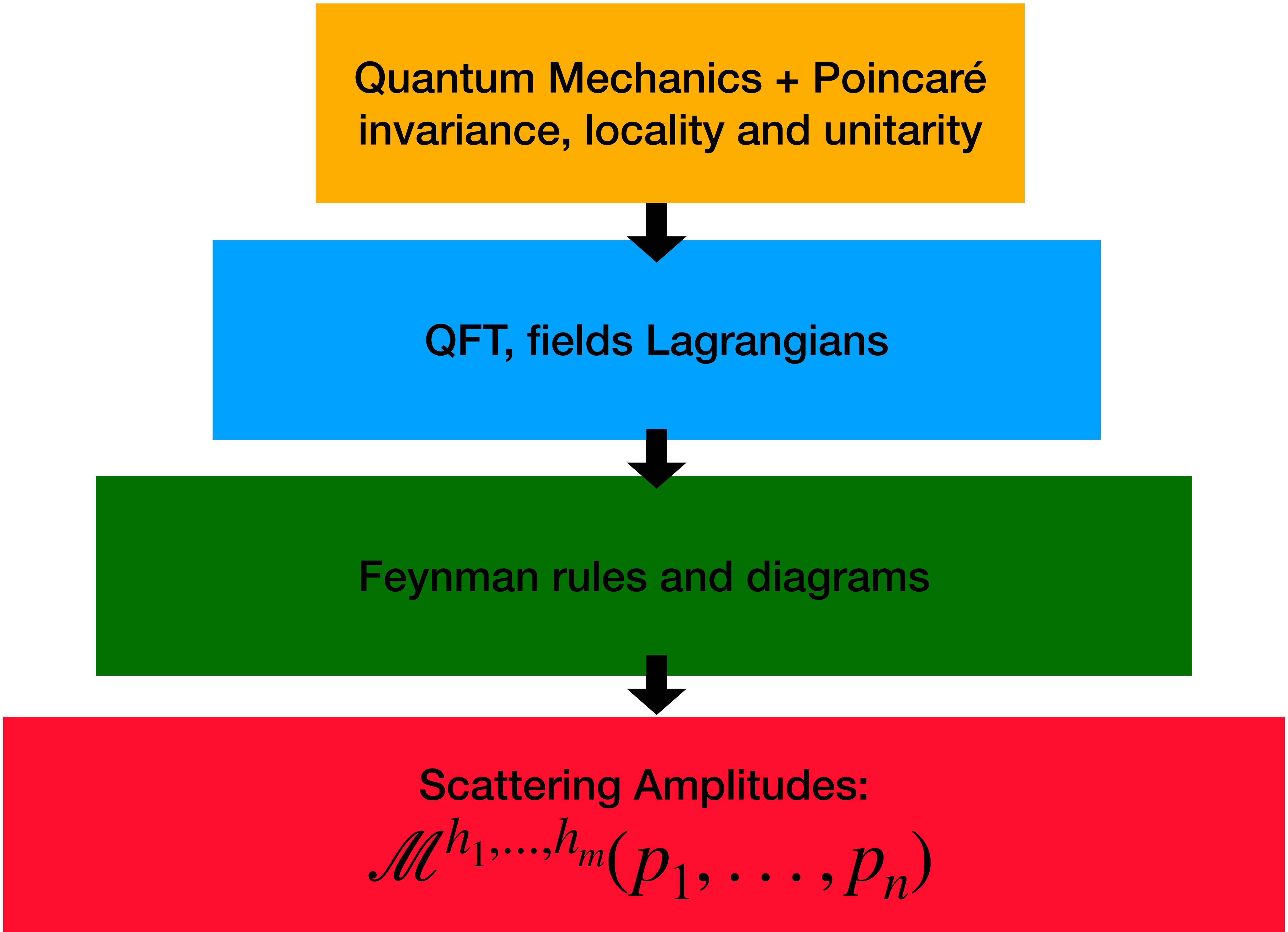
Adam Falkowski

A new perspective on amplitudes calculations and applications

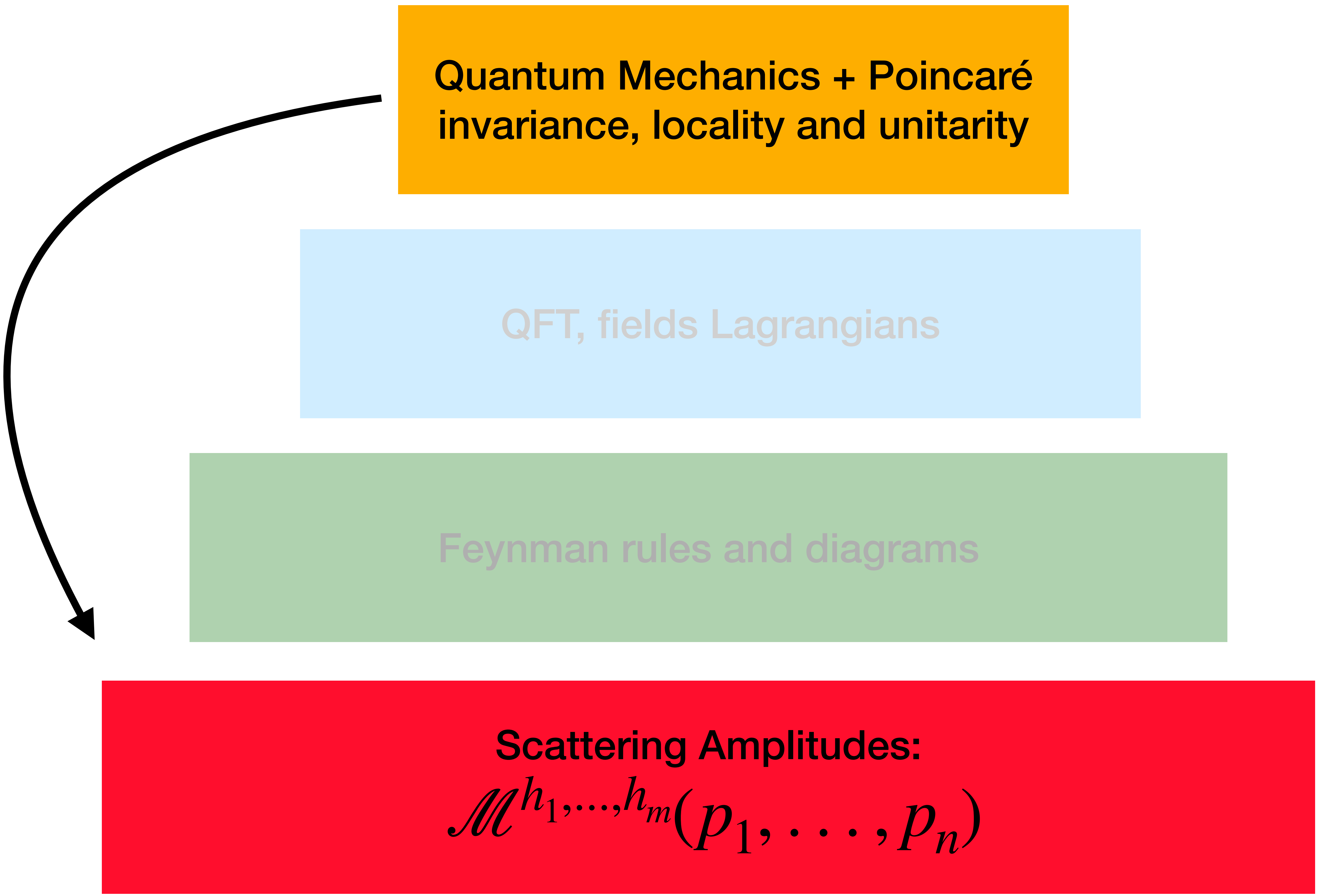
Discussion at AstroParticle Symposium 2024

