Numerical methods for black hole models

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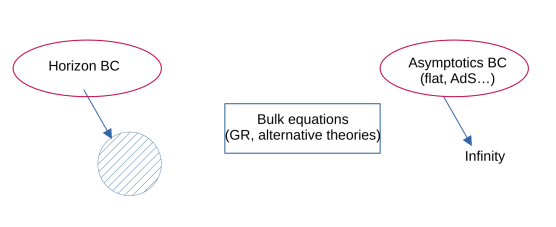
Introduction

Scientific goal

- Derive (and solve) a set of equations describing a spacetime containing one black hole in equilibrium.
- The starting point in GR but can be applied to alternative theories (hopefully).
- Especially useful for rotation black holes.
- Details in Grandclément and Nicoules (2022).



A simple example as appetizer

Schwarzschild spacetime in quasi-isotropic coordinates :

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + \Psi^4 f_{ij} \mathrm{d}x^i \mathrm{d}x^j.$$

Inner BC

$N = 0$ and $\partial_r \Psi + \Psi / 2a = 0$. ($a$ is the radius of the hole).

Outer BC

$$N = 1 \text{ and } \Psi = 1.$$

Bulk equations

$$\Delta \Psi = 0 \text{ and } \Delta N = -\frac{2}{\Psi} \partial_r \Psi \partial_r N.$$

Solutions

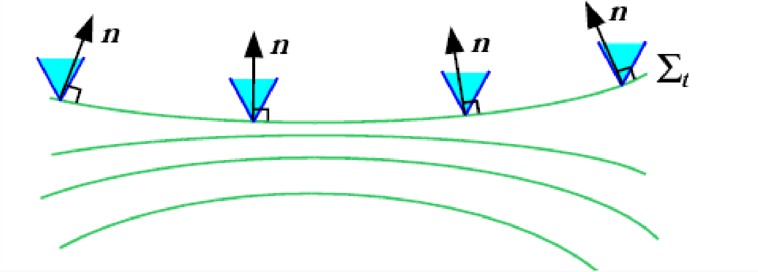
$$\Psi = 1 + \frac{a}{r} \text{ and } N = \frac{\left(1 - \frac{a}{r}\right)}{\left(1 + \frac{a}{r}\right)}.$$

3+1 formalism

Foliation of spacetime

The 3+1 formalism is the most widely used way to write Einstein equations for NR. It makes explicit the split between space and time (see for instance Gourgoulhon (2012)).

Spacetime is foliated by a family of spatial hypersurfaces Σ_t

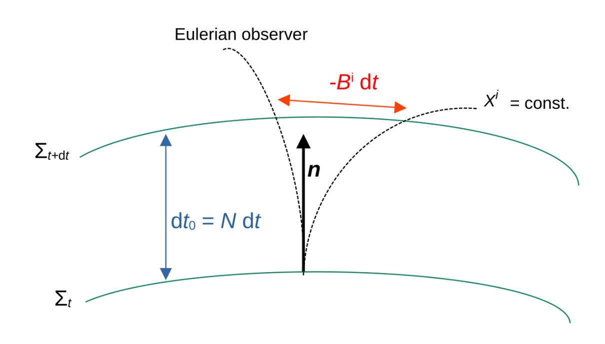


- Coordinate system of Σ_t : (x_1, x_2, x_3).
- Coordinate system of spacetime : (t, x_1, x_2, x_3).

Greek indices 4D $(0, 1, 2, 3)$ and Latin 3D $(1, 2, 3)$.

Unit normal

The unit normal can be written as $n^{\mu} = \left(\frac{1}{N}, -\frac{B^i}{N}\right)$.



- N, the lapse, defines the choice of time coordinate.
- B^i, the shift, defines the choice of spatial coordinates.

Projections

- Projection on the normal of a vector $\vec{V}$ is given by $n_\mu V^\mu$.
- Projection operator on the hypersurfaces $\gamma_\mu^\nu = g_\mu^\nu + n_\mu n^\nu$.
- Projection on the slices of a vector $\vec{V}$ is given by $\gamma_\mu^\nu V^\mu$.
- Projection the 4D metric $g_{\mu\nu}$ is

$$\gamma_\alpha^\nu \gamma_\beta^\mu g_{\mu\nu} = \gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta$$

- The induced metric γ_{ij} is the first fundamental form.

The 4D line-element reads

$$\mathrm{d}s^2 = -\left(N^2 - B^i B_i\right)\mathrm{d}t^2 + 2B_i \mathrm{d}t \mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j$$

3+1 quantities for Schwarzschild

Ingoing Eddigton-Finklestein coordinates :

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{4M}{r}\mathrm{d}r\mathrm{d}t + \left(1 + \frac{2M}{r}\right)\mathrm{d}r^2 + r^2\mathrm{d}\Omega$$

- $\gamma_{rr} = 1 + \dfrac{2M}{r}$.
- $\gamma_{\theta\theta} = r^2 \quad ; \quad \gamma_{\varphi\varphi} = r^2 \sin^2\theta$.
- $B_r = \dfrac{2M}{r} \Longrightarrow B^r = \dfrac{2M}{r + 2M}$.
- $-N^2 + B_i B^i = -\left(1 - \dfrac{2M}{r}\right) \Longrightarrow N = \dfrac{1}{\sqrt{1 + \dfrac{2M}{r}}}$.

Extrinsic curvature K_{ij}

- Describes the part of the geometry not accounted for by the induced metric.
- It describes the variation of normal projected on the hypersurface.

$$K_{ij} = -\gamma_i^{\mu} \gamma_j^{\nu} \nabla_{\mu} n_{\nu}$$

- In the 3+1 framework, it is given by

$$\left(\partial_t - \mathcal{L}_{\vec{B}}\right) \gamma_{ij} = -2 N K_{ij}$$

- It is known as the second fundamental form.

