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The $B \to \pi\pi\ell\bar{\nu}$ process on the lattice

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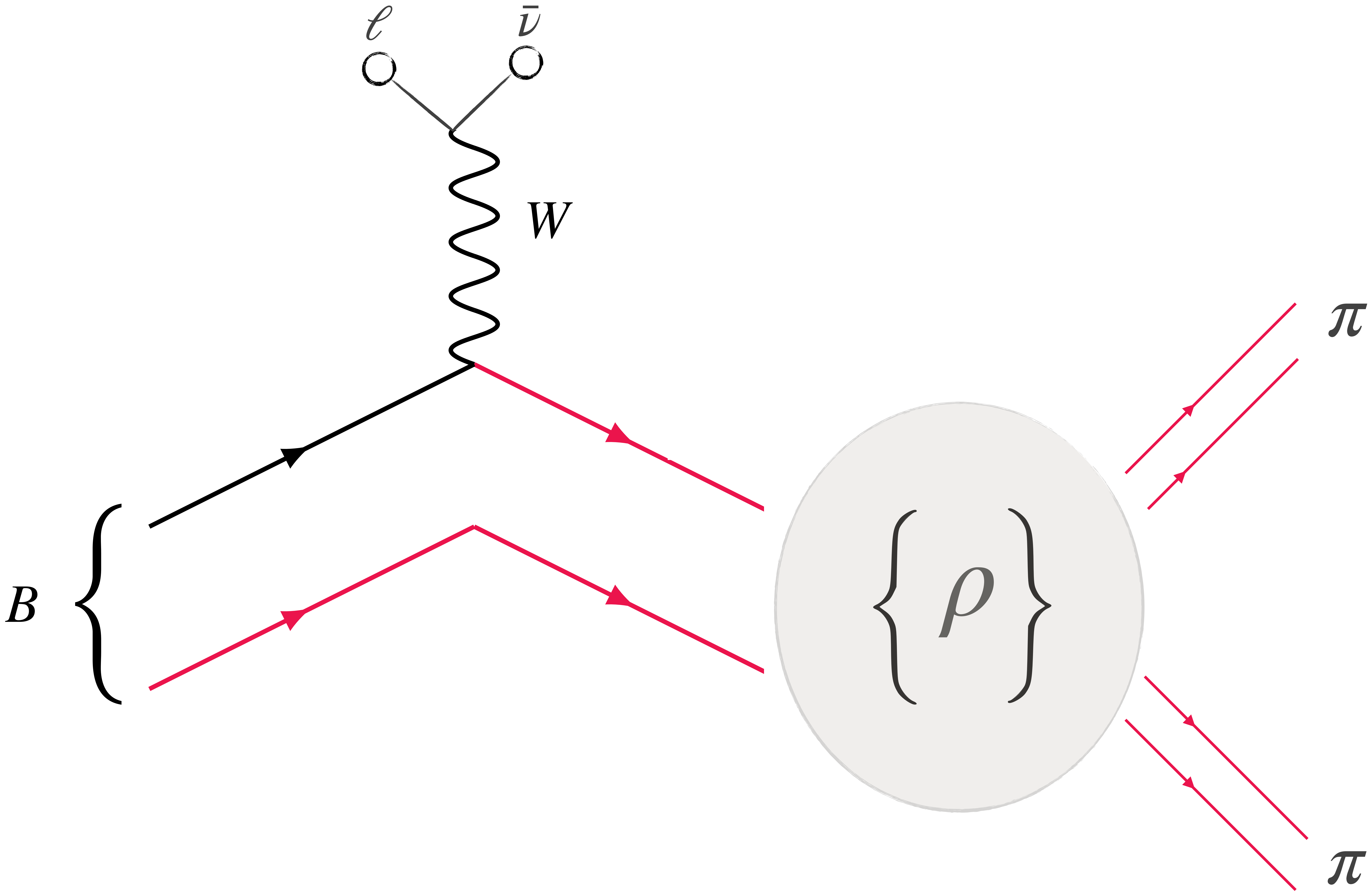
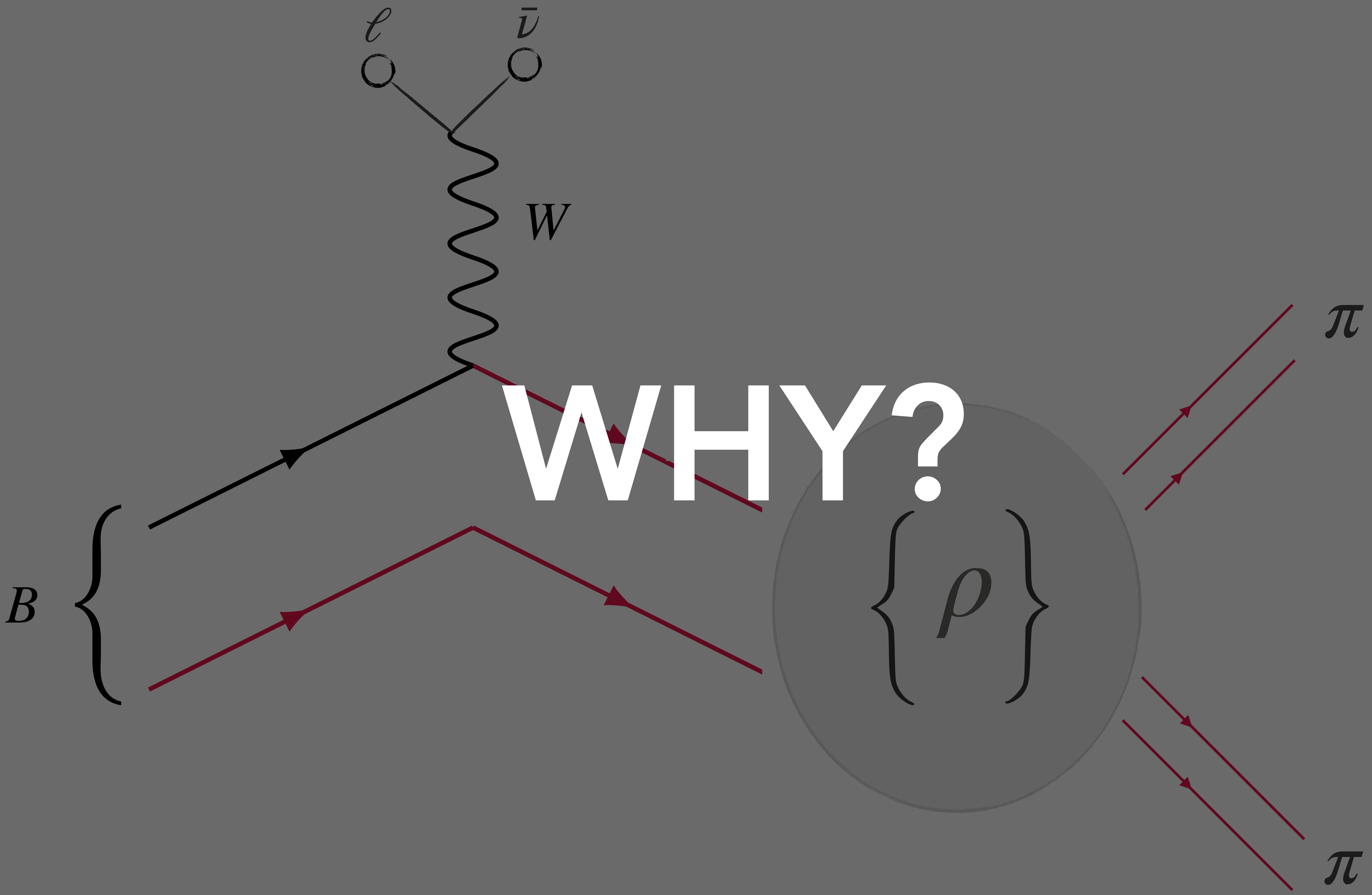


ℓ
$\bar{\nu}$
W
B
$\{\rho\}$
π
π


ℓ
$\bar{\nu}$
W
WHY?
B
ρ
π
π

the puzzle of $|V_{ub}|$

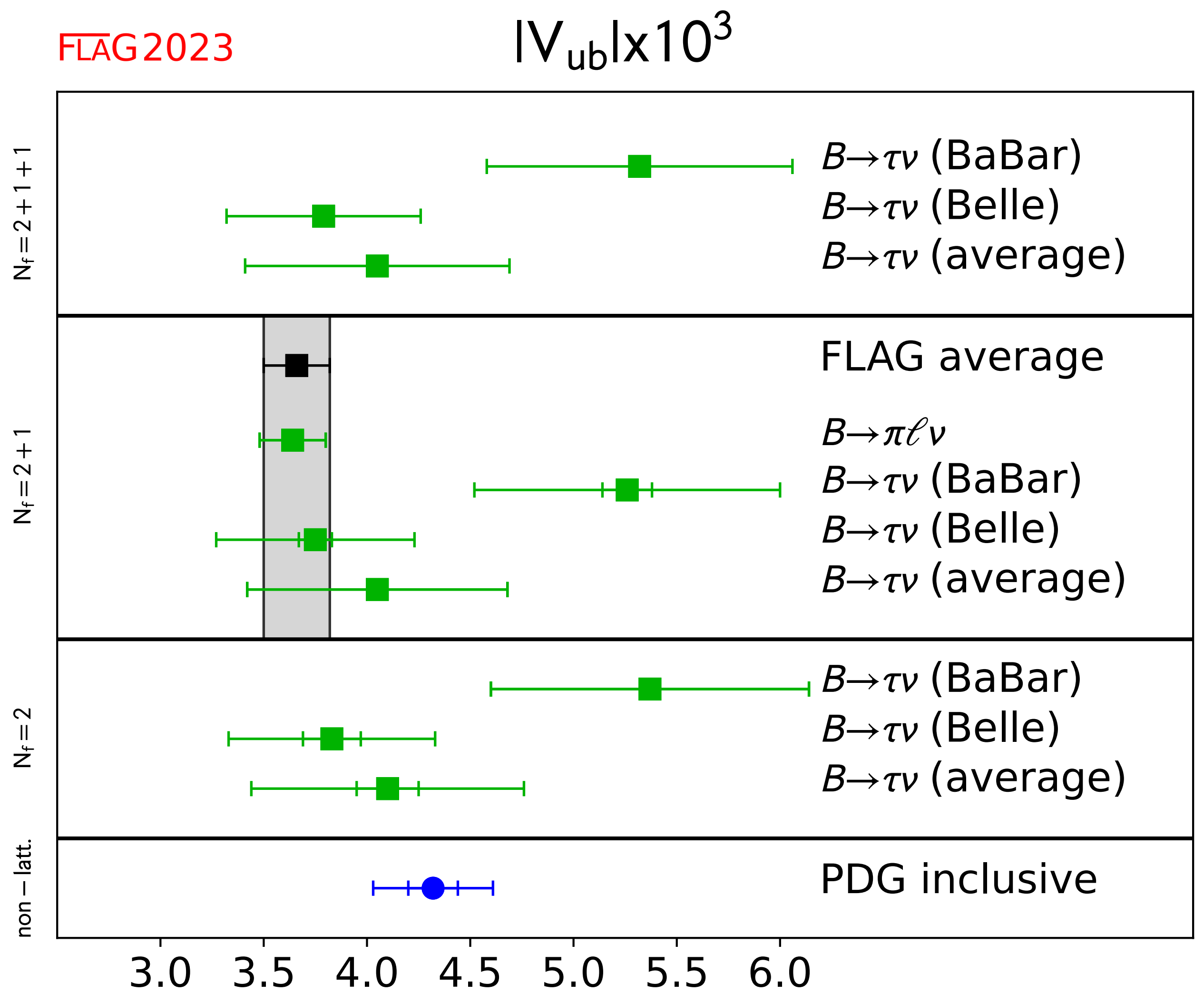
M. Hohmann, Challenges in Semileptonic B decays

Decay Channel	B^+ [%]	B^0 [%]
$B \to \pi \ell \nu$	0.152(3)	0.082(2)
$B \to \rho \ell \nu$	0.147(2)	0.082(1)
$B \to \omega \ell \nu$	0.127(2)	-
$B \to \eta \ell \nu$	0.127(4)	-
$B \to \eta' \ell \nu$	0.097(4)	-
$B \to x_u \ell \nu$	0.1030(5)	0.0540(4)

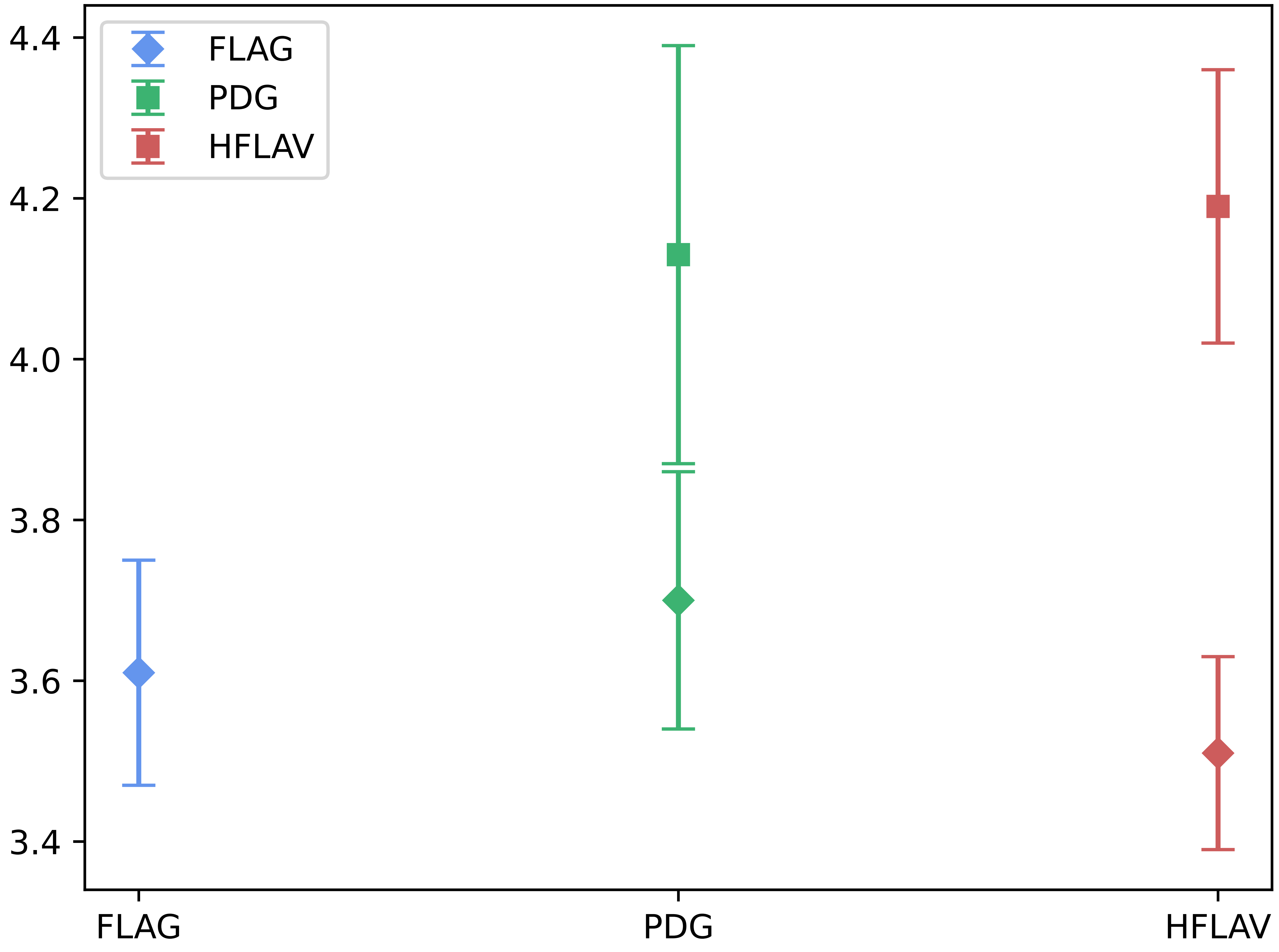
+

contributions from the V_{cb} processes

inclusive determination



exclusive determination



so how do we solve it?

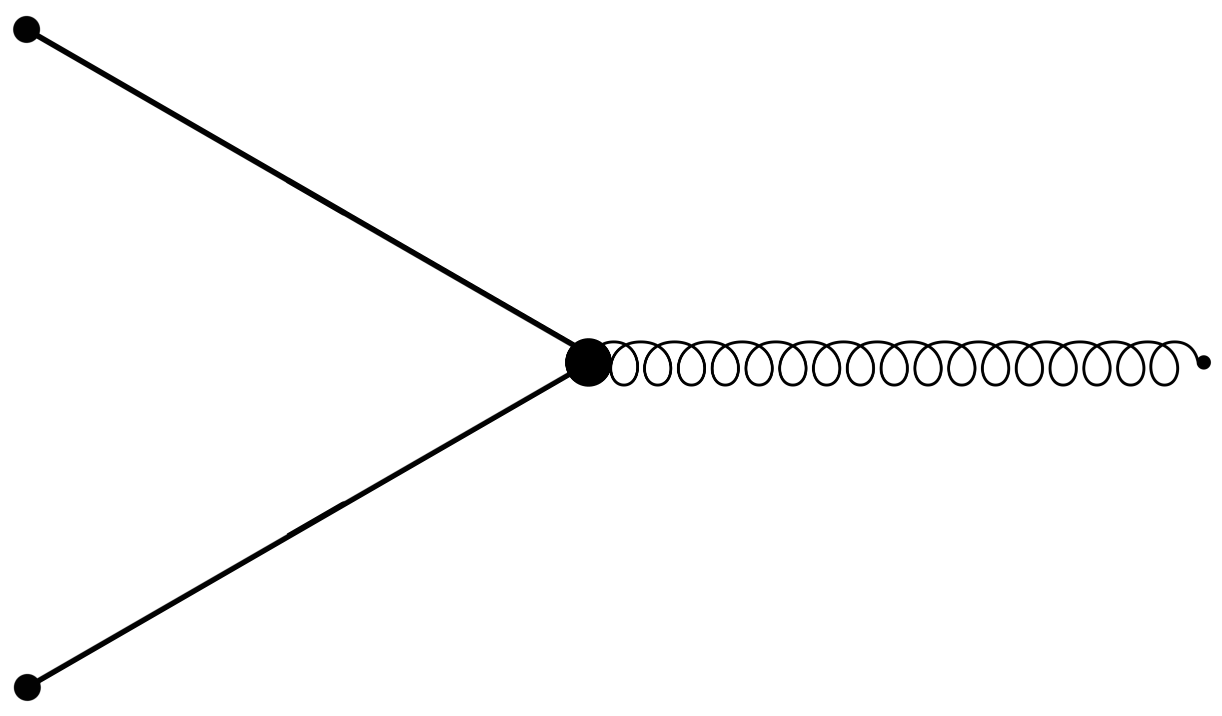
systematically!

