Cosmological Correlators under the Analytic Lamppost

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Cosmology **B**eyond the **A**nalytic **L**amppost

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w/ Yunjia Bao, Xingang Chen, Yanou Cui, Bingchu Fan, JiJi Fan, Soubhik Kumar, Yuanzhao Li, Haoyuan Liu, Tao Liu, Abraham Loeb, Qianshu Lu, Shiyun Lu, Zhehan Qin, Matthew Reece, Xi Tong, Lian-Tao Wang, Yi Wang, Jiayi Wu, Jiaju Zang, Hongyu Zhang, Yisong Zhang, Yiming Zhong

An appetizer: Roots of a higher-degree polynomial?

We learned from kindergarten how to solve a linear equation

$$ax + b = 0 \qquad\qquad x = -b/a$$

and a quadratic equation:

$$ax^2 + bx + c = 0 \qquad\qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Later we also learned roots of cubic and quartic equations

$$x^3 + px + q = 0 \qquad x = \sqrt[3]{-\tfrac{q}{2} + \sqrt{(\tfrac{q}{2})^2 + (\tfrac{p}{3})^3}} + \sqrt[3]{-\tfrac{q}{2} - \sqrt{(\tfrac{q}{2})^2 + (\tfrac{p}{3})^3}}, \quad \cdots$$

What about a quintic equation?

Roots of a quintic equation

- We all learned from Galois that quintic roots are not algebraically expressible, but this is not the story I'm going to tell today

- After all, the root must be a well-behaved function of coefficients of the quintic polynomial. What is this function?

Roots of a quintic equation

Let me find the quintet root for you with elementary calculus: $x^5 + x + t = 0$

Define $f(x) = x^5 + x$ Then we only need to invert $f(x) = -t$

Can be done with Lagrange inversion formula:

$$x = f^{(-1)}(-t) = \sum_{k=0}^{\infty} \binom{5k}{k} \frac{(-1)^{k+1} t^{4k+1}}{4k+1}$$

The series can be summed:

$$x = -t \times {}_4\mathrm{F}_3 \left[\left. \begin{matrix} \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \\ \frac{2}{4}, \frac{3}{4}, \frac{5}{4} \end{matrix} \right| -5 \left(\frac{5t}{4} \right)^4 \right]$$

Likewise, general sextic roots also expressible in hypergeometric functions, but with 2 variables (Kampé de Fériet function)

Hypergeometric functions

- Hypergeometric functions can be defined by series:

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix}\middle| z\right] \equiv \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!}. \qquad (a)_n = a(a+1)\cdots(a+n-1) $$

They are very general and very powerful. A lot of functional identities!

§15.4(i) Elementary Functions

The following results hold for principal branches when $|z| < 1$, and by analytic continuation elsewhere. Exceptions are (15.4.8) and (15.4.10), that hold for $|z| < \pi/4$, and (15.4.12), (15.4.14), and (15.4.16), that hold for $|z| < \pi/2$.

15.4.1 $$F(1,1;2;z) = -z^{-1}\ln(1-z),$$

15.4.2 $$F\left(\tfrac{1}{2},1;\tfrac{3}{2};z^2\right) = \frac{1}{2z}\ln\left(\frac{1+z}{1-z}\right),$$

15.4.3 $$F\left(\tfrac{1}{2},1;\tfrac{3}{2};-z^2\right) = z^{-1}\arctan z,$$

15.4.4 $$F\left(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};z^2\right) = z^{-1}\arcsin z,$$

15.4.5 $$F\left(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};-z^2\right) = z^{-1}\ln\left(z+\sqrt{1+z^2}\right).$$

15.4.6 $$F(a,b;a;z) = (1-z)^{-b}, \quad F(a,b;b;z) = (1-z)^{-a},$$

where the limit interpretation (15.2.6), rather than (15.2.5), has to be taken when the third parameter is a nonpositive integer. See the final paragraph in §15.2(ii).

15.4.7 $$F\left(a,\tfrac{1}{2}+a;\tfrac{1}{2};z^2\right) = \tfrac{1}{2}\left((1+z)^{-2a}+(1-z)^{-2a}\right),$$

15.4.8 $$F\left(a,\tfrac{1}{2}+a;\tfrac{1}{2};-\tan^2 z\right) = (\cos z)^{2a}\cos(2az).$$

15.4.9 $$F\left(a,\tfrac{1}{2}+a;\tfrac{3}{2};z^2\right) = \frac{1}{(2-4a)z}\left((1+z)^{1-2a}-(1-z)^{1-2a}\right),$$

15.4.10 $$F\left(a,\tfrac{1}{2}+a;\tfrac{3}{2};-\tan^2 z\right) = (\cos z)^{2a}\frac{\sin((1-2a)z)}{(1-2a)\sin z}.$$

15.4.11 $$F\left(-a,a;\tfrac{1}{2};-z^2\right) = \tfrac{1}{2}\left(\left(\sqrt{1+z^2}+z\right)^{2a}+\left(\sqrt{1+z^2}-z\right)^{2a}\right),$$

15.4.12 $$F\left(-a,a;\tfrac{1}{2};\sin^2 z\right) = \cos(2az).$$

§15.9(i) Orthogonal Polynomials

For the notation see §§18.3 and 18.19.

Jacobi

15.9.1 $$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}F\left({-n,n+\alpha+\beta+1 \atop \alpha+1};\frac{1-x}{2}\right).$$

Gegenbauer (or Ultraspherical)

15.9.2 $$C_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{n!}F\left({-n,n+2\lambda \atop \lambda+\frac{1}{2}};\frac{1-x}{2}\right).$$

15.9.3 $$C_n^{(\lambda)}(x) = (2x)^n\frac{(\lambda)_n}{n!}F\left({-\frac{1}{2}n,\frac{1}{2}(1-n) \atop 1-\lambda-n};\frac{1}{x^2}\right).$$

15.9.4 $$C_n^{(\lambda)}(\cos\theta) = e^{ni\theta}\frac{(\lambda)_n}{n!}F\left({-n,\lambda \atop 1-\lambda-n};e^{-2i\theta}\right).$$

Chebyshev

15.9.5 $$T_n(x) = F\left({-n,n \atop \frac{1}{2}};\frac{1-x}{2}\right).$$

15.9.6 $$U_n(x) = (n+1)F\left({-n,n+2 \atop \frac{3}{2}};\frac{1-x}{2}\right).$$

Legendre

15.9.7 $$P_n(x) = F\left({-n,n+1 \atop 1};\frac{1-x}{2}\right).$$

§15.8(i) Linear Transformations

All functions in this subsection and §15.8(ii) assume their principal values.

15.8.1 $$\mathbf{F}\left({a,b \atop c};z\right) = (1-z)^{-a}\mathbf{F}\left({a,c-b \atop c};\frac{z}{z-1}\right) = (1-z)^{-b}\mathbf{F}\left({c-a,b \atop c};\frac{z}{z-1}\right) = (1-z)^{c-a-b}\mathbf{F}\left({c-a,c-b \atop c};z\right),$$

$|\mathrm{ph}(1-z)| < \pi.$

15.8.2 $$\frac{\sin(\pi(b-a))}{\pi}\mathbf{F}\left({a,b \atop c};z\right) = \frac{(-z)^{-a}}{\Gamma(b)\Gamma(c-a)}\mathbf{F}\left({a,a-c+1 \atop a-b+1};\frac{1}{z}\right) - \frac{(-z)^{-b}}{\Gamma(a)\Gamma(c-b)}\mathbf{F}\left({b,b-c+1 \atop b-a+1};\frac{1}{z}\right),$$

$|\mathrm{ph}(-z)| < \pi.$

15.8.3 $$\frac{\sin(\pi(b-a))}{\pi}\mathbf{F}\left({a,b \atop c};z\right) = \frac{(1-z)^{-a}}{\Gamma(b)\Gamma(c-a)}\mathbf{F}\left({a,c-b \atop a-b+1};\frac{1}{1-z}\right) - \frac{(1-z)^{-b}}{\Gamma(a)\Gamma(c-b)}\mathbf{F}\left({b,c-a \atop b-a+1};\frac{1}{1-z}\right),$$

$|\mathrm{ph}(-z)| < \pi.$

15.8.4 $$\frac{\sin(\pi(c-a-b))}{\pi}\mathbf{F}\left({a,b \atop c};z\right) = \frac{1}{\Gamma(c-a)\Gamma(c-b)}\mathbf{F}\left({a,b \atop a+b-c+1};1-z\right) - \frac{(1-z)^{c-a-b}}{\Gamma(a)\Gamma(b)}\mathbf{F}\left({c-a,c-b \atop c-a-b+1};1-z\right),$$