Link between 4D and 3D quantities

In order to derive the 3+1 version of Einstein's equations, one needs to relate the 4D quantities to the 3D ones.

- 4D quantities : $g_{\mu\nu}$, n^α, ${}^4R^\alpha_{\beta\mu\nu}$, ∇_α...
- 3D quantities : γ_{ij}, K_{ij}, D_i, N, B^i ...

Some examples

- Gauss relation :

$$\gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\rho \gamma^\sigma_\delta \; {}^4R^\rho_{\sigma\mu\nu} = R^\gamma_{\delta\alpha\beta} + K^\gamma_\alpha K_{\delta\beta} - K^\gamma_\beta K_{\alpha\delta}.$$

- Codazzi relation :

$$\gamma^\gamma_\rho n^\sigma \gamma^\mu_\alpha \gamma^\nu_\beta \; {}^4R^\rho_{\sigma\mu\nu} = D_\beta K^\gamma_\alpha - D_\alpha K^\gamma_\beta.$$

Projection of Einstein's equations

The 3+1 equations are obtained by projecting $G_{\mu\nu} = 8\pi T_{\mu\nu}$ on n^α and on the hypersurfaces.

- projections onto $n^\mu n^\nu$, the Hamiltonian constraint:
$$R + K^2 - K_{ij}K^{ij} = 16\pi E.$$
- projection onto $n^\mu \gamma_i^\nu$, the momentum constraint:
$$D_j K_i^j - D_i K = 8\pi P_i.$$
- projection onto $\gamma_i^\nu \gamma_j^\mu$, the evolution equation:
$$\begin{aligned}\left(\frac{\partial}{\partial t} - \mathcal{L}_{\vec{B}}\right) \quad & K_{ij} = \\ -D_i D_j N \quad & +N\left(R_{ij} + KK_{ij} - 2K_{ik}K_j^k + 4\pi\left[\left(S-E\right)\gamma_{ij} - 2S_{ij}\right]\right).\end{aligned}$$
- E, P_i and S_{ij} are the various projections of $T_{\mu\nu}$.

Constraint equations

- The constraints are 4 equations, that do not contain ∂_t.
- Such equations are absent in Newtonian dynamics but not for Maxwell.

Type	Einstein	Maxwell
Constraints	Hamiltonian $R + K^2 - K_{ij}K^{ij} = 0$	$\nabla \cdot \vec{E} = 0$
	Momentum : $D_j K^{ij} - D^i K = 0$	$\nabla \cdot \vec{B} = 0$
Evolution	$\frac{\partial \gamma_{ij}}{\partial t} - \mathcal{L}_{\vec{B}} \gamma_{ij} = -2NK_{ij}$	$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon_0 \mu_0} \left(\vec{\nabla} \times \vec{B} \right)$
	$\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\vec{B}} K_{ij} = -D_i D_j N + N \left(R_{ij} - 2K_{ik}K^k_j + KK_{ij} \right)$	$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$

A two steps problem

Evolution problem

- Given initial value of $\gamma_{ij}\left(t=0\right)$ and $K_{ij}\left(t=0\right)$ use the evolution equations to determine the fields at later times.
- Similar to writing Newton's equation as $\partial_t x = v; \partial_t v = f/m$.
- Must ensure stability and accuracy.
- Must choose the lapse and shift in a clever way.

Initial data

- $\gamma_{ij}\left(t=0\right)$ and $K_{ij}\left(t=0\right)$ are not arbitrary but subject to the constraint equations.
- It is a set of four elliptic coupled equations.
- Needs to make the link between a given physical situation and the mathematical objects γ_{ij} and K_{ij}

Our case

Simplification

- Stationarity $\Longrightarrow \partial_t = 0$.
- Solve the ten 3+1 equations, for the quantities N, B^i and γ_{ij}.
- A set of ten partial differential, non-linear, coupled equations.

Coordinate choice

Gauge conditions: maximal slicing

- Einstein equations are not all independent, need to enforce gauge conditions (i.e. choice of coordinates).
- No explicit gauge choice (like quasi-isotropic).
- Differential gauges are more general.
- Choice of time : maximal slicing $K = 0$.
- Well tested in time-evolution codes, has some singularity avoidance properties.

Gauge conditions: spatial harmonic gauge

- Spatial harmonic gauge $V^i = \gamma^{kl}\left(\Gamma^i_{kl} - \bar{\Gamma}^i_{kl}\right) = 0$.
- $\bar{\Gamma}$ are the Christoffel's symbols of a reference metric (flat one for instance).
- $R_{ij} = -\frac{1}{2}\gamma^{kl}\bar{D}_k\bar{D}_l\gamma_{ij} + D_{(i}V_{j)} + \mathrm{first\,order}$
- If $V^i = 0$ then the Ricci tensor is close to a flat Laplacian (good for stability and convergence).
- The spatial harmonic gauge is the 3D equivalent of the Lorenz gauge used in the linearization of Einstein's equations.

Enforcing the gauges

In the 3+1 equations

- Remove all the occurrences of K and V^i in the equations.
- It gives a well-posed system of equations.
- Check, a posteriori, that $K = 0$ and $V^i = 0$ (very important test !).

