Lattice QCD and precision tests of the Standard Model

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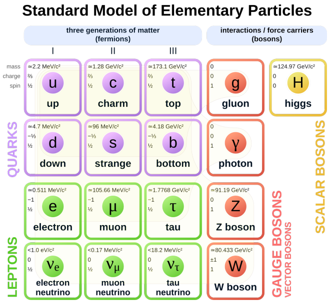
Outline

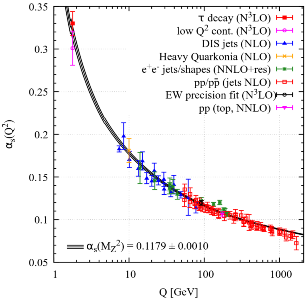
1. Introduction
2. Anomalous magnetic moment of the muon
3. Rare kaon decay $K_{\mathrm{L}} \to \mu^+\mu^-$

Introduction

The Standard Model of Particle Physics

- Many successful predictions with a handful parameters needed for the theory.
- Still some unanswered problems (e.g. neutrino masses) and tensions emerging with improved precisions.
- Theoretical challenge at low energies: non-perturbative Quantum Chromodynamics (QCD).





Lattice QCD

- Euclidean formulation of QCD regulated by the finite lattice spacing a (UV) and extent L (IR) with the $\mathrm{SU}(3)_{\mathrm{strong}}$ gauge field treated as a background.
- Positive (semi-)definite Boltzmann weight + gauge-invariant path-integral measure $\Rightarrow$ suitable for Monte Carlo-based methods:

$$\langle \mathcal{O} \rangle \equiv \int \mathcal{D}[U] e^{-S[U]} \mathcal{O}[U] \approx \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathcal{O}[U_n] \,.$$

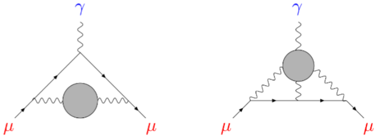
- Case with fermionic composite operators:

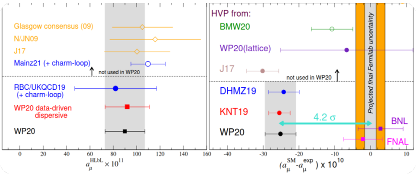
$$\begin{aligned} \langle \prod_n \psi_{i_n} \bar{\psi}_{j_n} \rangle =& \frac{1}{Z} \int \mathcal{D}[U] \Big(e^{-S_G[U]} \prod_{f\in \mathrm{flavor}} \det[D_f] \Big) \\ & \times \Big\{ \Big(\prod_n \frac{\partial}{\partial \eta_{j_n}} \frac{\partial}{\partial \bar{\eta}_{i_n}} \Big) \Big[\prod_{f\in \mathrm{flavor}} \exp \Big(\sum_{f\in \mathrm{flavor}} \bar{\eta}_f D_f^{-1} \eta_f \Big) \Big] \Big\} \Big|_{\eta=\eta'=0} , \end{aligned}$$

  $\Rightarrow$ each quark line leads to a D_f^{-1} evaluated in the gauge background.
- Connection to the physical world:
  - Setting the scale (a) to a physical value.
  - Determination of the "pion mass" of each ensemble.
  - Extrapolation to the physical point: $(a, L, m_\pi) \to (0, \infty, m_\pi^{\mathrm{Phys.}})$

The anomalous magnetic moment of the muon

- $(g\text{-}2)_e$: one of the first calculations to confirm the success of Quantum Electrodynamics (QED).
- $(g\text{-}2)_\mu$: more sensitive to hadronic physics. Experiment and theory are in 4.2-σ tension.

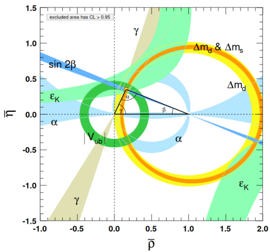




Source: A. El-Khadra's talk at the 2021 KEK g-2 workshop

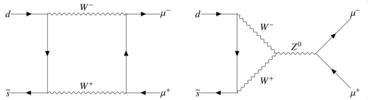
LQCD and flavor physics

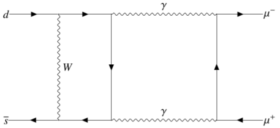
- Non-trivial phase of the CKM matrix: source of CP-violation in the SM.
- Success of LQCD in flavor physics: normalization of form factors, ϵ'/ϵ, ΔM_K etc.



