Cosmological Perturbations: A Wheeler–DeWitt Quantum-Gravity Perspective

Based on `arXiv:2412.19782` w/ Federico Piazza

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Perturbative vs. Non-Perturbative (WdW)

- Usually, perturbations are quantized on a fixed classical background $g^0_{\mu\nu}$ satisfying Einstein's equations

$$\hat{g}_{\mu\nu} = g^0_{\mu\nu} + \hat{\delta g}_{\mu\nu}$$

- The perturbation wavefunction satisfies a time-dependent Schrödinger equation:

$$i\,\partial_t \psi_P(\{\delta q\}, t) = \hat{H}(t)\psi_P(\{\delta q\}, t)$$

- Time t is defined by the background solution — it is **external** to the quantum system.

- Canonical quantization of gravity approach, quantize both geometry and matter $\rightarrow$ Wheeler–DeWitt equation

$$\hat{\mathcal{H}}\,\Psi[g_{\mu\nu}, \phi] = 0$$

- This is a constraint equation — not an evolution equation. It reflects full invariance under time reparametrizations.

- No external time, time emerges **relationally** (e.g., using a field as a clock).

System of interest

- Working in the **minisuperspace approximation**. Restrict to flat-FLRW metrics

$$ds^2 = -N^2 dt^2 + a^2(t)\delta_{ij} dx^i dx^j$$

- Integrating the Einstein-Hilbert action over a comoving spatial volume of size $\gtrsim H_\star^{-3}$

$$\frac{M_P^2}{2}\int d^4x \sqrt{-g} R + boundary\ terms \ \simeq \ \frac{1}{\alpha H_\star}\int dt\, a^3 \left(-\frac{3\dot{a}^2}{N}\right)$$

- With dimensionless "loop coupling" parameter

$$\alpha \ \equiv \frac{H_\star^2}{M_P^2}$$

- System with $d+1$ dynamical variables: $\boldsymbol{d}$ matter-fields, $\mathbf{1}$ metric-field,

$$I = \frac{1}{\alpha H_\star}\int dt \left(\frac{1}{2N} G_{\mu\nu}\dot{q}^\mu \dot{q}^\nu - N H_\star^2 U(q^\mu)\right)$$

Perturbation theory in minisuperspace and time invariance

- Working on shell of the constraint

$$N = \pm \sqrt{\frac{-G_{\mu\nu}\,\dot{q}^\mu \dot{q}^\nu}{2H_\star^2 U}}$$

- Inserting this back into the action gives

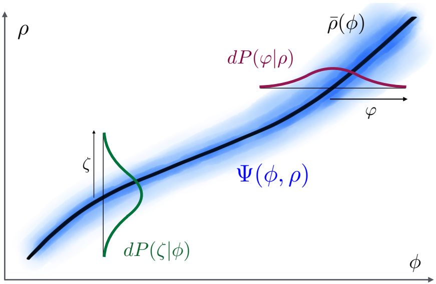
$$I = -\frac{\sqrt{2}}{\alpha}\int dt\,\sqrt{-G_{\mu\nu}\,\dot{q}^\mu \dot{q}^\nu\, U}$$

- Manifestly **invariant under $t \to \tilde{t}(t)$**; freedom to choose one of the field as time ($\tilde{t}(t) \equiv q^0$): the system has only d **DoFs**.
- Working at quadratic order in the zero-mode fluctuation $\varphi^i \equiv q^i - \bar{q}^i(t)$ yields a typical Schrödinger equation with time evolution t that can be written in different gauges.
- **Example:** $\rho = \ln a, \quad \phi$

Unitary gauge:	$\delta\phi = 0,\ \delta\rho = \zeta,$
Spatially-flat gauge:	$\delta\rho = 0,\ \delta\phi = \varphi$

WdW Gauge-fixing & relational time

- "Time" is just one choice of gauge—no absolute clock
- Probabilities are **conditional**: $P(q^i \,|\, q^0)$ = "value of field q^i *given that* the clock field reads q^0".
- Different gauges = different foliations of (q^0, q^i)-space