26 November 2024

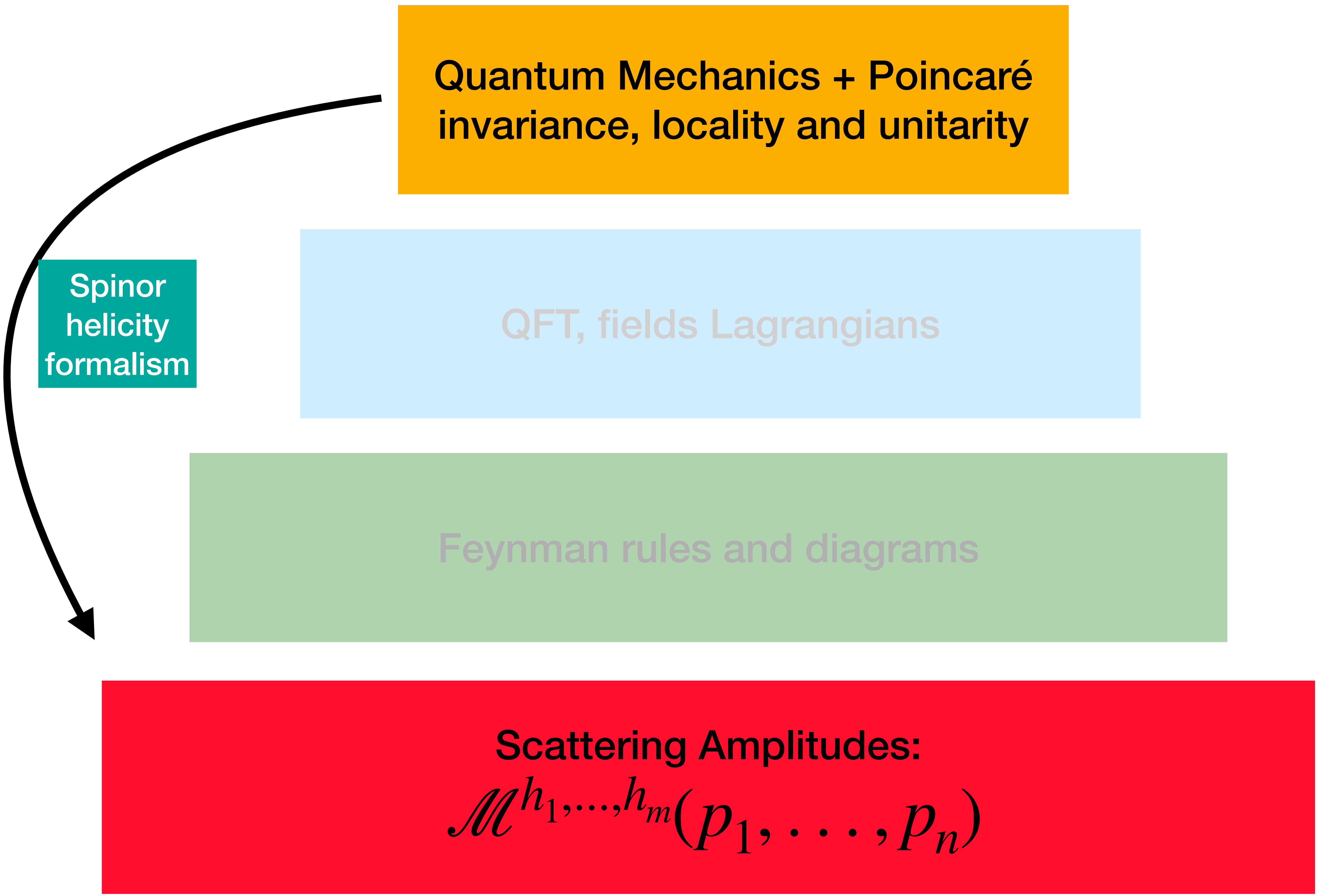
Usual (flat-space) QFT path



On-shell way


Quantum Mechanics + Poincaré invariance, locality and unitarity
QFT, fields Lagrangians
Feynman rules and diagrams
Scattering Amplitudes:
$\mathcal{M}^{h_1,\ldots,h_m}(p_1,\ldots,p_n)$

On-shell way



Quantum Mechanics + Poincaré invariance, locality and unitarity
Spinor helicity formalism
QFT, fields Lagrangians
Feynman rules and diagrams
Scattering Amplitudes:
$\mathscr{M}^{h_1,\ldots,h_m}(p_1,\ldots,p_n)$

Spinor
Helicity
Formalism

Spinor helicity formalism

Consider massless momentum p^μ, such that $p^2 = 0$

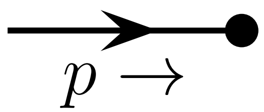
$$p\sigma \equiv p^\mu \sigma_\mu = \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix} \qquad \sigma^\mu = (1, \boldsymbol{\sigma})$$

$$\det[p\sigma] = 0 \quad \Rightarrow \quad [p\sigma]_{\alpha\dot{\beta}} = \lambda_\alpha \tilde{\lambda}_{\dot{\beta}} \qquad \alpha, \dot{\beta} \in 1,2$$

Helicity spinors associated with momentum p^μ

Spinor helicity formalism

If you think you've never seen helicity spinors...



$= u_{\mathsf{A}}(p, s) \qquad A = 1 \ldots 4$

Here u_A is a 4-component Dirac spinor describing polarization of an external fermion entering a Feynman diagram

In the helicity basis

$$u_A(p, -) = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix} \qquad u_A(p, +) = \begin{pmatrix} 0 \\ \tilde{\lambda}^{\dot{\beta}} \end{pmatrix}$$

For incoming real momenta, $E > 0$

$p^\mu = E(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$\tilde{\lambda}_{\dot{\alpha}} = \sqrt{2E}\begin{pmatrix} -e^{i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \quad \lambda_\alpha = \sqrt{2E}\begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

Spinor helicity formalism

$$p_i \sigma = \lambda_i \tilde{\lambda}_i$$

The decomposition of momentum into spinors is not unique because the transformation

$$\lambda_i \to z_i^{-1} \lambda_i \qquad \tilde{\lambda}_i \to z_i \tilde{\lambda}_i$$

does not change the momentum p_i^μ

Transformation corresponds to the little group associated with momentum p_i^μ

S-matrix elements or amplitude transform under the little group

$$\mathscr{M}[1^{h_1} 2^{h_2} \ldots n^{h_n}] \to z_1^{2h_1} z_2^{2h_2} \ldots z_n^{2h_n} \mathscr{M}[1^{h_1} 2^{h_2} \ldots n^{h_n}]$$

Weights of amplitude under little group transformations depend on particles' helicities

Spinor helicity formalism

$$p_1\sigma = \lambda_1\tilde{\lambda}_1 \qquad p_2\sigma = \lambda_2\tilde{\lambda}_2$$

Lorentz algebra is equivalent to $SU(2) \times SU(2)$

Spinors $\lambda_{i\alpha}$ transform in the spinor representation under the first SU(2) factor

Spinors $\tilde{\lambda}_{i\alpha}$ transform in the spinor representation under the second SU(2) factor

Lorentz invariants:

$$\langle 12 \rangle \equiv \epsilon^{\alpha\beta}\lambda_{1\beta}\lambda_{2\alpha} \equiv \lambda_1^\alpha \lambda_{2\alpha}$$

$$[12] \equiv \epsilon^{\dot\alpha\dot\beta}\tilde{\lambda}_{1\dot\alpha}\tilde{\lambda}_{2\dot\beta} \equiv \tilde{\lambda}_{1\dot\alpha}\tilde{\lambda}_2^{\dot\alpha}$$

But contracting twiddled and untwiddled spinors is illegal ! $\lambda_1^\alpha \tilde{\lambda}_{2\dot\alpha}$ ✗

Note also that $\langle ij \rangle = -\langle ji \rangle$ $[ij] = -[ji]$ which implies $\langle jj \rangle = [jj] = 0$

Momentum contractions can be traded for spinor contractions

$$2p_1p_2 = \mathrm{Tr}[p_1\sigma p_2\bar\sigma] = \lambda_{1\,\alpha}\tilde{\lambda}_{1\,\dot\beta}\tilde{\lambda}_2^{\dot\beta}\lambda_2^\alpha = (\lambda_2\lambda_1)(\tilde{\lambda}_1\tilde{\lambda}_2) \equiv \langle 21 \rangle [12]$$

$\bar\sigma^\mu = (1, -\boldsymbol{\sigma})$ $[p\bar\sigma]^{\dot\beta\alpha} = \tilde{\lambda}^{\dot\beta}\lambda^\alpha$ $\lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta$

Spinor helicity formalism

Consider massive momentum p^μ, such that $p^2 = m^2$

Massive spinors χ^J and $\tilde{\chi}^J$ are defined by the decomposition

$$p\sigma = \sum_{J=1}^{2} \chi^J \tilde{\chi}_J \qquad \chi^J \chi_K = m\delta^J_K \qquad \tilde{\chi}_J \tilde{\chi}^K = m\delta^K_J$$

Massive spinor transform under SU(2) little gauge group

$$\chi^J \to \Omega^J{}_K \chi^K \quad \tilde{\chi}^J \to \Omega^J{}_K \tilde{\chi}^K \qquad \chi_J \to \Omega^{\dagger K}_J \chi_K \quad \tilde{\chi}_J \to \Omega^{\dagger K}_J \tilde{\chi}_K$$

Massive Spin-S particle

$$\mathscr{M}[1^{J_1 \ldots J_{2S}} \ldots] \to \Omega^{J_1}_{K_1} \ldots \Omega^{J_{2S}}_{K_{2S}} \mathscr{M}[1^{K_1 \ldots K_{2S}} \ldots]$$

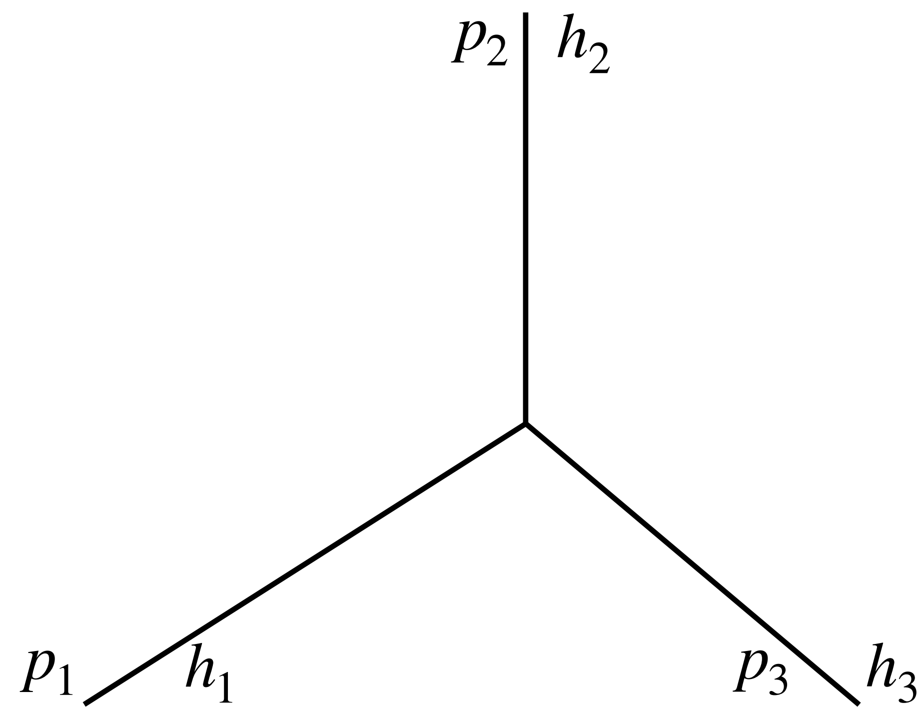
Little group indices fully symmetrized

On-shell

3-point

Amplitudes

On-shell 3-point kinematics



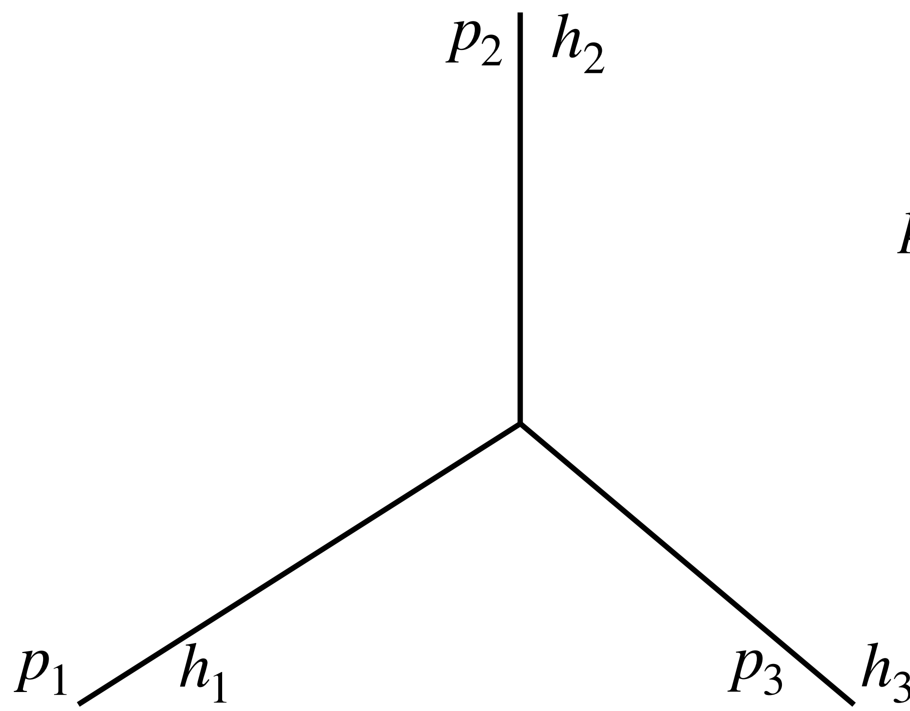
Assuming all particles are on-shell, incoming and massless