Source: PDG 2022

- $K_{\mathrm{L}} \to \mu^+\mu^-$: involving flavor-changing weak interaction, rare but known precisely ($\sim$1%).
- The short-distance contribution can probe new Physics beyond the Standard Model. The extraction requires knowledge of hadronic physics.





Anomalous magnetic moment of the muon

(Mainz/CLS)

a_μ: status and challenges

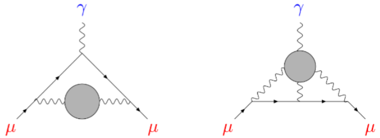
- Definition of a_μ from the Pauli form factor:

$$a_\mu = F_2(0)\,, \quad i\langle\mu^-(q_2)|j^\mu|\mu^-(q_1)\rangle = (-ie)\bar{u}(q_2)\left[\gamma^\mu F_1(p^2) + \frac{i\sigma^{\mu\nu}}{2m_l}p_\nu F_2(p^2)\right]u(q_1)\,.$$

- Currently, there is a 4.2σ-level discrepancy between the Standard Model prediction and experiment on a_μ [Bennett *et al* (BNL) '06, Abi *et al* (FNAL)'21] [Aoyama *et al* '20] :

$$a_\mu^{\mathrm{exp}} = 116\,592\,061(41)\times 10^{-11}\,,$$
$$a_\mu^{\mathrm{SM}} = 116\,591\,810(43)\times 10^{-11}\,,$$

- On the theory side, the uncertainties are dominated by hadronic contributions:



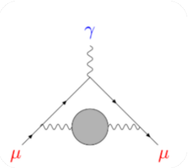
Hadronic vacuum polarization, HVP (left) vs. hadronic light-by-light scattering, HLbL (right).

- Theory estimates [Aoyama *et al* '20]:

$$a_\mu^{\mathrm{hvp}}(e^+e^-, \mathrm{LO} + \mathrm{NLO} + \mathrm{NNLO}) = 6845(40)\times 10^{-11}\,,$$
$$a_\mu^{\mathrm{hlbl}}(\mathrm{pheno.} + \mathrm{lattice} + \mathrm{NLO}) = 92(18)\times 10^{-11}\,.$$

$\Rightarrow$ target relative errors: $\mathrm{O}(10^{-3})$ for a_μ^{hvp} and $\mathrm{O}(10^{-1})$ for a_μ^{hlbl}.

$a_\mu^{\rm hvp}$: status and challenges



- **Data-driven approaches**:
  - $a_\mu^{\rm hvp}$ obtained from the R-ratio

$$a_\mu^{\rm hvp,LO} = \frac{\alpha^2}{3\pi^2}\int_{M_\pi^2}^{\infty}\frac{K(s)}{s}R(s)ds\,, \quad R(s) = \frac{\sigma^0(e^+e^- \to {\rm hadron}(+\gamma))}{4\pi\alpha^2/3s}\,.$$

  - e^+e^- collision data as inputs supplemented with perturbative-QCD calculation in the high-momentum regime [Davier *et al* '17, '20; Keshavarzi *et al*'18, '20].
  - Similar analysis for radiative corrections (NLO and NNLO in $\alpha_{\rm QED}$) also exist [Kurz *et al*]

  $\Rightarrow$ the uncertainty is dominated by the LO contribution.