Resulting system of equations

The modified 3+1 system

$$
\begin{aligned}
H \quad &: \quad R - D_k V^k - K_{ij} K^{ij} = 16\pi E \\
M_i \quad &: \quad D^j K_{ij} = 8\pi P_i \\
E_{ij} \quad &: \quad \mathcal{L}_{\vec{B}} K_{ij} - D_i D_j N + N \left(R_{ij} - \frac{1}{2} \left(D_i V_j + D_j V_i \right) - 2 K_{ik} K^k_j \right) \\
&= \quad 4\pi N \left(2 S_{ij} - \left(\gamma^{kl} S_{kl} - E \right) \gamma_{ij} \right).
\end{aligned}
$$

with

$$
K_{ij} = \frac{1}{2N} \left(D_i B_j + D_j B_i \right).
$$

$$
V^i = \gamma^{kl} \left(\Gamma^i_{kl} - \bar{\Gamma}^i_{kl} \right).
$$

Horizons

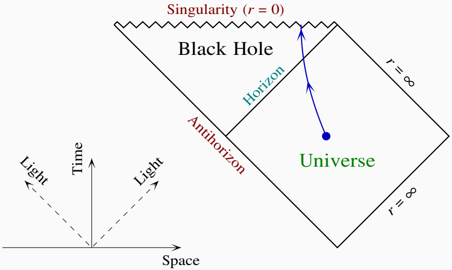
Event horizon

Boundary between two regions:

- *Outside* : where photons can reach infinity.
- *Inside* : where photons cannot !

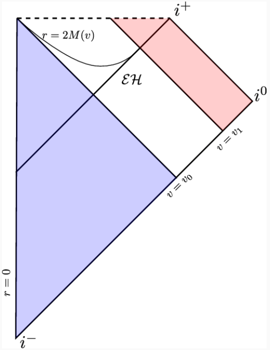
Properties

- Smooth null hypersurface.
- Weak cosmic censorship : any singularity is hidden inside an event horizon.

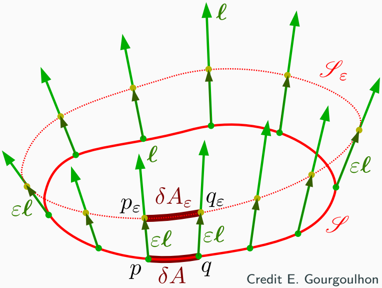


A non-local notion

- Need to know all the trajectories of photons, hence the whole spacetime.
- Does not necessarily track strong gravity.
- At a given point of spacetime, it is not possible to know if one is inside an event horizon : *eventhorizonmeters* cannot exist !
- The notion will not lead to boundary conditions, as needed by our program.



- Consider a spatial 2-surface with an induced metric q.
- Consider a vector field $\boldsymbol{\ell}$ normal to the surface.
- Consider a small displacement of the surface along $\boldsymbol{\ell}$.
- The expansion describes the variation of the surface area.



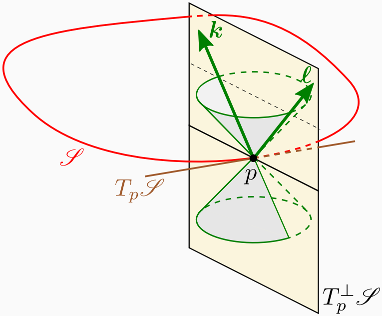
Credit E. Gourgoulhon

$$\theta^{(\boldsymbol{\ell})} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \frac{\delta A_\epsilon - \delta A}{\delta A} = \mathcal{L}_{\boldsymbol{\ell}} \ln \sqrt{q} = q^{\mu\nu} \nabla_\mu \ell_\nu$$

Expansion along null vectors.

- Consider a closed spatial 2-surface.
- Two null normals : an ingoing one $\boldsymbol{k}$ and an outgoing one $\boldsymbol{\ell}$.
- In flat spacetime :

$$\theta^{(\boldsymbol{k})} < 0 \quad \text{and} \quad \theta^{(\boldsymbol{\ell})} > 0.$$



Credit E. Gourgoulhon

Apparent horizon (1/2)

Definition

- A outer trapped surface is such that: $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} < 0$.
- A marginally outer trapped surface (MOTS) is such that:

$$\theta^{(\boldsymbol{k})} < 0 \quad \text{and} \quad \theta^{(\boldsymbol{\ell})} = 0.$$

- Essentially, an apparent horizon is the outermost MOTS (there is no trapped surface outside the apparent horizon).
- At the heart of Penrose (1965) (Nobel prize 2020).

Apparent horizon (2/2)

Some properties

- Apparent horizon traces strong gravitational fields.
- Their presence is linked to the existence of singularities (weak energy condition).
- When an apparent horizon exist it is inside an event horizon (cosmic censorship).
- Discussed in Hawking and Ellis (1973).
- The condition $\theta^{(\boldsymbol{\ell})} = 0$ is local.
- One could build an *apparenthorizonmeter* !
- The condition $\theta^{(\boldsymbol{\ell})} = 0$ leads to a boundary condition, as needed for our program.

Apparent and event horizons

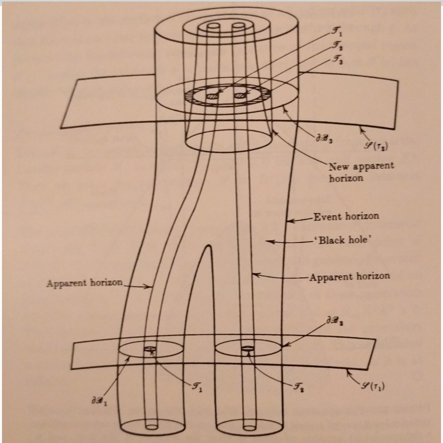


FIGURE 60. The collision and merging of two black holes. At time τ_1, there are apparent horizons $\partial\mathscr{T}_1$, $\partial\mathscr{T}_2$ inside the event horizons $\partial\mathscr{B}_1$, $\partial\mathscr{B}_2$ respectively. By time τ_2, the event horizons have merged to form a single event horizon; a third apparent horizon has now formed surrounding both the previous apparent horizons.

From Hawking and Ellis (1973)

For stationary situations, the two notions coincides !