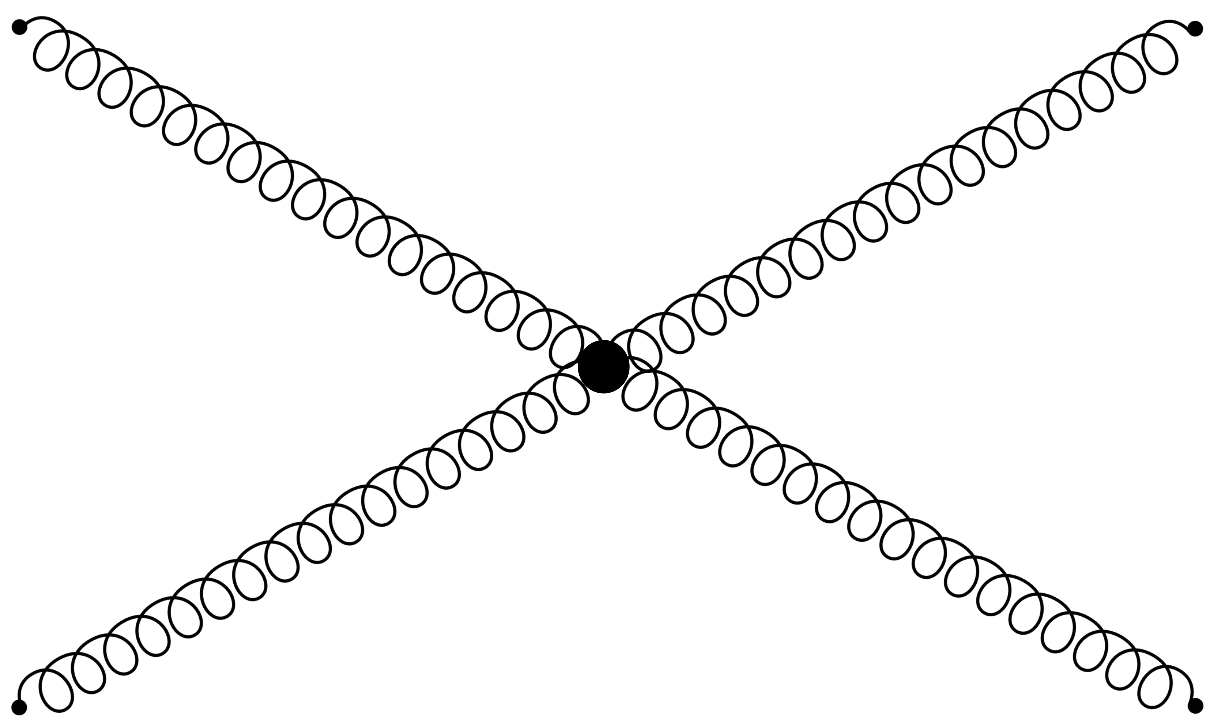
- improve inclusive methodology (great progress in inclusive decays on lattice: 2111.12774, 2405.06152)

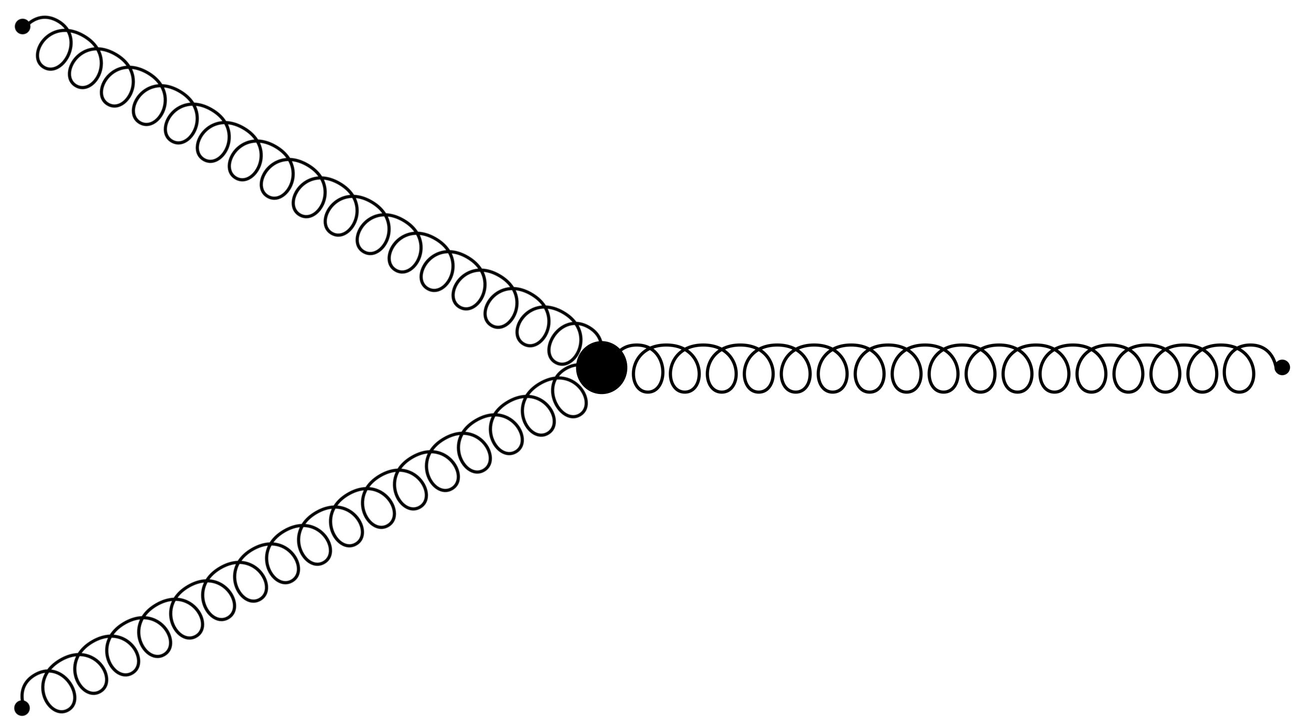
- **add exclusive processes (today)**

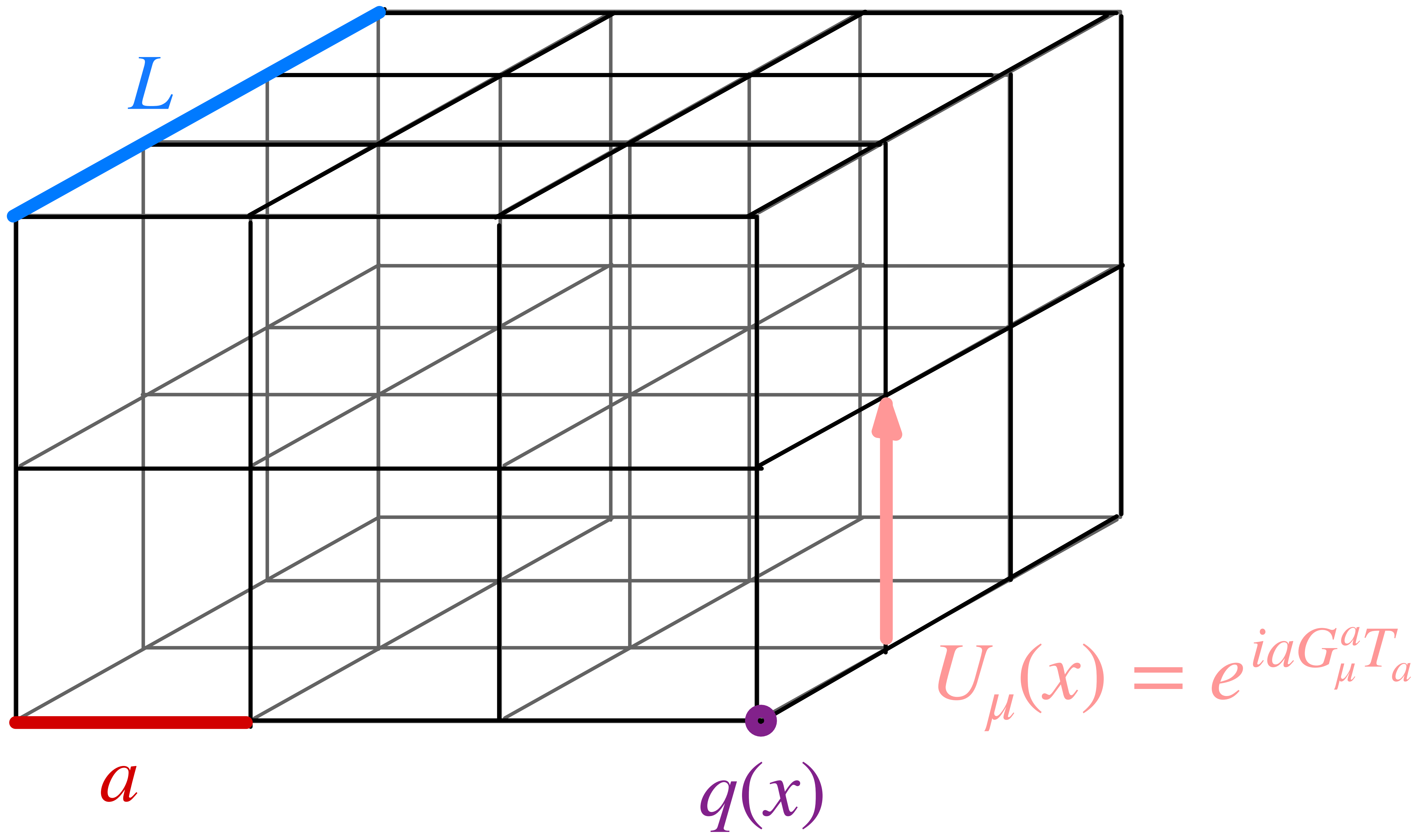
Quantum Chromodynamics

$$\mathscr{L}_{QCD} = \sum_q \left(\bar{q}_i i\gamma_\mu \left[\delta_{ij}\partial^\mu + i(G_a^\mu T^a)_{ij} \right] q_j - m_q \bar{q}_i q_i \right) - \frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu}$$



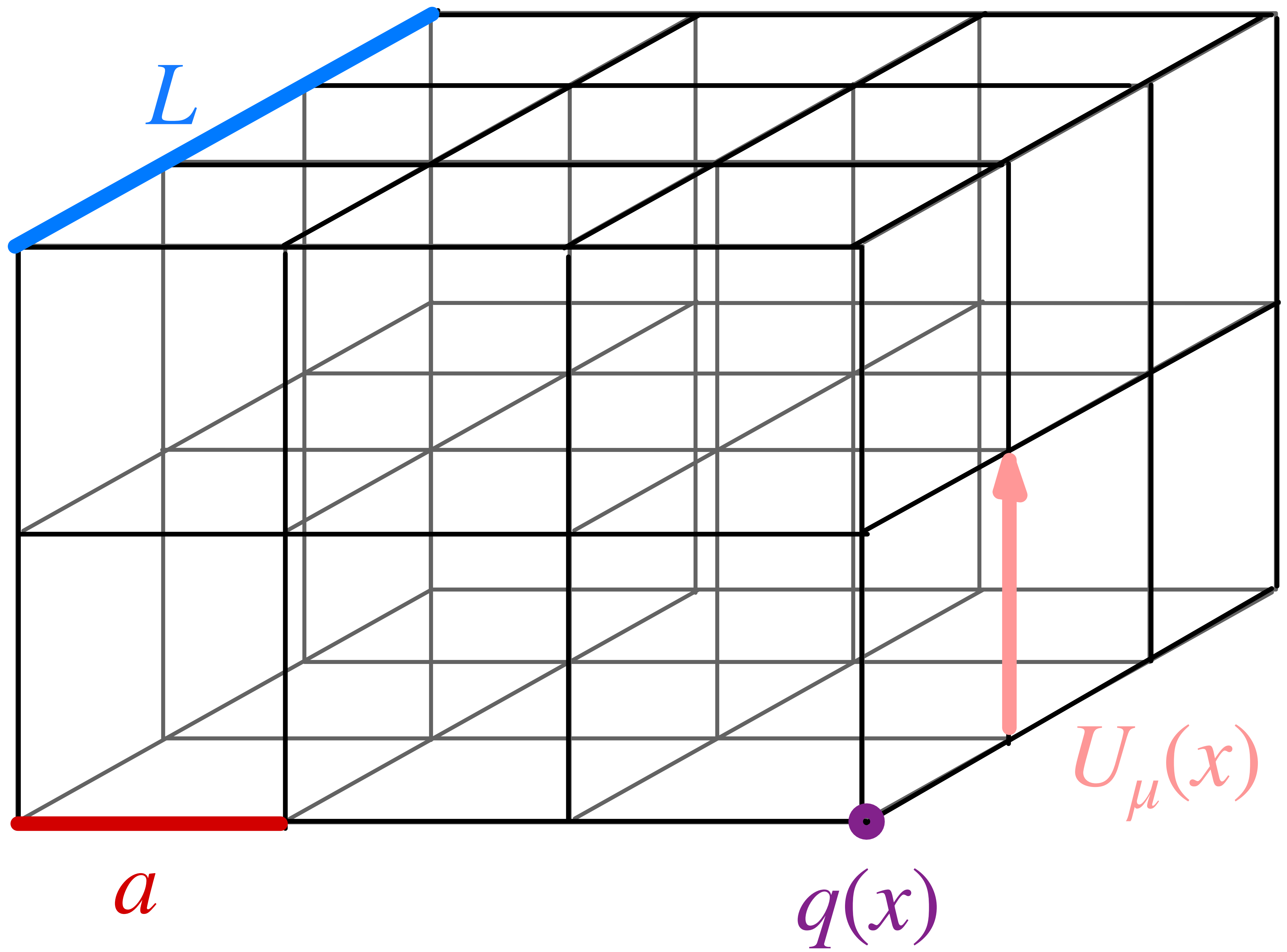




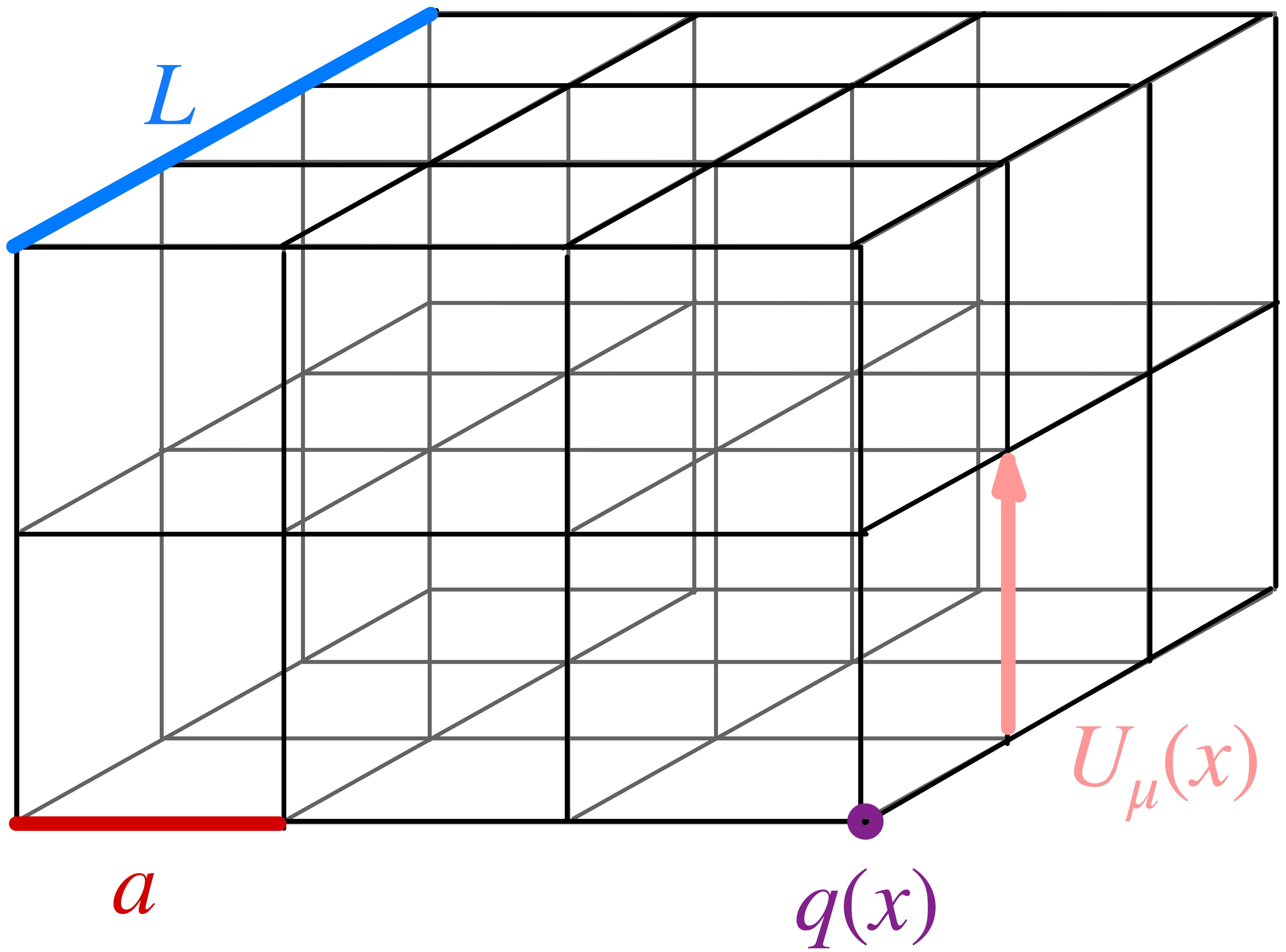


how do we add exclusive processes?

lattice QCD



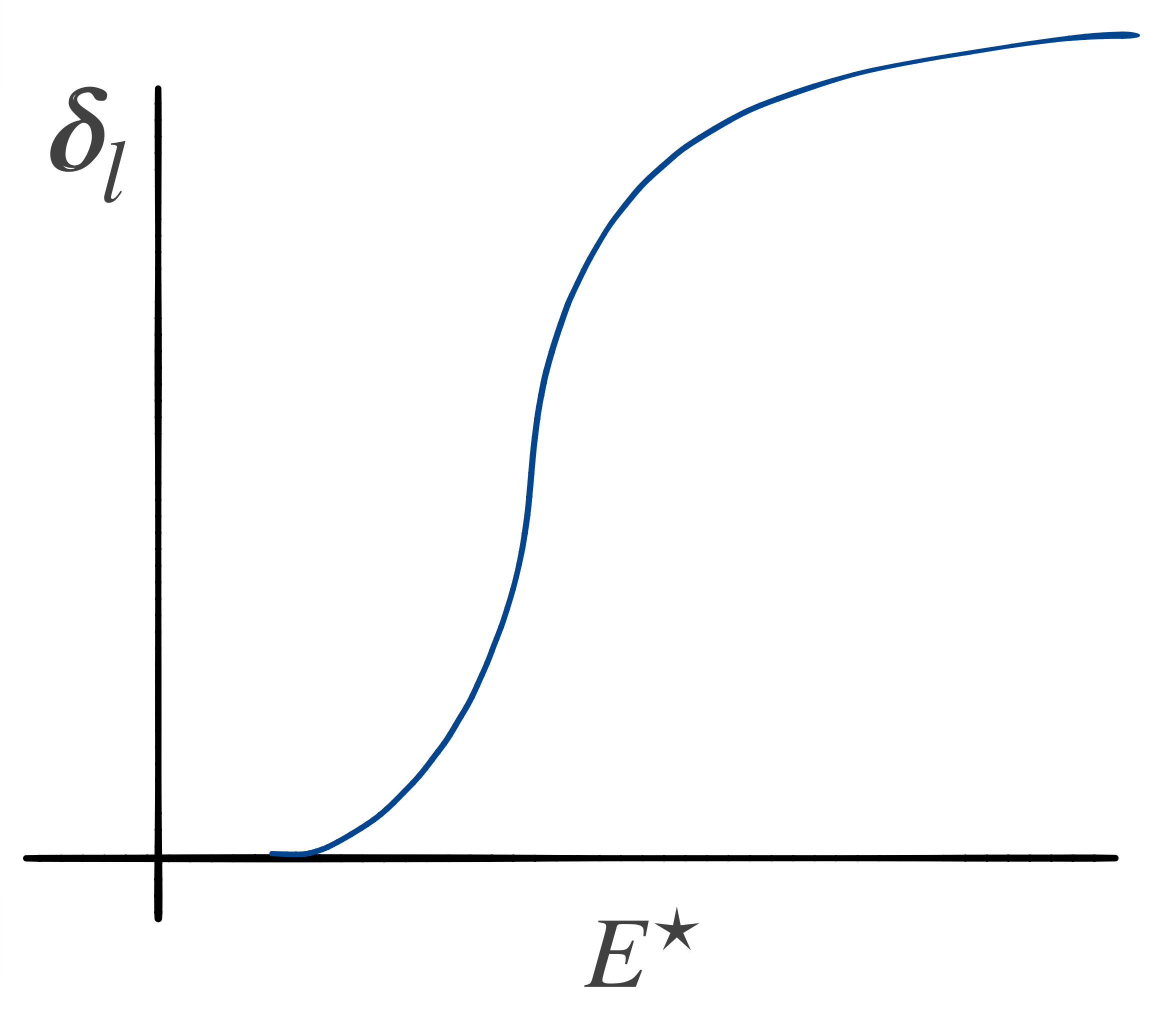
- $t \to i\,t\;(S \to iS)$
- $Z = \int \mathscr{D}U \mathscr{D}q \mathscr{D}\bar{q} e^{-S(q,\bar{q},U)}$
- discretize space time (a)
- place system in a finite-volume box (L)