$$p_1 + p_2 + p_3 = 0 \qquad p_1^2 = p_2^2 = p_3^2 = 0$$

It follows that all Mandelstam invariant vanish in this case

$$2p_i p_j = (p_i + p_j)^2 = p_k^2 = 0 \qquad i, j, k \in 1 \ldots 3 \quad i \neq j \neq k$$

On-shell 3-point kinematics



$$p_1 + p_2 + p_3 = 0 \qquad p_1^2 = p_2^2 = p_3^2 = 0$$

Still, using helicity spinors, there exist Lorentz invariants for this kinematics

Holomorphic kinematics

$$[jk] = 0 \qquad \langle jk \rangle \neq 0$$

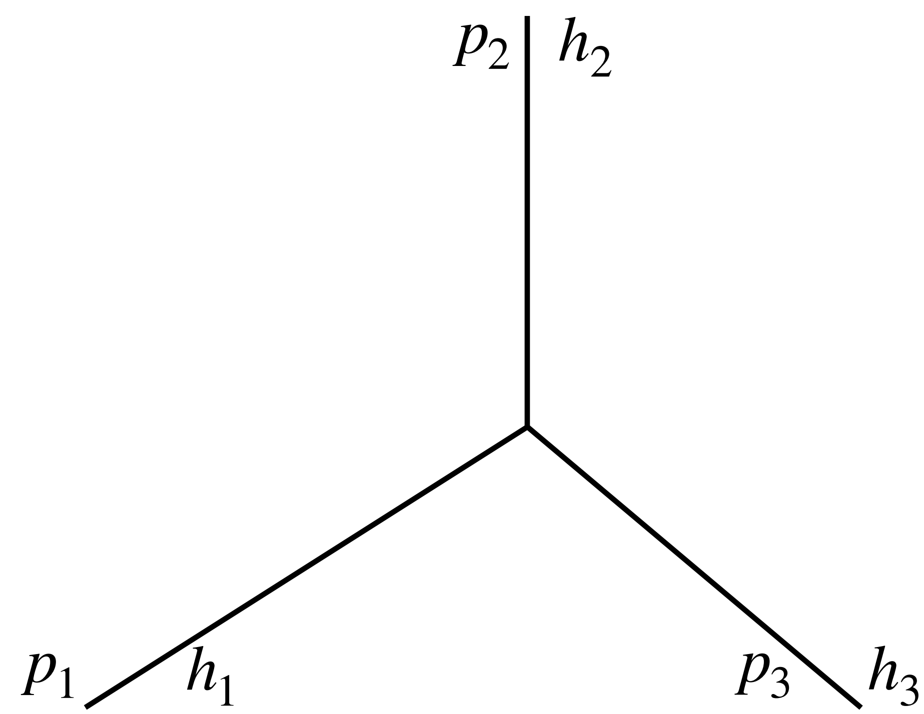
$$\tilde{\lambda}_i = -\frac{\langle jk \rangle}{\langle ik \rangle}\tilde{\lambda}_j \qquad i \neq j \neq k$$

Anti-holomorphic kinematics

$$\langle jk \rangle = 0 \qquad [jk] \neq 0$$

$$\lambda_i = -\frac{[jk]}{[ik]}\lambda_j \qquad i \neq j \neq k$$

On-shell 3-point amplitudes



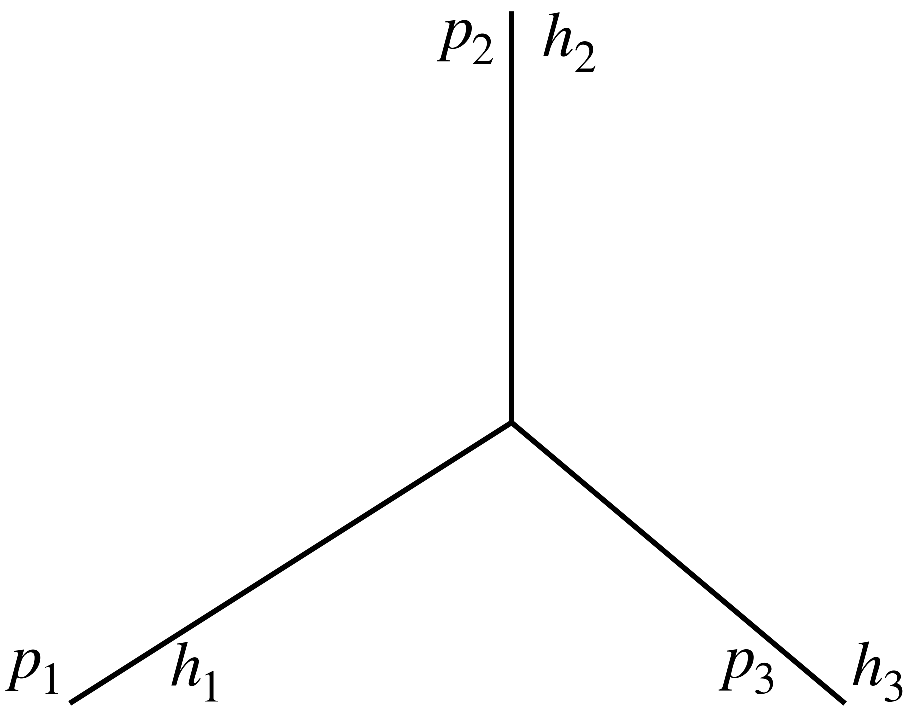
$$p_1 + p_2 + p_3 = 0 \qquad p_1^2 = p_2^2 = p_3^2 = 0$$

For a given helicity configuration, for H and AH kinematics each, there exist a single possible on-shell 3-point amplitude consistent with Lorentz invariance and little group scaling

$$\mathbf{H}: \quad \mathcal{M}[1^{h_1}2^{h_2}3^{h_3}] = g\langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1}$$

$$\mathbf{AH}: \quad \mathcal{M}[1^{h_1}2^{h_2}3^{h_3}] = \tilde{g}[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2}$$

On-shell 3-point amplitudes



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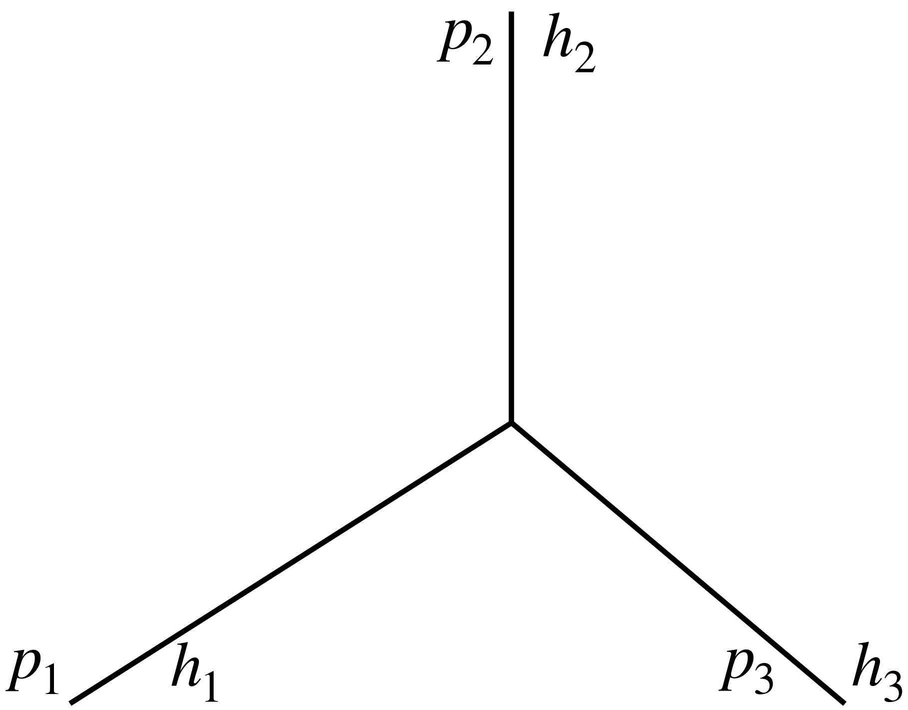
For 3 scalars, the only possible amplitude is a constant independent of kinematics

$$\mathcal{M}[1^0 2^0 3^0] = \kappa\Lambda$$

This corresponds to a Lagrangian with an interaction term $\mathcal{L} \supset \frac{\kappa\Lambda}{3!}\phi^3$