$|\mathrm{ph}\, z| < \pi, |\mathrm{ph}(1-z)| < \pi.$

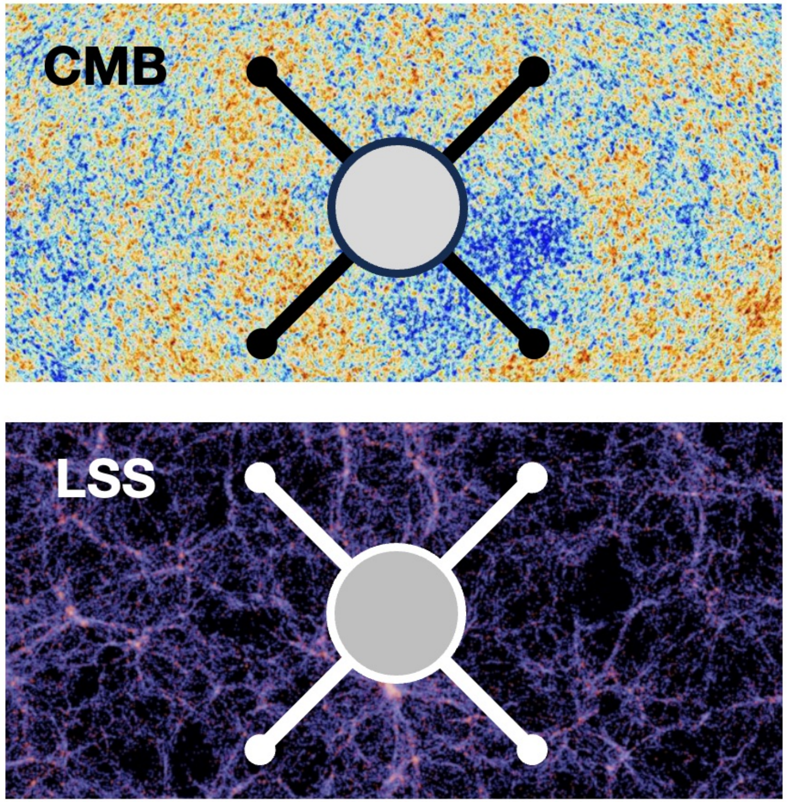
15.8.5 $$\frac{\sin(\pi(c-a-b))}{\pi}\mathbf{F}\left({a,b \atop c};z\right) = \frac{z^{-a}}{\Gamma(c-a)\Gamma(c-b)}\mathbf{F}\left({a,a-c+1 \atop a+b-c+1};1-\frac{1}{z}\right) - \frac{(1-z)^{c-a-b}z^{a-c}}{\Gamma(a)\Gamma(b)}\mathbf{F}\left({c-a,1-a \atop c-a-b+1};1-\frac{1}{z}\right),$$

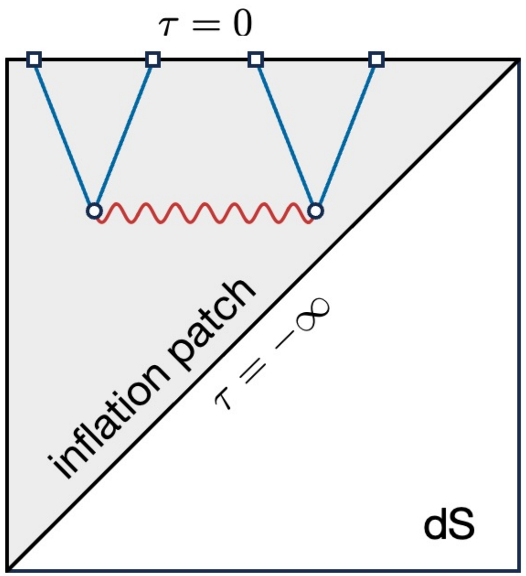
$|\mathrm{ph}\, z| < \pi, |\mathrm{ph}(1-z)| < \pi.$

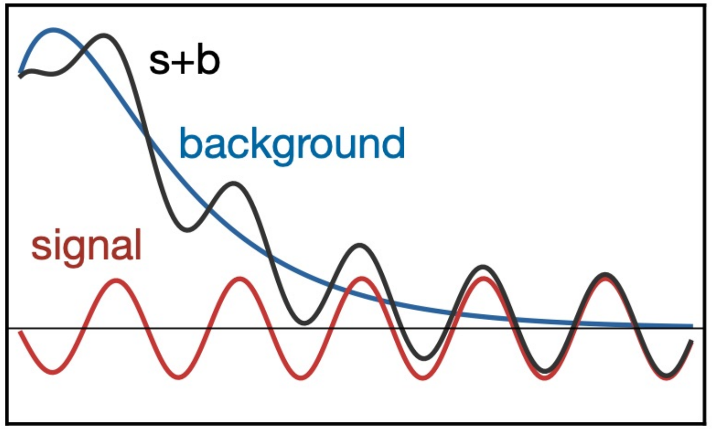
https://dlmf.nist.gov/ in case you are interested 😉

A Cosmological collider program

[Chen, Wang, 0911.3380; Arkani-Hamed, Maldacena, 1503.08043 and many more]







Inflation ~ dS

particle production

mass ~ 10^{14} GeV

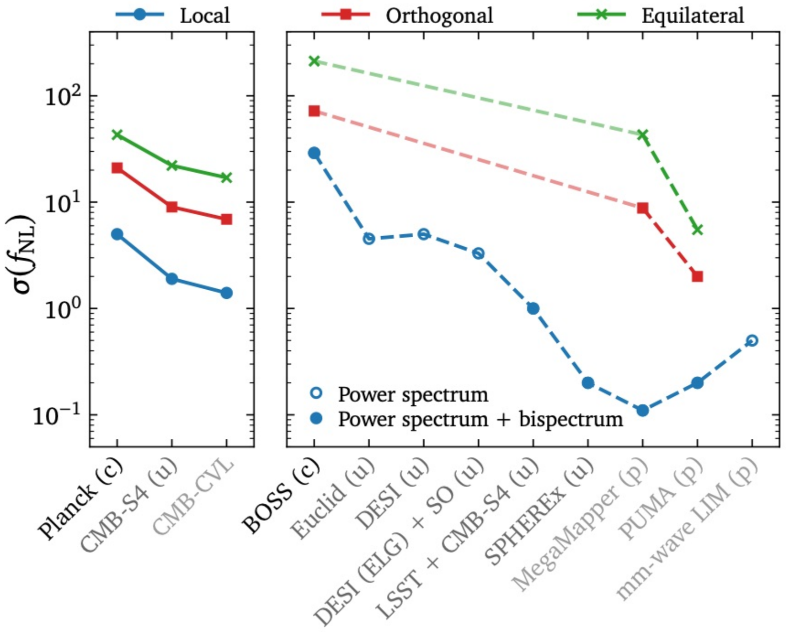
superhorizon resonance

mass, spin, coupling, etc

amplitude nonanalyticity

Data are coming in!

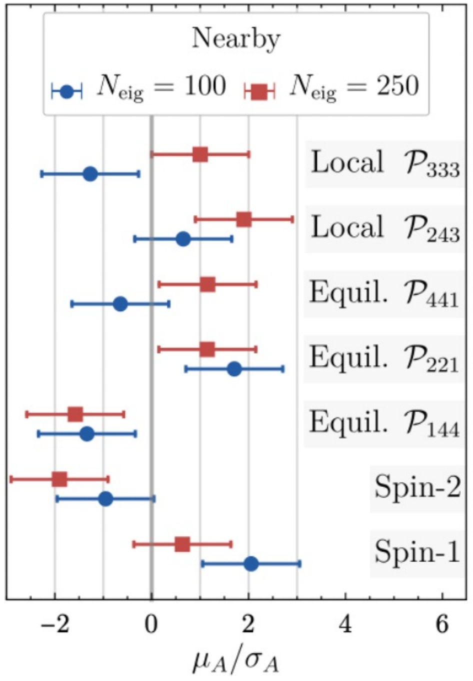
- ~ 2 orders in near future; ~ 4 ultimately with 21cm



[Snowmass 2021: 2203.08128]

- Searches from CMB [Sohn et al. 2404.07203] and LSS data [Cabass et al. 2404.01894]

- Realistic particle models
- Parity violation [Bao, Wang, ZX, Zhong, 2504.02931]
- Quasi-single field inflation meets CMB [Kumar, Lu, ZX, Zhang, to appear]



[Bao et al., 2504.02931]

