Modified gravity

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Introduction : Singularities & Quantum gravity

- Several deep issues of the high energy description of gravitational systems :
  - **Hawking-Penrose theorems** : Under realistic physical conditions, the solutions of General Relativity generically suffer from **geodesic incompleteness and curvature divergences** ;
    E.g. black holes and cosmological models :

$$\begin{aligned} ds^2 &= -\left(1-\frac{2M}{t}\right)dr^2 + \frac{dt^2}{1-\frac{2M}{t}} + t^2\left(d\theta^2+\sin^2\theta d\phi^2\right) \quad ; \quad t\to 0 \\ ds^2 &= -dt^2 + a_0^2\, t^{\frac{4}{3(w+1)}}\left(dx^2+dy^2+dz^2\right) \quad ; \quad t\to 0 \end{aligned} \tag{1}$$

  - **Information loss** : What happens to the information trapped inside black holes ? What is the final stage of Hawking evaporation ? Possible stable black hole remnants accounting for part of dark matter ;
  - **Quantum gravity** (QG) (or Emergent gravity, or modifications of quantum theory ?) : Gravity is perturbatively non-renormalizable, i.e. not possible to quantize like all the other fundamental fields
    - ☐ Path integral : Asymptotic Safety & Causal Dynamical Triangulations ; ... ?
    - ☐ Hamiltonian : Wheeler-DeWitt equation ; Loop Quantum Gravity ;
    - ☐ Higher dimensions : String theory ; Supergravity ; ... ?
    - ☐ Higher order gravities : Stelle gravity ; Conformal gravity ; Non-local gravity ; Horava-Lifshitz ;

Introduction : Modified gravity and infrared issues

- Whatever the high energy completion of GR is, it should have an expansion in terms of **high energy corrections** as an effective field theory ;
- To distinguish between those corrections, make use of known principles :
  - **Second order field equations** ; no ghosts & unitarity ;
  - Stability ;
  - Renormalizable/quantum gravity inspired corrections ;
  - **Symmetry** principles ; etc . . .
- **Other issues at large scales :**
  - Dark energy (Λ ?)
  - Dark matter (primordial black holes, right-handed neutrino ?)
  - Hubble tension

Formalisms of General Relativity : "Gravity = Geometry of spacetime"

- What action should be modified ? General Relativity is a peculiar theory which can be expressed in terms of completely different fields ;
- Assuming that gravity is described in terms of the geometry of space-time means that it depends on a metric field (or a frame) and an affine connection ;
- Theorem : The most general affine connection decomposes as (cf Unicity of Levi-Civita connection) :

$$\bar{\Gamma}^{\sigma}{}_{\mu\nu} = \Gamma^{\sigma}{}_{\mu\nu} + K^{\sigma}{}_{\mu\nu} + L^{\sigma}{}_{\mu\nu} = \Gamma^{\sigma}{}_{\mu\nu} + \delta\Gamma^{\sigma}{}_{\mu\nu} \tag{2}$$

where

$$\begin{aligned} \Gamma^{\sigma}{}_{\mu\nu} &= \frac{1}{2} g^{\sigma\rho}\left(\partial_{(\mu} g_{\nu)\rho} - \partial_{\rho} g_{\mu\nu}\right) \quad \text{Levi-Civita connection} \\ K^{\sigma}{}_{\mu\nu} &= \frac{1}{2}\left(T^{\sigma}{}_{\mu\nu} + T_{(\mu}{}^{\sigma}{}_{\nu)}\right) \quad \text{Contorsion} \\ L^{\sigma}{}_{\mu\nu} &= \frac{1}{2}\left(Q^{\sigma}{}_{\mu\nu} - Q_{(\mu}{}^{\sigma}{}_{\nu)}\right) \quad \text{Disformation} \end{aligned} \tag{3}$$

in terms of the fundamental geometric quantities (and metric field) :

$$\begin{aligned} T^{\sigma}{}_{\mu\nu} &= \bar{\Gamma}^{\sigma}{}_{[\mu\nu]} \quad \text{Torsion} \\ Q_{\sigma\mu\nu} &= \bar{\nabla}_{\sigma} g_{\mu\nu} \quad \text{Non-metricity} \end{aligned} \tag{4}$$

- These quantities give rise to the (post-Riemannian) curvature

$$\bar{R}^{\gamma}{}_{\sigma\mu\nu} = \partial_{[\mu}\bar{\Gamma}^{\gamma}{}_{\nu]\sigma} + \bar{\Gamma}^{\gamma}{}_{[\mu|\rho}\bar{\Gamma}^{\rho}{}_{|\nu]\sigma} = R^{\gamma}{}_{\sigma\mu\nu}(g) + \nabla_{[\mu}\delta\Gamma^{\gamma}{}_{\nu]\sigma} + \delta\Gamma^{\gamma}_{[\mu|\rho}\delta\Gamma^{\rho}{}_{|\nu]\sigma} \tag{5}$$

- Geometrical interpretations :
  - Riemann Curvature R : Given three vectors U, V, W : parallel transport W along closed parallelogram $\{U, V\}$, then

$$\delta W^{\mu} = R_{\mu\nu\rho\sigma}W^{\nu}U^{\rho}V^{\sigma} \tag{6}$$

  - Torsion (% Lie Derivative) : Given two vectors U, V : the parallelogram made of parallel transports (% flows) closes only up to a translation ;

$$T(U, V) = \bar{\nabla}_U V - \bar{\nabla}_V U - [U, V] \tag{7}$$

  - Non-metricity : parallel transport changes length :

$$U^{\mu}\bar{\nabla}_{\mu}(V^{\nu}V_{\nu}) = -V^{\mu}V^{\nu}U^{\sigma}Q_{\sigma\mu\nu} \neq 0 \tag{8}$$