On-shell 3-point amplitudes



$$p_1 + p_2 + p_3 = 0 \qquad p_1^2 = p_2^2 = p_3^2 = 0$$

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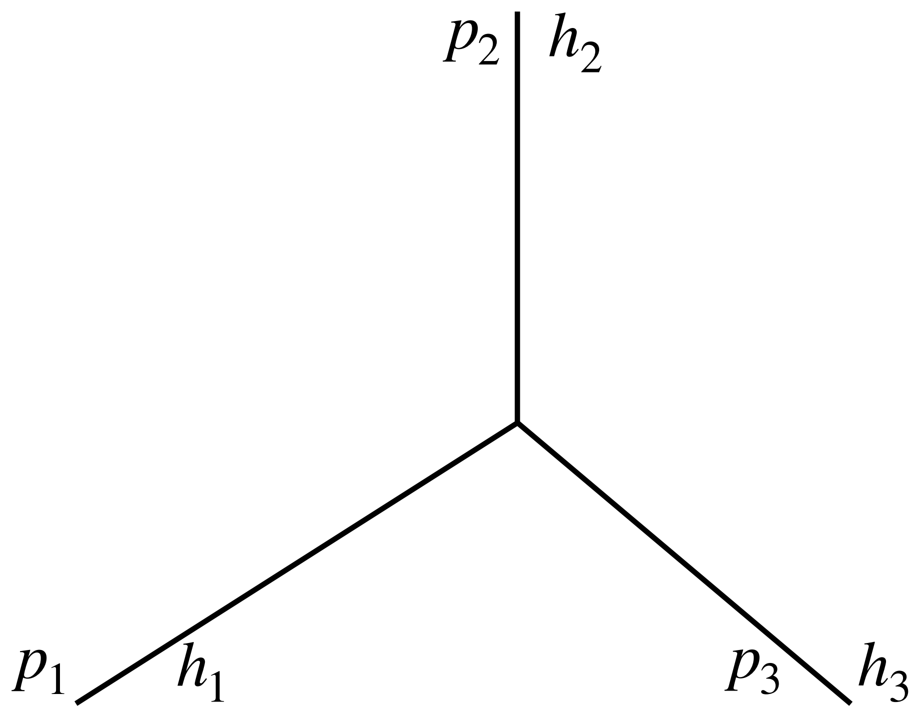
For 2 fermions and one scalar:

$$\mathcal{M}[1^{-1/2}2^{-1/2}3^{0}] = y\langle 12\rangle$$

$$\mathcal{M}[1^{+1/2}2^{+1/2}3^{0}] = \tilde{y}[12]$$

This corresponds to a Lagrangian with a Yukawa interaction term $\mathcal{L} \supset \frac{y}{2}\phi\psi^2 + \mathrm{h.c.}$

On-shell 3-point amplitudes



$$p_1 + p_2 + p_3 = 0 \qquad p_1^2 = p_2^2 = p_3^2 = 0$$

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For 3 photons:

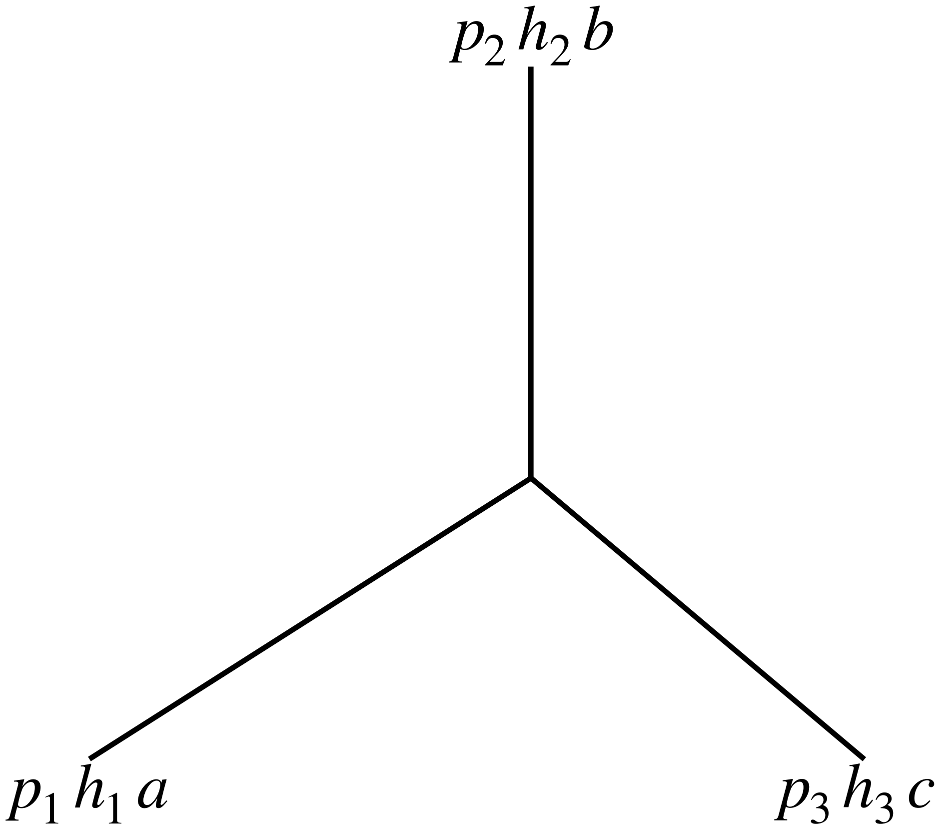
$$\mathcal{M}[1^{-1}2^{-1}3^{+1}]? = ?g\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle}$$

This however has wrong spin statistics !

One concludes 3-photon interaction is not possible (also for other helicity configurations)

$$\mathcal{M}[1^{-1}2^{-1}3^{+1}] = 0$$

On-shell 3-point amplitudes



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For 3 Yang Mills gauge bosons:

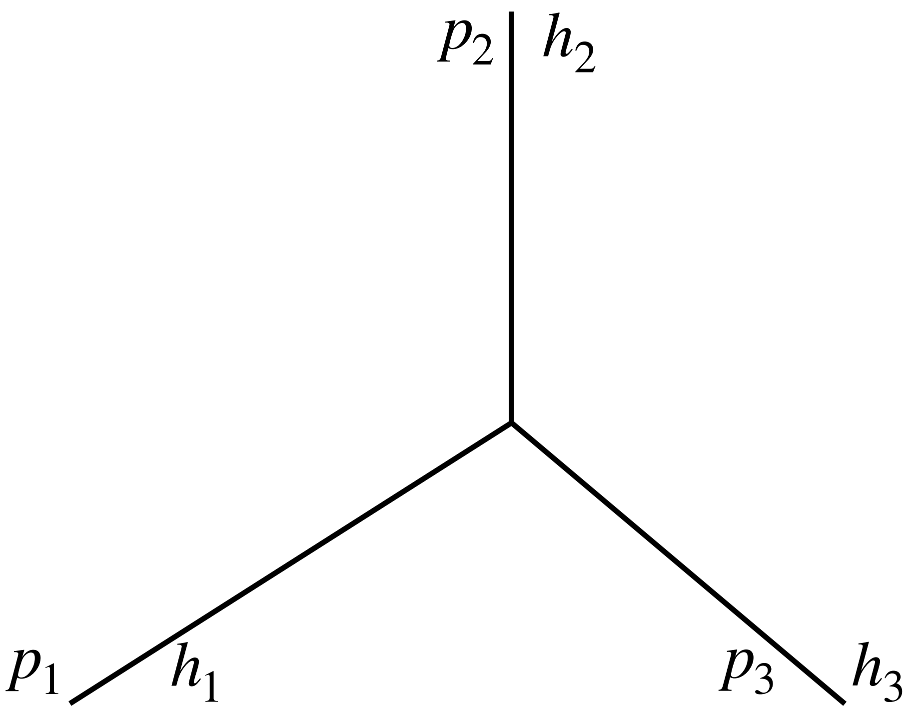
$$\mathscr{M}[1_a^{-1}2_b^{-1}3_c^{+1}] = i\sqrt{2}gf^{abc}\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle}$$

On-shell approach immediately says f^{abc} has to anti-symmetric in all 3 indices

This corresponds to YM Lagrangian interaction $gf^{abc}\partial_\mu G_\nu^a G_\mu^b G_\nu^c$

where f^{abc} is the structure constant of a Lie algebra

On-shell 3-point amplitudes



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For 3 gravitons:

$$\mathscr{M}[1^{-2}2^{-2}3^{+2}] = -\frac{1}{M_{\mathrm{Pl}}}\frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 23\rangle^2}$$

This corresponds to one page of Lagrangian obtained by expanding the Einstein-Hilbert Lagrangian $\mathscr{L} = \frac{M_{\mathrm{Pl}}^2}{2}\sqrt{-g}R$ **to cubic order around the flat metric,** $g_{\mu\nu} = \eta_{\mu\nu} + \frac{2}{M_{\mathrm{Pl}}}h_{\mu\nu}$