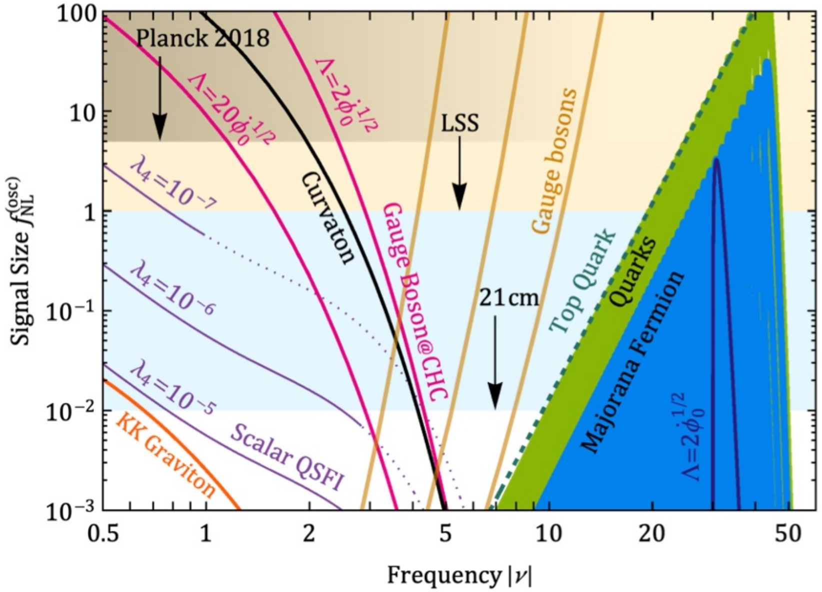
Big questions

- Cosmological collider signals are cool, but:

- Can they be true?

- How to find them?

Can they be true? --- Particle Phenomenology



[Lian-Tao Wang, ZX, 1910.12876]

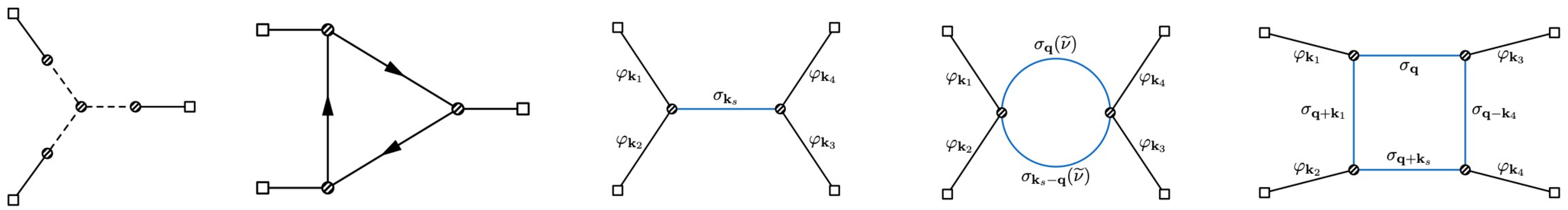
Over the years, many particle models identified in SM/BSM, with naturally large signals

Many fascinating stories which are still ongoing

The CC signals can be there, and deserve to be treated seriously

How to find them? --- Theory templates

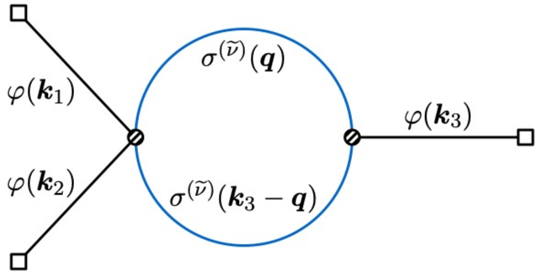
- Behind the CC signals are “simple” Feynman graphs in the inflationary background:



- To look for CC signals in real data, we need a template bank
- Not a kinematic point, but the full shape; not for a parameter; but a multi-dim parameter grid
- We’d better compute them with precision and efficiency
- They may be hard, but let’s not complain; Let’s do it, analytically
- Developing fast! Many computations considered impossible a few years ago are now done

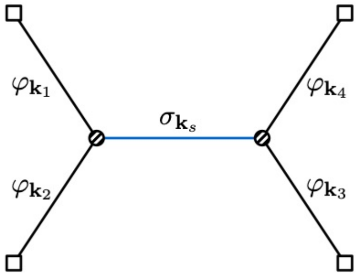
Why analytic?

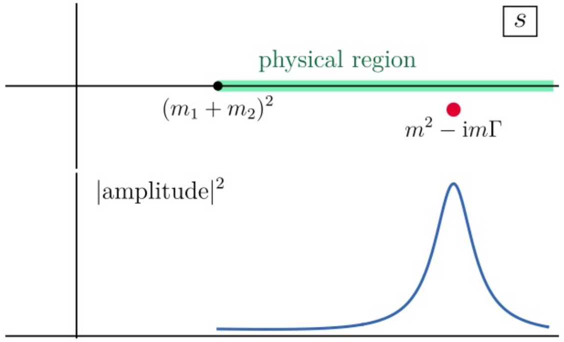
- Data-wise: good analytical strategy speeds up numerical computation
  Example: 3pt massive bubble: numerical [O(10^5) CPU hrs] vs. analytical [O(10s) @ laptop]
  [Wang, ZX, Zhong, 2109.14635] [Liu, Qin, ZX, 2407.12299]

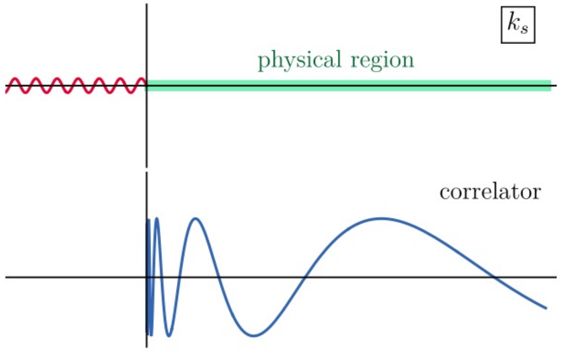


$$\mathcal{J}^{0,-2}(u) = Cu^3 - \frac{u^4}{128\pi\sin(2\pi \mathrm{i}\tilde\nu)}\sum_{n=0}^{\infty} \frac{(3+4\mathrm{i}\tilde\nu+4n)(1+n)_{\frac12}(1+2\mathrm{i}\tilde\nu+n)_{\frac12}}{(\frac12+\mathrm{i}\tilde\nu+n)_{\frac12}(\frac32+\mathrm{i}\tilde\nu+n)_{\frac12}}$$
$$\times\left\{ {}_2\mathcal{F}_1\left[\begin{matrix}2+2\mathrm{i}\tilde\nu+2n,\ 4+2\mathrm{i}\tilde\nu+2n\\ 4+4\mathrm{i}\tilde\nu+4n\end{matrix}\middle| u\right] u^{2n+2\mathrm{i}\tilde\nu} - {}_3\mathcal{F}_2\left[\begin{matrix}1,2,4\\ 1-2n-2\mathrm{i}\tilde\nu,\ 4+2n+2\mathrm{i}\tilde\nu\end{matrix}\middle| u\right]\right\}$$
$$+ (\tilde\nu\to-\tilde\nu)$$

- Theory-wise: good lessons about QFT in dS from analytical structures of correlators
  Whenever a correlator becomes singular, there is a physical reason



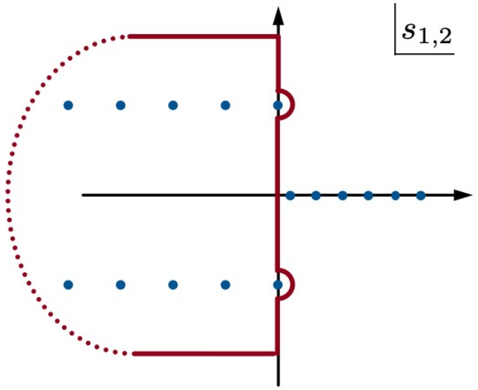




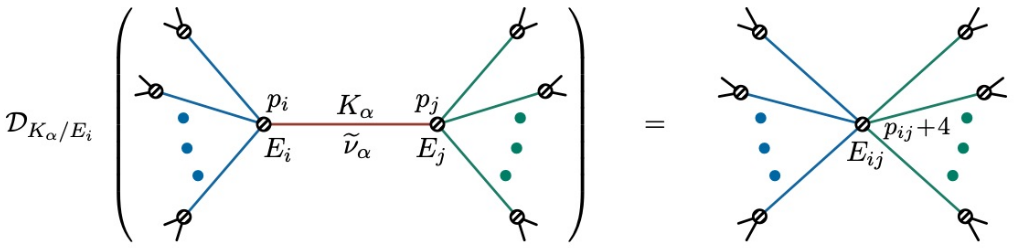
[Qin, ZX, 2308.14802]

Analytical methods

Partial Mellin-Barnes representation [Resolve!]