Formalisms of General Relativity

- Further motivations : If gravity is emergent (cf no sense to quantize Navier-Stokes), similar to elastic media with microstructure. E.g. In continuum theory of lattice defects (see Hehl (94)) :
  - Non-metricity emerges from density of point defects ;
  - Torsion emerges from density of line defects ;
- In order to know which theory to modify, need to make a choice between the following equivalent formalisms :
- Metric formalism (historical) : Gravity = Riemann Curvature ;

$$I\left[g_{\mu\nu}\right] = \int d^4x\sqrt{-g}\ \left(R - 2\Lambda\right), \tag{9}$$

- Palatini formalism (metric-compatible) ;

$$I\left[g_{\mu\nu}, \bar{\Gamma}^{\sigma}_{\mu\nu}\right] = \int d^4x\sqrt{-g}\ \left(g^{\sigma\nu}\delta^{\mu}_{\gamma}\bar{R}^{\gamma}{}_{\sigma\mu\nu} - 2\Lambda\right), \tag{10}$$

  - Enjoy an additional local symmetry % GR, projective invariance :

$$\bar{\Gamma}^{\sigma}_{\mu\nu} \to \bar{\Gamma}^{\sigma}_{\mu\nu} + A_{\mu}\delta^{\sigma}_{\nu}\ , \qquad \bar{R}_{\mu\nu} \to \bar{R}_{\mu\nu} + F_{\mu\nu}\left(A\right) \tag{11}$$

  - Necessary (in first order formalism) in Loop Quantum Gravity (Immirzi parameter $\gamma^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{R}_{\mu\nu\rho\sigma}$) ;

- Teleparallel equivalents of GR (∥ transport does not depend on path) :

$$I_{\mathrm{GR}}\left[g_{\mu\nu},\bar{\Gamma}^{\sigma}_{\mu\nu}\right]=\int d^4x\sqrt{-g}\left(\bar{R}+\delta\Gamma\right)=\int d^4x\sqrt{-g}\left(R+\nabla_{\mu}\left(\delta\Gamma^{\mu\nu}{}_{\nu}-\delta\Gamma_{\nu}{}^{\nu\mu}\right)\right) \tag{12}$$

where

$$\delta\Gamma=\delta\Gamma^{\mu\nu}{}_{\sigma}\delta\Gamma^{\sigma}{}_{\mu\nu}-\delta\Gamma^{\sigma}{}_{\sigma\nu}\delta\Gamma^{\nu\mu}{}_{\mu}\ ,\qquad \delta\Gamma^{\sigma}{}_{\mu\nu}=K^{\sigma}{}_{\mu\nu}+L^{\sigma}{}_{\mu\nu}=\frac{1}{2}\left(T^{\sigma}{}_{\mu\nu}+T_{(\mu}{}^{\sigma}{}_{\nu)}+Q^{\sigma}{}_{\mu\nu}-Q_{(\mu}{}^{\sigma}{}_{\nu)}\right) \tag{13}$$

  - □ Teleparallel equivalent of GR : Gravity = Torsion ($\bar{R}=Q=0$) ;
  - □ Symmetric Teleparallel equivalent of GR : Gravity = Non-metricity ($\bar{R}=T=0$) ;
  - □ Features :
    - ⋆ Quadratic actions not unique ;

$$L=\alpha^{\mu\nu\alpha\beta\gamma\delta}(g)\,Q_{\mu\nu\alpha}Q_{\beta\gamma\delta}+\beta^{\mu\nu\alpha\beta\gamma\delta}(g)\,T_{\mu\nu\alpha}T_{\beta\gamma\delta} \tag{14}$$

    - ⋆ The actions are Lorentz invariant only up to boundary terms ;
    - ⋆ Coupling with matter field : More complicated prescription are needed ;

- Purely affine (Eddington (1923)) (Q=0) : Solve $g(\Gamma)$ in Palatini ;

$$I\left[\Gamma^{\sigma}_{\mu\nu}\right]=\frac{1}{\Lambda}\int d^4x\sqrt{-\det\left(\bar{R}_{(\mu\nu)}\right)} \tag{15}$$

Modified gravity

- General action for modified gravity ;

$$I = \int \underbrace{d^D x}_{\text{Dimensionality}} \; L \left[\underbrace{\{g_{\mu\nu}, \Gamma^{\alpha}_{\mu\nu}\}}_{\text{Geometrical fields}}, \underbrace{\{\psi, \phi, A_\mu, A^a_\mu\}}_{\text{Matter fields}}, \underbrace{\{\phi, v_\mu, f_{\mu\nu}\}}_{\text{Background structures}}, \underbrace{\{\lambda\}}_{\text{Lagrange multipliers}}, \underbrace{\{\partial^n\}}_{\text{Higher-order \& non-locality}} \right] \tag{16}$$

- Geometrical fields : metric & affine connection ;
- Matter fields : scalar, vector, gauge field, spinor, higher rank tensors, other spins ;
- Background structures : time coordinate, aether, fiducial metric ;
  - ☐ Can break Lorentz & diffeomorphism invariances ;
  - ☐ Although it is possible to restore symmetries by Stuckelberg fields (cf artificial/substantial gauge symmetries, Dirac (55), Francois (2023)) ;
- Notice that same form of corrections in different formalisms can be widely inequivalent $f(R)$, $f(Q)$, $f(T)$, Born-Infeld type action :

$$I\left[g_{\mu\nu}, \Gamma^{\sigma}_{\mu\nu}\right] = \int d^4x \sqrt{-\det\left(g_{\mu\nu} + \bar{R}_{(\mu\nu)}\right)} \tag{17}$$

which is Ghost free in Palatini and has a ghost in the metric formalism ;

Theoretical criterions

- Unicity
- Stability : positive energy (no ghosts : $a > 0$) and no gradient instability ($b > 0$):

$$L = a\dot{\phi}^2 - bg^{ij}\partial_i\phi\partial_j\phi \tag{18}$$

- All classical field theories in physics have second order equations of motion
- Have some kind of internal symmetries or participate (gauge fields, charged particle, Higgs, etc). In particular Conformal Invariance :

$$I\left[g_{\mu\nu}\right] = I\left[e^{\phi}g_{\mu\nu}\right] \tag{19}$$

- Unification : cf metric affine, Kaluza Klein
- Similar exact solutions structure as GR & primary % secondary hair
- Simplicity : scalar fields are good theoretical labs because they are the simplest possible modifications : just 1 additional degree of freedom % GR.