- **Lattice QCD methods**:
  - Analytic QED-kernel + the Vacuum Polarization function/tensor on the lattice in a QCD-background [Blum '03, Bernecker & Meyer '11, Chakraborty *et al* '14]
- Discrepancy between $a_\mu^{\rm hvp}$ obtained from the lattice and from data-driven at the 2.1σ-level [Aoyama *et al* '20, Borsanyi *et al* '21]

$$a_\mu^{{\rm hvp},e^+e^-} = 693.1(4.0)\times 10^{-10}\,,$$
$$a_\mu^{\rm hvp,BMW} = 707.5(5.5)\times 10^{-10}$$

- Scrutinize $a_\mu^{\rm hvp}$ from different channels with the *window quantities* [Blum *et al* '18].

a_μ^{W} from the CCS method

Formalism

- The time-momentum representation (TMR) for the leading-order a_μ^{hvp}

$$a_\mu^{\mathrm{hvp}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\; K(t, m_\mu)\mathcal{G}(t)\,, \quad \mathcal{G}(t)\delta_{ij} = -\int d^3x G_{ij}(t, \vec{x})\,,$$

$$G_{\mu\nu}(x) \equiv \langle j_\mu(x) j_\nu(0)\rangle\,, \quad j_\mu = \sum_f \mathcal{Q}_f \bar{\psi}_f \gamma_\mu \psi_f\,.$$

- The window quantities a_μ^{W}: replace the kernel with a smeared version

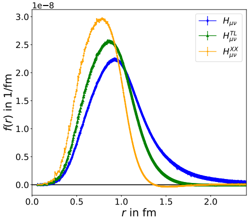
$$K_{\mathrm{W}}(t, m_\mu) = \left[\Theta_\Delta(t - t_0) - \Theta_\Delta(t - t_1)\right] K(t, m_\mu)\,, \quad \Theta_\Delta(t) \equiv \frac{1}{2}\left(1 + \tanh\left(\frac{t}{\Delta}\right)\right)\,.$$

- The intermediate window: $(t_0, t_1, \Delta) = (0.4, 1.0, 0.15)$ [fm], where precise comparisons between LQCD and data-driven methods can be made.

- Alternative based on the O(4)-covariant coordinate-space (CCS) formalism [Chao, Meyer & Parrino '23]

$$a_\mu^{\mathrm{W}} = \int d^4x\; H_{\mu\nu}(x) G_{\mu\nu}(x) = \int_0^\infty dr\; f(r)\,.$$

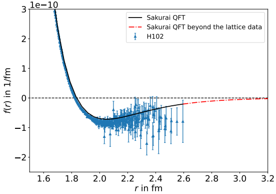
- The kernel is defined up to a total derivative term $\Rightarrow$ freedom to modify the shape of the radial integrand f.

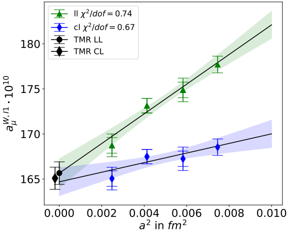


a_μ^{W} from the CCS method

Result at a reference point [Chao, Meyer & Parrino '23]

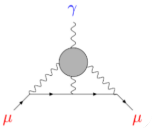
- Goal: compute the $I=1$ and the strange connected contribution to a_μ^{W} at $(m_\pi, m_K) = (350, 450)$ MeV.
- Finite-size effect corrections based on a ρ-γ mixing model, good description of our lattice data [Jegerlehner & Szafron '11].
- Consistent with the Mainz 2022 TMR-based result interpolated to the same reference point with different choices of discretized currents (**C**onserved vs **L**ocal) [Cè *et al* '22].



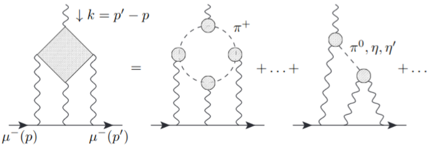


	isovector		strange	
Id	(LL)	(CL)	(LL)	(CL)
CCS ($H_{\mu\nu}^{\mathrm{TL}}$)	165.75(158)	164.69(156)	32.61(24)	32.38(23)
TMR	165.66(125)	165.09(123)	32.26(32)	32.11(31)

a_μ^{hlbl}: status and challenges



- The ABJ anomaly [Adler '69; Bell & Jackiw '69]: parametrically small but numerically large contribution from $\gamma^*\gamma^{(*)} \to$ hadrons [Hayakawa *et al* '95; Kinoshita *et al*'85; Bijnens *et al* '95].



Chiral expansion of the LbL amplitude: the $\pi^\pm$-loop and the π^0-pole [Nyffeler '19].