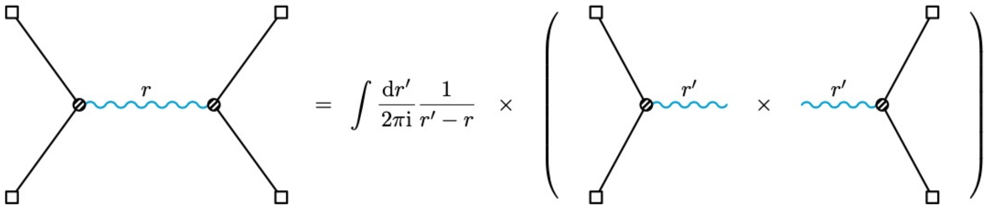


• Differential equations [Pinch!]



Family tree decomposition [Flip!]

• Dispersion relations [Glue!]



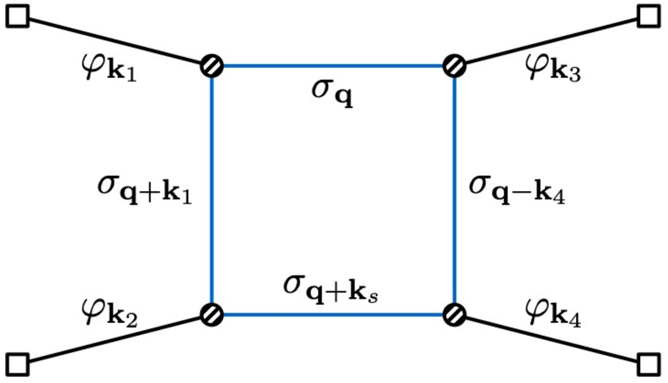
Massive inflation correlators

[See Chen, Wang, ZX, 1703.10166 for a review]

$$\mathcal{T}(\{\boldsymbol{k}\}) \sim \int \mathrm{d}\tau \int \mathrm{d}^d\boldsymbol{q} \times (-\tau)^p \times e^{\mathrm{i}E\tau} \times \mathrm{H}_{\mathrm{i}\tilde{\nu}}\Big[-K(\boldsymbol{q},\boldsymbol{k})\tau\Big] \times \theta(\tau_i - \tau_j)$$

vertex int   loop int   ext line   bulk line

- Massless / conformal external lines + (principal) massive internal lines
- Challenges:
    - Mode functions (Hankel, Whittacker, …)
    - Loop momentum integrals
    - Nested time integrals
- Complexity increases with # of loops and # of vertices



Partial Mellin-Barnes representation

[Qin, ZX, 2205.01692, 2208.13790]

MB rep for all bulk lines; Resolving special functions into power functions

For example: Massive scalar propagator [Hankel function]

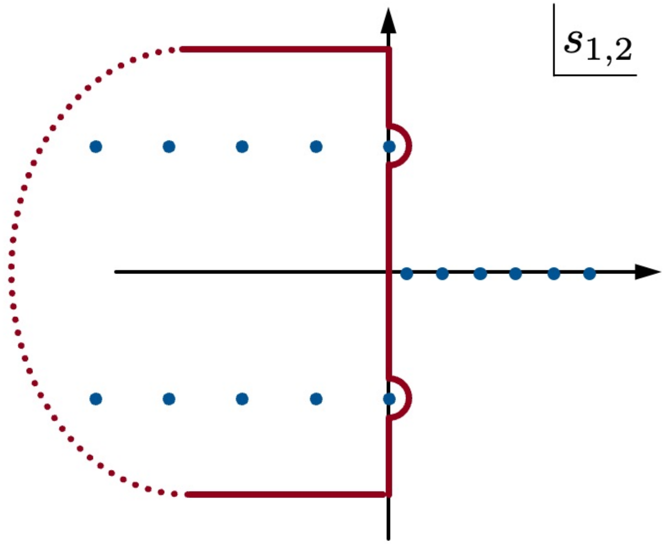
$$\mathrm{H}_\nu^{(1)}(-k\tau) = \frac{1}{\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\left(\frac{k}{2}\right)^{-2s}(-\tau)^{-2s}e^{(2s-\nu-1)\pi\mathrm{i}/2}\Gamma\left[s-\frac{\nu}{2},s+\frac{\nu}{2}\right]$$

Expanding in dilatation eigenmode, but no dilatation or boost symmetry required

Time and loop momentum integrals factorized, enabling separate treatments

$$\mathcal{T}(\{\boldsymbol{k}\}) \sim \int \mathrm{d}s \times \mathcal{G}(s) \times \left[\int \mathrm{d}^d\boldsymbol{q} K(\boldsymbol{q},\boldsymbol{k})^\alpha\right] \quad \text{Loop int}$$

$$\times \left[\int \mathrm{d}\tau e^{\mathrm{i}E\tau} \times (-\tau)^\beta \times \theta(\tau_i-\tau_j)\right] \quad \text{Time int}$$



[See also Sleight 1907.01143 etc.]

Family tree decomposition

[ZX, Zang, 2309.10849]

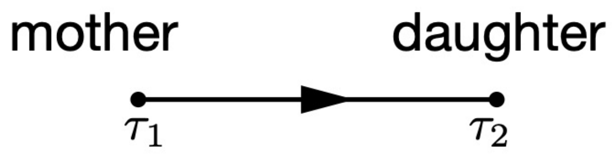
Family tree decomposition: flip the directions such that all graphs are partially ordered

$$\theta(\tau_1-\tau_2)+\theta(\tau_2-\tau_1)=1$$

Partial order:

A mother can have any number of daughters

but a daughter must have only one mother



Every resulting nested graph can be interpreted as a maternal family tree

sisters

A notation for FTs: $$\big[12(34\cdots)(5\cdots)\big]=\int_{-\infty}^{0}\prod_{i=1}^{N}\Big[\mathrm{d}\tau_i(-\tau_i)^{q_i-1}e^{\mathrm{i}\omega_i\tau_i}\Big]\theta_{21}\theta_{32}\theta_{52}\theta_{43}\cdots$$

mother-daughter

$$\big[\mathscr{P}(\widehat{1}2\cdots N)\big]=\frac{(-\mathrm{i})^N}{(\mathrm{i}\omega_1)^{q_{1\cdots N}}}\sum_{n_2,\cdots,n_N=0}^{\infty}\Gamma(q_{1\cdots N}+n_{2\cdots N})\prod_{j=2}^{N}\frac{(-\omega_j/\omega_1)^{n_j}}{(\widetilde{q}_j+\widetilde{n}_j)n_j!}$$

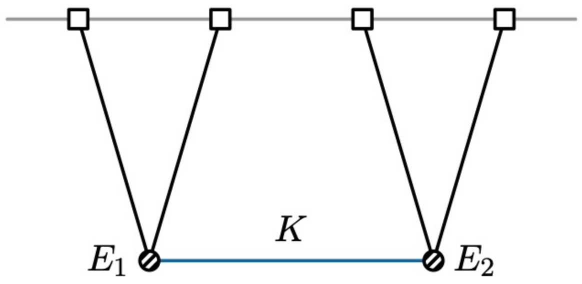
What can PMB + FTD do?

- Tree level: A trivial procedure to get analytical results for all trees [solved]
  PMB => FTD => Collecting MB poles => Solutions in hypergeo series [ZX, Zang, 2309.10849]

- A byproduct: complete analytical answer for all conformal scalar tree amplitudes in power-law FRW space [automatically solve the kinematic flow diff eqs] [Fan, ZX, 2403.07050]

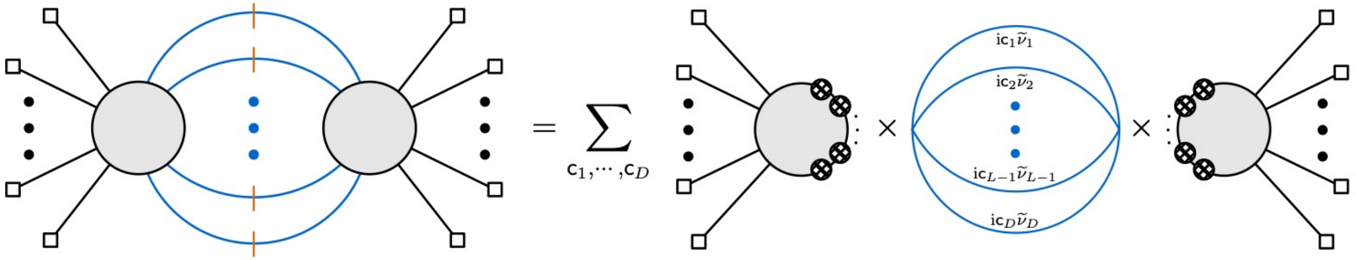
- Beyond tree level: Full computation remains challenging
  However, very useful for studying analytical structure of arbitrary loop graphs