Theoretical criterion : Lovelock unicity theorem

- Lovelock-Lanczos gravity (LLG) is the only purely metric (UV) extension of GR w. diffeomorphism invariance and 2^{sd} order field equations ;

$$I = \frac{1}{2\kappa} \int_{\mathcal{M}} d^D x \sqrt{-g} \sum_{p=0}^{\lfloor D/2 \rfloor} \alpha_p \mathcal{L}_p \ , \qquad \mathcal{L}_p = \frac{1}{2^p} \delta^{\mu_1 \nu_1 \dots \mu_p \nu_p}_{\sigma_1 \rho_1 \dots \sigma_p \rho_p} \prod_{r=1}^{p} R^{\sigma_r \rho_r}_{\mu_r \nu_r} \ , \tag{20}$$

where $R^{\mu\nu}_{\sigma\rho} = g^{\gamma\mu} R_{\sigma\rho\gamma}{}^{\nu}$ and $\delta^{\mu_1 \dots \mu_p}_{\sigma_1 \dots \nu_p} = \delta^{\mu_1}_{[\nu_1} \dots \delta^{\mu_p}_{\nu_p]}$,

$$\mathcal{L}_0 = 1 \ , \qquad \mathcal{L}_1 = R \ , \qquad \mathcal{L}_2 = R^2 - 4 R^{\mu}_{\nu} R^{\nu}_{\mu} + R^{\sigma\rho}_{\mu\nu} R^{\mu\nu}_{\sigma\rho} \ , \tag{21}$$

- ☐ Topological in $D = 2p$, because $\propto$ to the Euler characteristic $\chi(\mathcal{M})$ (= genus of closed orientable manifolds) ; Trivial in $D < 2p$;
- ☐ Unique purely gravitational theory whose Palatini formalism always admits a Levi-Civita solution ;
- ☐ Appears in the low-energy effective action of String Theory ;
- ☐ Related to Chern-Simons and Born-Infeld theories of gravity ;

Theoretical criterions : quantum gravity predictions

Although we do not have a consensus on a quantum gravity theory, there are many results regarding the description of gravitating systems at small scales which can be obtained in quantum field theory in curved spacetimes and semi-classical gravity ;

- What is the gravitational field of an electron, of an atom, etc... ?
- What is the effect of quantum fields on black hole horizons, on an expanding Universe ?
  - Hawking radiation : black hole evaporate ;
  - Particle production in an expanding universe ;
- Semi-classical gravity $\Longleftrightarrow$ (Classical gravity + Quantum matter + backreaction)

$$G_{\mu\nu} = \frac{8\pi G}{c^4} \langle \hat{T}_{\mu\nu} \rangle \tag{22}$$

Valid for $E \ll E_{\text{Planck}} \approx 10^{19}$GeV and small quantum fluctuations.

Two predictions for high energy gravity

- QFTCST and semi-classical gravity yields the Trace Anomaly :
  - Quantum Conformal Field Theories (CFT) are rigorously defined QFTs (cf. "cut-off free") ; Quantum Conformal Invariance implies :

$$\eta^{\mu\nu}\langle 0|\hat{T}_{\mu\nu}|0\rangle_{\text{ren}} = 0 \tag{23}$$

  - When considered in (classical) $(d = 2p)$-dimensional curved spacetime, a so-called Trace Anomaly appears :

$$g^{\mu\nu}\langle 0|\hat{T}_{\mu\nu}|0\rangle_{\text{ren}} = a\,\mathcal{E}_p + \cdots = a\left(\frac{1}{2^p}\delta^{\mu_1\nu_1\ldots\mu_p\nu_p}_{\sigma_1\rho_1\ldots\sigma_p\rho_p}\prod_{r=1}^{p} R^{\sigma_r\rho_r}_{\mu_r\nu_r}\right) + \ldots \tag{24}$$

    The Quantum Conformal Invariance is broken by the so-called *Euler densities/Critical Lovelock gravity* ;
  - In two spacetime dimensions, the Hawking effect is due entirely to the trace anomaly, [Christensen and Fulling, Phys.Rev.**D**15(1977)].

Therefore, the (quantum) effective equation for gravity should account for that term :

$$G_{\mu\nu} + \zeta\mathfrak{G}_{\mu\nu} + \cdots = \kappa T_{\mu\nu}\ ,\quad g^{\mu\nu}\mathfrak{G}_{\mu\nu} = \mathcal{E}_p \overset{4D}{=} \mathcal{G} = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{25}$$

Universal correction to the Bekenstein-Hawking Entropy found in many QG approaches :

- Perturbative quantum gravity [Solodukhin, Phys.Rev.**D**51, 618 (1995)]
- String theory [Maldacena, Strominger, Witten, JHEP 9712:002,(1997)]
- Loop quantum gravity [Engle, Noui, Perez, Phys.Rev.Lett. 105, (2010)].

$$\boxed{S = \frac{A}{4} + \zeta \log\left(A\right) + \ldots} \tag{26}$$

Is it possible to find local effective field theories of d-dimensional gravity with these (quantum) properties ?