- Data-driven methods:
  - Form factors for $\gamma^*\gamma^{(*)} \to$ hadrons in the spacelike region [Hoferichter *et al* '18a, '18b; Masjuan & Sánchez-Puertas '17; Colangelo *et al* '14, 17; Roig & Sánchez-Puertas '20; Pauk & Vanderhaeghen '14; Pascalutsa *et al* '12; Dankilkin & Vanderhaeghen '17].
  - Short distance effects are handled by perturbative QCD [Melnikov & Vainshtein '04; Bijnens *et al* '19; Colangelo *et al* '20].
- Lattice QCD:
  - The first lattice QCD determination of a_μ^{hlbl} is done by the RBC/UKQCD collaboration in a lattice QCD+QED setup [Blum *et al* '20].
    $\Rightarrow$ Strong finite-size effects due to the massless photons.
  - An alternative pioneered by the Mainz group: treat QED in the continuum and infinite-volume and the QCD effects on the lattice [Asmussen *et al* '16, '18; Blum *et al* '17].

x-space approach to $a_\mu^{\rm hlbl}$

Formalism for $a_\mu^{\rm hlbl}$

- Master formula of the Mainz x-space approach:

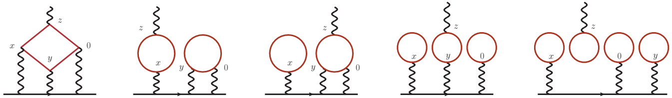
$$\boxed{a_\mu^{\rm hlbl} = \frac{m_\mu e^6}{3}\int_{x,y}\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\, i\hat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)\,,}$$

$$i\hat{\Pi}_{\rho;\mu\nu\lambda\sigma} = -\int_z z_\rho \tilde{\Pi}_{\mu\nu\sigma\lambda}\,, \quad \tilde{\Pi}_{\mu\nu\sigma\lambda}(x,y,z) \equiv \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0)\rangle_{\rm QCD}\,,$$

where $j_\mu = \sum_{i=u,d,s} Q_i \bar{\psi}_i \gamma_\mu \psi_i$ in an $N_f = 2+1$ theory.

- The QED-kernel $\mathcal{L}$ is computed semi-analytically in the continuum and infinite-volume. [Asmussen *et al* '23]

- 5 Wick-contraction topologies while computing the QCD 4-point function:
  - **Leading topologies**: fully-connected, (2+2)
  - **Sub-leading topologies**: (3+1), (2+1+1), (1+1+1+1)



a_μ^{hlbl} from LQCD
The Mainz/CLS results

- Results at the physical point with dynamical u, d, s-quarks [Chao, Hudspith *et al* '21]
    - $a \in [0.039, 0.099]$ fm and $m_\pi \in [200, 420]$ MeV, allowing for a robust chiral and continuum extrapolation.
    - All the subleading topologies are included for the first time in the literature.
- Estimate for the charm-quark [Chao, Hudspith *et al* '23]
    - Quenched c-quark as the sea effect is suppressed by $\alpha_{\mathrm{s}}(m_c)$.
    - Extrapolating from the $\mathrm{SU}(3)_{\mathrm{f}}$-symmetric point.

Contribution	Value$\times 10^{11}$
Light-quark fully-conn. and $(2+2)$	$107.4(11.3)_{\mathrm{stat.}}(9.2)_{\mathrm{syst.}}(6.0)_{\mathrm{chiral}}$
Strange-quark fully-conn. and $(2+2)$	$-0.6(2.0)$
$(3+1)$	$0.0(0.6)$
$(2+1+1)$	$0.0(0.3)$
$(1+1+1+1)$	$0.0(0.1)$
charm-quark conn.	3.1(4)
charm-quark (2+2)	-0.30(23)
Total	109.6(15.9)

Concluding remarks

Summary

- a_μ^{W} from the CCS method
  - We have performed a proof-of-concept calculation for a_μ^{W} based on an alternative LQCD approach at heavier-than-physical pion mass.
  - Successful consistency check for the Mainz 2022 determination on a_μ^{W} $\Rightarrow$ confidence building for the LQCD calculations based on the TMR method.
- Position-space calculation of a_μ^{hlbl}
  - We provide a LQCD determination of the *u*-, *d*-, *s*- and *c*-quark contributions to a_μ^{hlbl} at the physical pion mass.
  - The precision requirement for a_μ^{hlbl} is fulfilled. The resolution of the tension between theory and experiment relies solely on the improvement on a_μ^{hvp} or the appearance of new physics.