- Generally, a (tree or loop) correlator can exhibit singular behavior (branch point) at:
  ☐ Nonlocal signal branch points (soft momentum limit)
  $K \to 0$
  ☐ Local signal branch points (hard energy limit)
  $E_1 \to \infty$ or $E_2 \to \infty$
  ☐ Partial energy branch points (zero energy sum limit)
  $E_1 + K \to 0$ or $E_2 + K \to 0$ or $E_1 + E_2 \to 0$



Singularity structure / factorization theorems / cutting rules

- Take the nonlocal signal as an example (very relevant to CC pheno)
- The nonlocal signal is factorized (and thus cut) and computable to the leading order in the soft momentum but to all loop orders [Qin, ZX, 2304.13295; 2308.14802]



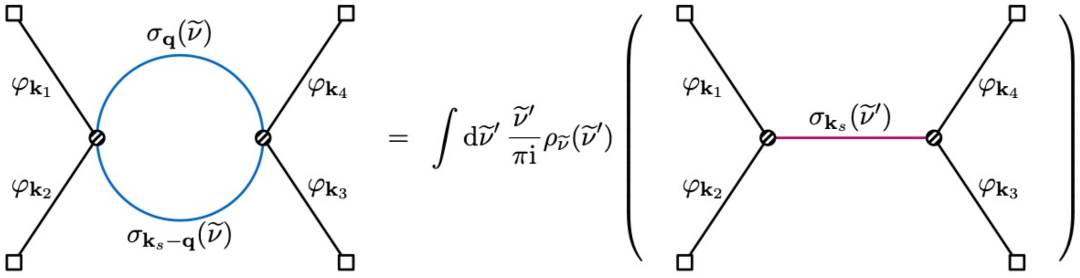
$$\mathfrak{M}_{c_1\cdots c_D}(P) \equiv \frac{P^{3(D-1)}}{(4\pi)^{(5D-3)/2}}\Gamma\begin{bmatrix} -\sum_{i=1}^{D} c_i \mathrm{i}\widetilde{\nu}_i - \frac{3}{2}(D-1) \\ \frac{3}{2}D + \sum_{i=1}^{D} c_i \mathrm{i}\widetilde{\nu}_i \end{bmatrix} \prod_{\ell=1}^{D}\left\{\Gamma\left[\frac{3}{2} + c_\ell \mathrm{i}\widetilde{\nu}_\ell, -c_\ell \mathrm{i}\widetilde{\nu}_\ell\right]\left(\frac{P}{2}\right)^{2\mathrm{i}c_\ell\widetilde{\nu}_\ell}\right\}$$

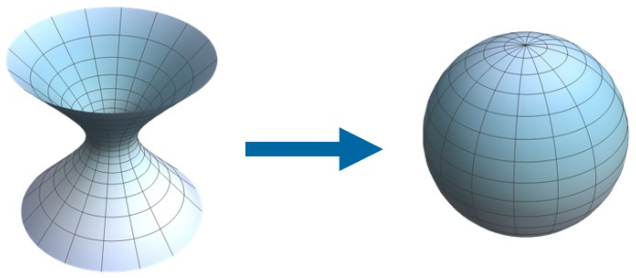
- Similar factorization and cutting rules hold for the local signal [Qin, ZX, to appear] and partial-energy limit [Wu, ZX, Zhang, to appear]
- In a sense, the nonanalytic part is always “simpler” than the analytic part

Spectral decomposition of loops

[ZX, Zhang, 2211.03810; Zhang, to appear]

Loops greatly simplified with new strategies in certain cases: spectral decomposition

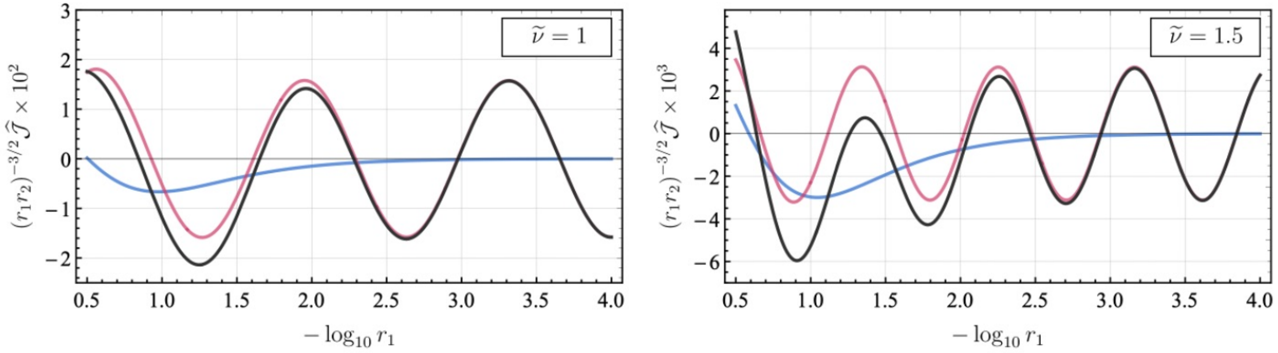




Rewrite bubble 1-loop as linear superposition of tree graphs with all possible masses.

The spectral density obtainable by Wick-rotating dS to sphere or AdS

With spectral method, we get the first and hitherto only known complete analytical result for massive 1-loop processes

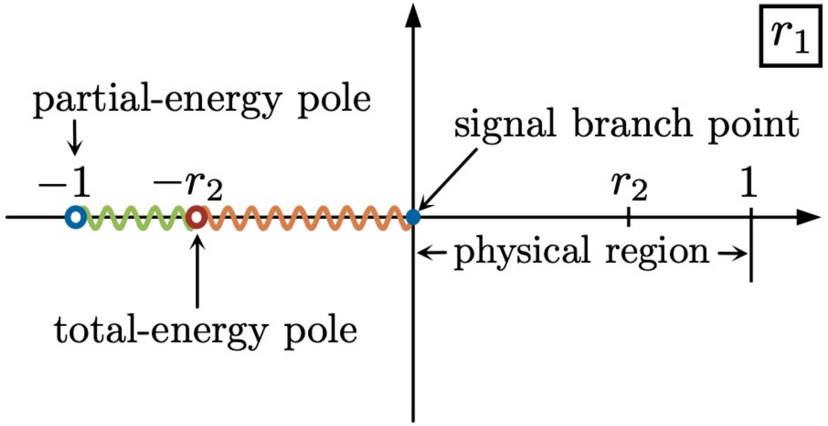


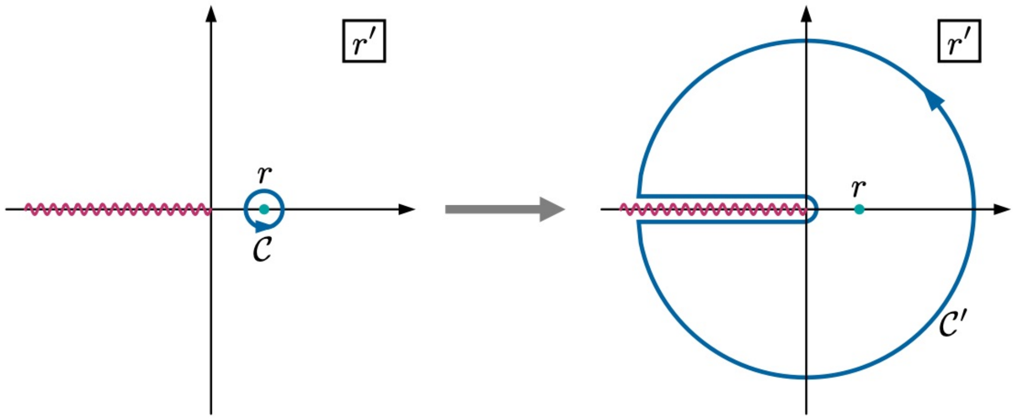
A dispersive boostrap

[Liu, Qin, ZX, 2407.12299; Liu, Qin, Wu, ZX, Zhang, to appear]

The study of analyticity allows us to locate all singularities on the complex plane

=> Bootstraping complex graphs by gluing simpler ones. The glue: dispersion integral





Dispersion integrals are insensitive to UV (local) physics

New and much simplified analytical expression for loops; UV and IR neatly separated

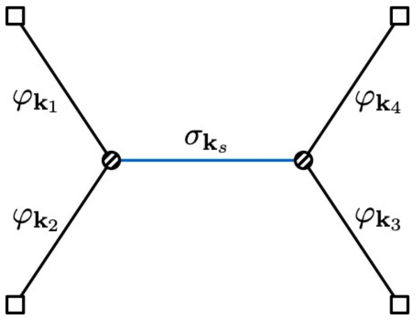
In particular: we identify an "irreducible background" demanded by analyticity