2^{sd} order field equations : 1. Unicity theorems - Horndeski

- Scalar-tensor theory : Horndeski (1974)

$$I_{\text{Horndeski}}\left[g_{\mu\nu},\phi\right] = \int d^4x\sqrt{-g}\sum_{n=2}^{5}\mathcal{L}_n^{\text{H}} \qquad X = -g^{\mu\nu}\phi_\mu\phi_\nu\,, \quad \phi_\mu = \nabla_\mu\phi\,, \quad \phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi \tag{27}$$

$$\begin{aligned}&\mathcal{L}_2^{\text{H}} := G_2\left(\phi,X\right)\ , \quad \mathcal{L}_3^{\text{H}} := G_3\left(\phi,X\right)\Box\phi\,, \quad \mathcal{L}_4^{\text{H}} := G_4\left(\phi,X\right)R + G_{4,X}\delta^{\alpha\beta}_{\mu\nu}\phi^\mu_\alpha\phi^\nu_\beta\\ &\mathcal{L}_5^{\text{H}} := G_5\left(\phi,X\right)G^\mu_\nu\phi^\nu_\mu - \frac{1}{6}G_{5,X}\delta^{\alpha\beta\rho}_{\mu\nu\sigma}\phi^\mu_\alpha\phi^\nu_\beta\phi^\sigma_\rho\,.\end{aligned} \tag{28}$$

- U(1)-tensor theory : Horndeski (1976)

$$I_{\text{U(1)}}\left[g_{\mu\nu},A_\mu\right] = \int d^4x\sqrt{-g}\left(R + \mathscr{L}\left(F_{\mu\nu}F^{\mu\nu},{}^*F_{\mu\nu}F^{\mu\nu}\right) + \gamma\ {}^*F_{\sigma\rho}{}^*F^{\mu\nu}R^{\sigma\rho}_{\mu\nu}\right), \tag{29}$$

where ${}^*F^{\rho\sigma} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}$ and $R^{\mu\nu}_{\sigma\rho} = g^{\gamma\mu}R_{\sigma\rho\gamma}{}^{\nu}$

Horndeski theories accounting for the trace anomaly : 2D

- In two dimensions, a scalar-tensor theory with these properties has been found from a *dimensional regularization of Einstein gravity* (the first Critical Lovelock theory) ; (Mann, (1993)) :

Consider a conformal metric $\bar{g}_{\mu\nu} = e^{\phi} g_{\mu\nu}$,

$$\mathfrak{L}_{(1)} = \lim_{d \to 2} \frac{\sqrt{-\bar{g}}\bar{R} - \sqrt{-g}R + \dots}{d-2} = \frac{\phi}{2}\left(R - \frac{1}{2}\Box\phi\right) \tag{30}$$

so that half-onshell, it reduces to the famous two-dimensional *Polyakov quantum gravity action* (Polyakov, (1981))

$$2\frac{\delta\mathfrak{L}_1}{\delta\phi} = R - \Box\phi = 0 \implies \phi = \frac{1}{\Box}R \implies \mathfrak{L}_1 = \frac{1}{4}R\frac{1}{\Box}R \tag{31}$$

This theory accounts for the trace anomaly :

$$\mathcal{E}_1 = R = \kappa\, T_{\mu\nu}g^{\mu\nu} \tag{32}$$

Horndeski theories accounting for the trace anomaly : 4D

- In four dimensions, a scalar-tensor theory with these properties has been found from :
  - ⋆ Renormalization Group flows Komargodski (2011)
  - ⋆ Compactification of higher-dimensional Lovelock gravity Van Acoleyen (2011), Charmousis (2015)
  - ⋆ Dimensional regularization of Gauss-Bonnet gravity Hennigar (2020), Fernandes (2020)

The theory is given by the (ghost-free) theory (w. $\phi_\mu = \partial_\mu \phi$) :

$$I = \int d^4x\sqrt{-g}\,\left[R + \zeta\left(\frac{\phi}{2}\mathcal{L}_2 - G^{\mu}_{\nu}\phi_\mu\phi^\nu - \frac{1}{2}\phi^\zeta\phi_\zeta\Box\phi - \frac{\left(\phi^\zeta\phi_\zeta\right)^2}{8}\right)\right] \tag{33}$$

and account from the trace anomaly and the logarithmic correction to the entropy :

$$R + \frac{\zeta}{2}\mathcal{E}_2 = \kappa T_{\mu\nu}g^{\mu\nu}\,, \quad S = \frac{A}{4} + 2\pi\zeta\log\frac{A}{A_0} \tag{34}$$

2^{sd} order field equations : Conformally invariant theories in 4D

- Exact solutions, identical to that of Lovelock (for FLRW, spherically symmetric and slowly rotating black hole) but in lower dimensions, e.g. for regularization of Gauss-Bonnet gravity,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \,, \quad f(r) = 1 + \frac{r^2}{2\alpha}\left(1 - \sqrt{1 + 4\alpha\left(\frac{2M}{r^3} + \frac{\Lambda}{3}\right)}\right) \,,$$
$$\phi_\pm(t,r) = qt - 2\log\left(\frac{r}{r_0}\right) \pm \int dr \frac{\sqrt{q^2 r^2 + 4f(r)}}{r f(r)} \longrightarrow \text{secondary hair } q \tag{35}$$