3+1 expression for the expansion

- Assume that the apparent horizon is a sphere.
- In orthonormal spherical tensorial basis : $s_i = (1, 0, 0)$ is normal to the sphere but not normalized.
- The unit spatial normal is : $\tilde{s}^i = \frac{s^i}{\left(s_k s^k\right)^{1/2}}$.
- The outgoing null normal is then $\ell^\mu = n^\mu + \tilde{s}^\mu$.
- From the 3+1 expressions (especially of the extrinsic curvature tensor), one can show that $\theta^{(\boldsymbol{\ell})} = N\left(D_i \tilde{s}^i + K_{ij}\tilde{s}^i\tilde{s}^j - K\right)$.

This gives us a inner boundary condition on the sphere :

$$D_i \tilde{s}^i + K_{ij}\tilde{s}^i\tilde{s}^j = 0.$$

See Gourgoulhon and Jaramillo (2006)

Additional conditions

Coordinates stationary wrt to the horizon

- Ensures that the horizon stays at constant spatial location.
- Equivalent to the condition $B^i \tilde{s}_i = N$ on the horizon.

Tangential part

- Demand that the shear of the null outgoing rays vanishes (equilibrium).
- The tangential part of the shift must be a conformal Killing vector of the sphere.
- One can choose : $B^i_{\parallel} = -\Omega \left(\partial_\varphi\right)^i$
- Ω encodes the rotation state of the black hole.

All that leads to a boundary condition on the shift : $B^i = N\tilde{s}^i - \Omega \left(\partial_\varphi\right)^i$

See Cook and Pfeiffer (2004) and Vasset et al. (2009)

Freedom on the coordinates

Differential gauges

- The gauges are differential, in the sense that they involve (first) derivatives of the fields :

$$K \equiv \frac{D_k B^k}{N} = 0 \quad \text{and} \quad V^i \equiv \gamma^{kl} \left(\Gamma^i_{kl} - \bar{\Gamma}^i_{kl} \right) = 0.$$

- It follows that the coordinates are defined up to some boundary conditions.
- This differs, from instance, from the QI isotropic gauge where components of the metric are set to zero directly.

Example on the time-coordinate

- Start from a spherically symmetric solution with $K \neq 0 :\ N, B^i, \gamma_{ij}$.
- Consider the variable change : $\mathrm{d}t = \mathrm{d}t' + \alpha\,(r)\,\mathrm{d}r$.
- The new 3+1 quantities can be obtained as a function of the old ones, for instance

$$\gamma'_{rr} = \gamma_{rr} + 2B_r\alpha - \left(N^2 B_i B^i\right)\alpha^2.$$

- The new quantities depend on α but not on its derivative.
- So, $K' = 0$ is an equation that involves the first derivative of α.
- It requires a single boundary condition on the horizon.
- This boundary condition on α can be used to specify the value of N on the horizon.

Additional boundary condition on the horizon $N = N_0\,(\theta, \varphi)$.

Choice of spatial coordinates

- Consider a change of spatial coordinates $x'^i = x^i + \xi^i \left(x^j\right)$.
- The condition $V^i = 0$ leads to a second order equation on $\vec{\xi}$.
- It must be solved using two boundary conditions.
- At infinity, one requires that $\vec{\xi} = 0$ (to recover the usual asymptotic behaviors).
- The boundary conditions on the horizon seem that it can be used to freely specify three quantities.
- However the location of the horizon is fixed $\Longrightarrow$ one must have $\xi^r = 0$.
- This implies that one can choose freely two components of γ on the horizon.

Two additional boundary conditions on the horizon $\gamma_{r\theta} = f\left(\theta, \varphi\right)$ and $\gamma_{r\varphi} = g\left(\theta, \varphi\right)$.

Singular behavior of the equations

Degenerate equations

Example in 1D

- Consider the equation $a\,(r)\,f'' + b\,(r)\,f' + c\,(r)\,f = s\,(r)$.
- $a\,(r) = 0$ at the inner boundary.
- One can not impose any boundary condition on f at the inner boundary.
- The equation is its own boundary condition : $b\,(r)\,f' + c\,(r)\,f = s\,(r)$.

The vanishing factor

- On the horizon, given that $\tilde{s}^i = \dfrac{s^i}{\left(s_k s^k\right)^{1/2}}$ with $s_i = \ (1, 0, 0)$, one can show that $\tilde{s}^r = \sqrt{\gamma^{rr}}$.
- The boundary condition on the shift is $B^i = N\tilde{s}^i - \Omega \left(\partial_\varphi\right)^i$ so that $B^r = N\sqrt{\gamma^{rr}}$.
- From all that, it follows that $g^{rr} = \left(\gamma^{rr} - \dfrac{B^r B^r}{N^2}\right)$ vanished on the horizon.

The term g^{rr} is in front of many second order radial derivative terms...

Application to the 3+1 system

- One looks at the terms involving ∂_r^2 for all the unknown fields, in all the equations.
- Some terms are multiplied by $\left(N^2 - B_i B^i\right)$ which vanishes on the horizon.

Example, the component E_{rr} of the evolution equation contains the following second order radial derivative terms :

$$
\begin{aligned}
E_{rr} = \quad - & \quad \partial_r^2 N + \frac{1}{N} B^r \gamma_{rr} \partial_r^2 B^r + \frac{1}{N} B^r \gamma_{r\theta} \partial_r^2 B^\theta \\
+ & \quad \frac{1}{N} B^r \gamma_{r\varphi} \partial_r^2 B^\varphi - N/2 g^{rr} \partial_r^2 \ \gamma_{rr} + \ldots
\end{aligned}
$$

Order of degeneracy

Matrix description

- Each line corresponds to one equation (H, M^i, E_{ij}).
- Each column corresponds to one unknown (N, B^i, γ_{ij}).
- Each term in the factor of ∂_r^2, for the given equation and given unknown.
- One can show that the null eigenvalue has a multiplicity 3.
- The three degenerate equations correspond to the angular components of the evolution equation.

Three additional boundary conditions $E_{\theta\theta} = 0$, $E_{\theta\varphi} = 0$ and $E_{\varphi\varphi} = 0$.