Lesson: UV div/regularization artificial and avoidable; but renormalization physical

[See also: Werth, 2409.02072]

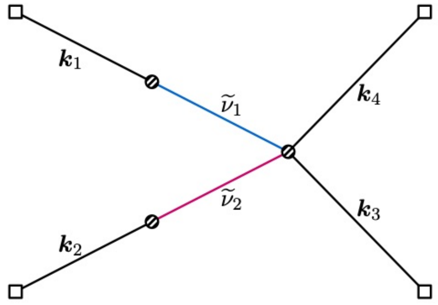
Differential equations

- "Old" technique, first used in cosmo correlators as a "bootstrap" equation
- However, much easier to derive and to generalize in the bulk



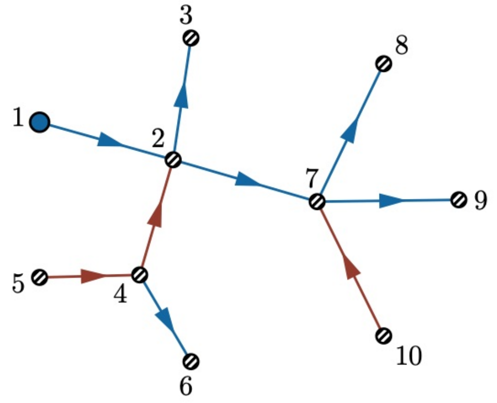
1 exchange (2018)

"Cosmological bootstrap"
[Arkani-Hamed, Baumann, Lee, Pimentel, 1811.00024]



2 exchanges (2024)

[ZX, Zang, 2309.10849]
[Aoki, Pinol, Sano, Yamaguchi, Zhu, 2404.09547]



Arbitrary exchanges (2023)

Partial Mellin-Barnes
[Qin, ZX, 2205.01692, 2208.13790]
Family tree [ZX, Zang, 2309.10849]

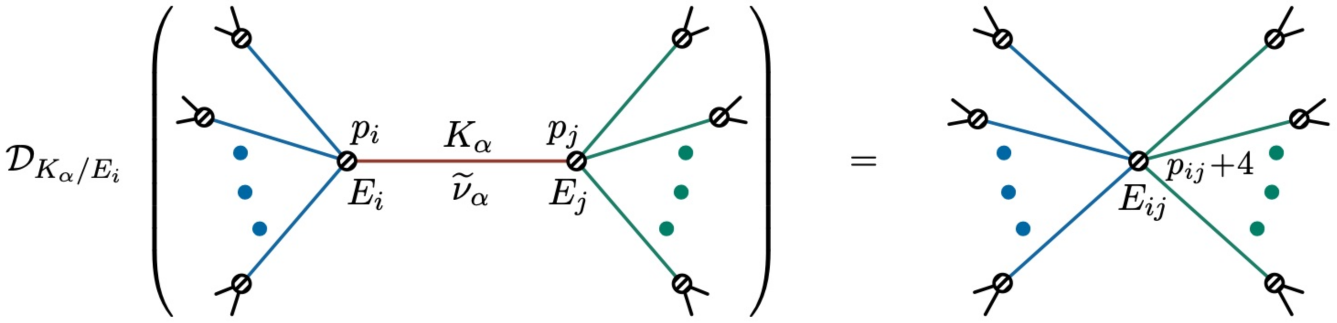
- Partial Mellin-Barnes + family-tree decomposition reduce all analytical computation to a trivial but tedious routine; The results involve too many layers of summations
- It'd be good to have a rule to write down the results without doing any computation

[See also Pimentel, Wang, 2205.00013, Qin, ZX, 2208.13790, 2301.07047, Jazayeri, Renaux-Petel, 2205.10340, Qin, Renaux-Petel, Tong, Werth, Zhu, 2506.01555, etc]

Differential equations for arbitrary massive trees

[Liu, ZX, 2412.07843]

- An internal line (bulk propagator) is collapsed to 0 or δ by a Klein-Gordon operator
- The KG operator can be pulled out of the integral with IBP at a given vertex
- We obtain a 2nd order diff eq for the graph by picking up a line + one of its two endpoint
- There are a total of 2*I* choices => 2*I* diff eqs for 2*I* indep energy ratios. A complete set!



$$\mathcal{D}_{(\alpha i)}\mathcal{G} = \frac{r_{(\alpha i)}^{p_j+4}\, r_{(\alpha j)}^{p_i+4}}{\left[r_{(\alpha i)}+r_{(\alpha j)}\right]^{p_{ij}+5}}\mathsf{C}_\alpha[\mathcal{G}],$$

$$\mathcal{D}_{(\alpha i)} \equiv \left(\vartheta_{(\alpha i)}-\frac{3}{2}\right)^2+\widetilde{\nu}_\alpha^2-r_{(\alpha i)}^2\left(\vartheta_{\{i\}}+p_i+2\right)\left(\vartheta_{\{i\}}+p_i+1\right)$$

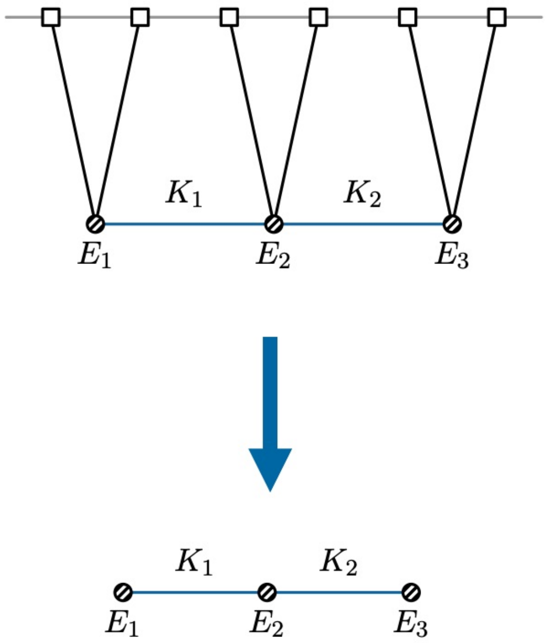
$$r_{(\alpha i)} = \frac{K_\alpha}{E_i} \qquad \vartheta_{(\alpha i)} \equiv r_{(\alpha i)}\frac{\partial}{\partial r_{(\alpha i)}} \qquad \vartheta_{\{i\}} \equiv \sum_{\beta\in\mathcal{N}(i)}\vartheta_{(\beta i)}$$

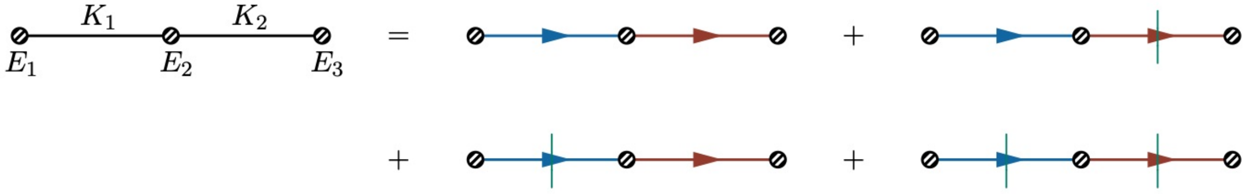
Complete solution

[Liu, ZX, 2412.07843]

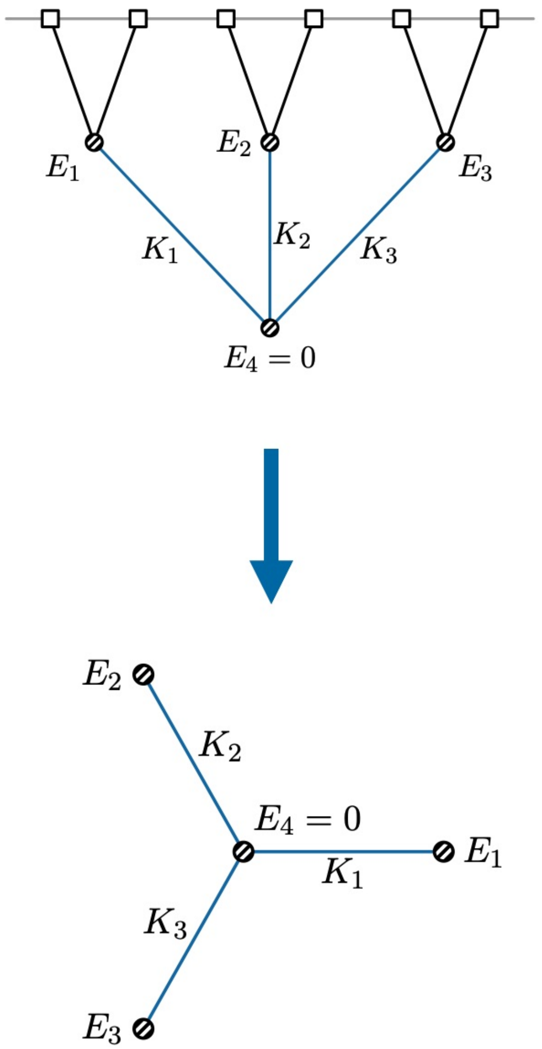
$$\mathcal{G} = \sum_{i\in 2^{\{K\}}} \underset{i}{\mathrm{Cut}}\left[\mathcal{G}\right]$$