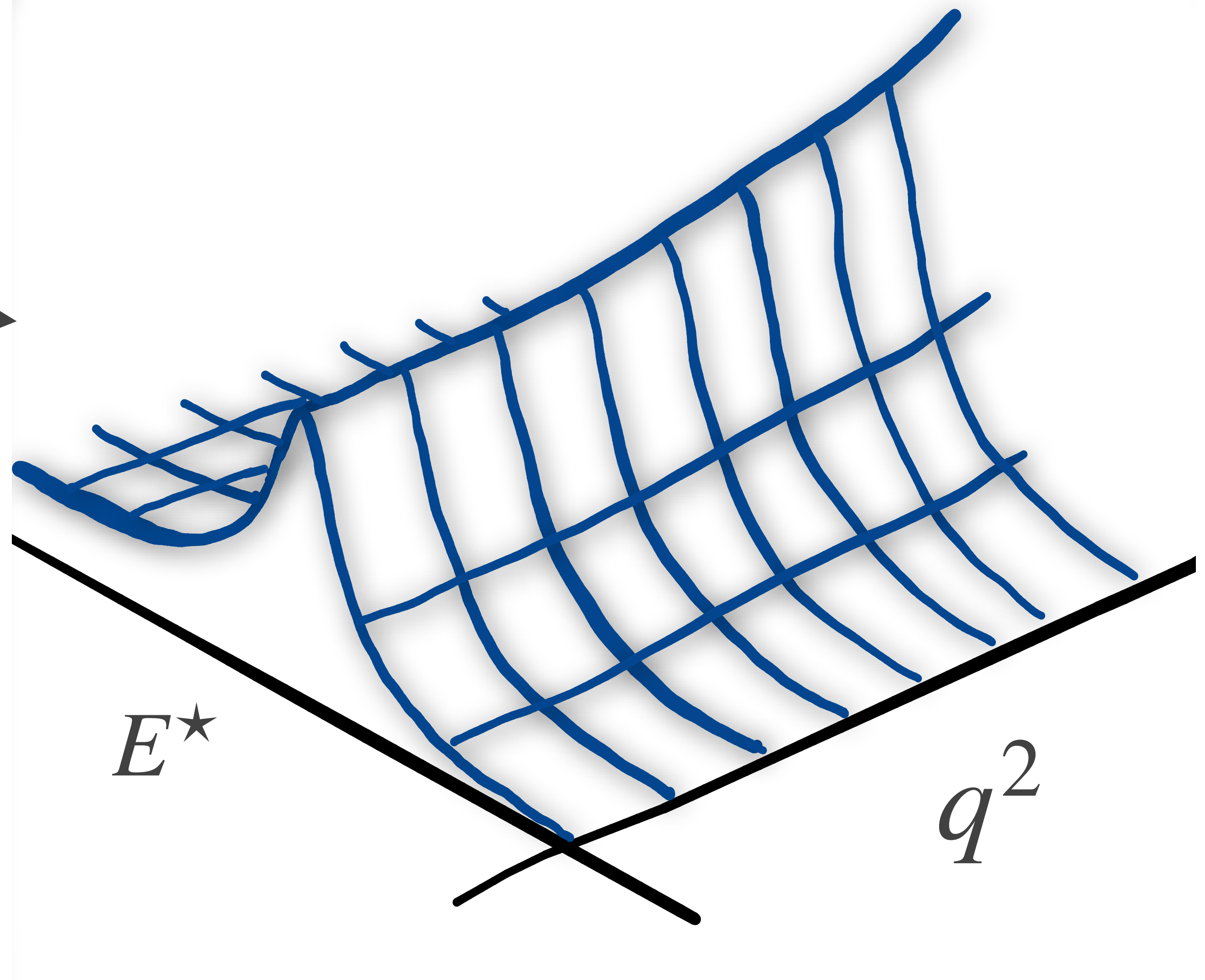


- Wilson-Clover fermions
- $a \approx 0.11403$ fm
- $m_\pi \approx 320$ MeV
- $32^3 \times 96$
- $N_{config} = 1039$

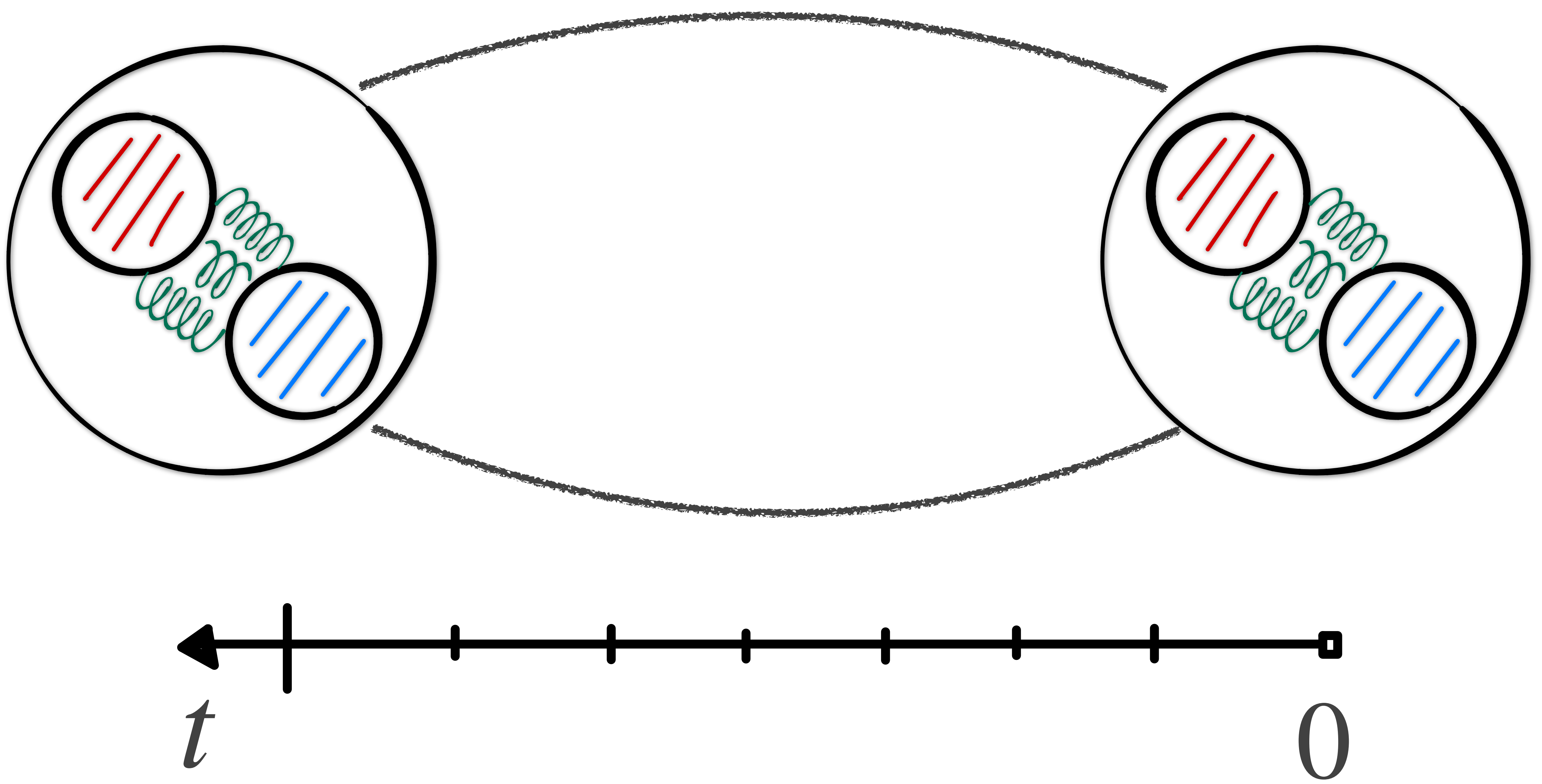
what can we calculate on the lattice?

- **1-point functions**
  (vevs)
- **2-point functions**
  (masses, scattering amplitudes,...)
- **3-point functions**
  (matrix elements of all kinds,...)
- **4-point functions**
  (DIS, spectral decompositions, ...)





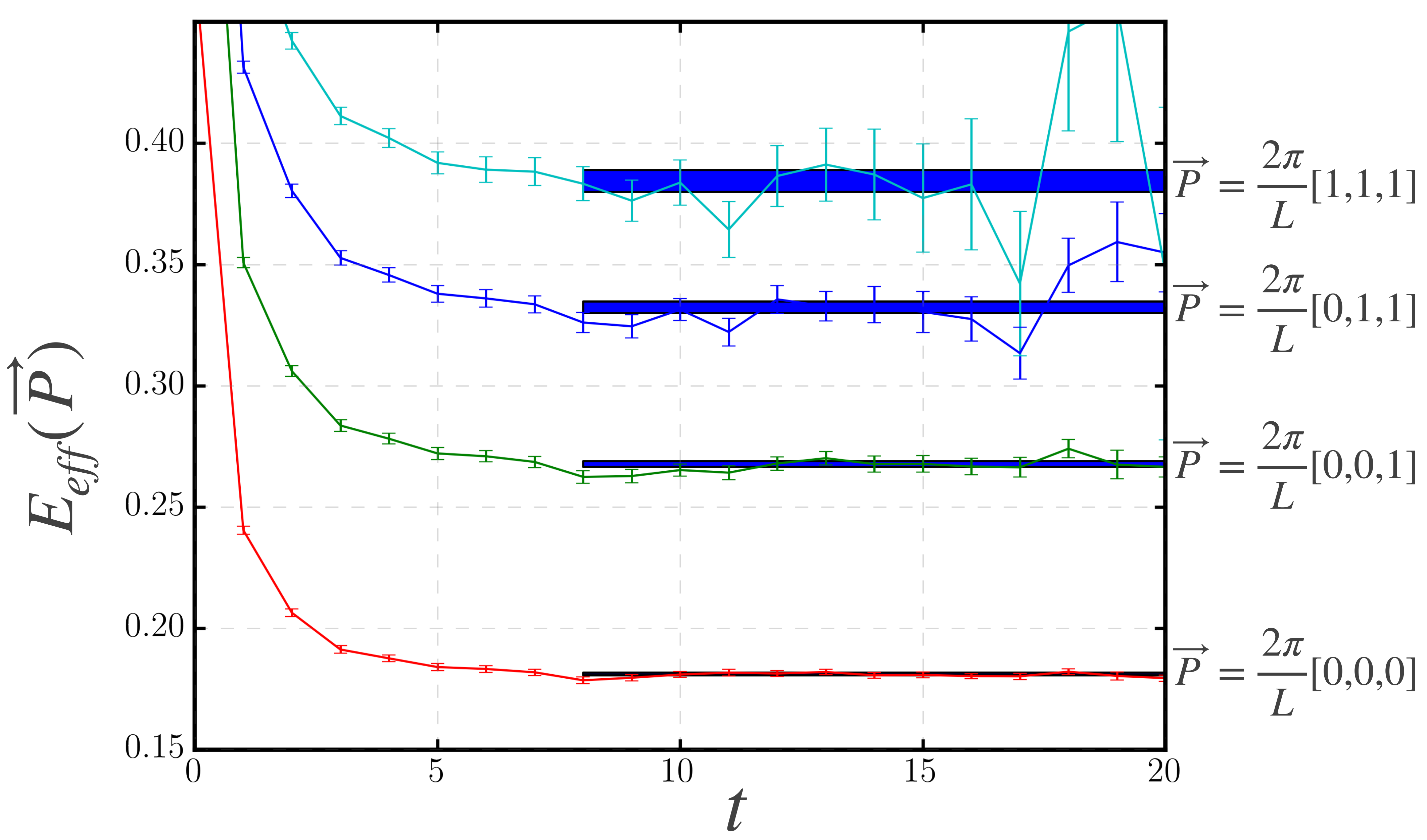
2-point functions: stable hadrons

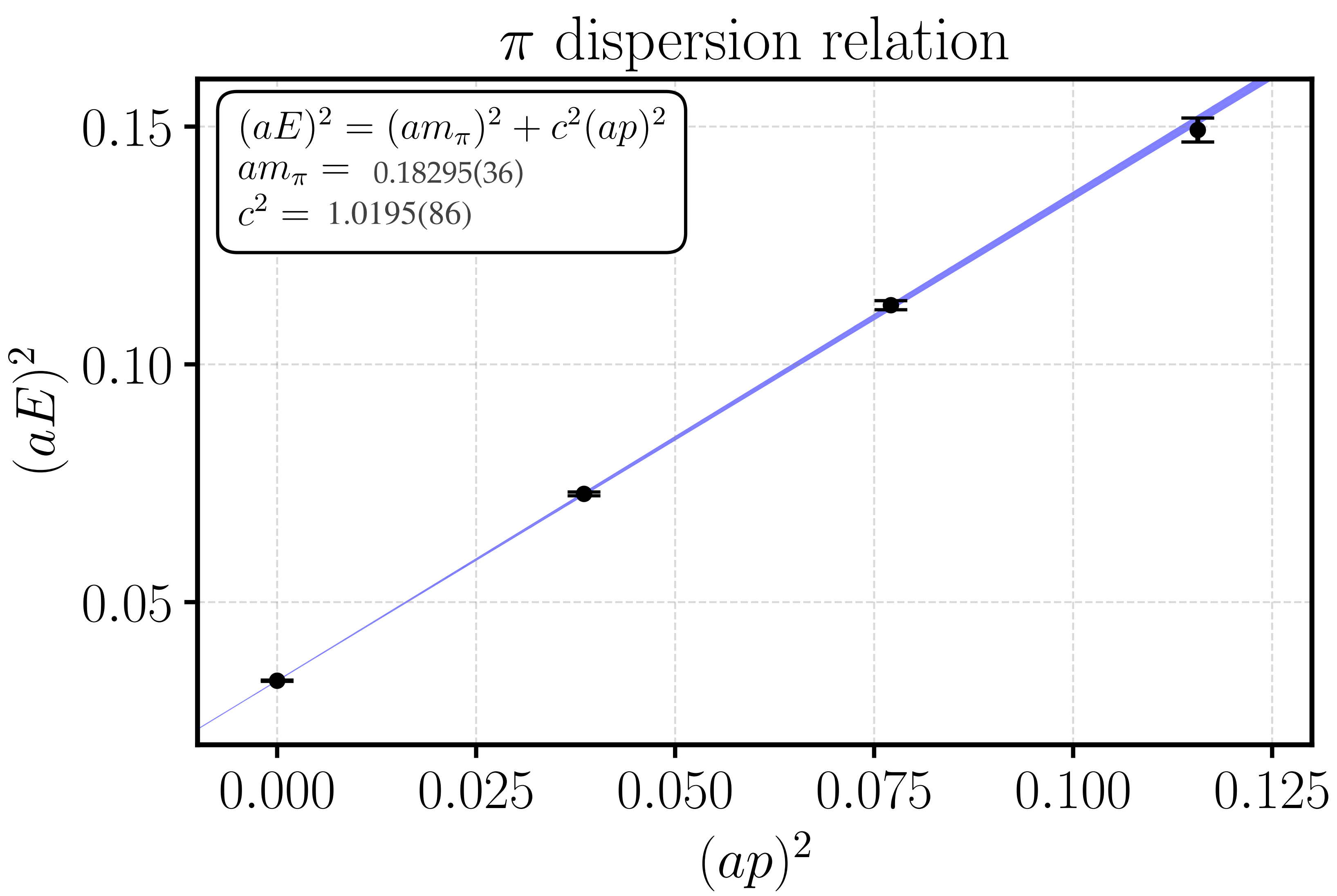


$$C_\pi^{\vec{P}}(t) = \sum_n \frac{|\langle 0|O_\pi(\vec{P})|n\rangle|^2}{2E_n} e^{-E_n t}$$

$t \to \infty$

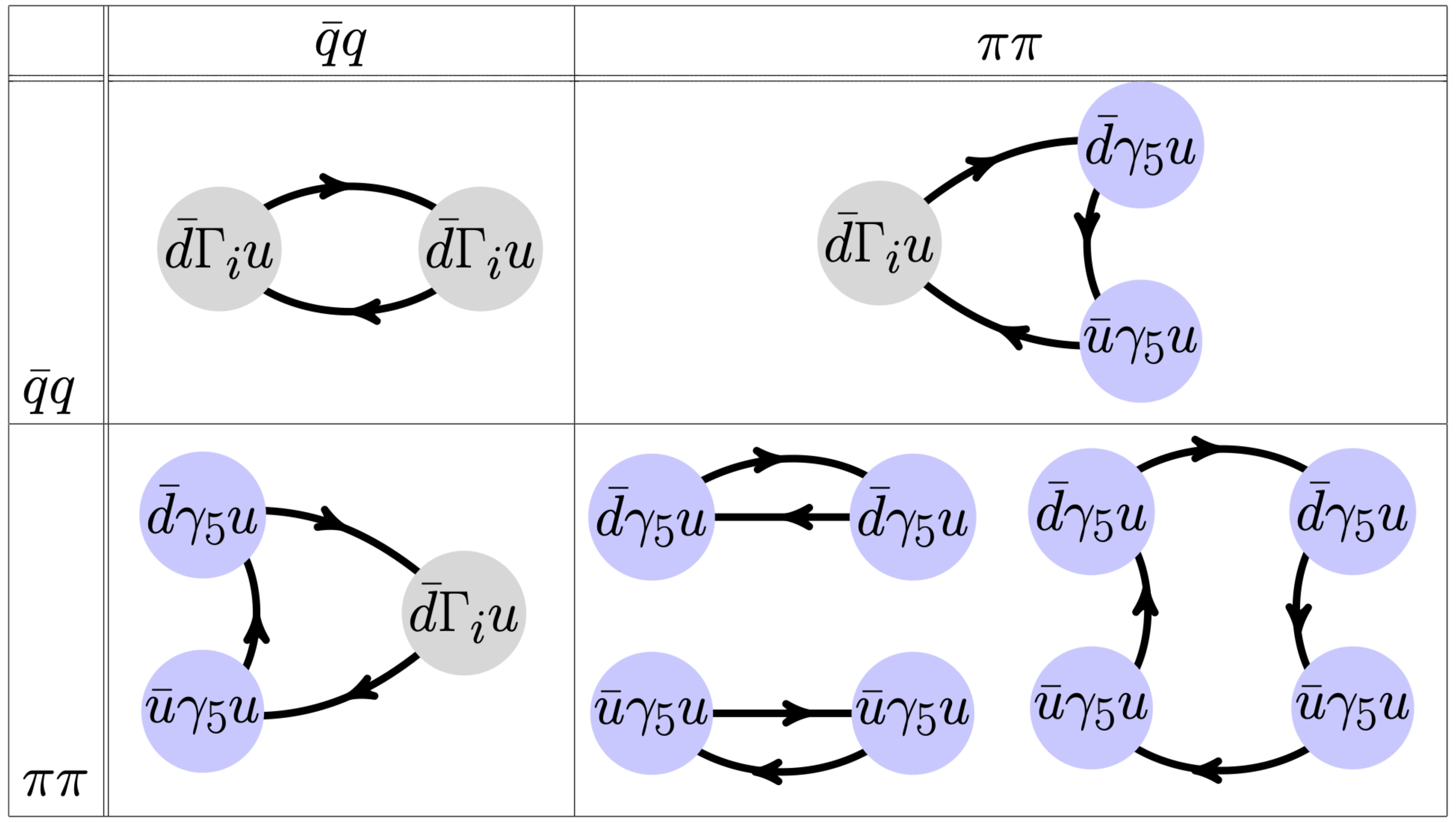
$$C_\pi^{\vec{P}}(t) = \frac{|\langle 0|O_\pi(\vec{P})|\pi\rangle|^2}{2E_\pi} e^{-E_\pi t}$$