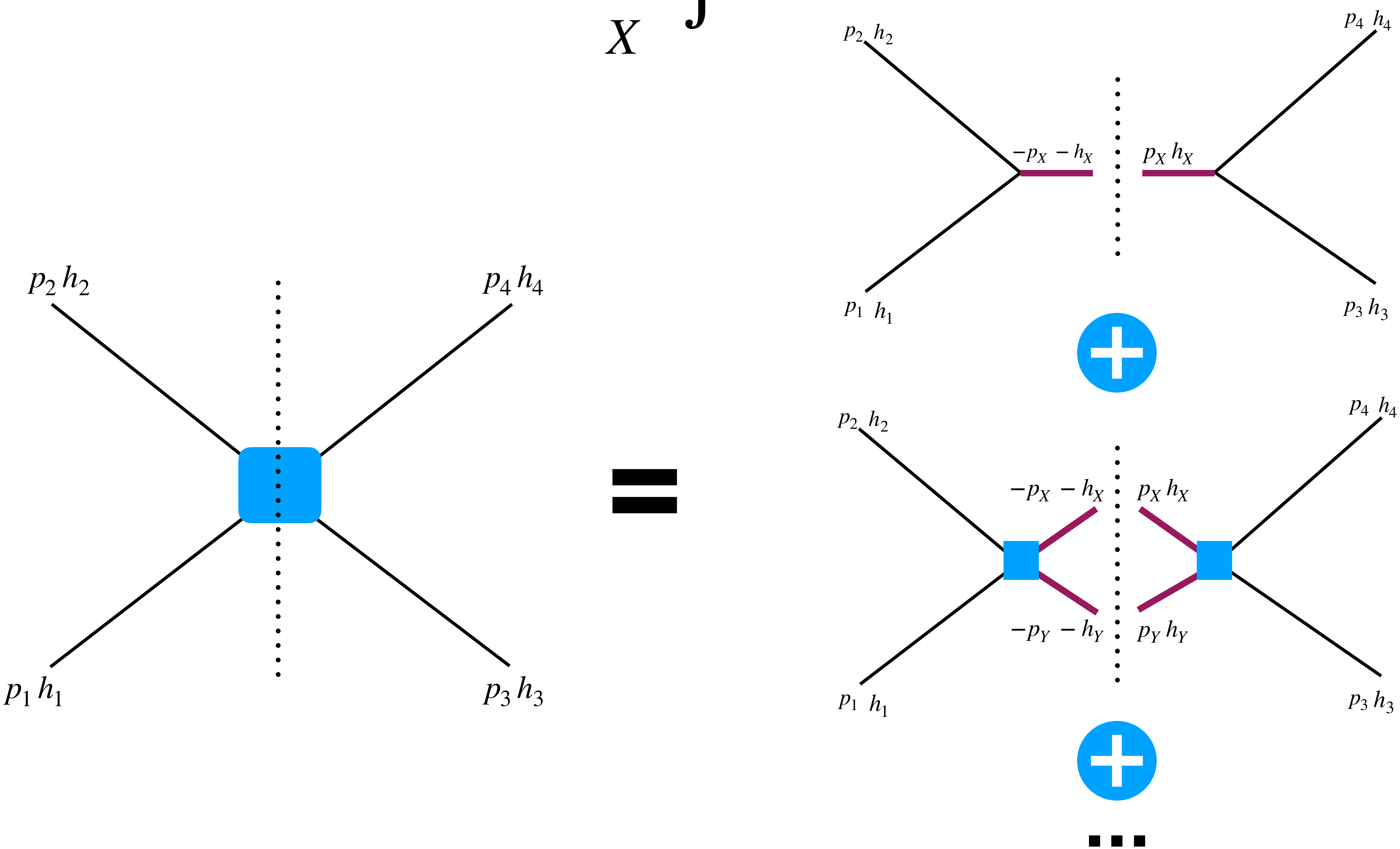
On-shell Higher-point Amplitudes

On-shell higher-point amplitudes

In the on-shell approach, 3-point amplitudes are the fundamental building blocks, from which all other amplitudes can be constructed using (generalized) unitarity

Master equation:

$$\mathrm{Disc}\mathscr{M}(\alpha \to \beta) = i \sum_X \int d\Pi_X \mathscr{M}(\alpha \to X)\mathscr{M}(X \to \beta)$$



On-shell higher-point amplitudes

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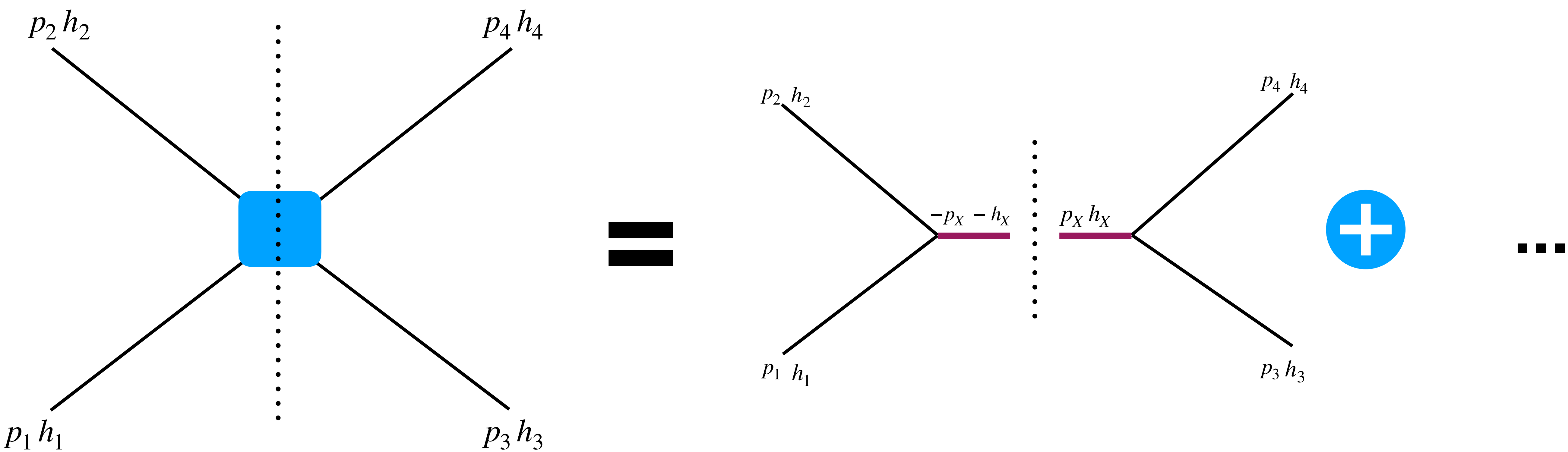


At tree level the master equation simplifies into the residue formula

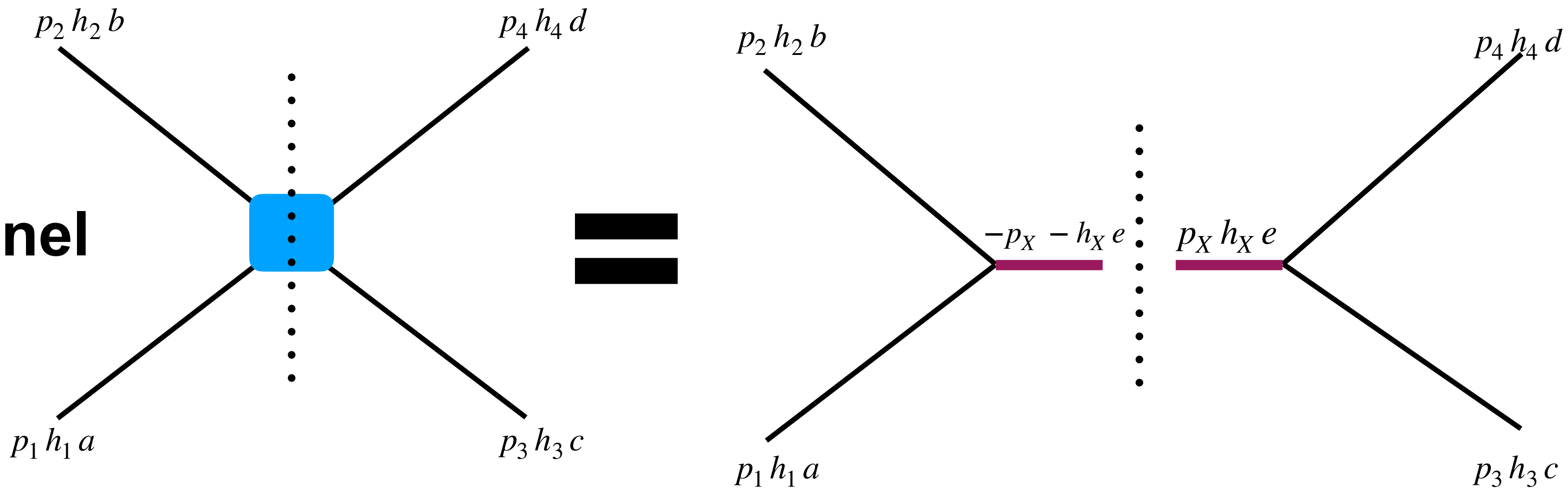
$$\lim_{p_\alpha^2 \to m_X^2} \mathscr{M}(\alpha \to \beta) = -\frac{1}{p_\alpha^2 - m_X^2 + i\epsilon}\mathscr{M}(\alpha \to X)\mathscr{M}(X \to \beta)$$

On-shell 4-point amplitudes

Consider tree-level 4-point Yang-Mills amplitude $\mathcal{M}[1_a^- 2_b^- \to (-3)_c^- (-4)_d^-]$

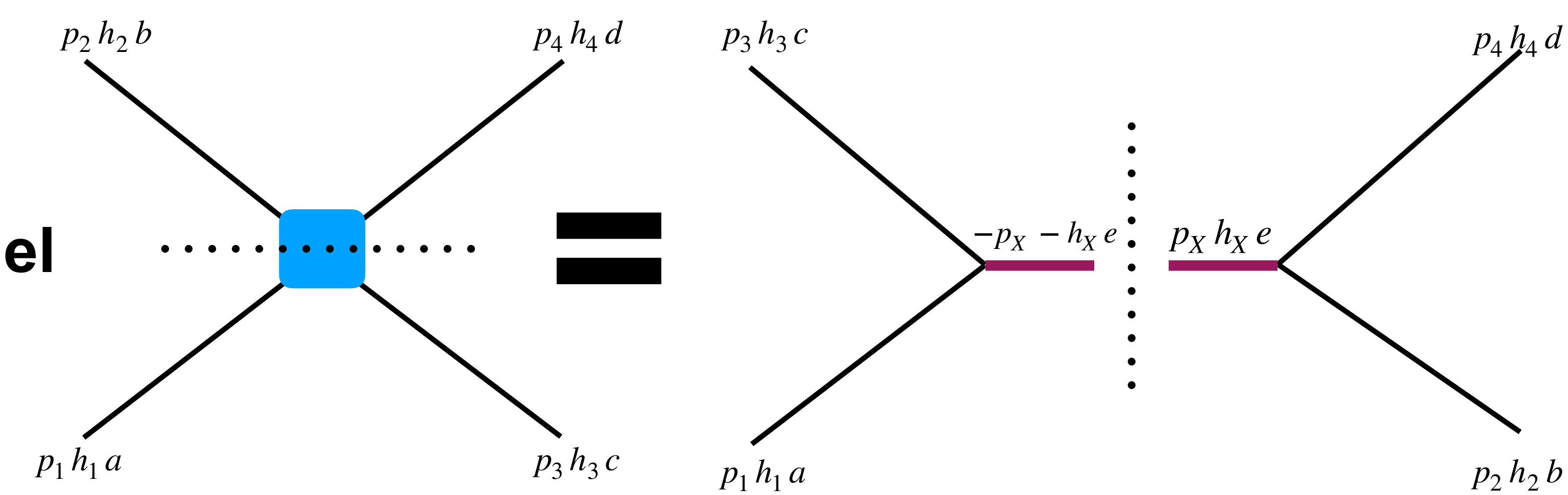
By crossing symmetry it is equivalent to $\mathcal{M}[1_a^- 2_b^- 3_c^+ 4_d^+]$