Additional and final complication

Boundary conditions so far...

- $B^i = N\tilde{s}^i - \Omega\left(\partial_\varphi\right)^i$ (horizon at fixed location, with no shear).
- $D_i\tilde{s}^i + K_{ij}\tilde{s}^i\tilde{s}^j = 0$ (definition of an apparent horizon).
- $N = N_0\left(\theta, \varphi\right)$ (time coordinate freedom).
- $\gamma_{r\theta} = f\left(\theta, \varphi\right)$ and $\gamma_{r\varphi} = g\left(\theta, \varphi\right)$ (spatial gauge freedom).
- $E_{\theta\theta} = 0$, $E_{\theta\varphi} = 0$ and $E_{\varphi\varphi} = 0$ (degenerate equations).

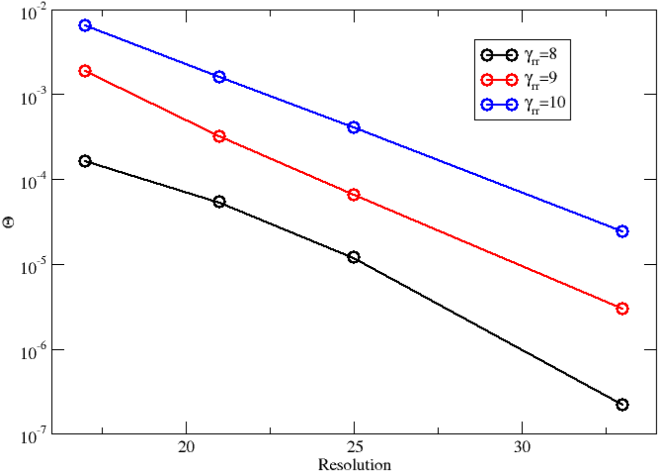
The right number of conditions, in some cases lead to correct solutions.

But convergence is not very good, which may indicate some additional difficulties...

Situation without rotation

- Convergence is much better by replacing $\theta^{(\boldsymbol{\ell})} = 0$ by $\gamma_{rr} = \gamma_0$.
- One can check that indeed, the solutions fulfill $\theta^{(\boldsymbol{\ell})} = 0$.
- They correspond to Schwarzschild solutions with different masses.

Convergence of the expansion



Analytic proof

- In spherical symmetry one has :
  $N(r)$, $B^i = (w(r), 0, 0)$ and $\gamma_{ij} = A(r)\,\mathrm{d}r^2 + B(r)\,\mathrm{d}\Omega^2$.
- $\theta \propto rBw\frac{\partial A}{\partial r} + 2\,rAB\frac{\partial w}{\partial r} + 2\left(rN\frac{\partial B}{\partial r} + 2\,BN\right)\sqrt{A}$
- $K \propto rBw\frac{\partial A}{\partial r} + 2\,rAw\frac{\partial B}{\partial r} + 2\,rAB\frac{\partial w}{\partial r} + 4\,ABw$
- Inject K in $\theta \Longrightarrow \theta \propto \left(N\sqrt{A} - Aw\right)\left(2r\frac{\partial B}{\partial r} + 4B\right)$
- On the horizon (BC for the shift), one has $w = \dfrac{N}{\sqrt{A}}$
- so $K = 0 \Longrightarrow \theta = 0$.
- and $K = 0$ is ensured by the bulk equations.

What about rotation ?

- Enforcing $\gamma_{rr} = \gamma_0$ is too strong.
- It leads to solutions for which $\theta^{(\boldsymbol{\ell})} \neq 0$.
- But one needs to have continuity with the spherically symmetric case.
- Proposal : enforce the value of γ_{rr} only for the $l = m = 0$ harmonic.

Final set of boundary conditions

- $B^i = N\tilde{s}^i - \Omega\left(\partial_\varphi\right)^i$ (horizon at fixed location, with no shear).
- $N = N_0\left(\theta, \varphi\right)$ (time coordinate freedom).
- $\gamma_{r\theta} = f\left(\theta, \varphi\right)$ and $\gamma_{r\varphi} = g\left(\theta, \varphi\right)$ (spatial gauge freedom).
- $E_{\theta\theta} = 0$, $E_{\theta\varphi} = 0$ and $E_{\varphi\varphi} = 0$ (degenerate equations).
- $\gamma_{rr} = \gamma_0$ for $l = m = 0$ and $\theta^{(\boldsymbol{\ell})} = 0$ otherwise.

Suggested values $N_0 = 1/2$, $f = g = 0$ and $\gamma_0 = 8$.

Asserting the validity of a solution

Direct error indicators

- Gauge quantities : $K = 0$ and $V^i = 0$.
- Spherical part of $\theta^{(\boldsymbol{\ell})} = 0$.

On global quantities

- Equality of the ADM and Komar masses :

$$M_{\mathrm{ADM}} = \frac{1}{16\pi} \int_{r=\infty} f^{ik} f^{jl} \left(\bar{D}_j \gamma_{kl} - \bar{D}_k \gamma_{jl} \right) \mathrm{d}S$$

$$M_{\mathrm{Komar}} = \frac{1}{4\pi} \int_{r=\infty} \left(\tilde{s}^i D_i N - K_{ij} \tilde{s}^i \tilde{s}^j \right) \mathrm{d}S.$$

- Equality of the angular momentum and spin :

$$J = \frac{1}{8\pi} \int_{r=\infty} K_{ij} \left(\partial_\varphi \right)^i \tilde{s}^j \mathrm{d}S.$$

$$S = \frac{1}{8\pi} \int_{r=r_H} \sqrt{(}q) K_{ij} \left(\partial_\varphi \right)^i \tilde{s}^j \mathrm{d}S.$$

The Kadath tool

`KADATH` **is a library that implements spectral methods in the context of theoretical physics.**

- It is written in C++, making extensive use of object oriented programming.
- Versions are maintained via git.
- Website : *www.kadath.obspm.fr*
- The library is described in Grandclement (2010).
- Designed to be very modular in terms of geometry and type of equations.
- LateX-like user-interface.
- More general than its predecessor LORENE.