- The complete solution to arbitrary massive tree is the sum of the CIS (completely inhom sol) and all of its cuts.
- CIS => massive family tree
- Cuts => "tuned" (♯ or ♭) massive family trees





$$\mathcal{G}_3 = \llbracket 123\rrbracket + \llbracket 1^{\sharp_1}\rrbracket\Big(\llbracket 2^{\sharp_1}3\rrbracket + \llbracket 2^{\flat_1}3\rrbracket\Big) + \llbracket 12^{\sharp_2}\rrbracket\Big(\llbracket 3^{\sharp_2}\rrbracket + \llbracket 3^{\flat_2}\rrbracket\Big)$$
$$+ \llbracket 1^{\sharp_1}\rrbracket\Big(\llbracket 2^{\sharp_1\sharp_2}\rrbracket + \llbracket 2^{\flat_1\sharp_2}\rrbracket\Big)\Big(\llbracket 3^{\sharp_2}\rrbracket + \llbracket 3^{\flat_2}\rrbracket\Big) + \text{shadows}$$



$$\mathcal{G}'_4 = \mathrm{CIS}\left[\mathcal{G}'_4\right] + \sum_{\alpha=1}^{3} \underset{K_\alpha}{\mathrm{Cut}}\left[\mathcal{G}'_4\right] + \sum_{\alpha\neq\beta} \underset{K_\alpha,K_\beta}{\mathrm{Cut}}\left[\mathcal{G}'_4\right] + \underset{K_1,K_2,K_3}{\mathrm{Cut}}\left[\mathcal{G}'_4\right]$$

$$\mathrm{CIS}\left[\mathcal{G}'_4\right] = [\![1\not{4}(2)(3)]\!],$$

$$\sum_{\alpha=1}^{3} \underset{K_\alpha}{\mathrm{Cut}}\left[\mathcal{G}'_4\right] = [\![1^{\sharp_1}]\!]\Big([\![2\not{4}^{\sharp_1}3]\!] + [\![2\not{4}^{\flat_1}3]\!]\Big) + [\![1\not{4}^{\sharp_2}3]\!]\Big([\![2^{\sharp_2}]\!] + [\![2^{\flat_2}]\!]\Big)$$

$$+ [\![1\not{4}^{\sharp_3}2]\!]\Big([\![3^{\sharp_3}]\!] + [\![3^{\flat_3}]\!]\Big) + \text{shadows},$$

$$\sum_{\alpha\neq\beta} \underset{K_\alpha,K_\beta}{\mathrm{Cut}}\left[\mathcal{G}'_4\right] = [\![1^{\sharp_1}]\!]\,[\![2^{\sharp_2}]\!]\Big([\![3\not{4}^{\sharp_1\sharp_2}]\!] + [\![3\not{4}^{\flat_1\sharp_2}]\!] + [\![3\not{4}^{\sharp_1\flat_2}]\!] + [\![3\not{4}^{\flat_1\flat_2}]\!]\Big)$$

$$+ [\![1^{\sharp_1}]\!]\Big([\![2\not{4}^{\sharp_1\sharp_3}]\!] + [\![2\not{4}^{\flat_1\sharp_3}]\!]\Big)\Big([\![3^{\sharp_3}]\!] + [\![3^{\flat_3}]\!]\Big)$$

$$+ [\![1\not{4}^{\sharp_2\sharp_3}]\!]\Big([\![2^{\sharp_2}]\!] + [\![2^{\flat_2}]\!]\Big)\Big([\![3^{\sharp_3}]\!] + [\![3^{\flat_3}]\!]\Big) + \text{shadows},$$

$$\underset{K_1,K_2,K_3}{\mathrm{Cut}}\left[\mathcal{G}'_4\right] = [\![1^{\sharp_1}]\!]\,[\![2^{\sharp_2}]\!]\,[\![3^{\sharp_3}]\!]\Big([\![\not{4}^{\sharp_1\sharp_2\sharp_3}]\!] + [\![\not{4}^{\flat_1\sharp_2\sharp_3}]\!] + [\![\not{4}^{\sharp_1\flat_2\sharp_3}]\!] + [\![\not{4}^{\sharp_1\sharp_2\flat_3}]\!]$$

$$+ [\![\not{4}^{\flat_1\flat_2\sharp_3}]\!] + [\![\not{4}^{\flat_1\sharp_2\flat_3}]\!] + [\![\not{4}^{\sharp_1\flat_2\flat_3}]\!] + [\![\not{4}^{\flat_1\flat_2\flat_3}]\!]\Big) + \text{shadows}$$

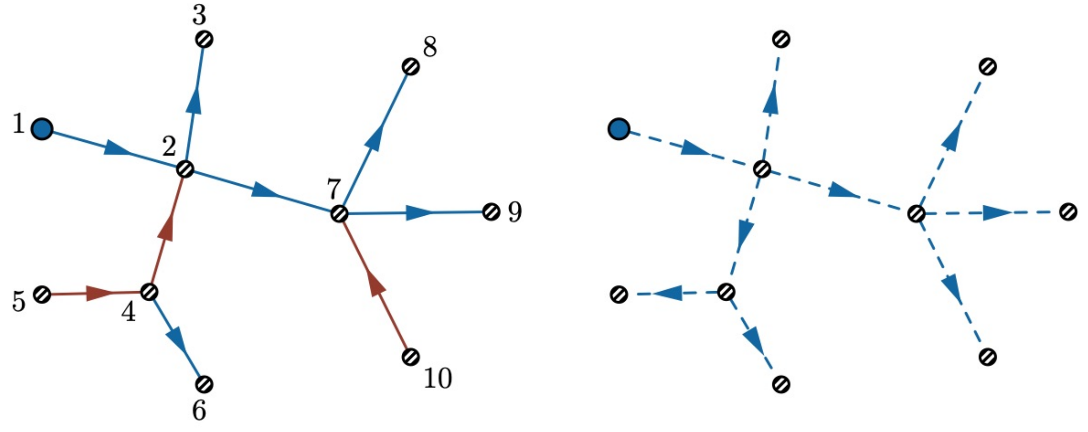
Completely inhomogeneous solution: massive family trees

- Quite remarkably, the CIS has a direct hypergeo rep: $\mathrm{CIS}\,[\mathcal{G}_V] = [\![\mathscr{P}(1\cdots V)]\!]$

- The solution expanded in the largest vertex energy ($1/E_1$), indep of the order of other energies

- Picking up a largest energy automatically generates a partial order: **massive family tree**
  q: a “family parameter” encoding the tree structure:

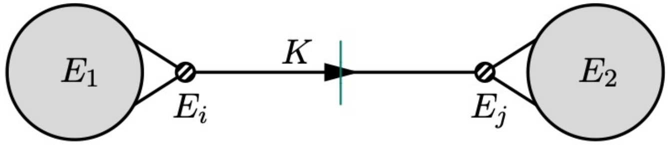
$$q_i \equiv \widetilde{\ell}_i + 2\widetilde{m}_i + \widetilde{p}_i + 4N_i$$



$$[\![1\cdots V]\!] = \sum_{\{\ell,m\}} 2^V \cos(\pi p_{1\cdots V}/2)\Gamma(q_1+p_1+1) \times \prod_{i=2}^{V} \frac{(-1)^{\ell_i}}{\ell_i!\left(\frac{\ell_i+q_i+p_i}{2}+\frac{5}{4}\pm\frac{\mathrm{i}\widetilde{\nu}_i}{2}\right)_{m_i+1}} \left(\frac{K_i}{2E_1}\right)^{2m_i+3} \left(\frac{E_i}{E_1}\right)^{\ell_i+p_i+1}$$

Homogeneous solutions: cuts of massive family trees

- The homogeneous solutions are obtained by executing appropriate cuts:



$$\underset{K_\alpha}{\mathrm{Cut}}\left[\widetilde{\mathcal{G}}_V\right] = [\![\widehat{1}\cdots i^\sharp \cdots V_1]\!]\left\{[\![(V_1+1)\cdots j^\sharp\cdots V]\!] + [\![(V_1+1)\cdots j^\flat\cdots V]\!]\right\} + \text{ c.c.}$$