s-channel



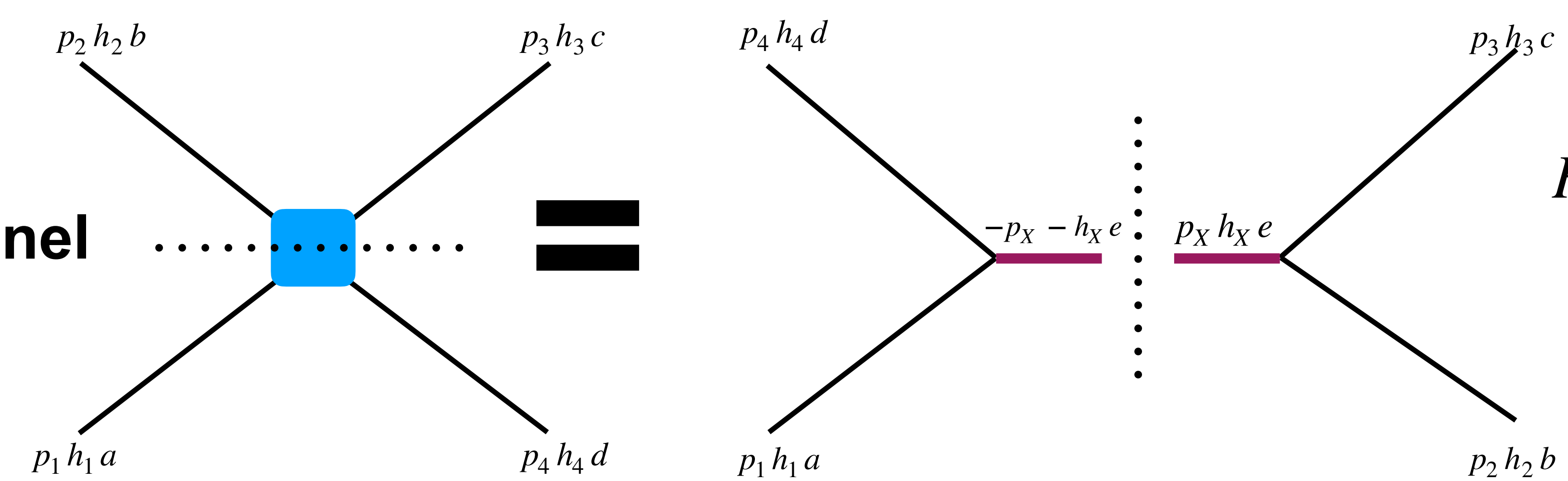
$$R_s \equiv \mathrm{Res}_{(p_1+p_2)^2 \to 0} \mathcal{M}[1_a^- 2_b^- 3_c^+ 4_d^+]$$
$$= -\mathcal{M}[1_a^- 2_b^- (-s)_e^+] \mathcal{M}[3_c^+ 4_d^+ s_e^+]$$

t-channel



$$R_t \equiv \mathrm{Res}_{(p_1+p_3)^2 \to 0} \mathcal{M}[1_a^- 2_b^- 3_c^+ 4_d^+]$$
$$= -\mathcal{M}[1_a^- (-t)_e^- 3_c^+] \mathcal{M}[4_d^+ t_e^+ 2_b^-]$$

u-channel

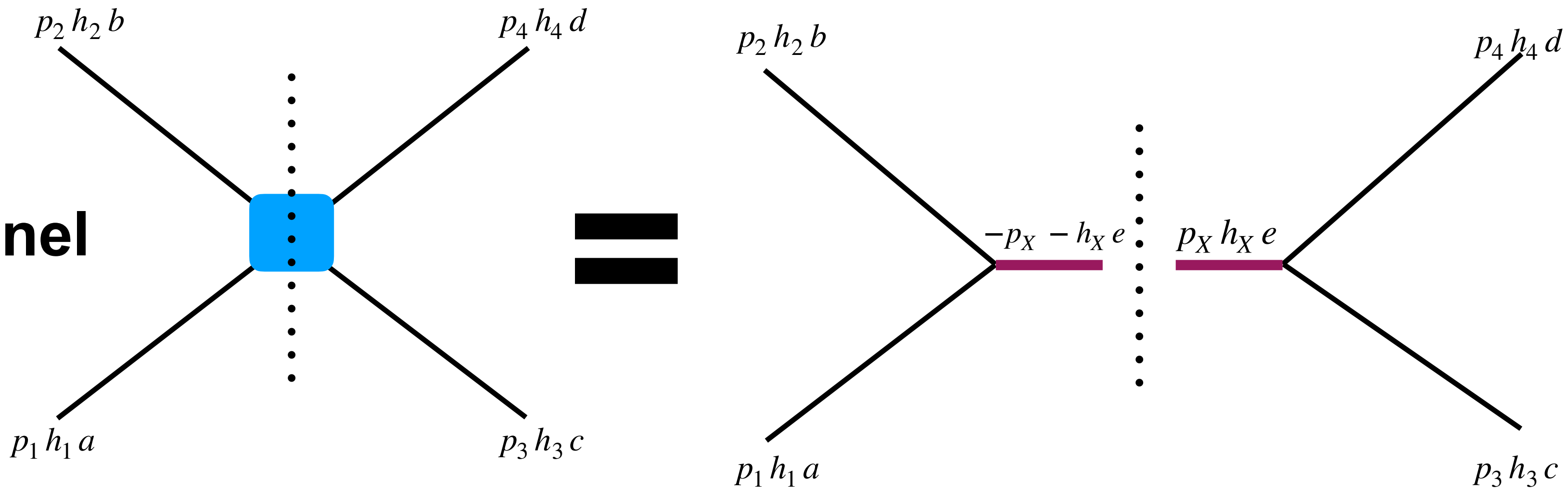


$$R_u \equiv \mathrm{Res}_{(p_1+p_4)^2 \to 0} \mathcal{M}[1_a^- 2_b^- 3_c^+ 4_d^+]$$
$$= -\mathcal{M}[1_a^- (-u)_e^- 4_d^+] \mathcal{M}[3_c^+ u_e^+ 2_b^-]$$

On-shell 4-point amplitudes

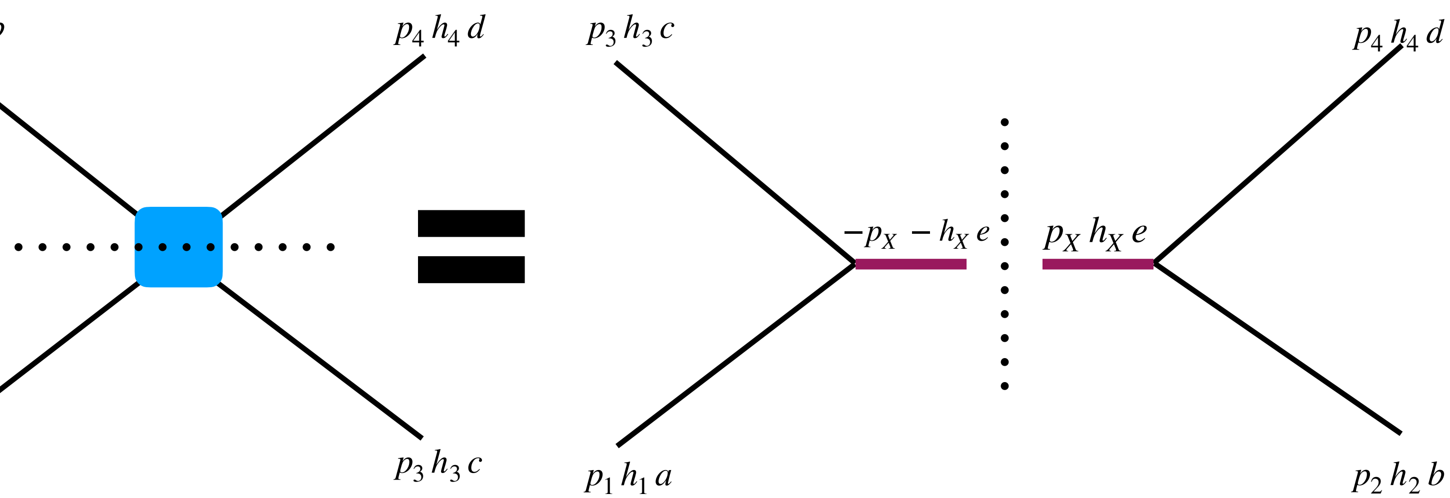
$$\mathscr{M}[1_a^- 2_b^- 3_c^+ 4_d^+]$$