- This theory is closely related to conformal invariance, assuming only conformal invariant field equation of the scalar field (Fernandes (2020)), the unique four-dimensional scalar-tensor action with second order equations of motion is :

$$I = \int d^4x \sqrt{-g}\,(R - 2\Lambda) + \int d^4x \sqrt{-\tilde{g}}\,(\beta \tilde{R} - 2\lambda) + \gamma \lim_{D\to 4} \frac{1}{D-4} \int d^D x \left(\sqrt{-\tilde{g}}\tilde{\mathcal{G}} - \sqrt{-g}\mathcal{G}\right)$$
$$= \int d^4x \left(R - 2\Lambda + \beta\left(\frac{R}{6}\phi^2 + g^{\mu\nu}\phi_\mu\phi_\nu\right) - 2\lambda\phi^4 + \alpha\left(\frac{\phi}{2}\mathcal{L}_2 - G^\mu_\nu\phi_\mu\phi^\nu - \frac{1}{2}\phi^\zeta\phi_\zeta\Box\phi - \frac{\left(\phi^\zeta\phi_\zeta\right)^2}{8}\right)\right) \quad (3$$

Higher-order : 1. UV improved theories

- One way to bypass Lovelock theorem is to allow interactions with higher (than two) order field equations ;
- Properties and features :
  - ☐ Yields additional degrees of freedom (more initial conditions needed) ;
  - ☐ Improves the UV behaviour of the fields (higher powers of momentum in integrals) ;
  - ☐ In some cases, equivalent to Pauli-Villars regularization ;
- Examples :
  - ☐ Stelle Gravity (77) : Renormalizable ; scale and conformal invariant UV corrections ;

$$I = \int d^4x \sqrt{-g} \left(R + \alpha R^2 + \beta C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} \right) \tag{37}$$

  - ☐ Bopp-Podolsky action (40, 42)

$$I_{BP} = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{a^2}{2} \partial_\mu F^{\mu\nu} \partial^\rho F_{\rho\nu} \right) \tag{38}$$

  - ☐ One photon and one ghost-like "Pauli-Villars" massive photon (a Proca field) of mass $1/a^2$
  - ☐ Non-singular modified Coulomb potential for specific boundary cdts : $V(r) = \frac{q}{r} \left(1 - e^{-r/a}\right)$

Higher-order : 2. Ostrogradski ghost

- Higher order derivatives generically implies additional degrees of freedom and usually have negative kinetic energy for these. Consider the Pais-Uhlenbeck oscillator,

$$L = \ddot{q}^2 + \alpha\dot{q}^2 - \beta q^2 \quad \Longleftrightarrow \quad \begin{aligned} L_{\text{equiv}} &= \phi^2 + \alpha\dot{q}^2 - \beta q^2 + \lambda\left(\ddot{q} - \phi\right) \\ &= \phi^2 + \alpha\dot{q}^2 - \beta q^2 - \left(\dot{\lambda}\dot{q} + \lambda\phi\right) + \text{b.t.} \end{aligned} \tag{39}$$

  - ☐ At the level of the Lagrangian, solving $\delta L_{\text{equiv}}/\delta\phi = 2\phi - \lambda = 0$, in ϕ and defining we obtain

$$L_{\text{equiv}}|_{\phi(\lambda)} = \alpha\dot{\gamma}^2 - \frac{\dot{\lambda}^2}{4\alpha} + V(\lambda,\gamma) \;, \quad \text{with} \quad \gamma = q - \frac{\lambda}{2\alpha} \tag{40}$$

  so the kinetic energy has the wrong sign ;
  - ☐ At the level of the Hamiltonian, considering instead

$$\tilde{L}_{\text{equiv}} = \dot{\phi}^2 + \alpha\phi^2 - \beta q^2 + \lambda\left(\dot{q} - \phi\right) \;, \quad \text{with} \quad \{p,\psi\} = \frac{\partial\tilde{L}_{\text{equiv}}}{\partial\partial_t\{q,\phi\}} \tag{41}$$

  we obtain,

$$H = p\dot{q} + \psi\dot{\phi} - \tilde{L}_{\text{equiv}} = -\frac{1}{4}\psi^2 + \phi p + \tilde{V}(q,\phi) \tag{42}$$

Higher-order : 3. Ostrogradski theorem and caveats

- Ostrogradski theorem : A non-degenerate theory with higher order equations of motion yields a Hamiltonian that is linear in one of the momenta and so is unbounded from below ;
- This does not necessarily yield classical instability :
- Theorem (Liapunov) : If $V(q, \dot{q})$ is an integral of motion with a strict minimum at the equilibrium position $q = \dot{q} = 0$, then this equilibrium point is stable.
- In some cases, the associated quantum theory has negative norm states and the theory suffers from a quantum instability ;
- However, we will see that there are some caveats :
  - ☐ In quantum field theories : examples of non-perturbative QED and Non-local gravity
  - ☐ Different Hamiltonians can yields same classical field equations : some can be bounded from below : $f(R)$ theory ;
  - ☐ More generally, the energy in (metric) gravity is a very subtle notion ;
  - ☐ Degenerate theories ;

Higher-order : 2. Ghosts in quantum field theories

- Klein-Gordon, propagator and particle production $I = \frac{1}{2}\int d^4x\phi\left(\Box - m^2\right)\phi \longrightarrow G(q) = \frac{1}{q^2-m^2}$
- Effective (quantum) actions are usually non-local, for example for QED :

$$\Gamma_0 = -\frac{1}{4}\int d^4x F_{\mu\nu}P\left(\Box\right)F^{\mu\nu}\,, \qquad P\left(q^2\right) = 1 - \frac{\alpha}{3\pi}\log\left(\frac{q^2-4m_e^2}{4m_e^2}\right) \quad \text{at one loop} \tag{43}$$

It vanishes (Landau pole) at $q \approx 10^{227}$GeV ($q \approx 10^{34}$GeV for the standard model) : Perturbative theory breaks down. It can be seen as a ghost.