2-point functions: unstable hadrons

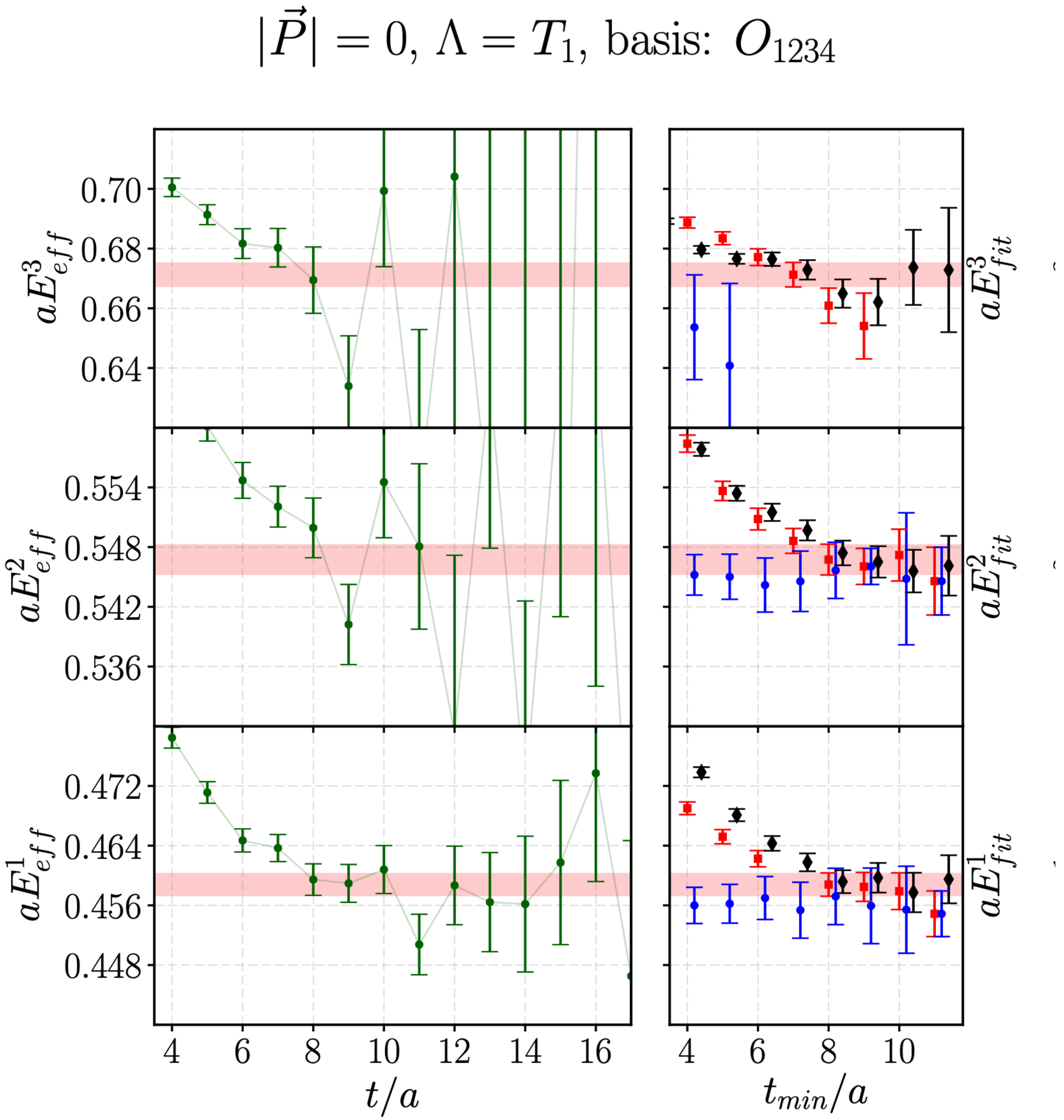
- use excited states!
- build a matrix of correlation functions



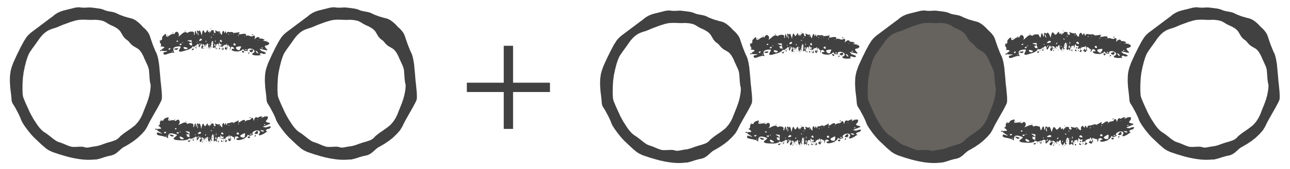
$$C_{ij}^{\vec{P},\Lambda}(t) = \sum_n \frac{\langle 0|O_i(\vec{P},\Lambda)|n,\vec{P},\Lambda\rangle\langle n,\vec{P},\Lambda|O_j(\vec{P},\Lambda)|0\rangle}{2E_n^{\vec{P},\Lambda}} e^{-E_n^{\vec{P},\Lambda}t}$$

- variational analysis:

$$C^{\vec{P},\Lambda}(t)v_n^{\vec{P},\Lambda}(t) = \lambda_n^{\vec{P},\Lambda}(t,t_0)C^{\vec{P},\Lambda}(t_0)v_n^{\vec{P},\Lambda}(t)$$

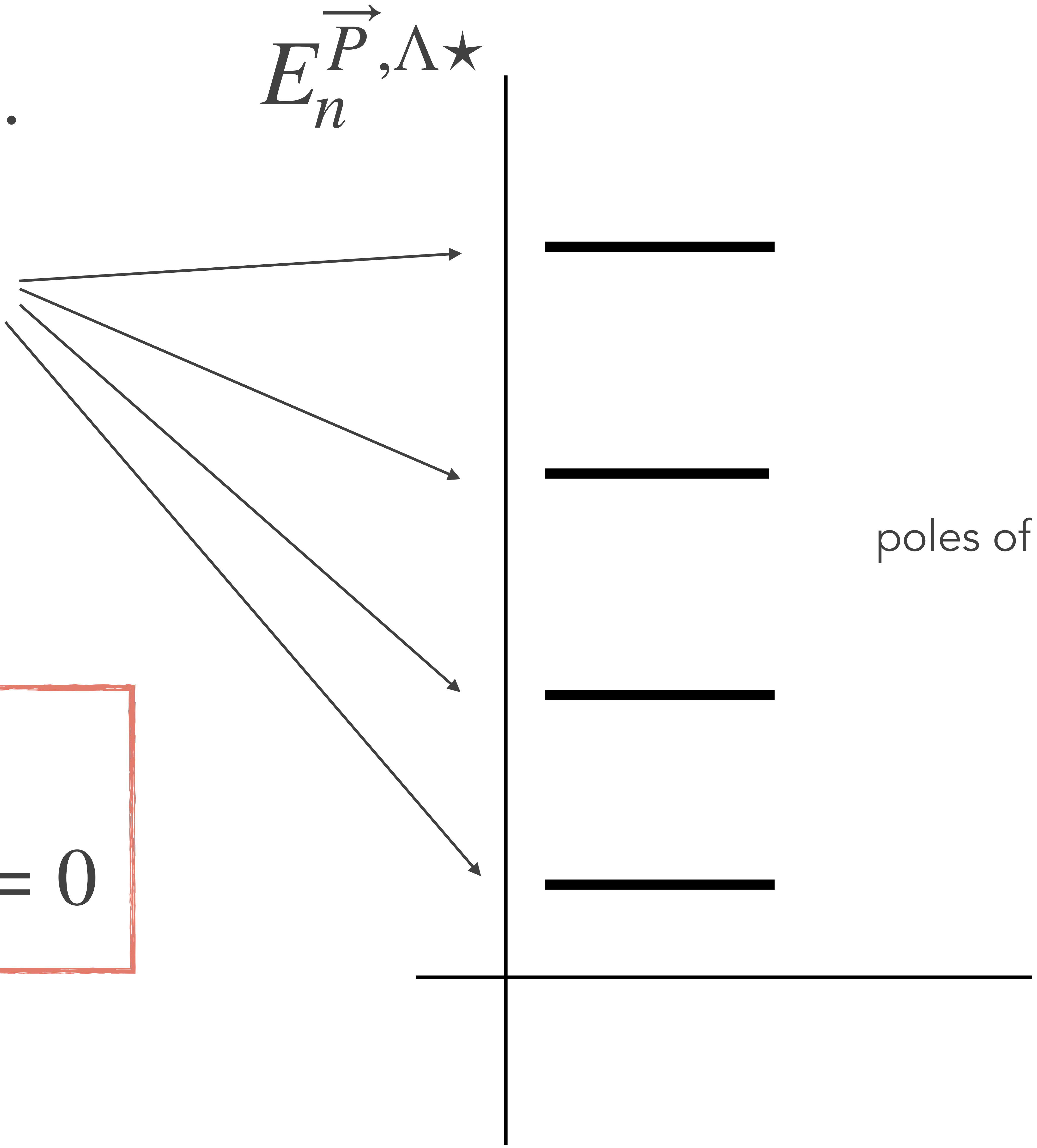


2-point functions: unstable hadrons

$C_L^{(2)} =$  $+ \ldots$

$$C_L^{(2)} = C_\infty^{(2)} - A' \frac{1}{F^{-1}(E^\star) + T(E^\star)} A$$

$$C_L^{(2)} = \sum_n \frac{R_n}{E^\star - E_n^\star} + \mathcal{O}(\text{reg})$$



for each $\vec{P}$ and Λ:

$$\det\left[F^{-1}(E^\star) + T(E^\star)\right] = 0$$

Luscher NPB354
Rummukainen, Gottlieb hep-lat/9503028
Kim, Sharpe, Sachrajda hep-lat/0507006
Leskovec, Prelovsek 1202.2145
Briceño 1401.3312
Briceño, Dudek, Young 1706.06223
Woss, Wilson, Dudek 2001.08474
[and many more]

scattering on the lattice

$$\det\left[F^{-1}(E^\star)+T(E^\star)\right]\Big|_{E^\star=E^\star_n}=0$$

$$F_{lM,l'M'}(E^\star)=\frac{ik}{8\pi E}\left[\delta_{MM'}\delta_{ll'}+\ i\sum_{\bar{l},\bar{m}}\sqrt{\frac{(2l+1)(2\bar{l}+1)}{4\pi(2l'+1)}}\ \langle lM,\bar{l}\bar{m}_l\,|\,l'M'\rangle\langle l0,\bar{l}0\,|\,l'0\rangle\ \frac{(4\pi)^2}{\gamma L k^{\bar{l}+1}}\left(\frac{2\pi}{L}\right)^{\bar{l}-2}Z_{\bar{l}\bar{m}}(k^2)\right]$$

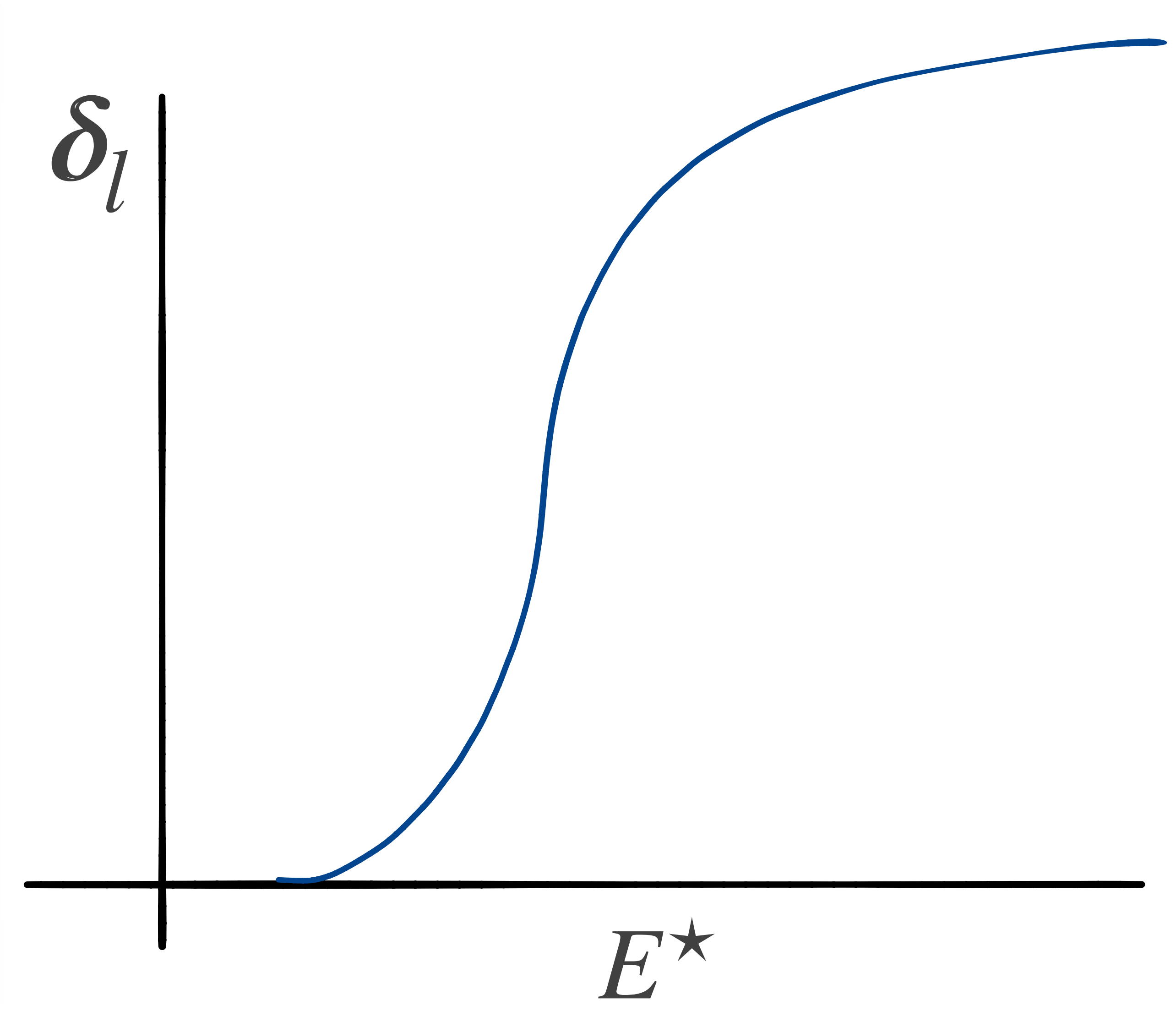
Clebsch-Gordan coefficients

the Lüscher Zeta function

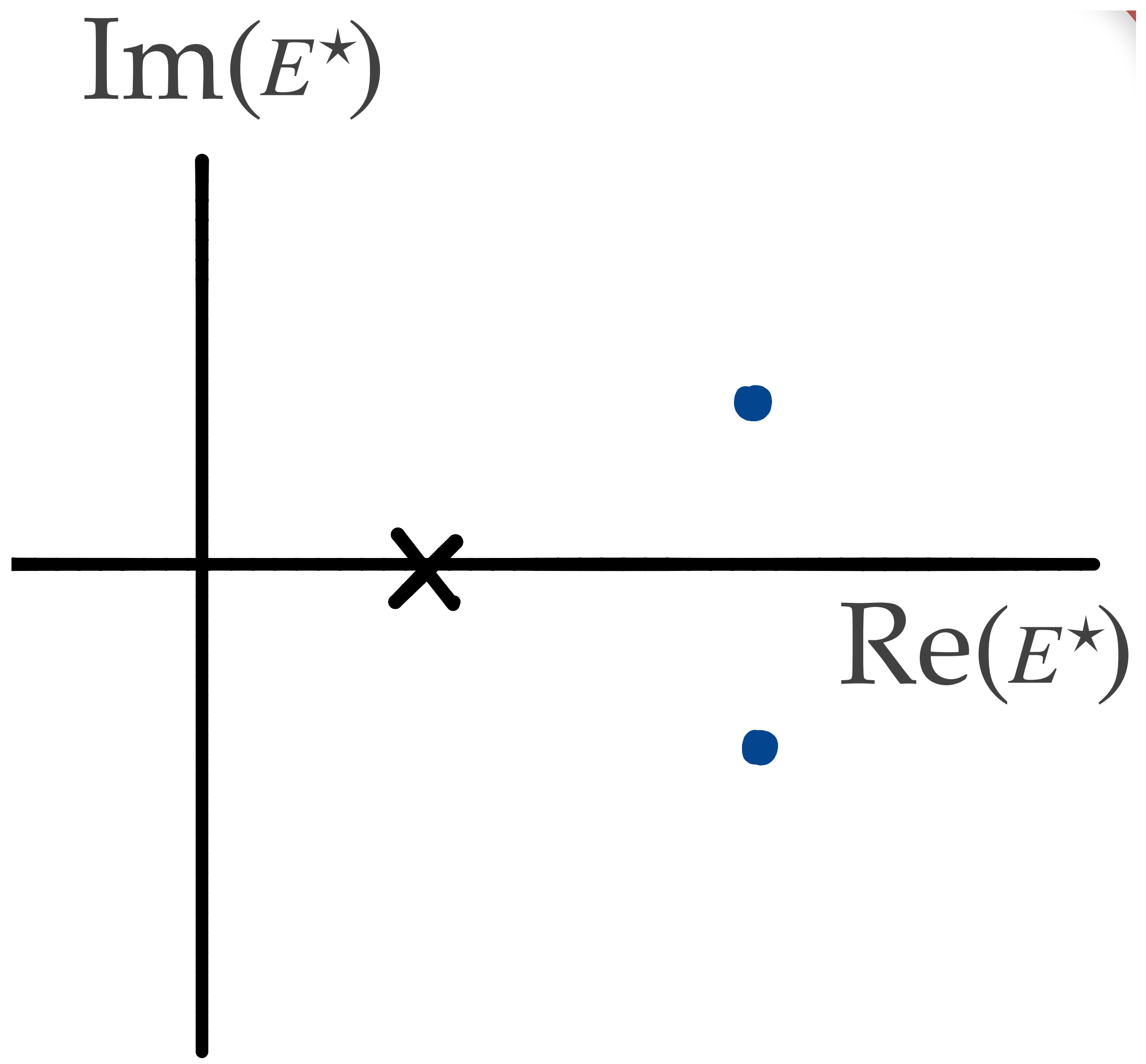
convention from Briceño, Dudek, Young 1706.06223