Outlook

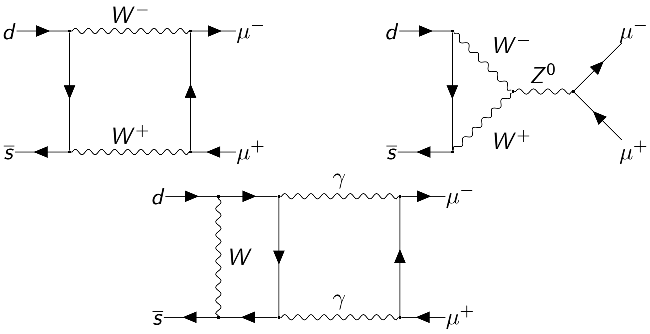
- Computing the NLO HVP from the forward LbL amplitude [Biloshytskyi *et al* '23].

Rare kaon decay $K_{\mathrm{L}} \to \mu^+\mu^-$

(RBC/UKQCD)

Motivation

- In the Standard Model, $K_{\mathrm{L}} \to \mu^+\mu^-$ comes in at one-loop level with exchange of two W-bosons or two W- and a Z-boson (short-distance contribution, SD).
- Precisely measured $\mathrm{Br}(K_{\mathrm{L}} \to \mu^+\mu^-) = 6.84(11) \times 10^{-9} \Rightarrow$ good test for the SM and potential interest for the physics beyond the SM. [BNL E871 Collab., PRL '00]
- Current theory limitation is the long-distance contribution (LD) involving two-photon exchange entering at $\mathrm{O}(G_{\mathrm{F}}\alpha_{\mathrm{QED}}^2)$, parametrically comparable to the SD contribution: the real part of the amplitude is not well understood.

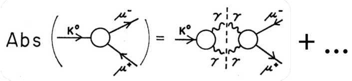


Introduction

Various estimates

$$\boxed{\mathrm{Br}(K_\mathrm{L} \to \mu^+\mu^-) = 6.84(11) \times 10^{-9}}$$

- SD contribution computed with RG technique [Buchalla & Buras '94], known to NNLO with the charm quark effect included: $0.79(12) \times 10^{-9}$ [Gorbahn & Haisch '06]
- LD absorptive (imaginary) part from optical theorem
  - The 2γ cut dominates over other channels [Martin et al, PRD '70]



  - Estimate with the most recent $\Gamma(K_\mathrm{L} \to \gamma\gamma)$ saturates the experimental KL2mu decay rate: $\mathrm{Br}(K_\mathrm{L} \to \mu^+\mu^-) = 6.59(5) \times 10^{-9}$ [Ceccucci '17]

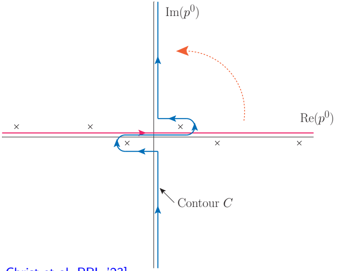
  $\Rightarrow$ *unitary bound* for the LD amplitude.
- Phenomenological attempts for the dispersive (real) part (+large-N_c)
  - Chiral perturbation with $\pi^0/\eta/\eta'$ pole [Dumm & Pich, PRL '98] $\Rightarrow$ GMO-suppressed, needs to go beyond $SU(3)_f$ and include mixings.
  - Paramtrization of the $K_\mathrm{L} \to \gamma^*\gamma^*$ transition form factor [Knecht et al, PRL '99][Hoferichter et al, '23]

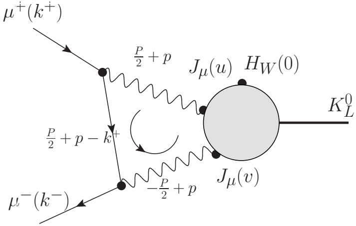
Formalism

- Strategy: perturbatively expanded kernel function in G_{F} and α_{QED} + hadronic correlation function computed on the lattice.