- However, non-perturbative (resummation of loops) form-factor $P\left(\Box\right)$ can solve the issue and be ghost-free, even though they have ghosts (artifacts) in perturbation (see Platania 2020)
- Generalization for gravity : Non-local gravity (see Beneito 2022)

$$I = \int d^4x\sqrt{-g}\left(R + R\gamma_0\left(\Box\right)R + R_{\mu\nu}\gamma_1\left(\Box\right)R^{\mu\nu}\right) \tag{44}$$

- Finally, some theories with Ostrogradski ghost can be quantized and are unitary (no ghost) (see Donoghue 2021)

$$I = \frac{1}{2}\int d^4x\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 - \frac{1}{M^2}\left(\Box\phi\right)^2\right) \tag{45}$$

Higher-order : 3. The case of $f(R)$

- Not all higher order theories have ghosts, even if (one of the many forms of) their Hamiltonian is linear in one of the momenta ;
- Consider the case of $f(R)$ gravity :

$$S = \int d^4x\sqrt{-g}f(R) \equiv \int d^4x\sqrt{-g}\left(f(\phi) + \frac{df(\phi)}{d\phi}(R-\phi)\right) \tag{46}$$

whose field equations are indeed higher order :

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = 0\,, \qquad F(R) = \partial f/\partial R \tag{47}$$

- Under a conformal transformation, it reduces to a ghost-free minimally coupled scalar-tensor theory ;
- For instance, the Starobinsky model of inflation given by

$$S = \int d^4x\sqrt{-g}\left(R + \frac{\lambda}{16}R^2\right) \tag{48}$$

yields after the above redefinition followed by a conformal transformation :

$$S \equiv \int d^4x\sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right)\,, \quad V(\phi) = \frac{1}{\lambda}\left(1 - \exp\left(-\phi/\sqrt{3}\right)\right)^2 \tag{49}$$

Higher-order : 4. Energy of (metric) gravitational theories

- The gravitational energy (Hamiltonian of GR) is vanishing onshell for manifold without boundary ;
- Hamiltonian formalism of General Relativity : For any non-null vector n (ie $g_{\mu\nu}n^\mu n^\nu \neq 0$),

$$R + 2\delta^{\mu\nu}_{\sigma\rho}\nabla_\mu\left[\frac{n^\sigma\nabla_\nu n^\rho}{n_\gamma n^\gamma}\right] = \delta^{\xi\mu\nu}_{\zeta\sigma\rho}\left(\frac{n_\xi n^\zeta}{n_\gamma n^\gamma}\right)\left[\frac{1}{2}R^{\sigma\rho}_{\mu\nu} + 2\frac{\nabla_\mu n^\sigma\nabla_\nu n^\rho}{n_\gamma n^\gamma}\right] \tag{50}$$

- If we define the projector orthogonal to n,

$$h^\mu_\nu = \delta^\mu_\nu - \frac{n^\mu n_\nu}{n^\gamma n_\gamma}\,, \quad \text{which implies} \quad h_{\mu\nu}n^\nu = 0\,, \quad \text{and} \quad \delta^{\xi\mu\nu}_{\zeta\sigma\rho}\left(\frac{n_\xi n^\zeta}{n_\gamma n^\gamma}\right) = h^{\mu\nu}_{\sigma\rho} = h^\mu_\sigma h^\nu_\rho - h^\mu_\rho h^\nu_\sigma \tag{51}$$

- With this method, the extrinsic curvature appears naturally as an algebraic object. Defining

$$K^\mu_\nu = h^\mu_\alpha h^\beta_\nu \nabla_\beta n^\alpha\,, \tag{52}$$

and noting that $G^\xi_\zeta = -\frac{1}{4}\delta^{\xi\mu\nu}_{\zeta\sigma\rho}R^{\sigma\rho}_{\mu\nu}$, we obtain

$$R + 2\delta^{\mu\nu}_{\sigma\rho}\nabla_\mu\left[\frac{n^\sigma\nabla_\nu n^\rho}{n_\gamma n^\gamma}\right] = -2\left(\frac{n^\mu n^\nu}{n_\gamma n^\gamma}\right)G_{\mu\nu} + \frac{2}{n_\gamma n^\gamma}h^{\mu\nu}_{\sigma\rho}K^\sigma_\mu K^\rho_\nu \tag{53}$$

- We obtain for a manifold without boundary

$$I=\int d^4x\sqrt{-g}R=\int d^4x\sqrt{-g}\left(-H_\perp+2K^\mu_\nu\pi^\nu_\mu\right) \tag{54}$$

where

$$H_\perp=2\left(\frac{n^\mu n^\nu}{n_\gamma n^\gamma}\right)G_{\mu\nu}\quad,\qquad \pi^\mu_\sigma=\frac{1}{n_\gamma n^\gamma}h^{\mu\nu}_{\sigma\rho}K^\rho_\nu\iff K^\rho_\nu=n_\gamma n^\gamma h^{(-1)\,\sigma\rho}_{\mu\nu}\pi^\mu_\sigma \tag{55}$$

- Introducing a ADM foliation adapted to the vector n, ie $M=R\times\Sigma$ where $h_{\mu\nu}$ is the projector to Σ and n_μ is its normalized normal (set $n^2=-1$)

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=-N^2dt^2+h_{ab}\left(dx^a+N^a dt\right)\left(dx^b+N^b dt\right)\ ,\quad n_\mu=-\frac{\delta^0_\mu}{N} \tag{56}$$

  - Latin indices are spatial $a,b=1,2,3$
  - N and N^a are respectively the lapse and the shift
  - D the covariant derivative compatible with h_{ab}