scattering in the infinite volume

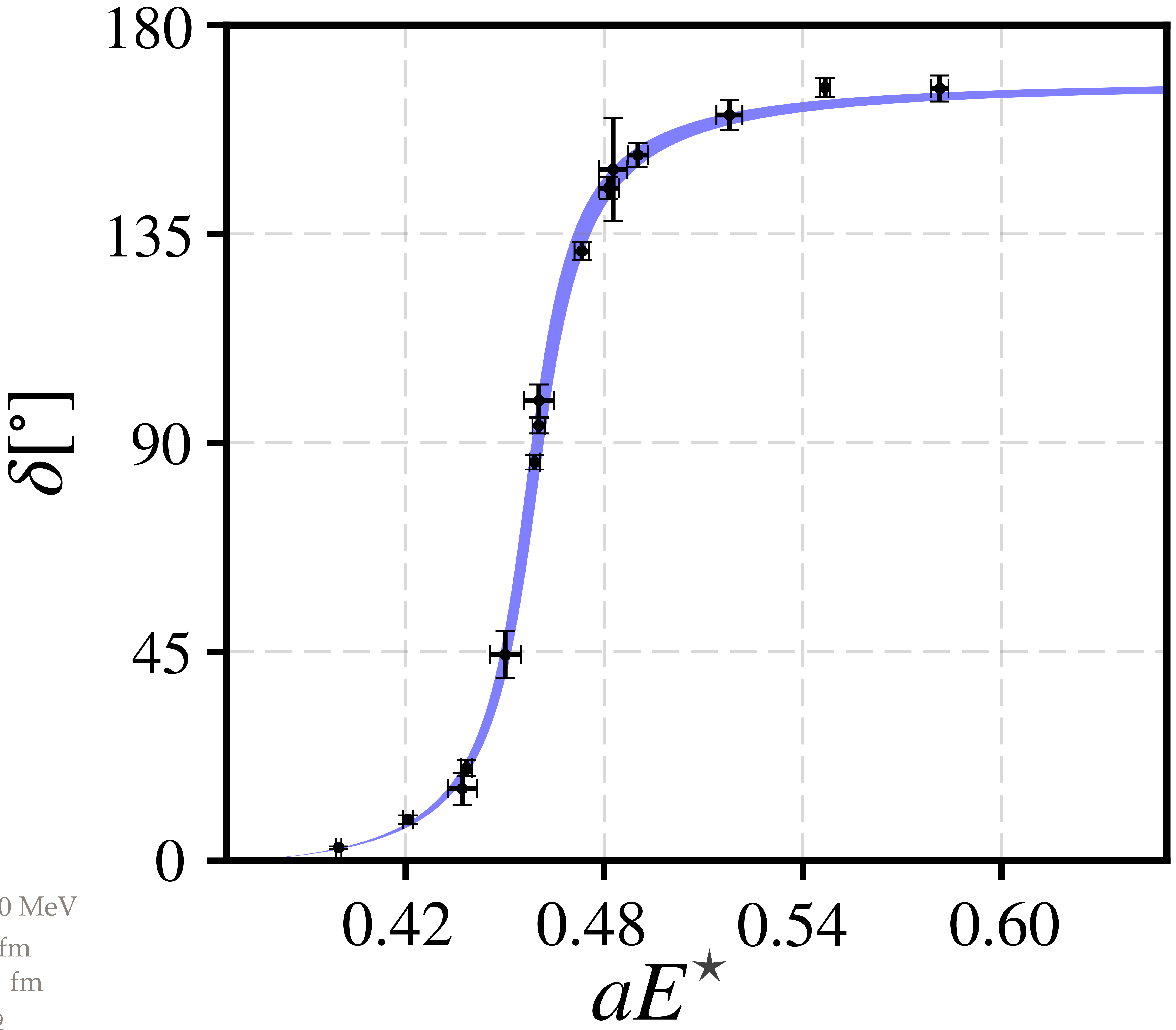
$$\det\left[F^{-1}(E^\star)+T(E^\star)\right]\Big|_{E^\star=E_n^\star}=0$$



$$T_{lM,l'M'}(E^\star)=\delta_{ll'}\delta_{MM'}\frac{16\pi E^\star}{ik}\frac{1}{\cot\delta_l(E^\star)-i}$$



ρ from lattice QCD



discrete data points:

- lattice QCD
- $\vec{P}, \Lambda, n$
- 15 data points

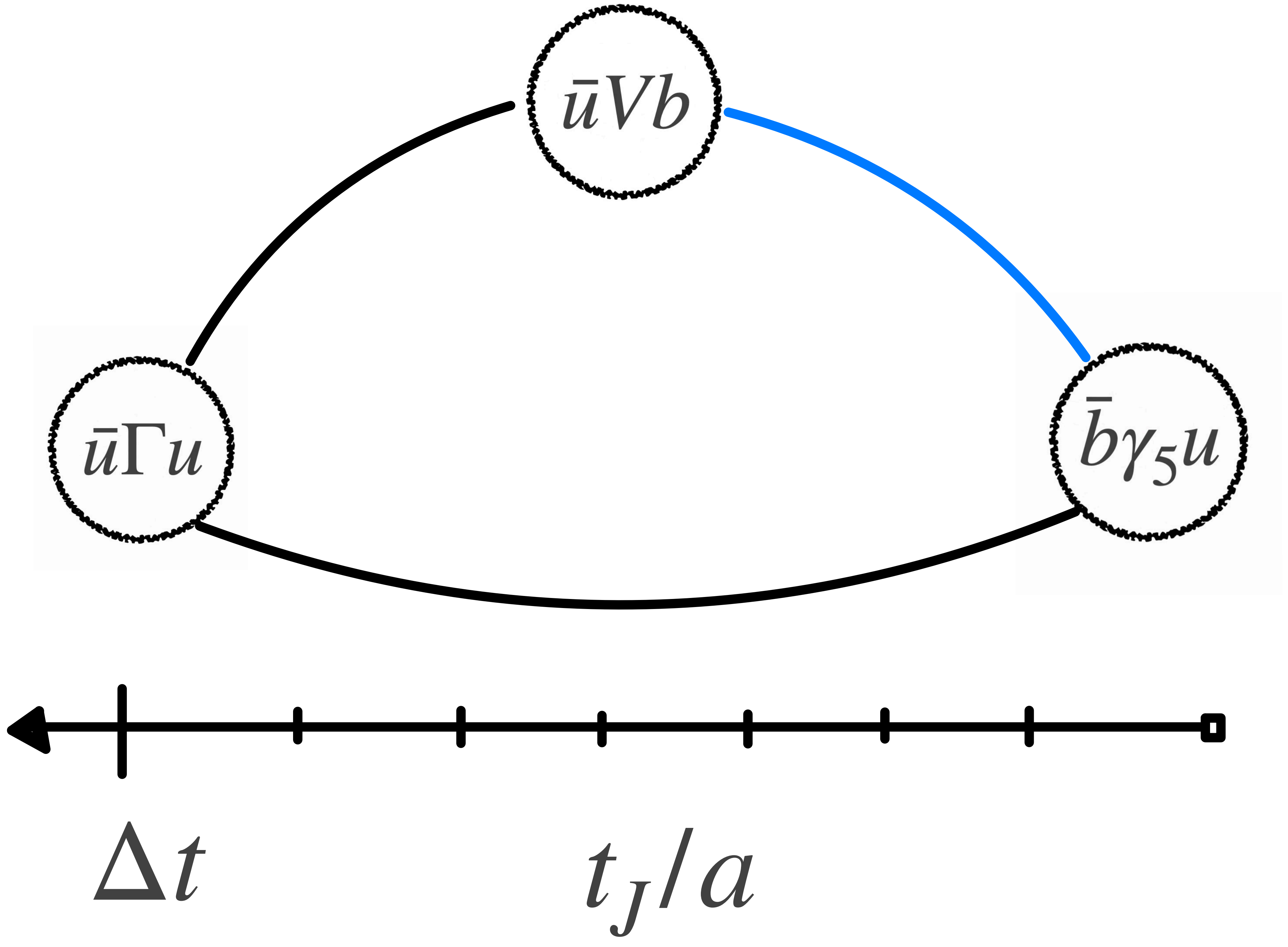
shaded region

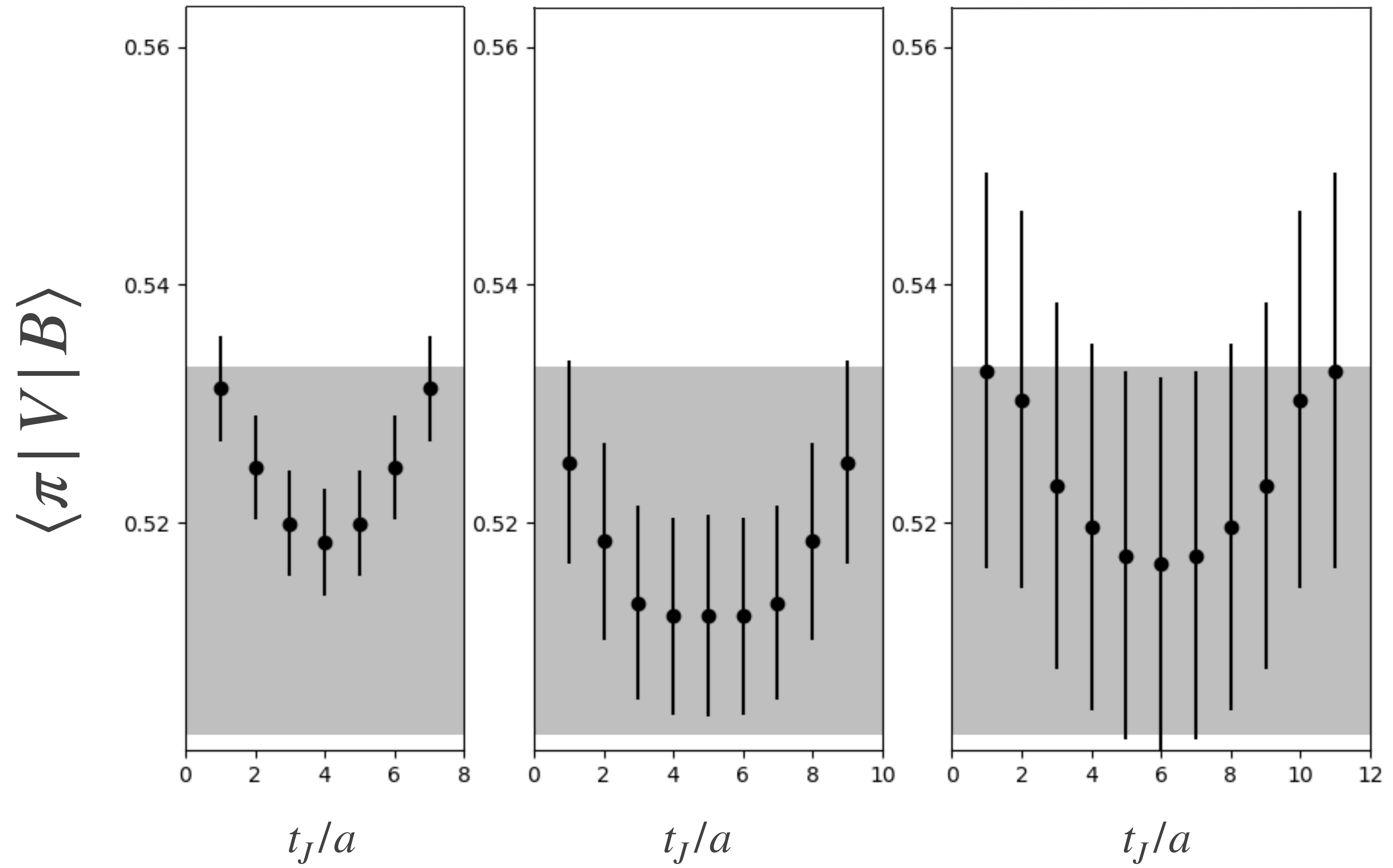
- Breit-Wigner for ρ
- mass and decay width

$m_\pi \approx 320$ MeV
$L \approx 3.6$ fm
$a \approx 0.11$ fm
1704.05439

$$T(E^\star) \propto \frac{1}{\cot\delta - i}$$

3-point functions: stable hadrons

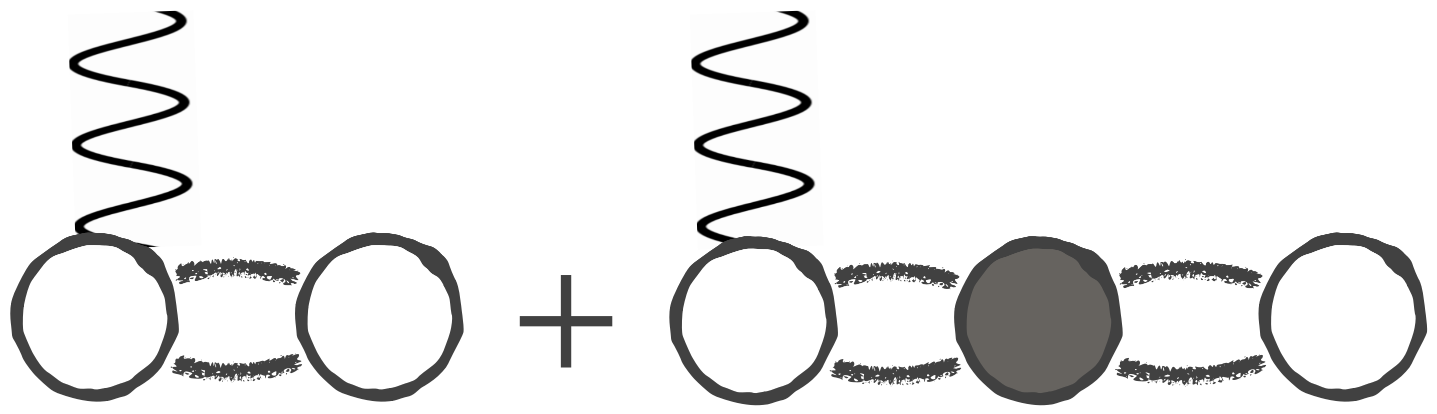




$$\lim_{\Delta t \to \infty} C_3 = Z_\pi \langle \pi | V | B \rangle Z_B \frac{e^{-E_\pi(\Delta t - t_J)} e^{-E_B t_J}}{2E_\pi 2E_B}$$

$$\langle \pi | V | B \rangle = \sum_i K_i f_i(q^2)$$

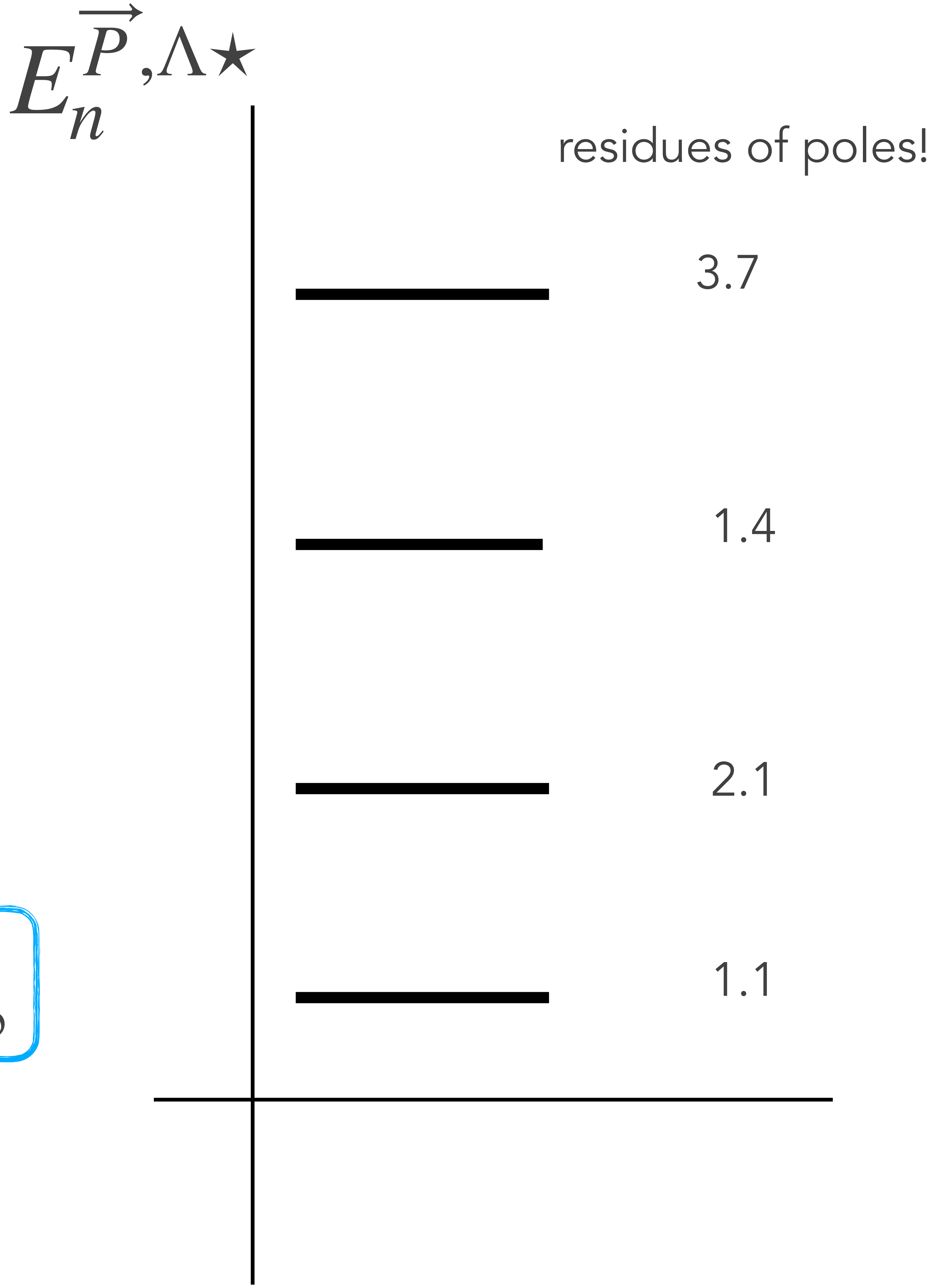
3-point functions: unstable hadrons

$C_L^{(3)} =$  $+ \ldots$

$$C_L^{(3)} = C_\infty^{(3)} - A'RA$$

$$R = \lim_{E \to E_n} \frac{E - E_n}{F^{-1} + T}$$

residue of pole!