s-channel



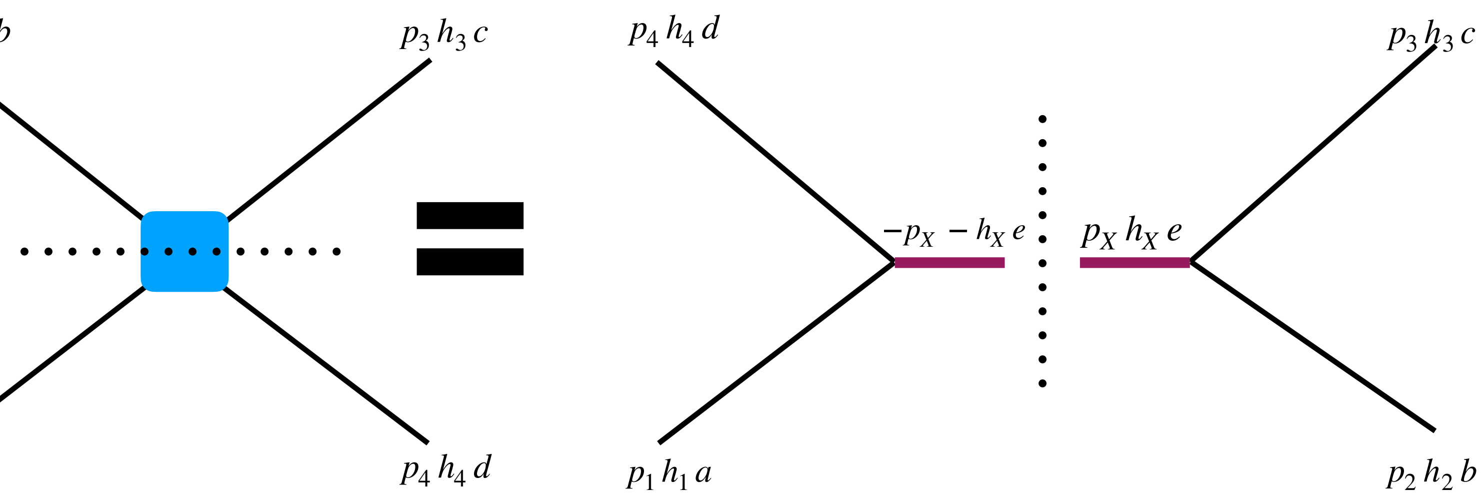
$$R_s = -2g^2 f_{abe} f_{cde} \frac{\langle 12\rangle^2 [34]^2}{t}$$

t-channel



$$R_t = -2g^2 f_{ace} f_{bde} \frac{\langle 12\rangle^2 [34]^2}{s}$$

u-channel



$$R_u = -2g^2 f_{ade} f_{bce} \frac{\langle 12\rangle^2 [34]^2}{s}$$

On-shell 4-point amplitudes

$$R_s = -2g^2 f_{abe} f_{cde} \frac{\langle 12\rangle^2 [34]^2}{t}$$

Reconstructing $\mathscr{M}[1_a^- 2_b^- 3_c^+ 4_d^+]$ from its residues:

$$R_t = -2g^2 f_{ace} f_{bde} \frac{\langle 12\rangle^2 [34]^2}{s}$$

$$R_u = -2g^2 f_{ade} f_{bce} \frac{\langle 12\rangle^2 [34]^2}{s}$$

$$\mathscr{M}[1_a^- 2_b^- 3_c^+ 4_d^+] = -2g^2 \langle 12\rangle^2 [34]^2 \left[\frac{f_{ace} f_{bde}}{st} + \frac{f_{ade} f_{bce}}{su} \right]$$

This has manifestly the correct residue in the t and u channels

It also has the correct residue in the s channel provided the structure functions obey the Jacobi identity

$$f_{abe} f_{cde} - f_{ace} f_{bde} + f_{ade} f_{bce} = 0$$

On-shell methods know about the Lie algebra structure of the theory, even though gauge invariance was never introduced!

On-shell 4-point amplitudes

$$R_s = -2g^2 f_{abe} f_{cde} \frac{\langle 12 \rangle^2 [34]^2}{t}$$

Reconstructing $\mathscr{M}[1_a^- 2_b^- 3_c^+ 4_d^+]$ from its residues:

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$$R_u = -2g^2 f_{ade} f_{bce} \frac{\langle 12 \rangle^2 [34]^2}{s}$$

More general amplitude consistent with the above residues

$$\mathscr{M}[1_a^- 2_b^- 3_c^+ 4_d^+] = -2g^2 \langle 12 \rangle^2 [34]^2 \left[\frac{f_{ace} f_{bde}}{st} + \frac{f_{ade} f_{bce}}{su} + F_{abcd}[s,t,u] \right]$$

This is the so-called contact term ambiguity
It is physical, and parametrises the freedom of adding higher-dimensional operators to the Lagrangian

Has no kinematic poles

For example, $F_{abcd} = C_{G^4} \frac{f_{ace} f_{bde} + f_{ade} f_{bce}}{\Lambda^4}$ corresponds to adding dimension-8 operator $\Delta \mathscr{L} \sim [G^a_{\mu\nu} G^a_{\mu\nu}]^2$ to the Lagrangian

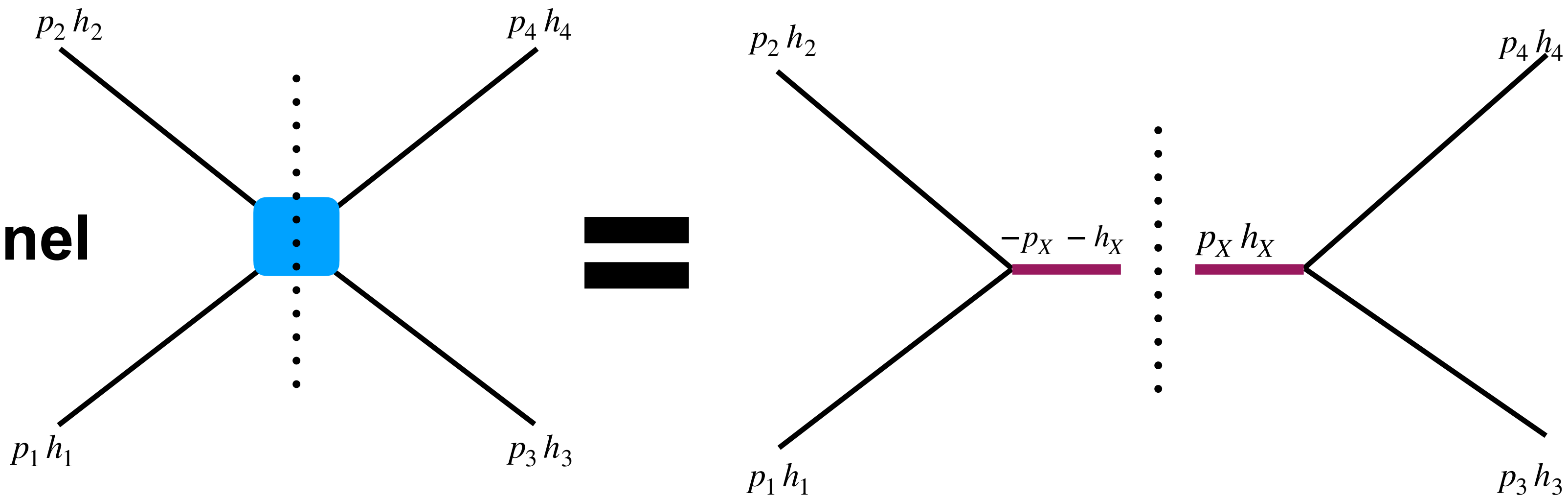
These should be systematically included when doing YM-EFT, or dropped if one insists on the usual renormalizable YM

In some class of theories (QED, QCD, GR, Supergravity), contact term ambiguity can be circumvented by using recursion relations

On-shell 4-point amplitudes

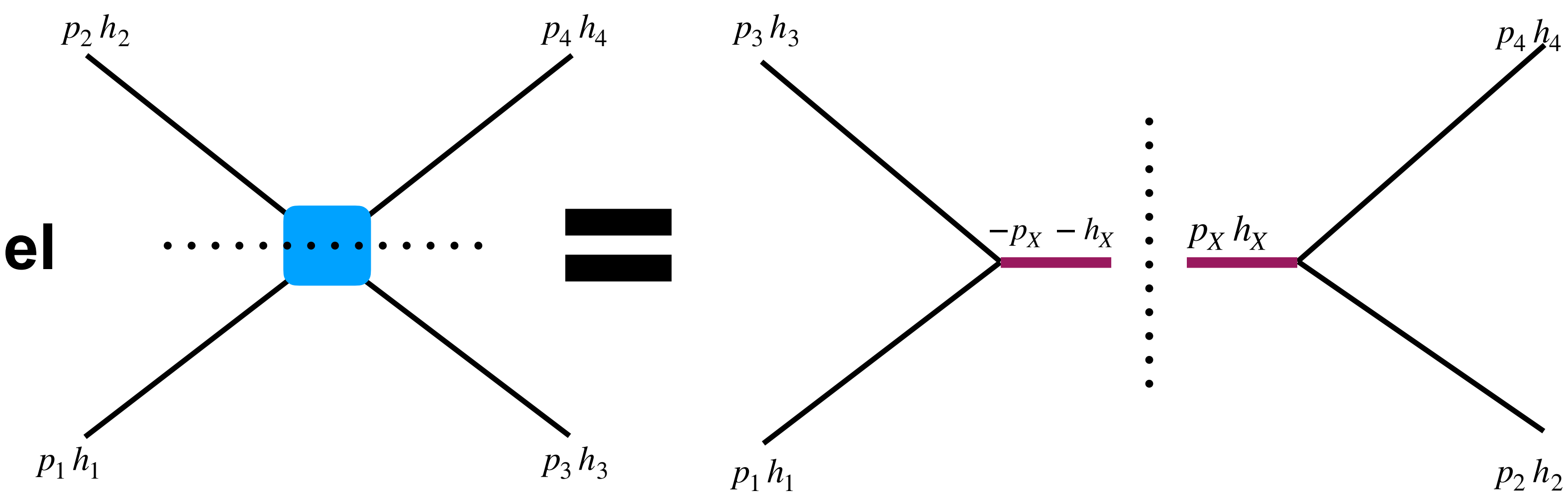
Graviton 2-to-2 scattering in GR $\mathscr{M}[1^-2^-3^+4^+]$