- Using the decompositions of extrinsic curvature and 4D determinant,

$$K_{ab} = \frac{1}{2N}\left(\partial_0 h_{ab} - D_{(a}N_{b)}\right)\,, \qquad \sqrt{-g} = N\sqrt{|h|} \tag{57}$$

we obtain

$$\begin{aligned} I &= \int d^4x\sqrt{-g}\left(-H_\perp + 2K^\mu_\nu \pi^\nu_\mu\right) \\ &= \int dt \int_\Sigma d^3x\sqrt{|h|}\left[\pi^{ab}\partial_0 h_{ab} - NH_\perp - N_a H^a \underbrace{-D_{(a}\left(N_{b)}\pi^{ab}\right)}_{\text{Spatial boundary term}}\right] = \int dtL \end{aligned} \tag{58}$$

where the vector constraint and scalar constraint (using the Gauss-Codazzi equation $(*)$) are given by

$$H_a = -2D_b\pi^b_a\,, \quad H_\perp = 2\left(\frac{n^\mu n^\nu}{n_\gamma n^\gamma}\right)G_{\mu\nu} = -\frac{1}{2}h^{\mu\nu}_{\sigma\rho}R^{\sigma\rho}_{\mu\nu} \overset{*}{=} -R^{(3)} + h^{\mu\nu}_{\sigma\rho}K^\sigma_\mu K^\rho_\nu = H_\perp\left[h_{ab}, \pi^{ab}\right] \tag{59}$$

Therefore, π_{ab} is the momentum density, N, N^a are Lagrange multipliers and the Hamiltonian vanishes onshell,

$$H = \int_\Sigma d^3x\left(\sqrt{h}\pi^{ab}\partial_0 h_{ab} - L\right) = \int_\Sigma d^3x\sqrt{h}\left[NH_\perp + N_a H^a\right] \overset{\text{onshell}}{=} 0 \tag{60}$$

Degenerate theories : Invertible disformal transformations

- It is possible to obtain equivalent theories from the previous second order Lagrangians by performing any invertible field transformations :

$$\tilde{g}_{\mu\nu} = \Phi g_{\mu\nu} + f_{(\mu\nu)}\,, \quad \text{with} \quad \det\left(\frac{\delta \tilde{g}_{\rho\sigma}}{\delta g_{\mu\nu}}\right) \neq 0 \tag{61}$$

- Although, the Einstein and Jordan frames are equivalent : eg

$$\int d^4x\sqrt{-g}\,(R - 2\Lambda) + I_{\text{M}}\,[\psi, g] \equiv \int d^4x\sqrt{-\tilde{g}}\,\left(\tilde{R} - 2\Lambda\right) + I_{\text{M}}\,[\psi, \tilde{g}] \tag{62}$$

- This is no longer the case when the matter action couples to the Einstein frame metric :

$$\int d^4x\sqrt{-\tilde{g}}\,\left(\tilde{R} - 2\Lambda\right) + I_{\text{M}}\,[\psi, g] \tag{63}$$

- Disformal transformation : contains derivative of a scalar and so generates higher-order equations of motion while preserving the degrees of freedom :

$$\tilde{g}_{\mu\nu} = \Phi\,(\phi, X)\,g_{\mu\nu} + \Psi\,(\phi, X)\,\phi_\mu\phi_\nu \tag{64}$$

- In addition, it is a very useful solution-generating-technique enabling to obtain exact solutions in highly non-trivial theories (see eg Babichev (2020)) ;

Degenerate theories : Pais-Uhlenbeck oscillator

- Why these higher order field equations have no ghost ? : Degenerate coupled Pais-Uhlenbeck oscillator

$$L = \ddot{q}^2 + \alpha\dot{q}^2 + a\dot{Q}^2 + 2b\ddot{q}\dot{Q} + c\dot{q}\dot{Q} - V(q,Q) \\ \Longleftrightarrow \\ L_{\text{equiv}} = \dot{\phi}^2 + \alpha\phi^2 + a\dot{Q}^2 + 2b\dot{\phi}\dot{Q} + c\phi\dot{Q} - V(q,Q) + \lambda(\dot{q}-\phi) \tag{65}$$

- Phase space : $\{p,q\} = \{\psi,\phi\} = \{P,Q\} = 1$ and momenta :

$$p = \frac{\partial L_{\text{equiv}}}{\partial\partial_t q} = \lambda\,, \quad \psi = \frac{\partial L_{\text{equiv}}}{\partial\partial_t \phi} = 2\left(b\dot{Q} - \dot{\phi}\right)\,, \quad P = \frac{\partial L_{\text{equiv}}}{\partial\partial_t Q} = 2\left(a\dot{Q} + b\dot{\phi}\right) + c\phi \tag{66}$$

  - □ If $a + b^2 \neq 0$, it is possible to express the velocities in terms of the momenta

$$\lambda = p\,, \quad \dot{Q} = \frac{1}{2(a+b^2)}\left(P + b\psi - c\phi\right)\,, \quad \dot{\phi} = \frac{1}{2(a+b^2)}\left(bP - a\psi - bc\phi\right) \tag{67}$$

    so we end up with 3 degrees of freedom with a ghost.
  - □ If $a + b^2 = 0$, we have a constraint among the momenta and the Lagrangian becomes

$$C = P + b\psi - c\phi = 0\,, \quad L_{\text{equiv}} = \left(\dot{\phi} - b\dot{Q}\right)^2 + c\phi\dot{Q} - V(q,Q) + \lambda(\dot{q}-\phi) + \alpha\phi^2 \tag{68}$$

- 6-dimensional phase space : $\{p,q\}=\{\psi,\phi\}=\{P,Q\}=1$, with coordinate q_i and momenta p_i, and primary constraint :

$$p=\lambda\,,\quad \psi=2\left(b\dot{Q}-\dot{\phi}\right)\,,\quad C=P+b\psi-c\phi\approx 0 \tag{69}$$

- Adding the constraint to the theory via a Lagrange multiplier γ enables invert the velocities

$$\begin{aligned} H_{\text{tot}} &= p\dot{q}+\psi\phi+P\dot{Q}-L_{\text{equiv}}+\left(\gamma-\dot{Q}-\frac{1}{4b^2}C\right)C \\ &= -\frac{1}{4}\psi^2+p\phi-a\phi^2+V\left(q,Q\right)+\gamma C=H_0+\gamma C \end{aligned} \tag{70}$$