Concept in 1D

Given a set of orthogonal functions Φ_i on an interval Λ, spectral theory gives a recipe to approximate f by

$$f \approx I_N f = \sum_{i=0}^{N} a_i \Phi_i$$

Properties

- the Φ_i are called the basis functions.
- the a_i are the coefficients : it is the quantity stored on the computer.
- Multi-dimensional generalization is done by direct product of basis.
- The computation of the a_i comes from the Gauss quadratures.

Coefficient and configuration spaces

There exist $N+1$ point x_i in Λ such that

$$f(x_i) = I_N f(x_i)$$

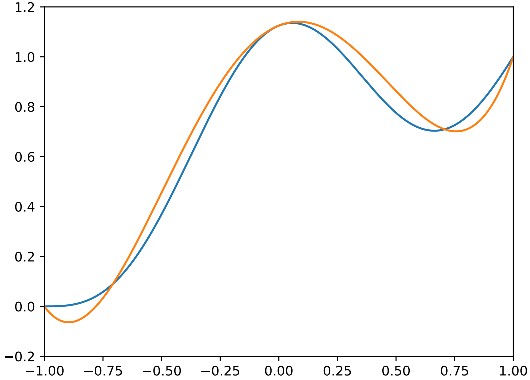
Two equivalent descriptions

- Formulas relate the coefficients a_i and the values $f(x_i)$.
- Complete duality between the two descriptions.
- One works in the coefficient space when the a_i are used (for instance for the computation of f').
- One works in the configuration space when the $f(x_i)$ are employed (for the computation of $\exp(f)$)

Spectral convergence

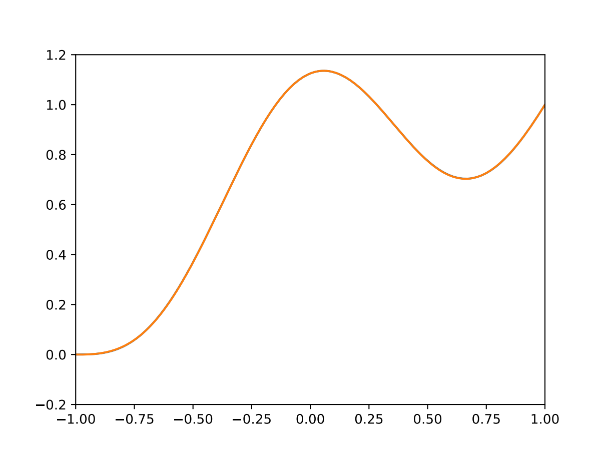
- If f is $\mathcal{C}^\infty$, then $I_N f$ converges to f faster than any power of N.
- For functions less regular (i.e. not $\mathcal{C}^\infty$) the error decrease as a power-law.

Example of interpolant for $N = 4$



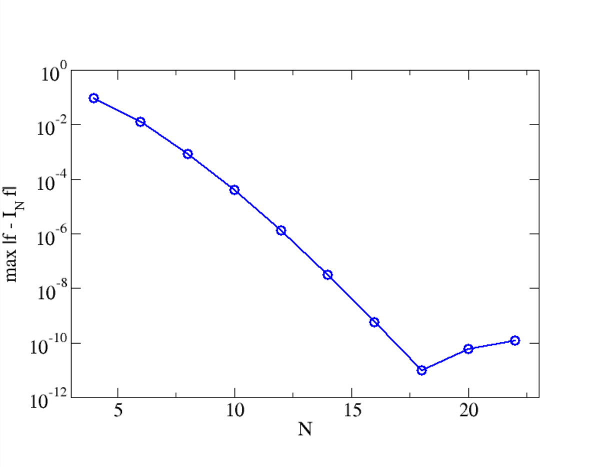
blue curve $f(x) = \cos^3(\pi x/2) + (x+1)^3/8$; *orange* : $I_4 f$.

Example of interpolant for $N = 8$



blue curve $f(x) = \cos^3(\pi x/2) + (x+1)^3/8$; *orange* : $I_8 f$.

Spectral convergence



Multi-domain setting

Numerical coordinates

- Space is divided into several numerical domains.
- In each domain there is a link between the physical coordinates X and the numerical ones $X^\star$.
- Spectral expansion is performed with respect to $X^\star$.
- Non-periodic coordinates are expanded wrt to polynomials.
- Periodic coordinates (i.e. angles) are described by trigonometrical functions.

Weighted residual method

Consider a field equation $R = 0$ (ex. $\Delta f - S = 0$). The discretization demands that

$$(R, \xi_i) = 0 \quad \forall i \leq N$$

Properties

- $(,)$ is the same scalar product as the one used for the spectral approximation.
- the ξ_i are called the test functions.
- For the τ-method, the ξ_i are the basis functions.
- Amounts to cancel the coefficients of R.
- Some equations are relaxed and must be replaced by appropriate boundary and matching conditions.

The discrete system

Original system

- Unknowns : tensorial fields.
- Equations : partial derivative equations.

Discretized system

- Unknowns : coefficients $\vec{u}$.
- Equations : algebraic system $\vec{F}(\vec{u}) = 0$.

Properties

- For a linear system $\vec{F}(\vec{u}) = 0 \Longleftrightarrow A^i_j u^j = S^i$
- In general $\vec{F}(\vec{u})$ is even not known analytically.
- $\vec{u}$ is sought numerically.

Newton-Raphson iteration

Given a set of field equations with boundary and matching equations, `KADATH` translates it into a set of algebraic equations $\vec{F}(\vec{u}) = 0$, where $\vec{u}$ are the unknown coefficients of the fields.