normalization of finite-volume states

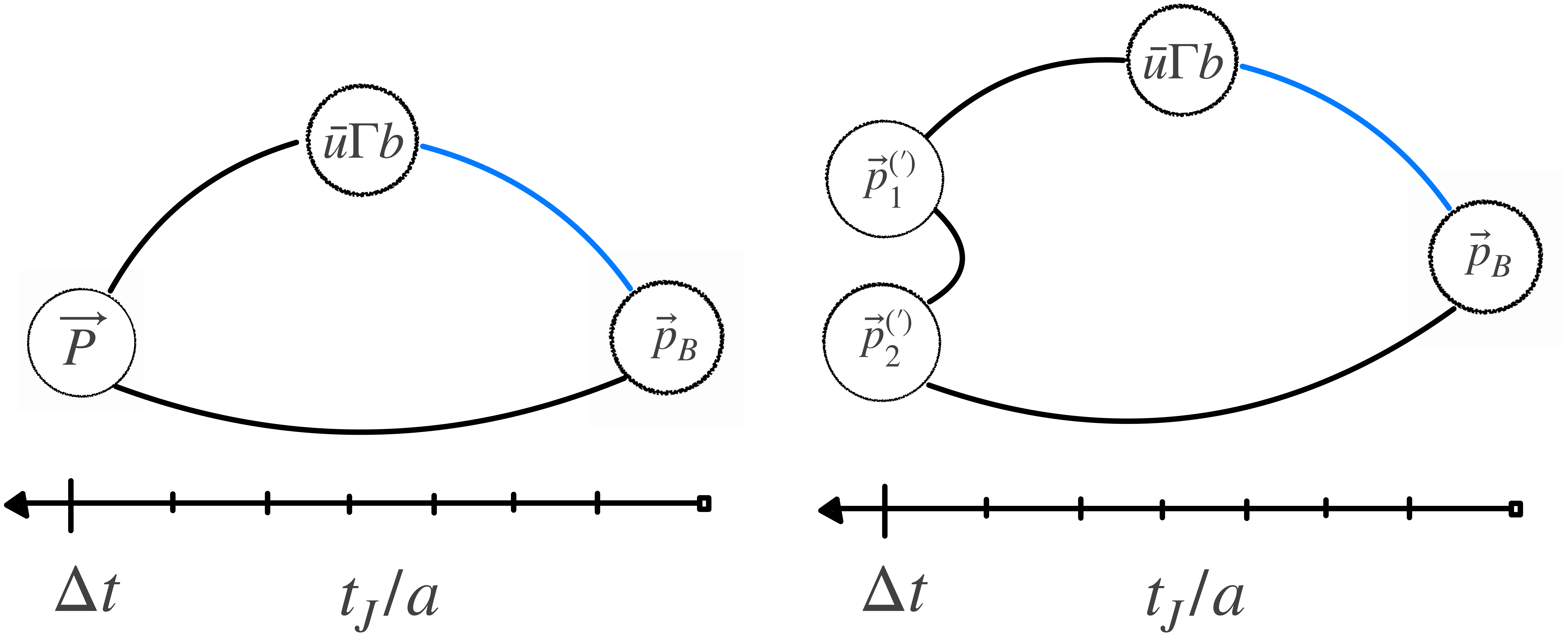
$$|E_n^{\vec{P},\Lambda\star}\rangle_L \sim \sqrt{R}\,|p_1 p_2(E^\star = E_n^{\vec{P},\Lambda\star})\rangle_\infty$$

the "Lellouch-Lüscher" factor

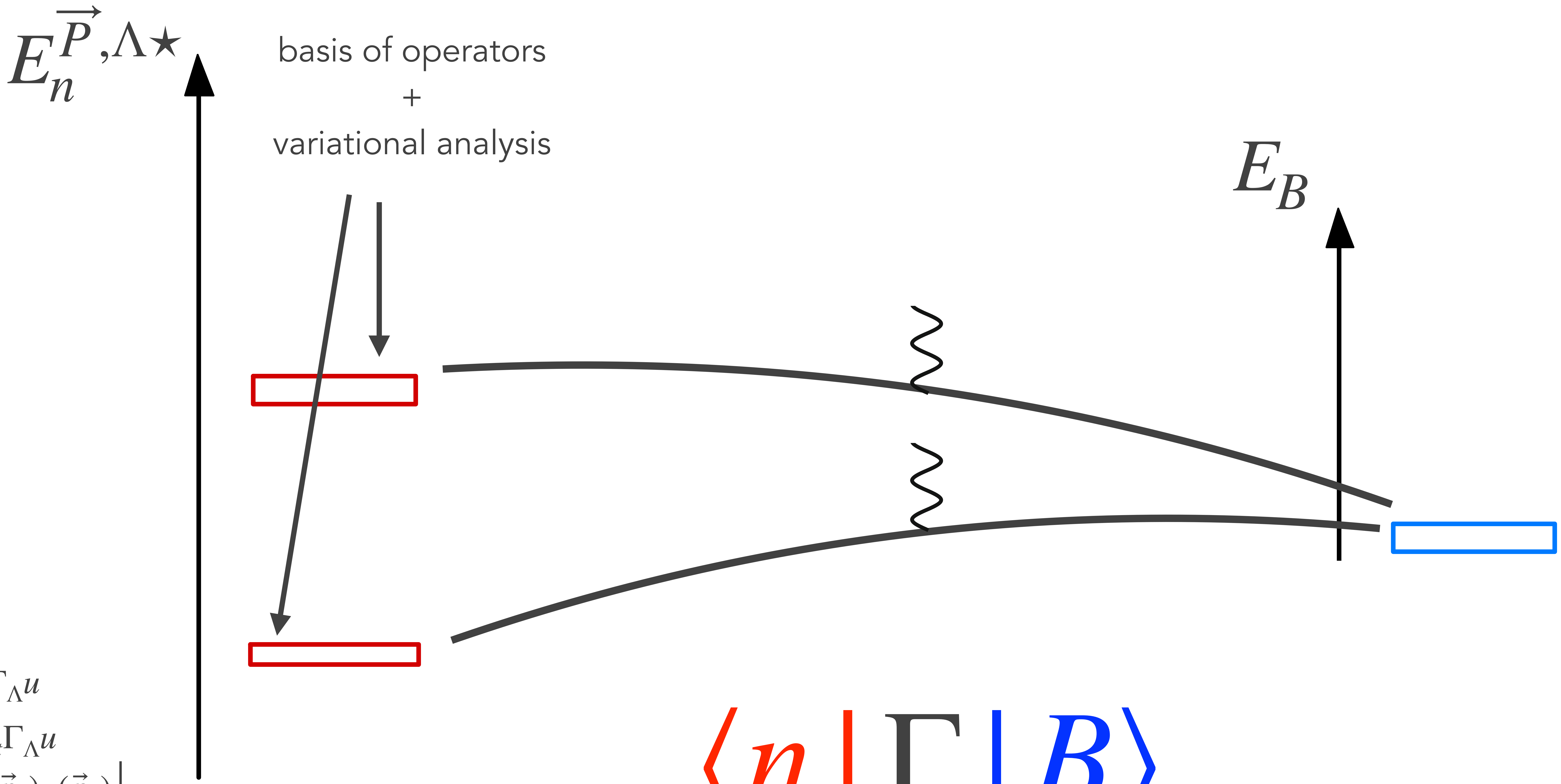
Lellouch, Luscher hep-lat/0003023
Lin, Sachrajda, Testa hep-lat/0104006
…
Briceno, Hansen, Walker-Loud 1406.5965
Briceno, Hansen 1502.04314
Briceno, Dudek, LL 2105.02017

3-point functions

$$C_{3,i} = \langle\, O_i(\vec{P}, \Lambda)\;\; \Gamma\;\; O_B(\vec{p}_B)\, \rangle$$



matrix elements



$$\langle n | \Gamma | B \rangle$$

$O_1 = \bar{u}\Gamma_\Lambda u$

$O_2 = \bar{u}\gamma_t\Gamma_\Lambda u$

$O_3 = \pi(\vec{p}_1)\pi(\vec{p}_2)\Big|_\Lambda$

$O_4 = \pi(\vec{p}_1')\pi(\vec{p}_2')\Big|_\Lambda$

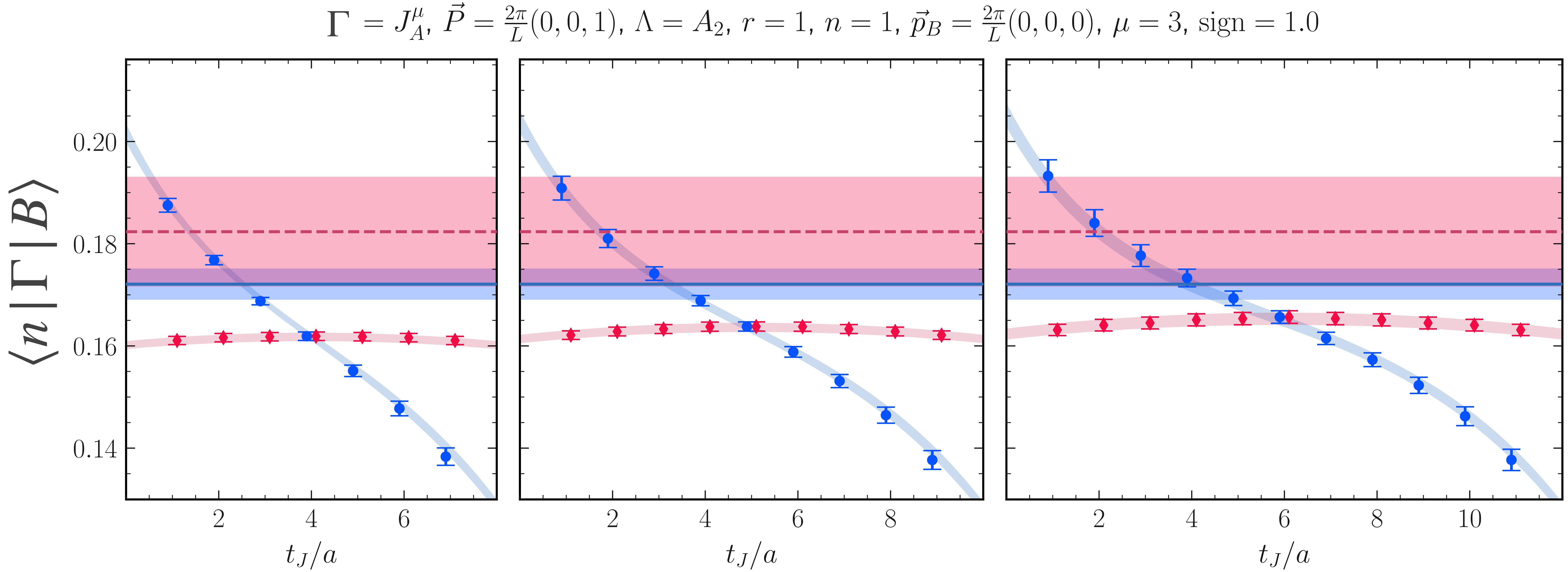
$\vec{p}_1^{(')} + \vec{p}_2^{(')} = \vec{P}$

state projection

$$C_{3,i} = \sum_{n\in[\pi\pi]} Z_i^n \langle n|\Gamma|B\rangle Z_B \frac{e^{-E_n(\Delta t - t)} e^{-E_B t}}{2E_n 2E_B}$$

$$C_3^n = u_i^n C_{3,i}$$

$$u_i^n Z_i^m = 2E_n \, e^{E_n t_0} \, \delta^{nm}$$



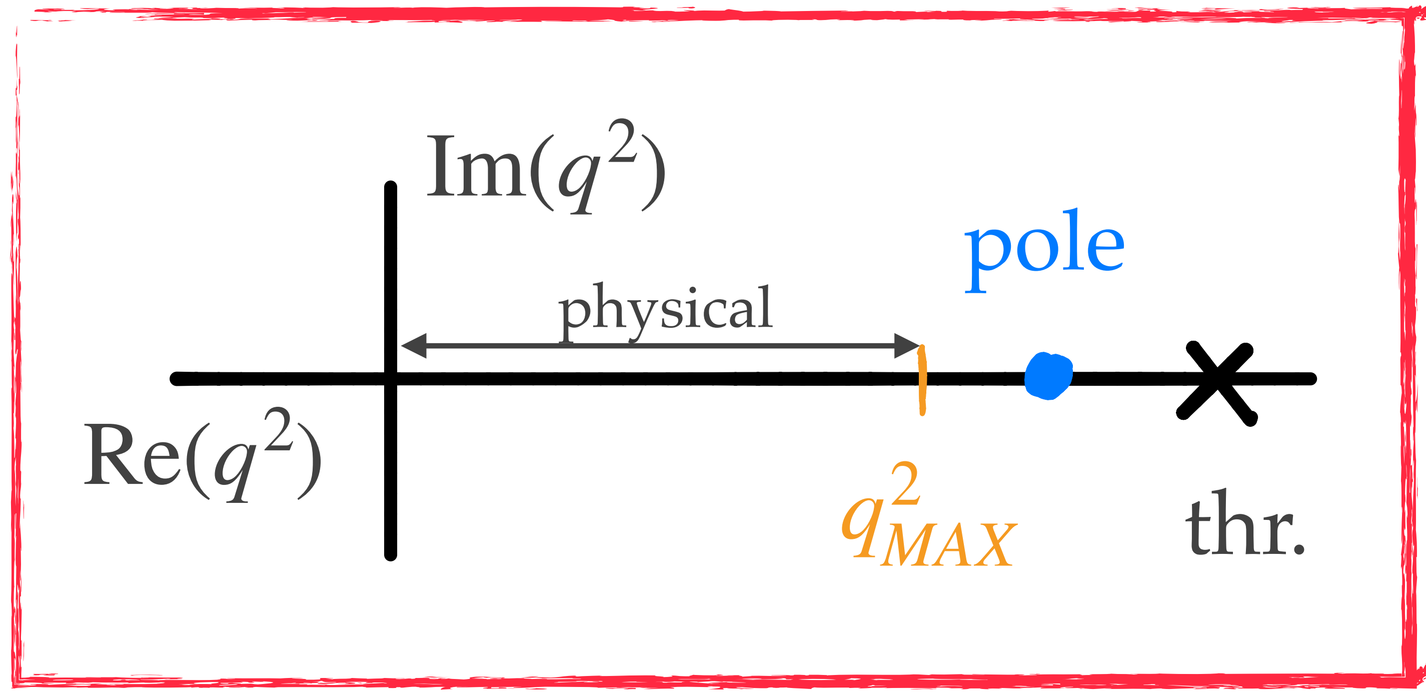
non-physical
kinematics
$q^2 < (m_B - E^\star)^2$
$(aq)^2$
$aE^\star$
$2m_\pi$
$m_B - m_\ell - m_\nu$
0.5
1.0
1.5
2.0
2.5
3.0
0
1
2
3
4
5
6
7
0.40
0.45
0.50
0.55
0.60
5.50
5.75
6.00
6.25
6.50
6.75
7.00
7.25

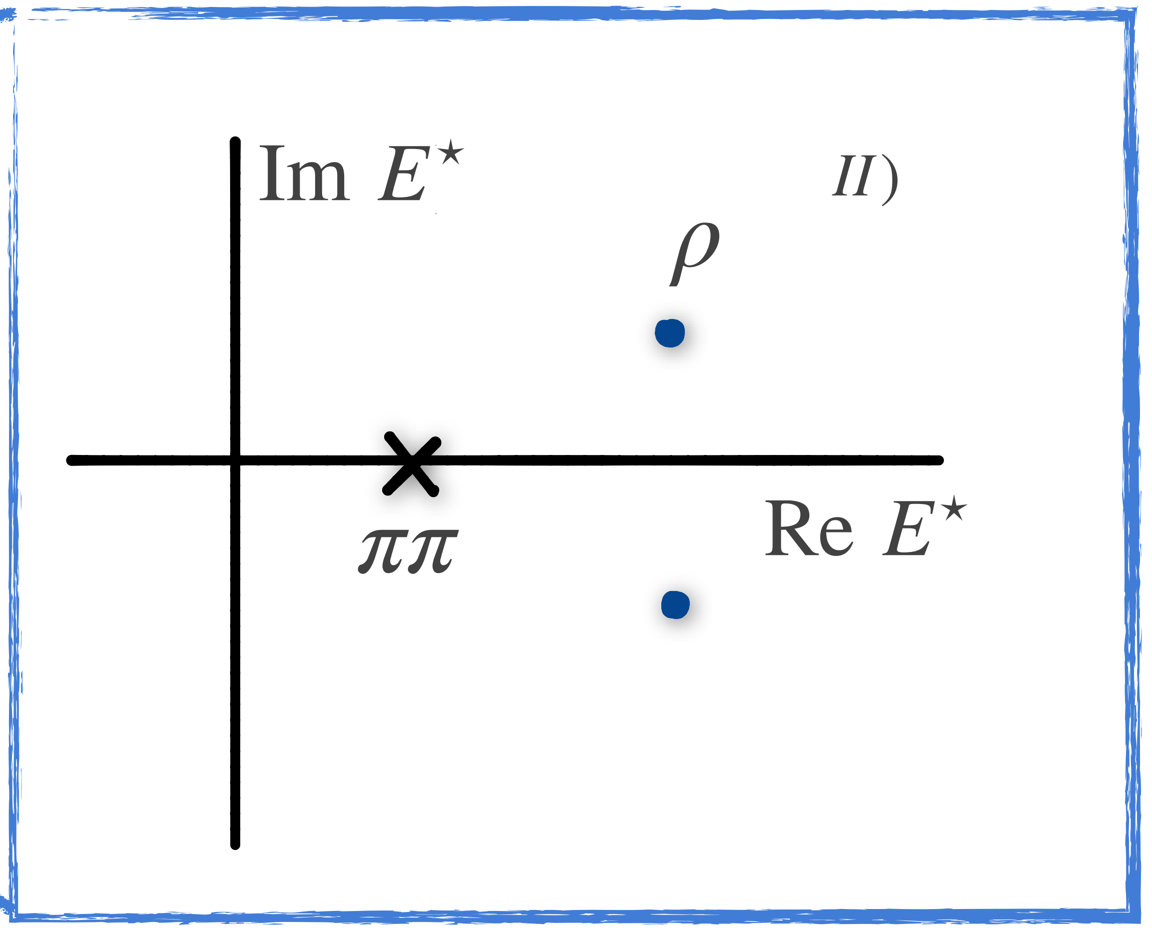
transition amplitude

Boyd, Grinstein, Lebed hep-ph/9412324
Bourrely, Caprini, Lellouch 0807.2722
Alexandrou, LL, Meinel et al. 1807.08357

$$\langle \pi\pi, E^\star, \epsilon \,|\, \Gamma^\mu \,|\, B, p_B \rangle_\infty = \sum_i K_i^\mu(P, p_B, \epsilon)\, f^{(i)}(q^2, s)$$

$$f^{(i)}(E^\star, q^2) = F_i(E^\star, q^2)\, \frac{T(E^\star)}{k}$$





$$\langle n \,|\, V \,|\, B \rangle_L = \sqrt{R_n}\; \langle \pi\pi, E^\star \,|\, V \,|\, B, p_B \rangle_\infty$$