- The secondary and tertiary constraint (time evolution of the primary) yields,

$$\begin{aligned} \dot{C} &= \{C,H_{\text{tot}}\}=\sum_{i=1}^{3}\left(\frac{\delta C}{\delta q_i}\frac{\delta H_{\text{tot}}}{\delta p_i}-\frac{\delta C}{\delta p_i}\frac{\delta H_{\text{tot}}}{\delta q_i}\right)=-bp+\frac{c}{2}\psi+2\alpha b\phi-V_Q\approx 0 \\ \ddot{C} &= \{\dot{C},H_{\text{tot}}\}=\{\dot{C},H_0\}+\gamma\{\dot{C},C\}\approx 0 \end{aligned} \tag{71}$$

where

$$\{\dot{C},H_{\text{tot}}\}=-\frac{c}{2}p-\alpha b\psi+\alpha c\phi\,,\qquad \{\dot{C},C\}=2\alpha b^2+\frac{c^2}{2}-V_{QQ} \tag{72}$$

- ☐ If $\{\dot{C},C\}\neq 0$, this fixes the Lagrange multiplier γ. (Otherwise keep going with quaternary etc)

- Using the secondary constraint, we obtain the physical Hamiltonian which is quadratic in the momentum :

$$H_{\text{physical}} = -\frac{P^2}{4b^2} + \left(\alpha + \frac{c^2}{4b^2}\right)\phi^2 - \frac{\phi V_Q}{b} + V \tag{73}$$

- The theory possesses $(6-2)/2 = 2$ degrees of freedom and the Ostrogradski ghost has disappeared.

Degenerate theories : Classifications of higher order theories

- Non-perturbative one-loop slowly varying approximation in QED : Euler-Heisenberg Lagrangian and its extensions ;
- Scalar-tensor (DHOSTs) Langlois, Noui (2016), B. Achour (2016)

$$I_{\text{quadratic}} = \int d^4x\sqrt{-g}\left(f\left(\phi, X\right)R + C^{\mu\nu\rho\sigma}\left(g, \phi, \phi_\alpha\right)\phi_{\mu\nu}\phi_{\rho\sigma}\right) \tag{74}$$

$$I_{\text{cubic}} = \int d^4x\sqrt{-g}\left(h\left(\phi, X\right)\phi^\mu\phi^\nu G_{\mu\nu} + C^{\mu\nu\rho\sigma\gamma\delta}\left(g, \phi, \phi_\alpha\right)\phi_{\mu\nu}\phi_{\rho\sigma}\phi_{\gamma\delta}\right) \tag{75}$$

- U(1)-tensor theory :

$$I_{\text{quadratic}} = \int d^4x\sqrt{-g}\left(\mathscr{L}\left(F^2, {}^*FF\right) + \frac{1}{4}\mathscr{A}^{\mu\nu\rho\sigma}\left(F, g\right)R_{\mu\nu\rho\sigma} + \mathscr{B}^{\gamma\mu\nu,\delta\rho\sigma}\left(F, g\right)\nabla_\gamma F_{\mu\nu}\nabla_\delta F_{\rho\sigma}\right), \tag{76}$$

where $\mathscr{A}$ and $\mathscr{B}$ are the most general tensors built from $F_{\rho\sigma}$ with the corresponding symmetries.

Degenerate theories

- For quadratic scalar-tensor, degeneracy conditions obtained imposing

$$\mathcal{L}_{\text{kin}} = \mathcal{K}^{ijkl} K_{ij} K_{kl} + \mathcal{B}^{ij} \ddot{\phi} K_{ij} + \mathcal{A} \ddot{\phi}^2 + V\left(\dot{\phi}\right) = \mathcal{K}^{ijkl} \left(K_{ij} + \mathcal{E}_{ij} \ddot{\phi}\right) \left(K_{kl} + \mathcal{E}_{kl} \ddot{\phi}\right) + V\left(\dot{\phi}\right) \tag{77}$$

  where $\{\mathcal{K}, \mathcal{B}, \mathcal{A}, \mathcal{E}\}$ come from $\{f, C\}$ and depend on $\phi, \dot{\phi}, \partial_i \phi$.
- Similarly for quadratic U(1)-tensor, degeneracy conditions obtained imposing

$$\begin{aligned} \mathcal{L}_{\text{kin}} &= \mathcal{K}^{ijkl} K_{ij} K_{kl} + \mathcal{B}^{ijk} \dot{E}_k K_{ij} + \mathcal{A}^{ij} \dot{E}_i \dot{E}_j + V\left(E, B\right) \\ &= \mathcal{K}^{ijkl} \left(K_{ij} + \mathcal{F}^a_{ij} \dot{E}_a\right) \left(K_{kl} + \mathcal{F}^b_{kl} \dot{E}_b\right) + V\left(E, B\right) \end{aligned} \tag{78}$$

  where $\{\mathcal{K}, \mathcal{B}, \mathcal{A}, \mathcal{F}\}$ and depend on E, B.
- Beyond Horndeski (obtained from disformal transformation of the Horndeski action)

$$\int d^4x \sqrt{-g} F_4\left(\phi, X\right) \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma}{}_{\sigma} \phi_\mu \phi_\alpha \phi_{\nu\beta} \phi_{\rho\gamma} \tag{79}$$

- Lovelock's unique purely metric action with third order field equations

$$\int d^4x \sqrt{-g} \star R^{\mu\nu}_{\alpha\beta} \star R^{\rho\sigma}_{\mu\nu} \star R^{\alpha\beta}_{\rho\sigma} \tag{80}$$

Thank you for your attention !