s-channel



$$R_s = \frac{\langle 12\rangle^4[34]^4}{M_{\rm Pl}^2 tu}$$

t-channel



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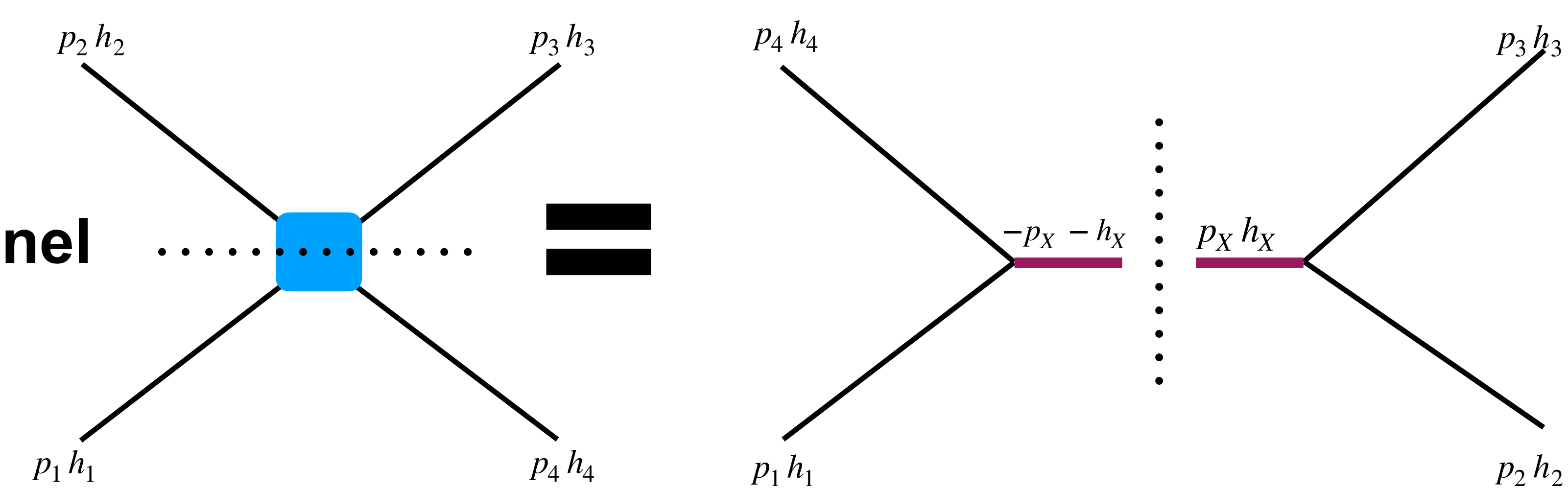


$$R_u = \frac{\langle 12\rangle^4[34]^4}{M_{\rm Pl}^2 st}$$

On-shell 4-point amplitudes

Reconstructing $\mathscr{M}[1^-2^-3^+4^+]$ from its residues

$$R_s = \frac{\langle 12\rangle^4[34]^4}{M_{\rm Pl}^2 tu}$$

$$R_t = \frac{\langle 12\rangle^4[34]^4}{M_{\rm Pl}^2 su}$$

$$R_u = \frac{\langle 12\rangle^4[34]^4}{M_{\rm Pl}^2 st}$$

$$\mathscr{M}[1^-2^-3^+4^+] = \frac{\langle 12\rangle^4[34]^4}{M_{\rm Pl}^2 stu}$$

up to a contact term ambiguity,
which corresponds to the possibility of adding
$R^4_{\mu\nu\alpha\beta}$ higher-dimensional operators to the GR Lagrangian

Some

Applications

QFT simpler and more intuitive

Many QFT facts become more transparent in the on-shell light

- **Lack of 3-photon interactions**
- **Emergence of a gauge group structure for theories of massless spin-1 bosons with cubic interactions**
- **The need for equivalence theorem in theories with a massless spin-2 boson**
- **The need for supergravity in theories with massless spin-3/2 fermions**
- **The impossibility of massless particles with spin > 2**
- **The structure of minimal coupling to electromagnetism or gravity for massive higher-spin particles**
- **Maximum possible cutoff scale for theories with massive higher-spin particles**
- **Yang theorem (a massive spin-1 particle cannot decay into a pair of identical massless spin-1 particles)**

QFT simpler and more intuitive

Yang theorem

Decay of massive vector to photons described by the amplitude $\mathscr{M}[1^{JK}2_\gamma^{h_2}3_\gamma^{h_3}]$

For both photons of negative helicity
the most general on-shell 3-point amplitude consistent with little group invariance is

$$\mathscr{M}[1^{JK}2_\gamma^{-}3_\gamma^{-}]? = ?\frac{g}{\Lambda^2}(\chi_1^J\lambda_2)(\chi_1^K\lambda_3)(\lambda_2\lambda_3)$$

But that has wrong spin statistics under $(2 \leftrightarrow 3)$ **so impossible !**

Same conclusion for other helicity configurations

RG running on-shell

- Perhaps the most spectacular application of on-shell methods is for calculating RG running of higher-dimensional operators in EFTs
- Working on shell greatly simplifies the calculations and makes them more transparent
- It also elucidates the structure of the anomalous dimension matrix, in particular it allows one to understand some of the``magic zeros" that appear mysterious from the point of view of standard calculations
- Especially for gravitational theories, on-shell methods offer a lot of extra mileage
- On-shell methods can also be readily extended beyond one loop, avoiding lots of complications of the standard approach (such as e.g. evanescent operators)

RG running on-shell

General structure of 4-point amplitudes up to one loop:

$$\mathscr{M}_4 = \mathscr{M}_4^{(0)} + \sum_{x=s,t,u} c_2^x I_2^x + \text{triangles} + \text{boxes} + \text{rational}$$

At one loop, only bubbles matter for calculation of anomalous dimensions, because scalar triangle and box integrals, as well as rational terms have no UV divergences

Boxes do have UV divergences and contribute to anomalous dimensions

$$I_2^{p^2} \equiv \int \frac{d^d k}{i(2\pi)^d} \frac{1}{k^2(k+p)^2} = \frac{1}{16\pi^2}\left[\frac{1}{\epsilon} + \log\left(-\frac{\mu^2}{p^2}\right) + 2\right]$$

Unitarity master equation: $\text{Disc}_2^s \mathscr{M}[1234] = i \sum \int d\Pi_{XY} \mathscr{M}[12XY]\mathscr{M}[(-X)(-Y)34]$

allows one to extract bubble coefficients: $c_2^s = 8\pi \mathscr{R}\left\{ \sum \int d\Pi_{XY} \mathscr{M}[12XY]\mathscr{M}[(-X)(-Y)34] \right\}$

Drop all logs and IR divergences

$$\frac{\partial}{\partial \log \mu} \mathscr{M}_4^{(0)} = -\frac{c_2^s + c_2^t + c_2^u}{8\pi^2} + \gamma_{\text{coll}} \mathscr{M}_4^{(0)}$$

Classical phenomena

- The KMOC formalism allows one to connect classical observables (momentum or spin kick, waveforms for emitted radiation, etc) to quantum amplitudes in classical (soft momentum exchange) limit

- Classical gravitational observables can be calculated from amplitudes of scattering of point-like massive particles coupled to gravity (possibly in a non-minimal way). The observables are obtained in the post-Minkowskian (PM) expansion, where loop expansion of amplitudes corresponds to expansion in inverse powers of M_{Pl} on classical side

- Observables are naturally obtained in a Lorentz-invariant form with resummed velocity expansion, thus containing information about infinite orders in post-Newtonian expansion, typically employed by classical GR practitioners

- Spin effects can be easily included by considering scattering of massive particles with spin

- On-shell methods provide a lot of mileage for calculating gravitational amplitudes, pushing the current state of the art to 5PM for some observables

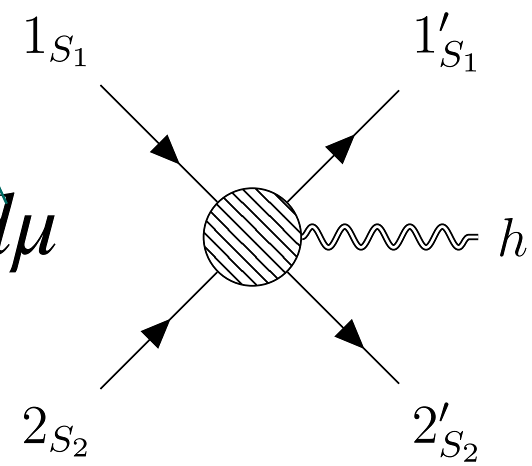
Classical phenomena

Example: gravitational waveforms at leading PM order

spectral waveform

on-shell measure

$$f_h(\omega) = \frac{1}{64\pi^3 m_1 m_2} \int d\mu$$



strain

$$W_h(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f_h(\omega)$$

In fact, the waveform can be expressed in terms of a single residue of the 5-point amplitude

$$f_h(\omega) = -\frac{1}{64\pi^2 m_1 m_2 \sqrt{\gamma^2-1}} \int_{-\infty}^{\infty} \frac{dz e^{i\omega[b_1 n + z(\hat{u}_1 n) b]}}{\sqrt{z^2+1}} \mathscr{R}\left\{ R\left(w_2 \to \omega(\hat{u}_1 n)\left[\gamma \hat{u}_2 - \hat{u}_1 + z\tilde{b} + i\sqrt{z^2+1}\,\tilde{\mathrm{v}}\right]\right) \right\}$$
$$+(1 \leftrightarrow 2)$$

$$R \equiv \mathrm{Res}_{w_2^2 \to 0} \mathscr{M}[(p_1+w_1)_{\Phi_1}(p_2+w_2)_{\Phi_2}(-p_1)_{\bar{\Phi}_1}(-p_2)_{\bar{\Phi}_2}(-\omega n)_h^s]$$

In GR, at leading order in spin expansion

$$W_h^{(0)}(t) = -\frac{m_1 m_2}{512\pi^2 b M_{\mathrm{Pl}}^3 \sqrt{\gamma^2-1}(\hat{u}_1 n)^2} \frac{1}{\sqrt{z^2+1}} \mathscr{R}\left\{ \frac{\left(\Lambda[\hat{u}_1,\hat{u}_2] - \Lambda[z\tilde{b} + i\sqrt{z^2+1}\tilde{\mathrm{v}}, 2\gamma\hat{u}_1 - \hat{u}_2]\right)^2 + \left(\Lambda[\hat{u}_1,\hat{u}_2] - \Lambda[z\tilde{b} + i\sqrt{z^2+1}\tilde{\mathrm{v}}, \hat{u}_2]\right)^2}{\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + z(\tilde{b} n) + i\sqrt{z^2+1}(\tilde{\mathrm{v}} n)} \right\}_{|z = \frac{t-(b_1 n)}{(\hat{u}_1 n) b}}$$
$$+(1 \leftrightarrow 2)$$

$$\Lambda[a,b] \equiv (\lambda_n a \sigma b \bar{\sigma} \lambda_n)$$

See talk of Panagiotis Marinellis tomorrow for application to scalar-tensor theories