"Lellouch-Lüscher"
factor

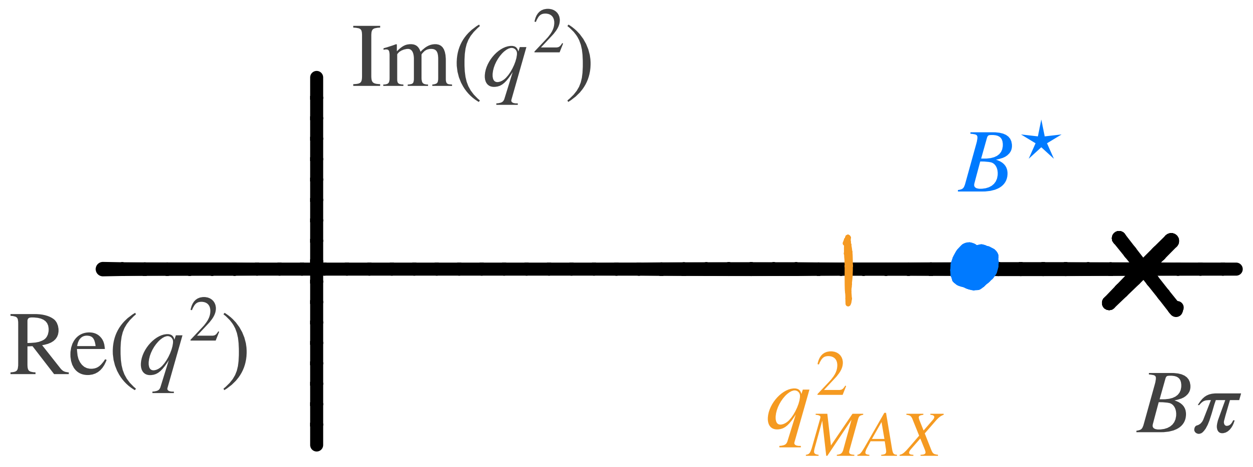
transition amplitude - Vector Current

$$f^{(V)}(E^\star, q^2) = F_V(E^\star, q^2)\ \frac{T(E^\star)}{k}$$

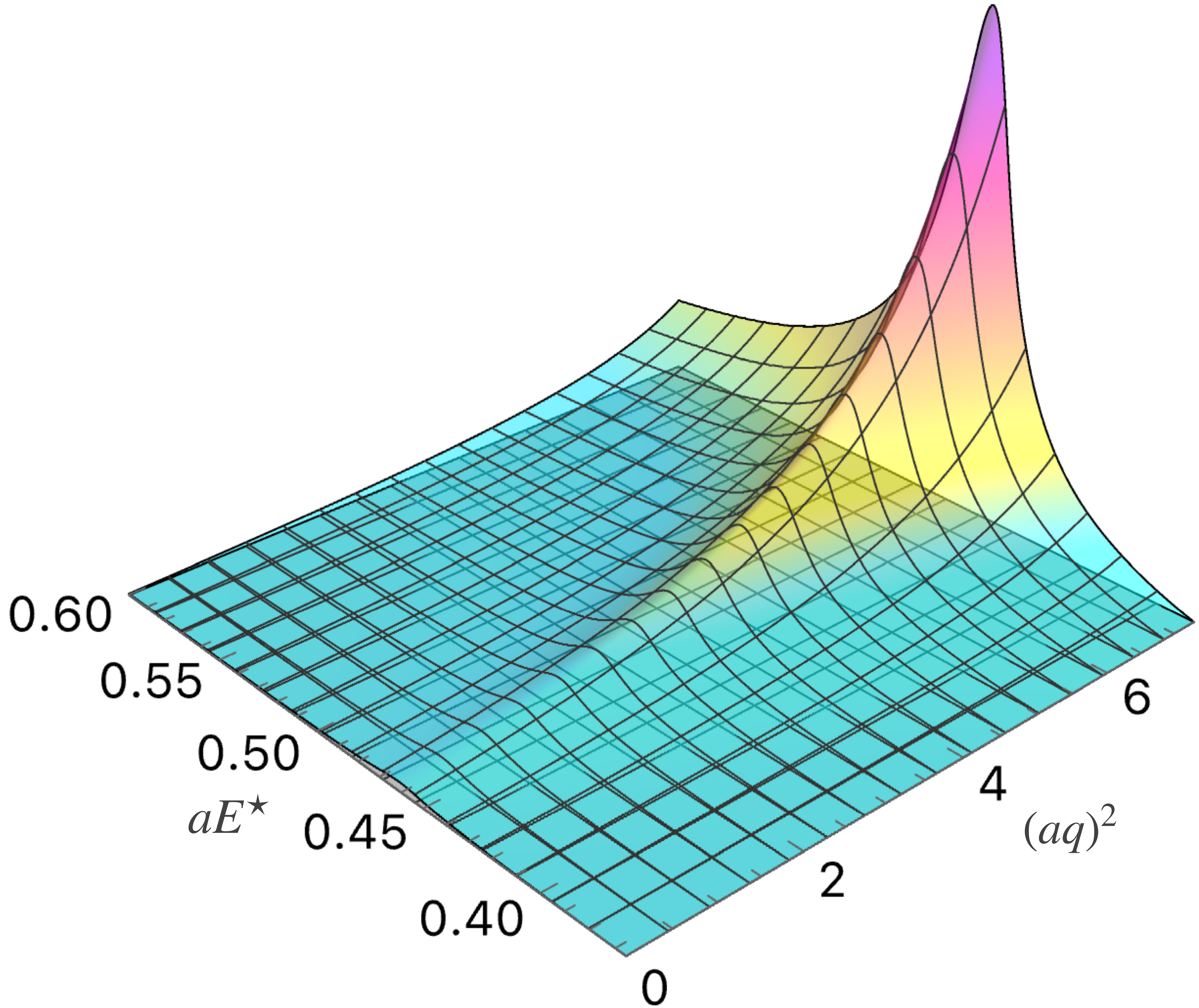
$$K_V^\mu(P, p_B, \epsilon) = \frac{2i}{m_B + E^\star}\varepsilon^{\mu\nu\alpha\beta}\epsilon_\nu(P, m)P_\alpha p_{B\beta}$$

$$F_V(q^2, s) = \frac{a_0^{(V)} + a_1^{(V)} z(q^2)}{1 - \frac{q^2}{m_{B^\star}^2}} \qquad T(E^\star) = \frac{E^\star \Gamma}{m_R^2 - E^{\star 2} - iE^\star\Gamma}$$

$$\Gamma = \frac{g_{\rho\pi\pi}^2}{6\pi}\frac{p^3}{E^{\star 2}}$$



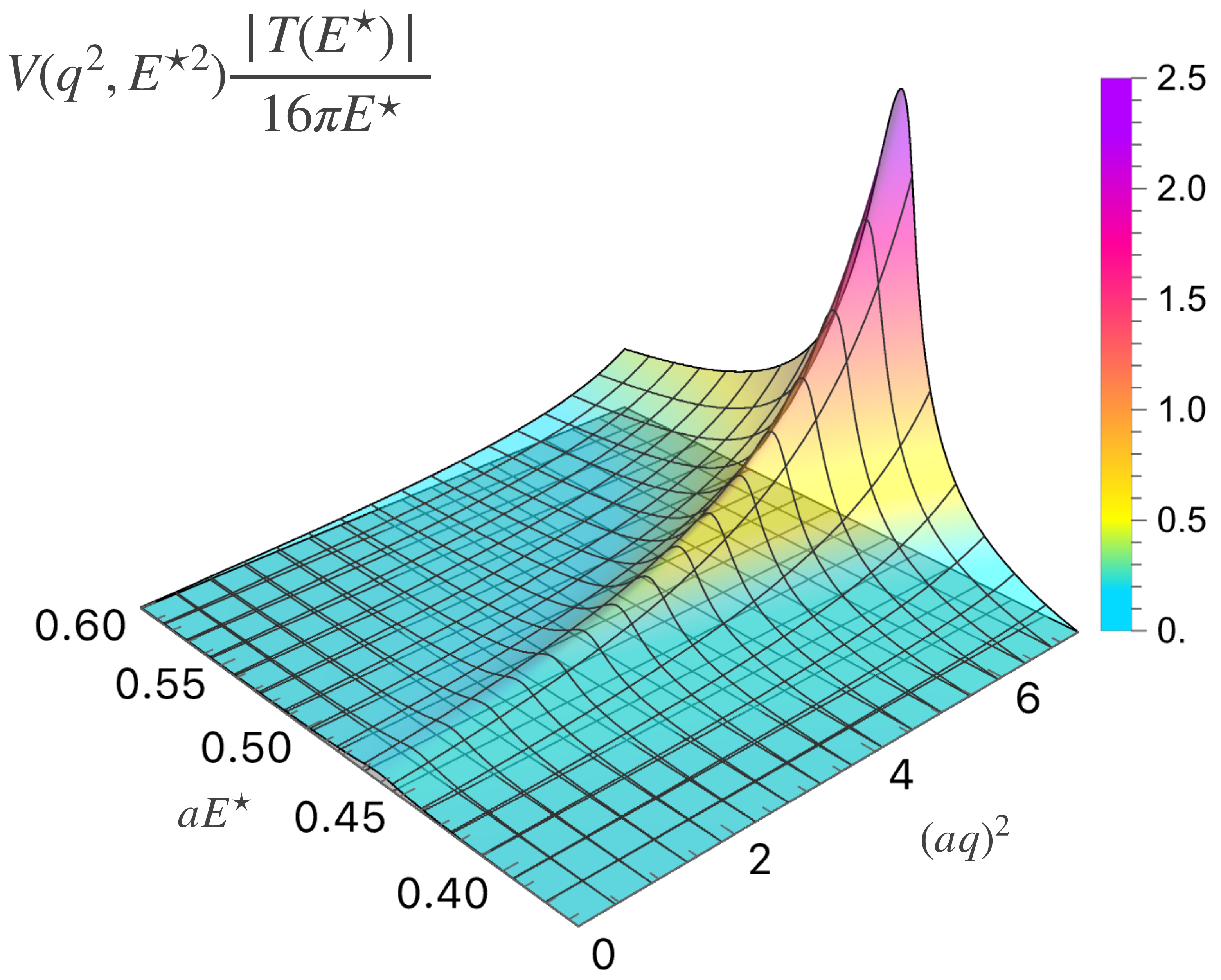
$$\frac{\chi^2}{\mathrm{dof}} = \frac{72.1}{64-2} = 1.16$$



the observable part

- example!
  (similar to 2311.00864)
- only vector form factor shown
- differential branching fraction

$$\left.\frac{d^2\Gamma}{dE^{\star 2}dq^2}\right|_{V,\ell=1} = \frac{G_F^2\left|V_{ub}\right|^2}{(4\pi)^5}\frac{2}{3}\frac{\lambda^{3/2}(m_B^2, E^{\star 2}, q^2)q^2}{m_B^3 E^\star (m_B + E^\star)^2}\left|T(E^\star)\right|^2\left|V(q^2, E^{\star 2})\right|^2$$

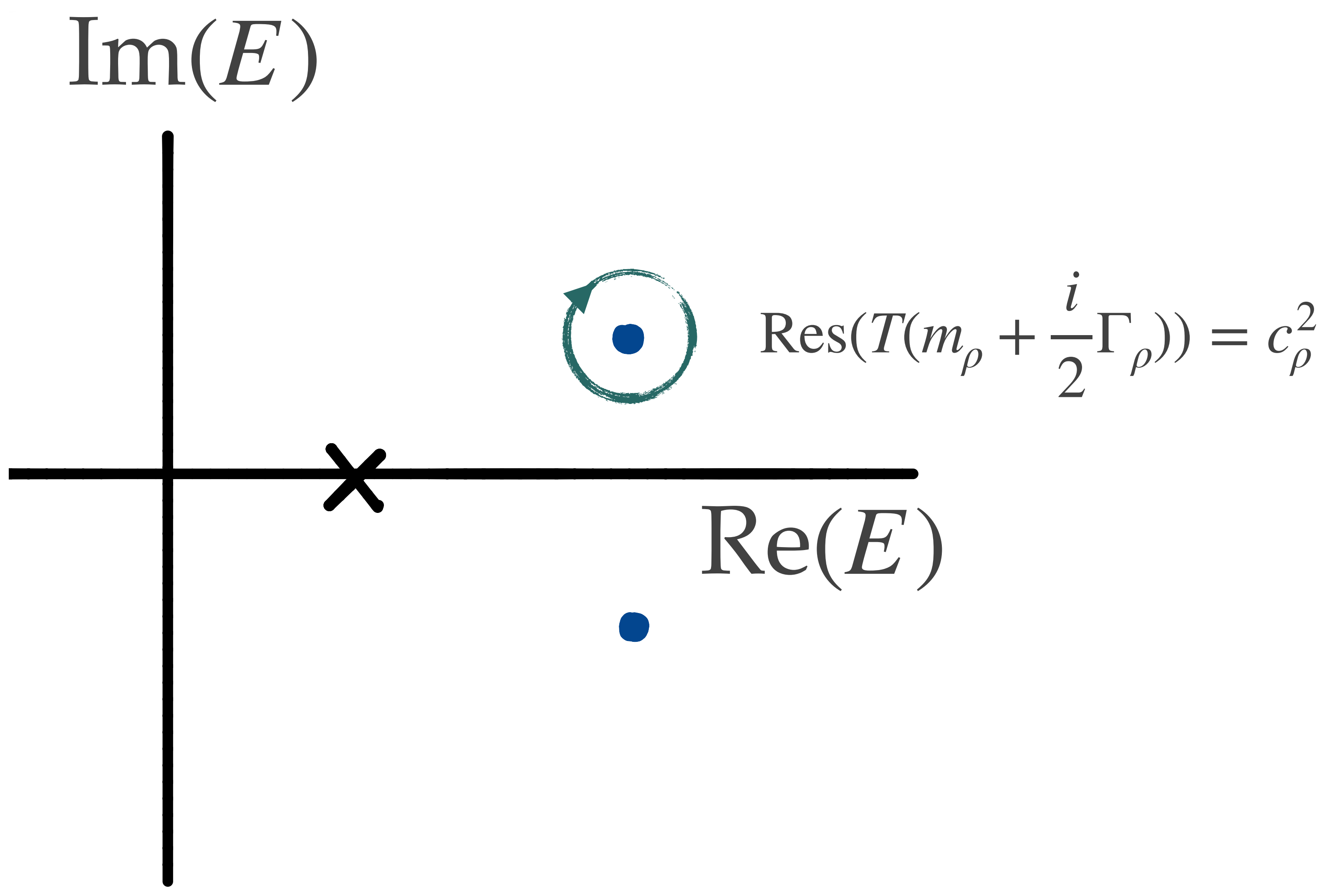


as a resonance?

- ρ as a pole of $\pi\pi$ scattering

$$m_\rho + \frac{i}{2}\Gamma_\rho \leftrightarrow k_\rho$$

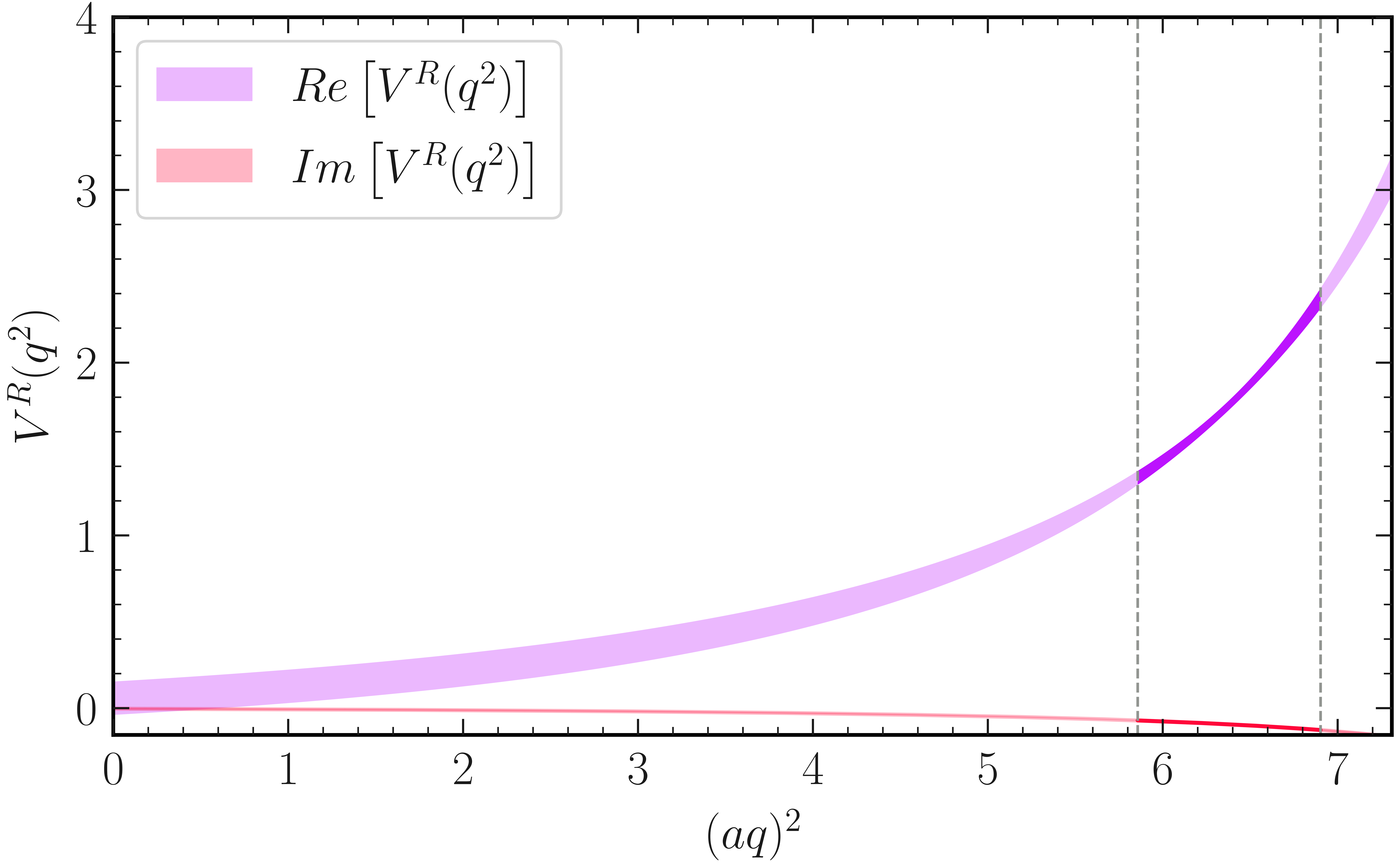
$$V^R(q^2) = \frac{c_\rho}{k_\rho} F_V\left(q^2, E^\star = m_\rho + i\Gamma_\rho/2\right)$$



- reduce $E^\star$ dependence into a resonance ("narrow width approx.")

- differential branching fraction (going to the B rest frame and simplifying)

$$\left.\frac{d^2\Gamma}{dE^{\star 2}dq^2}\right|_{V,\ell=1} = \frac{G_F^2 \left|V_{ub}\right|^2}{12\pi^3} |\vec{q}\,|^3_{B-\mathrm{RF}}\, q^2 \underbrace{\frac{1}{\pi}\frac{E^\star\Gamma(E^\star)}{\left(E^{\star 2} - m_\rho^2\right)^2 + E^{\star 2}\Gamma^2(E^\star)}}_{\to\delta\left(E^{\star 2} - m_\rho^2\right)} \frac{1}{\left(m_B + E^\star\right)^2} \underbrace{\frac{16\pi E^{\star 2}\Gamma(E^\star)}{k^3}\left|V\left(q^2, E^\star\right)\right|^2}_{V^R(q^2, E^\star)}$$



$Re\left[V^R(q^2)\right]$
$Im\left[V^R(q^2)\right]$
$V^R(q^2)$
$(aq)^2$
0
1
2
3
4
5
6
7
0
1
2
3
4

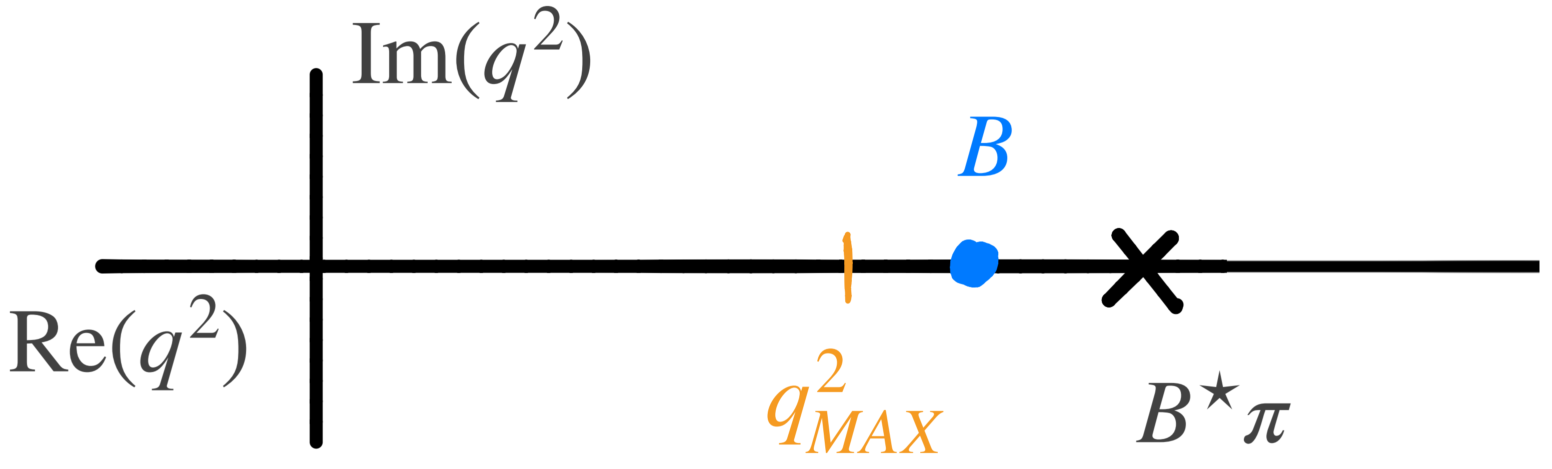
transition amplitude - Axial Current

$$K^{\mu}_{A_0}(P, p_B, \epsilon) = 2E^{\star}\ \epsilon(P, m)\cdot q\frac{q^{\mu}}{q^2}$$

$$K^{\mu}_{A_1}(P, p_B, \epsilon) = (m_B + E^{\star})\left[\epsilon(P, m)^{\mu} - \epsilon(P, m)\cdot q\frac{q^{\mu}}{q^2} - \epsilon(P, m)\cdot q\frac{q^2(m_B^2 - E^{\star 2} - q^2)}{\lambda(m_B^2, E^{\star 2}, q^2)}(P^{\mu} + p_B^{\mu} - (m_B^2 - E^{\star 2})\frac{q^{\mu}}{q^2}\right]$$