The non-linear system is solved by Newton-Raphson iteration

- Initial guess $\vec{u}_0$.
- Iteration :
  - Compute $\vec{s}_i = \vec{F}(\vec{u}_i)$
  - If $\vec{s}_i$ if small enough $\Longrightarrow$ solution.
  - Otherwise, one computes the Jacobian : $\mathbf{J}_i = \frac{\partial \vec{F}}{\partial \vec{u}}(\vec{u}_i)$
  - One solves : $\mathbf{J}_i \vec{x}_i = \vec{s}_i$.
  - $\vec{u}_{i+1} = \vec{u}_i - \vec{x}_i$.

Convergence is very fast for good initial guesses.

Computation of the Jacobian

Explicit derivation of the Jacobian can be difficult for complicated sets of equations.

Automatic differentiation

- Each quantity x is supplemented by its infinitesimal variation δx.
- The dual number is defined as $\langle x, \delta x \rangle$.
- All the arithmetic is redefined on dual numbers. For instance $\langle x, \delta x \rangle \times \langle y, \delta y \rangle = \langle x \times y, x \times \delta y + \delta x \times y \rangle$.
- Consider a set of unknown $\vec{u}$, and a its variations $\delta \vec{u}$. When $\vec{F}$ is applied to $\langle \vec{u}, \delta \vec{u} \rangle$, one then gets : $\left\langle \vec{F}\left(\vec{u}\right), \delta \vec{F}\left(\vec{u}\right) \right\rangle$.
- One can show that

$$\delta \vec{F}\left(\vec{u}\right) = \mathbf{J}\left(\vec{u}\right) \times \delta \vec{u}$$

The full Jacobian is generated *column by column*, by taking all the possible values for $\delta \vec{u}$, at the price of a computation roughly twice as long.

Numerical resources

Consider N_u unknown fields, in N_d domains, with d dimensions. If the resolution is N in each dimension, the Jacobian is an $m \times m$ matrix with:

$$m \approx N_d \times N_u \times N^d$$

For $N_d = 5$, $N_u = 5$, $N = 20$ and $d = 3$, one reaches $m = 200\,000$

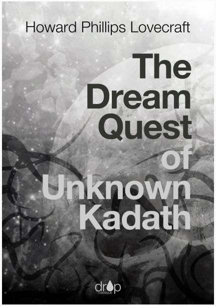
Solution

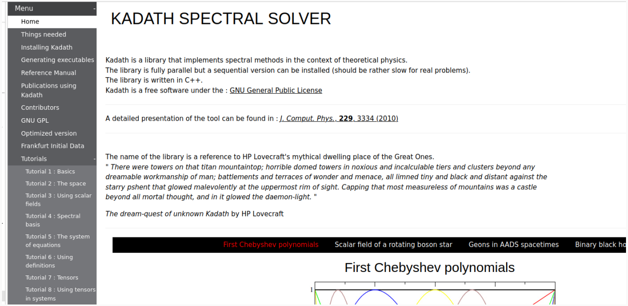
- The matrix is distributed on several processors.
- Easy because the Jacobian is computed column by column.
- The library `SCALAPACK` is used to invert the distributed matrix.

- $d = 1$ problems : sequential.
- $d = 2$ problems : 100 processors (mesocenters).
- $d = 3$ problems : 1000 processors (national supercomputers).

Try it...

Kadath website (https://kadath.obspm.fr) has some tutorials... Have fun...



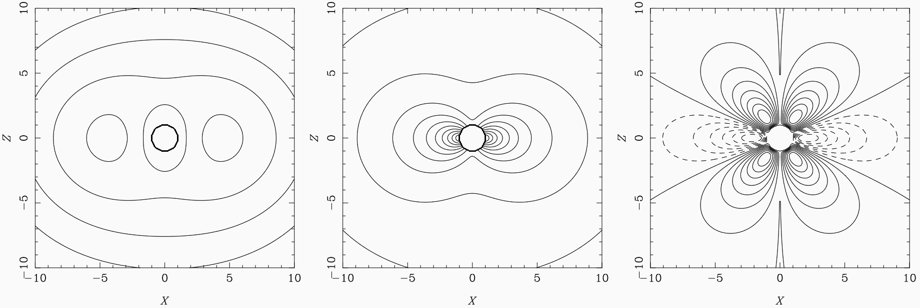


Some applications

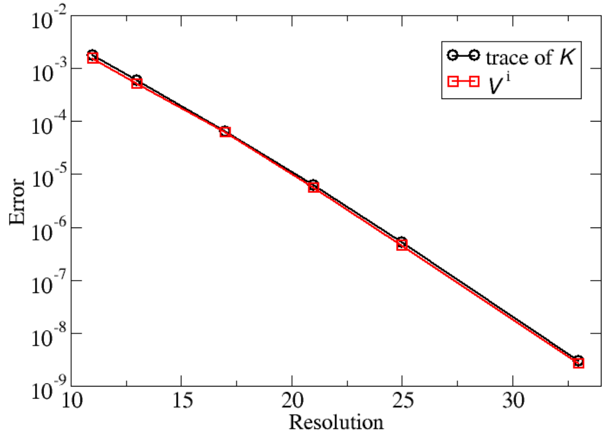
Kerr solution

Direct application of the equations, in the vacuum case

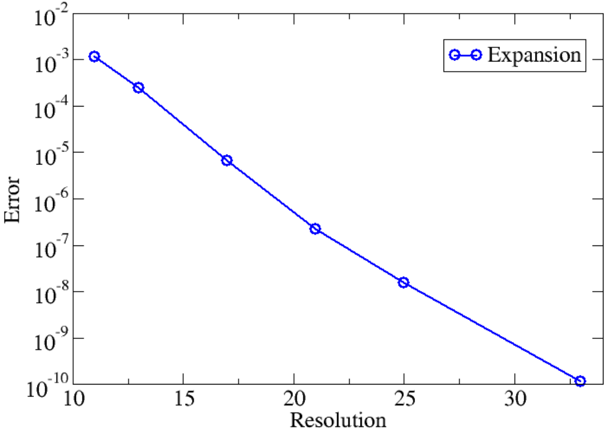
Plots of N, B^r and $\gamma_{r\varphi}$ for $a/M \approx 0.99$.