- The cut involves certains dressings of massive family trees: augmentation and flattening:

$$[\![\cdots i^\sharp\cdots]\!] \equiv \sum_{m=0}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\Gamma(-m-\mathrm{i}\widetilde{\nu}_\alpha)}{m!}\left(\frac{K_\alpha}{2E_i}\right)^{2m+\mathrm{i}\widetilde{\nu}_\alpha+3/2}[\![\cdots i\cdots]\!]_{p_i\to p_i+2m+\mathrm{i}\widetilde{\nu}_\alpha+3/2},$$

$$[\![\cdots i^\flat\cdots]\!] \equiv \sum_{m=0}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\Gamma(-m+\mathrm{i}\widetilde{\nu}_\alpha)}{m!}\left(\frac{K_\alpha}{2E_i}\right)^{2m-\mathrm{i}\widetilde{\nu}_\alpha+3/2}\left\{\frac{\cos\left[\frac{\pi(p_{\mathrm{tot}}+2\mathrm{i}\widetilde{\nu}_\alpha)}{2}\right]}{\cos\left(\frac{\pi p_{\mathrm{tot}}}{2}\right)}[\![\cdots i\cdots]\!]\right\}_{p_i\to p_i+2m-\mathrm{i}\widetilde{\nu}_\alpha+3/2}$$

Where are we now?

- Massive tree graphs: solved; WYSIWYG solutions, in hypergeo series
- Loop level: simple 1-loop graphs (massive bubbles) computed, also in hypergeo series
- Analytical structures largely known for all trees and many loops: only poles / branch points of finite degrees
- Conjecture: Any graphic contribution to a renormalized massive cosmological correlator is a multivariate hypergeometric function with only power-law singularities (finite-deg poles or branch points)
- Most of these hypergeo functions are not yet named, and are like "black boxes"
- Then what does the analytical calculation mean other than giving correlators names?
- Why pFq / Appell / Lauricella look like black boxes to us, but sine and cosine do not?

What is analytical computation?

- Using series solutions to define, identify, and represent family trees (hypergeo functions)
- Using the flexibility of FTD to link different reps of family trees => Analytical continuation!

$$[12] = [12] \qquad \frac{1}{\omega_1^{q_{12}}}\,{}_2\mathcal{F}_1\left[\begin{array}{c|c} q_2, q_{12} & -\dfrac{\omega_2}{\omega_1} \\ q_2+1 & \end{array}\right] = \frac{\Gamma[q_2]}{\omega_{12}^{q_{12}}}\,{}_2\mathcal{F}_1\left[\begin{array}{c|c} 1, q_{12} & \dfrac{\omega_2}{\omega_{12}} \\ q_2+1 & \end{array}\right]$$

$$[12] + [21] = [1]\,[2] \qquad \frac{1}{\omega_1^{q_{12}}}\,{}_2\mathcal{F}_1\left[\begin{array}{c|c} q_2, q_{12} & -\dfrac{\omega_2}{\omega_1} \\ q_2+1 & \end{array}\right] + \frac{1}{\omega_2^{q_{12}}}\,{}_2\mathcal{F}_1\left[\begin{array}{c|c} q_1, q_{12} & -\dfrac{\omega_1}{\omega_2} \\ q_1+1 & \end{array}\right] = \frac{\Gamma[q_1, q_2]}{\omega_1^{q_1}\omega_2^{q_2}}$$

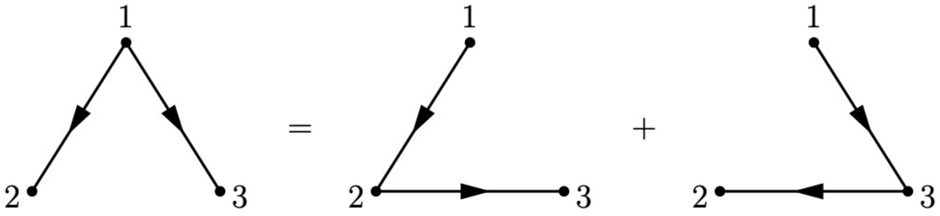
$$[123] + [2(1)(3)] = [1]\,[23] \qquad \frac{1}{\omega_1^{q_{123}}}\,{}_{2+1}\mathcal{F}_{1+1}\left[\begin{array}{c|c|c} q_{123}, q_{23} & \text{-}\ , q_3 & -\dfrac{\omega_2}{\omega_1}, -\dfrac{\omega_3}{\omega_1} \\ q_{23}+1 & \text{-}\ , q_3+1 & \end{array}\right] + \frac{1}{\omega_2^{q_{123}}}\mathcal{F}_2\left[\begin{array}{c|c|c} q_{123} & q_1, q_3 & -\dfrac{\omega_1}{\omega_2}, -\dfrac{\omega_3}{\omega_2} \\ & q_1+1, q_3+1 & \end{array}\right]$$

$$= \frac{\Gamma[q_1]}{\omega_1^{q_1}\omega_2^{q_{23}}}\,{}_2\mathcal{F}_1\left[\begin{array}{c|c} q_3, q_{32} & -\dfrac{\omega_3}{\omega_2} \\ q_3+1 & \end{array}\right]$$

- We are currently able to find hypergeo series reps for any family trees at all of their singular points [Fan, ZX, 250X.XXXXX]

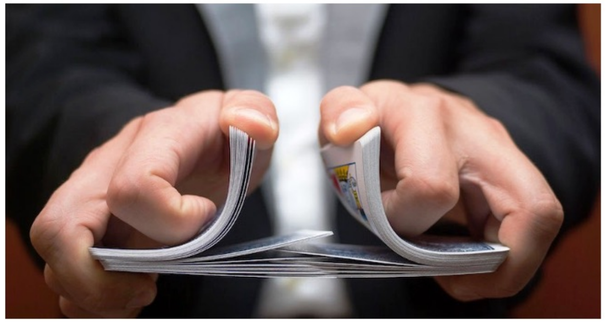
Family trees are further decomposible into chains [Fan, ZX, 2403.07050; 250X.XXXXX]

$$\theta_{21}\theta_{31}(\theta_{32}+\theta_{23}) = \theta_{32}\theta_{21} + \theta_{23}\theta_{31}$$



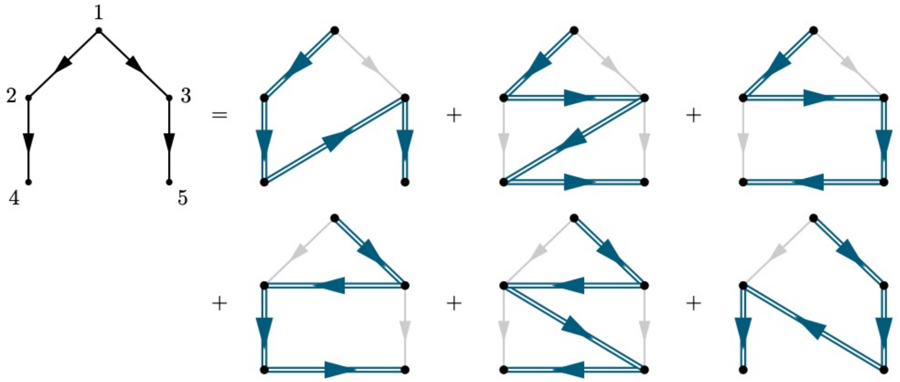
Shuffle product:

$$ab \sqcup\!\sqcup\, cd = abcd + acbd + acdb + cabd + cadb + cdab$$



Practically: taking shuffle product recursively among all subfamilies

$$[1(24)(35)] = \{1(24) \sqcup\!\sqcup\, (35)\} = \{12435\} + \{12345\} + \{12354\} + \{13245\} + \{13254\} + \{13524\}$$



Family chain: standard iterated integrals; Hopf algebra; transcendental weight; Higher weight functions cannot be fully reduced to lower weight functions

Final thoughts and outlooks

- We have found simple rules to identify & write down all hypergeo sols for any tree graphs
- Straightforward generalizations:
  ☐ degenerate limits ☐ boost-breaking dispersion ☐ loop integrands [ongoing]
- However, we have to deal with unfamiliar hypergeo functions. Two bold programs:
  ☐ Charting out all singularity structure of cosmological corrrelators
  ☐ Obtaining hypergeo series reps at all singularities (at least for trees)
- In the meantime, many classic pheno examples remain challenging (triple exchange / strong mixing / chemical potential loops), even numerically. We should work harder
- Thinking pheno-wise: all computations must be initiated analytically and finished numerically, the only question being where to execute the analytical-to-numerical transition
- We hope that some of the analytical progress can provide new insights and better answers!

Thank you!