$$\mathcal{A}_{ss'}(k^+,k^-) = e^4 \int d^4p \int d^4u \int d^4v \, e^{-i\left(\frac{P}{2}+p\right)u} e^{-i\left(\frac{P}{2}-p\right)v} \frac{1}{(\frac{P}{2}-p)^2+m_\gamma^2-i\varepsilon} \cdot \frac{1}{(\frac{P}{2}+p)^2+m_\gamma^2-i\varepsilon}$$
$$\times \frac{\overline{u}_s(k^-)\gamma_\nu\{\gamma\cdot(\frac{P}{2}+p-k^+)+m_\mu\}\gamma_\mu v_{s'}(k^+)}{(\frac{P}{2}+p-k^+)^2+m_\mu^2-i\varepsilon} \cdot \left\langle 0 \left| T\left\{J_\mu(u)J_\nu(v)\mathcal{H}_W(0)\right\}\right| K_L \right\rangle .$$

- Analytic continuation of the kernel: $\Rightarrow$ unphysical exponentially growing contribution from states lighter than the kaon at rest.
- Finite number of such states on a finite lattice $\Rightarrow$ explicit, precise subtraction of such is possible.



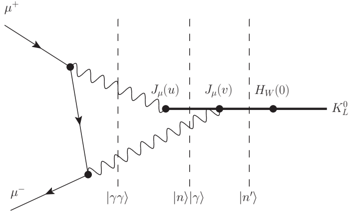


[cf. Christ et al, PRL '23]

Formalism

Time-ordering and Wick rotation

- Set an IR cutoff T and consider the possible intermediate states in the particular time-ordering $0 \leq v_0 \leq u_0$.



- The contribution from this time-ordering reads

$$\int_0^T du_0 \int_0^{u_0} dv_0 \int_{-\infty}^{\infty} dp_0 \; e^{i\left(\frac{M_K}{2}+p_0\right)u_0} e^{i\left(\frac{M_K}{2}-p_0\right)v_0}$$
$$\times \tilde{\mathcal{L}}^{\mu\nu}(p) e^{-iE_n u_0} e^{-i(E_{n'}-E_n)v_0} \langle 0\,| J_\mu(0)|n\rangle\langle n|J_\nu(0)|n'\rangle\langle n'|\mathcal{H}_W(0)|\, K_L\rangle\,.$$

- Under Wick rotation $u_0 \leftarrow -iu_0$, it converges at $T \to \infty$ iff

$$\boxed{E_n' > M_K \quad (i) \qquad \text{and} \qquad E_n + \sqrt{\vec{p}^{\,2} + m_\gamma^2} \geq M_K \quad (ii)}$$

  Otherwise, unphysical exponential terms appear.

- Repeating the above analysis for all possible time-ordering and intermediate state, the two sources for the exponential terms are

  1. π^0 with zero spatial momentum, coming from $K_{\rm L}$ turned into π^0 by the weak Hamiltonian.
  2. $\pi\pi(\gamma)$ states with low kinetic energy, propagating between the electromagnetic currents.

Numerical implementation

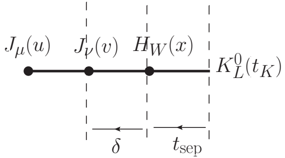
Parameter	Value
$L^3 \times T \times L_s$	$24^3 \times 64 \times 24$
m_π [MeV]	142
M_K [Mev]	515
a^{-1} [GeV]	1.023

- Lattice setup: Möbius Domain Wall fermion ensemble 24ID from the RBC/UKQCD collaboration.
- Master formula:

$$\mathcal{A}(t_{\rm sep}, \delta, x) \equiv \sum_{d \leq \delta} \sum_{u,v \in \Lambda} \delta_{v_0 - x_0, d} \; e^{M_K (v_0 - t_K)} \; K_{\mu\nu}(u - v) \left\langle J_\mu(u) J_\nu(v) \mathcal{H}_{\rm W}(x) K_{\rm L}(t_K) \right\rangle ,$$

$$\mathcal{H}_{\rm W}(x) = \frac{G_{\rm F}}{\sqrt{2}} V^*_{us} V_{ud} (C_1 Q_1 + C_2 Q_2) ,$$

$$Q_1 \equiv (\bar{s}_a \Gamma^L_\mu d_a)(\bar{u}_b \Gamma^L_\mu u_b) , \quad Q_2 \equiv (\bar{s}_a \Gamma^L_\mu d_b)(\bar{u}_b \Gamma^L_\mu u_a) .$$