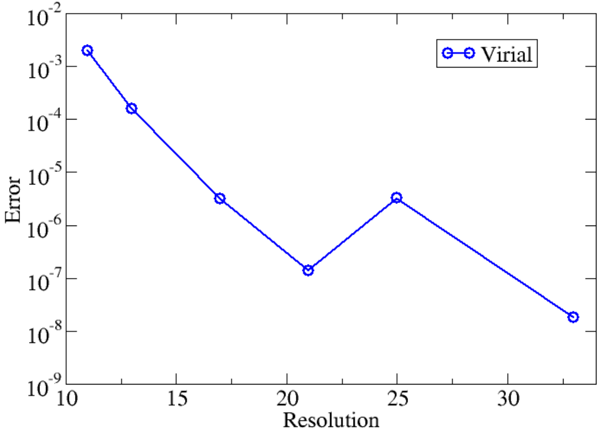
Gauge quantities



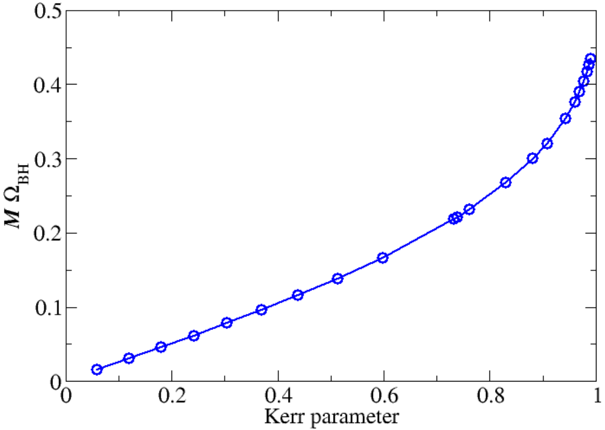
Error on the expansion



$M_{\text{ADM}} = M_{\text{Komar}}$



Some global quantities



Adding a scalar field

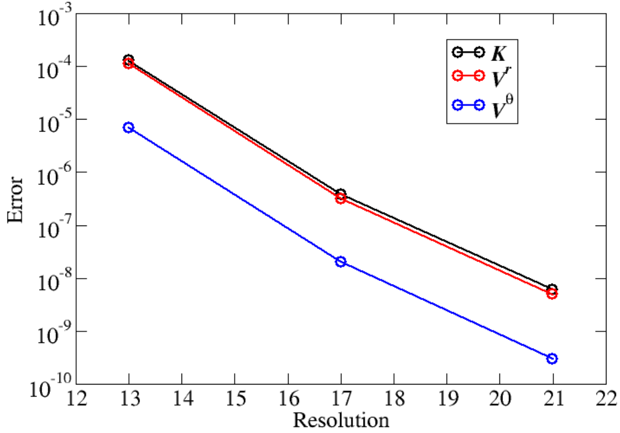
BH with complex scalar hairs

$$S = \int \frac{R}{16\pi} - \frac{1}{2}\left[\nabla_\mu \Phi \nabla^\mu \bar{\Phi} + V\left(|\Phi|^2\right)\right]\sqrt{-g}\mathrm{d}^4 x$$

- Ansatz for the field : $\Phi = \left(R_\Phi\left(r,\theta\right) + i I_\Phi\left(r,\theta\right)\right)\exp\left[i\left(\omega t - k\varphi\right)\right] = 0.$
- Given the form of the action, the metric is stationnary and axi-symmetric.
- The field must obey the KG equation : $\nabla_\mu \nabla^\mu \Phi = \frac{\mathrm{d}V}{\mathrm{d}\left|\Phi\right|^2}\Phi$, which contains both a real and imaginary part.
- Using QI isotropic coordinates, $I_\phi = 0$ but it is not the case in general.

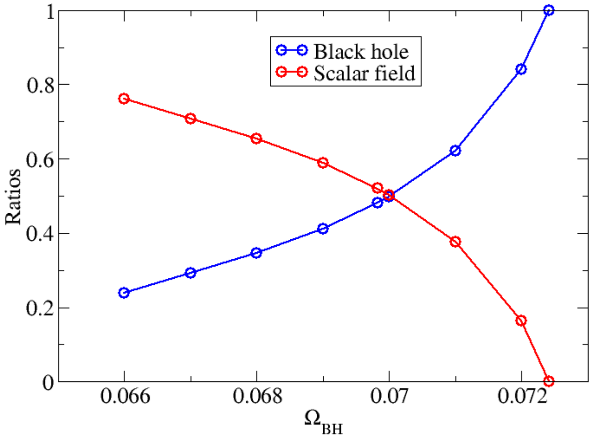
Given its expression, the KG equation is also degenerate on the horizon, the second radial derivative term being $g^{rr}\partial_r^2 \Phi$.

Gauge quantities



Convergence is only observed if the KG equation is correctly solved.

Mass contributions



Conclusion

Formalism

- Based on 3+1 formalism.
- Maximal slicing and spatial harmonic gauge.
- Inner boundary is an apparent horizon in equilibrium.

Applications

- Kerr black hole.
- Black holes with scalar hair : matter, minimally coupled.
- cubic Galileon : exotic matter.
- MTZ black hole : AADS spacetime, change topology of the horizon, no rotation.

Possible extensions

- Rotating horizon with non-usual topology.
- Non-minimally coupled theories.
- Influence of exotic matter, violating the energy conditions.
- Applications to binary systems.

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