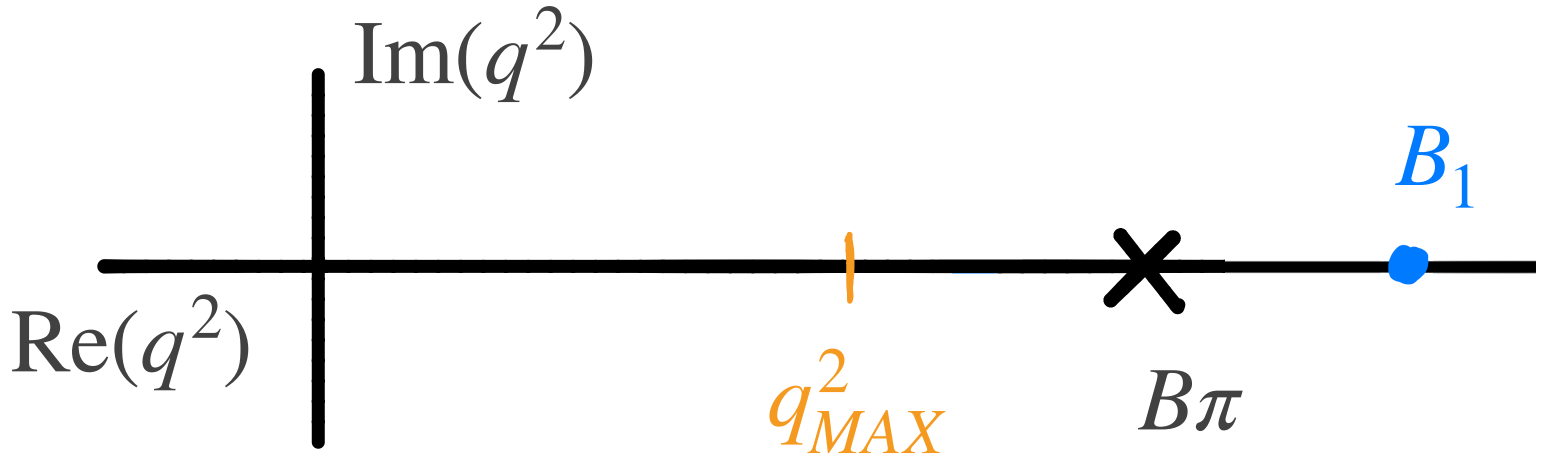
$$K^{\mu}_{A_{12}}(P, p_B, \epsilon) = \frac{16\ \epsilon(P, m)\cdot q\ m_B\ E^{\star 2}}{\lambda}\left[P^{\mu} + p_B^{\mu} - (m_B^2 - E^{\star 2})\frac{q^{\mu}}{q^2}\right]$$

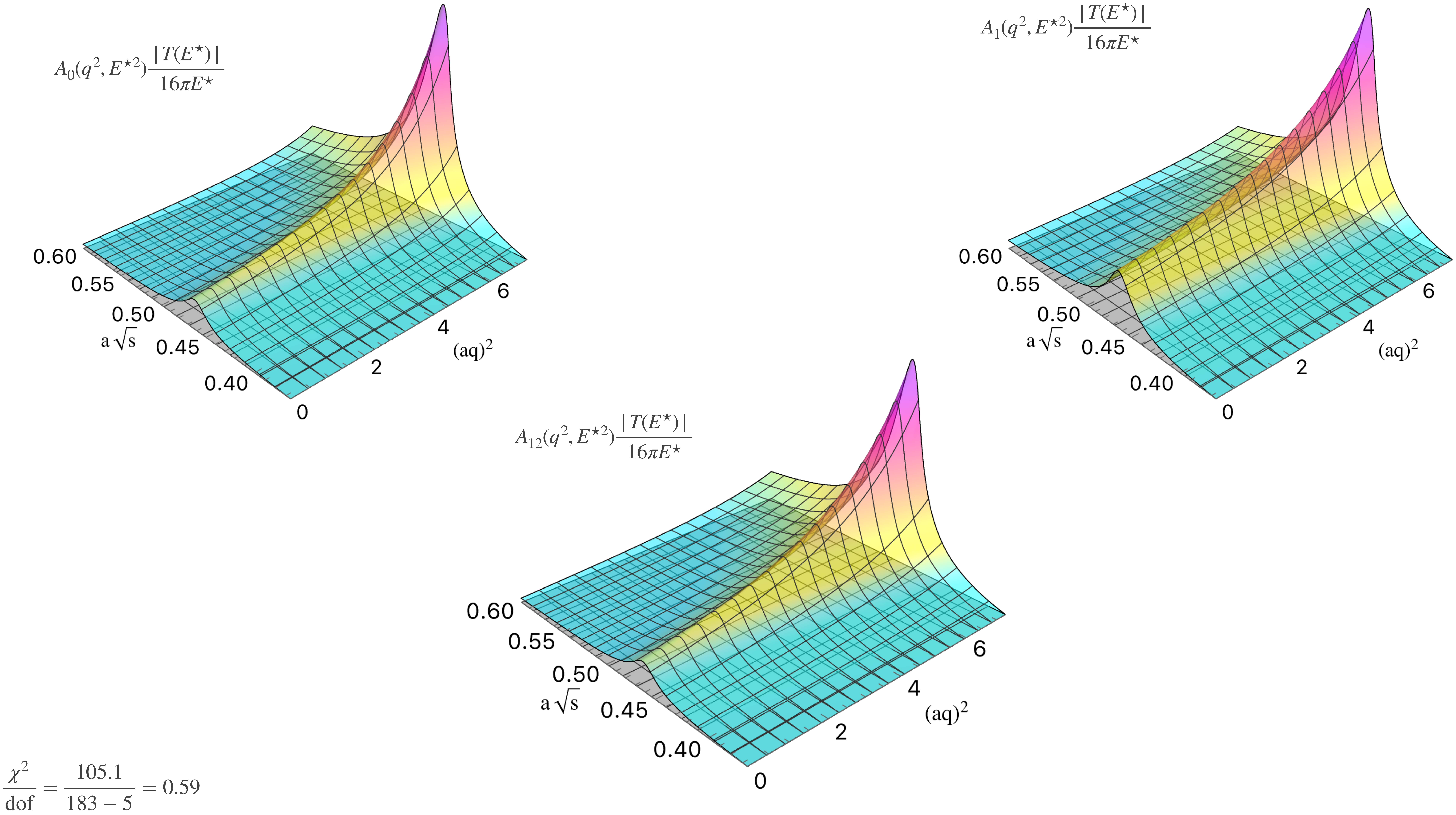
$$F_{A_0}(q^2, s) = \frac{a_0^{(A_0)} + a_1^{(A_0)}z(q^2)}{1 - \frac{q^2}{m^2_{B^{\star}}}}$$



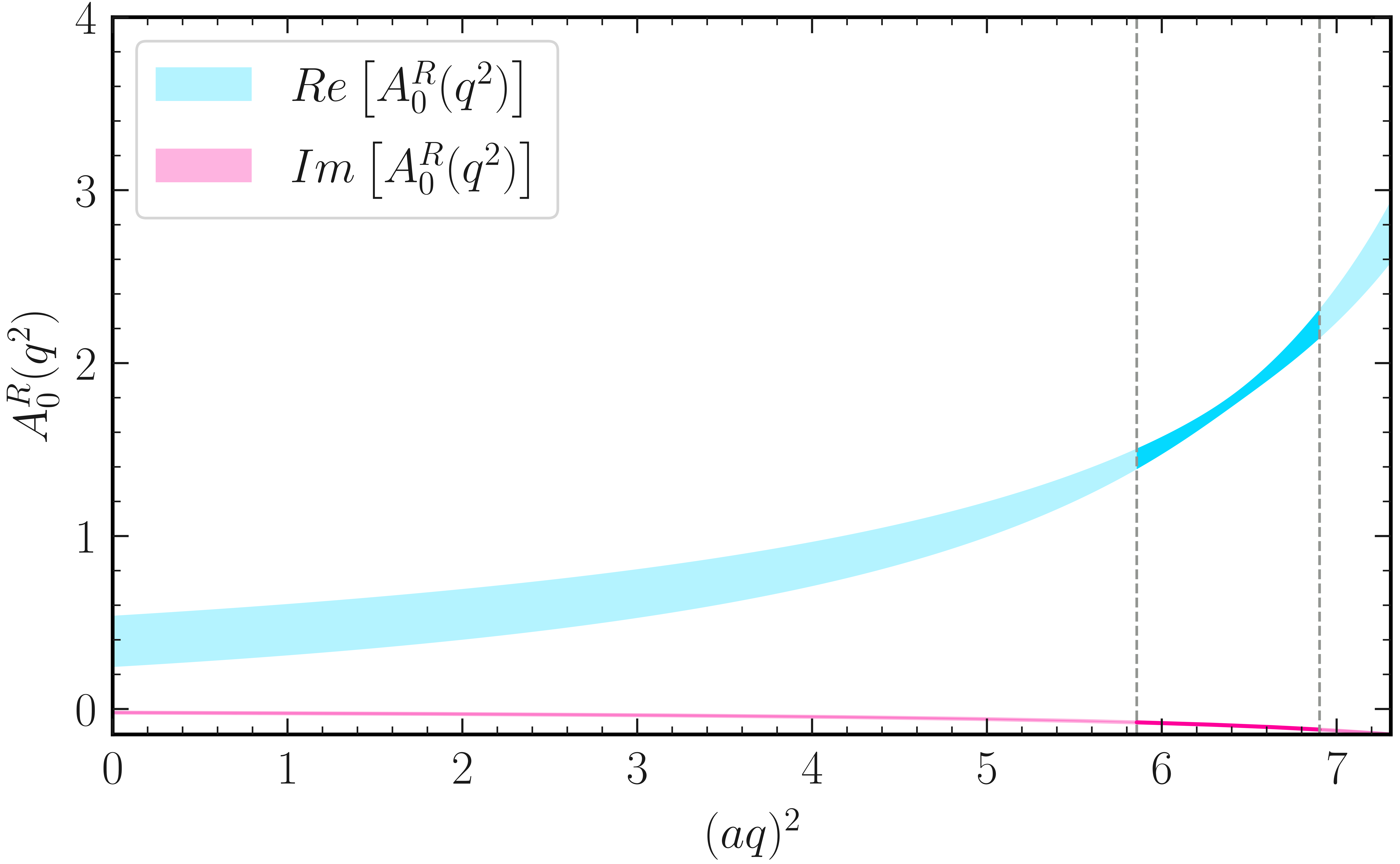
$$F_{A_1}(q^2, s) = \frac{a_0^{(A_1)} + a_1^{(A_1)}z(q^2)}{1 - \frac{q^2}{m^2_{B^{\star}}}}$$

$$F_{A_{12}}(q^2, s) = \frac{a_0^{(A_{12})} + a_1^{(A_{12})}z(q^2)}{1 - \frac{q^2}{m^2_{B^{\star}}}}$$

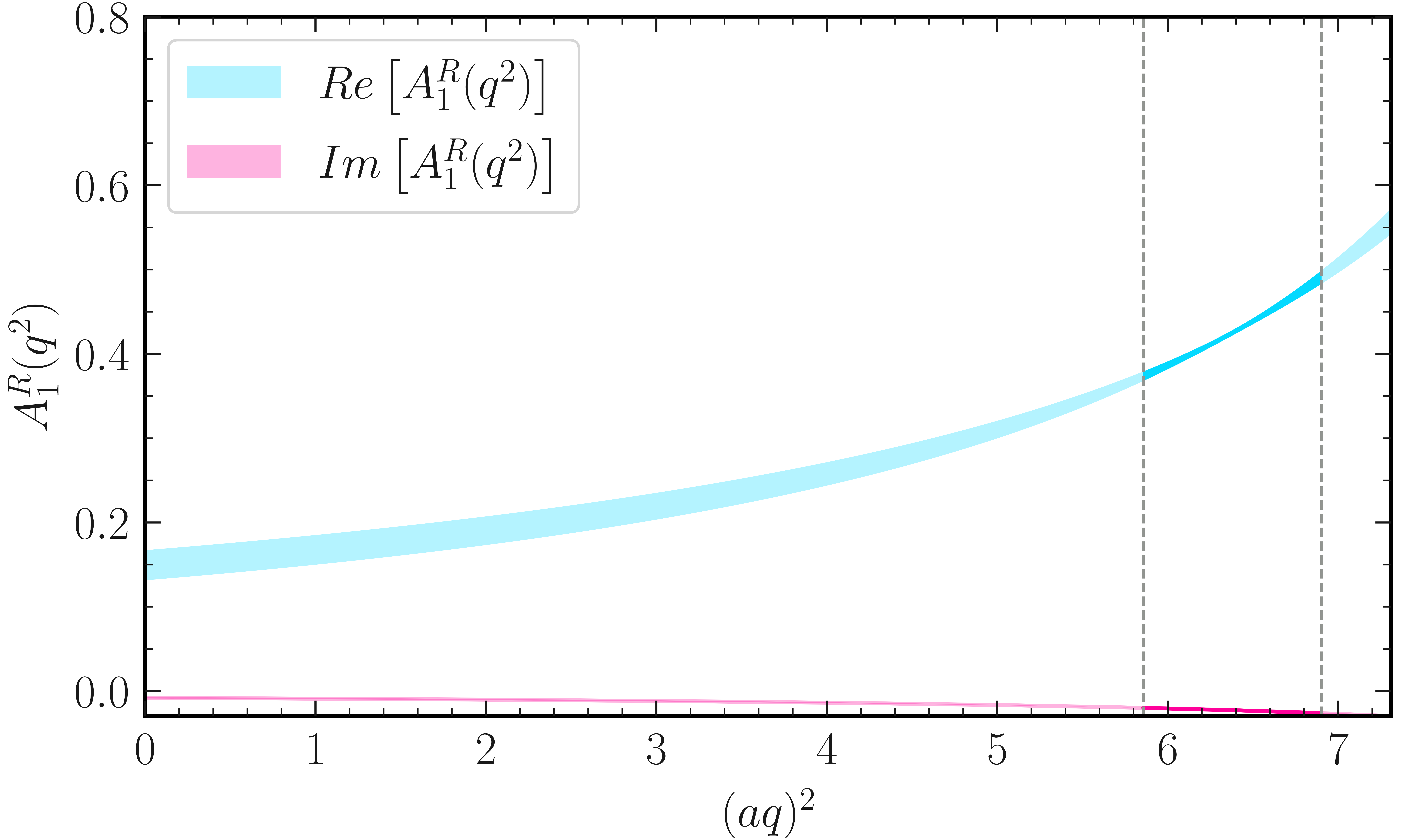




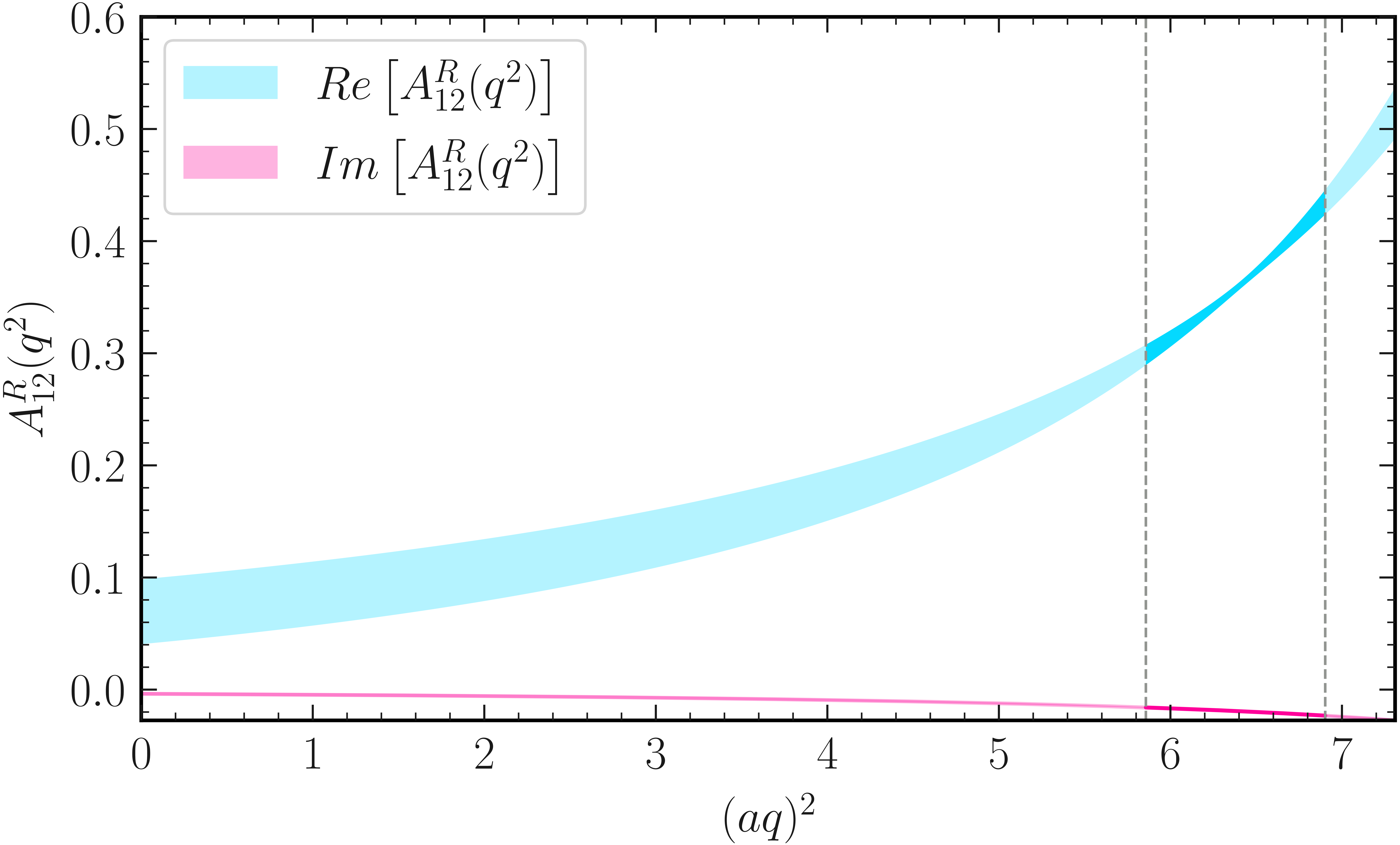
$A_0(q^2, E^{\star 2}) \frac{|T(E^\star)|}{16\pi E^\star}$
$A_1(q^2, E^{\star 2}) \frac{|T(E^\star)|}{16\pi E^\star}$
$A_{12}(q^2, E^{\star 2}) \frac{|T(E^\star)|}{16\pi E^\star}$
$a\sqrt{s}$
0.40
0.45
0.50
0.55
0.60
$(aq)^2$
0
2
4
6
$\frac{\chi^2}{\text{dof}} = \frac{105.1}{183-5} = 0.59$



$Re\left[A_0^R(q^2)\right]$
$Im\left[A_0^R(q^2)\right]$
$A_0^R(q^2)$
$(aq)^2$
0
1
2
3
4
5
6
7



$Re\left[A_1^R(q^2)\right]$
$Im\left[A_1^R(q^2)\right]$
$A_1^R(q^2)$
$(aq)^2$
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5
6
7



$Re\left[A_{12}^{R}(q^2)\right]$
$Im\left[A_{12}^{R}(q^2)\right]$
$A_{12}^{R}(q^2)$
$(aq)^2$
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
1
2
3
4
5
6
7

Summary

- first steps on
  the first process
- exciting opportunities ahead
- a "new" approach to flavor physics