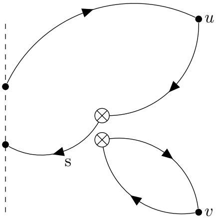
- Control of the contaminations from π^0 and low-energy $\pi\pi\gamma$ states:
  - The unphysical π^0 contribution can be measured and subtracted exactly

$$\frac{1}{2m_\pi} \sum_{\delta \geq 0} \sum_{u \in \Lambda} e^{(M_K - m_\pi)\delta} \left\langle 0 | J_\mu(u) J_\nu(v) | \pi^0 \right\rangle K_{\mu\nu}(u - v) \left\langle \pi^0 | \mathcal{H}_{\rm W}(v) | K_{\rm L} \right\rangle .$$

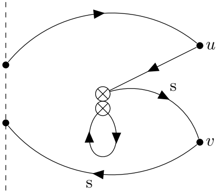
  - Control of the $\pi\pi\gamma$-intermediate state: use several kernels with different $|u - v| \leq R_{\rm max}$.

Contractions

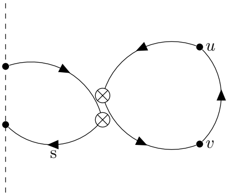
- Quark-connected Wick contractions for $\langle J_\mu(u) J_\nu(v) \mathcal{H}_\mathrm{W}(x) K_\mathrm{L}(t_K) \rangle$.
  Dashed line: $K_\mathrm{L}(t_K)$, crosses: $\mathcal{H}_\mathrm{W}(x)$, solid dots: $J_\mu(u)$ and $J_\nu(v)$



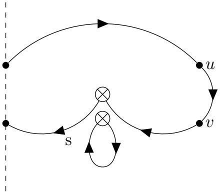
Type 1 diagram 1a



Type 2 diagram 1a



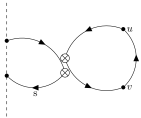
Type 3 diagram 1a



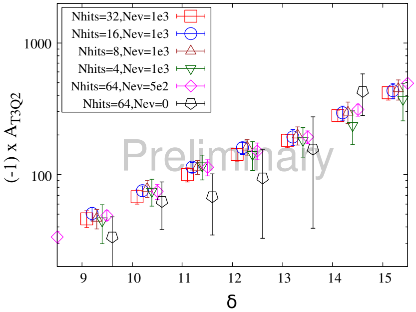
Type 4 diagram 1a

Preliminary result: type 3

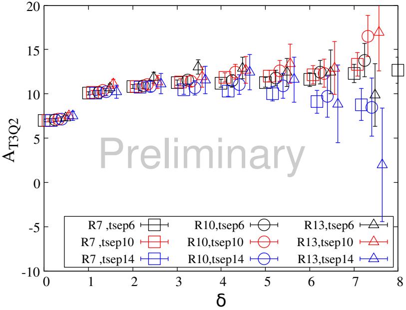
Removal of the unphysical π^0



- Expected exponentially-growing behavior due to the unphysical π^0 intermediate state.
- Plateau after subtracting the π^0 contamination. No strong sign of the $\pi\pi\gamma$ contamination by increasing $R_{\max}$.



$\mathcal{A}_{\mathrm{T3Q2}}$ in log scale with different hits and low modes.



$\mathcal{A}_{\mathrm{T3Q2}}$ after subtracting π^0

Conclusions and outlook

- A coordinate-space based lattice-QCD formalism for the $K_{\mathrm{L}} \to \mu^+\mu^-$ decay is proposed, enabling the determination of the phenomenologically inaccessible real part of the decay amplitude.
- Numerical strategies allowing to deal with different connected topologies have been developed, with possibility of keeping the $\pi\pi\gamma$ intermediate state under control.
- The so-far ignored disconnected part might not be negligible and can be much noisier due to the η intermediate state. More efficient sampling strategies will be needed.