- **Minisuperspace example:** $\rho = \ln a, \quad \phi$

Unitary gauge: $\delta\phi = 0,\ \delta\rho = \zeta,$

Spatially-flat gauge: $\delta\rho = 0,\ \delta\phi = \varphi$

WdW equation and the WKB ansatz

- Full Wheeler–DeWitt equation (minisuperspace)

$$\left(-\frac{\alpha^2}{2}\,\Box + U(q^\mu)\right)\Psi(q^\mu) = 0, \qquad \Box \equiv \frac{1}{\sqrt{-G}}\partial_\mu\left(G^{\mu\nu}\sqrt{-G}\partial_\nu\right)$$

- Semiclassical WKB ansatz

$$\Psi(q^\mu) = e^{\frac{i}{\alpha}S(q^\mu)}\,\psi(q^\mu) \quad \text{with } \alpha \ll 1$$

- Leading order wavefunction is peaked on classical flow

$$p_\mu\Psi = -i\partial_\mu\Psi \approx (\partial_\mu S)\,\Psi$$

- **Order $\boldsymbol{\alpha^0}$** Hamilton–Jacobi equation for S

$$\partial^\mu S\partial_\mu S + 2U(q^\mu) = 0$$

- **Higher orders** Schrödinger-like equation for ψ

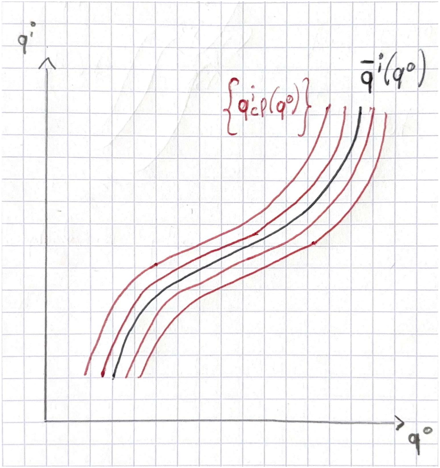
$$i\left(\partial^\mu S\,\partial_\mu\psi + \tfrac{1}{2}\,\Box\,S\cdot\psi\right) = -\tfrac{\alpha}{2}\,\Box\,\psi$$

Going with the classical flow

- Many solutions exist for the HJ function S.
- Work in Lorentzian regime $\Rightarrow$ choose $S \in \mathbb{R}$ so $\Psi = e^{iS/\alpha}\psi$ has oscillatory (not Euclidean) behavior.

Going with the classical flow

- Use the ansatz: $\Psi = e^{iS/\alpha}\psi$
- **Pick a congruence:** Embed a background $\bar{q}^{\mu}(q^0)$ into a surface-orthogonal bundle of classical solutions $q^{\mu}_{\text{cl}}(q^0)$.



Going with the classical flow

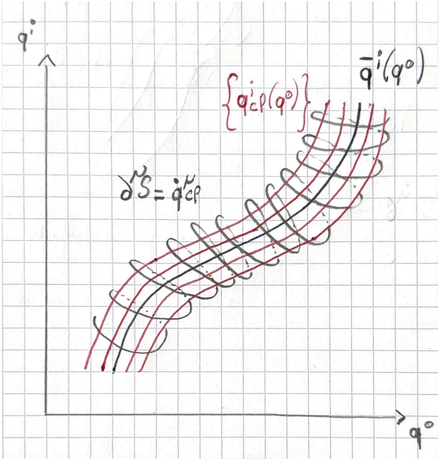
- Classical hamiltonian constraint yields

$$g_{\mu\nu}\,\dot{q}^{\mu}\dot{q}^{\nu} + 2H_{\star}^{2}U\,N^{2} = 0$$

- Inside the tube of solutions, impose:

$$\partial^{\mu}S \approx \frac{\dot{q}^{\mu}_{\mathrm{cl}}}{H_{\star}}$$

to align the WKB phase with classical motion $p^{\mu} \sim \dot{q}^{\mu}$ for $N = 1$.



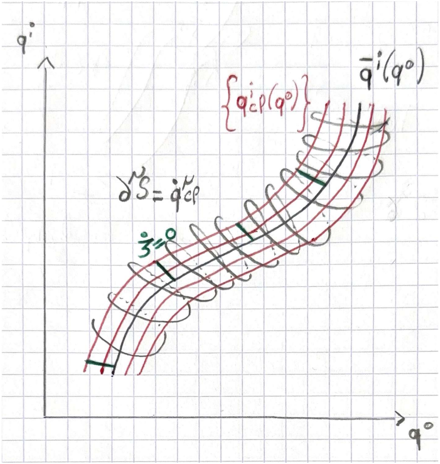
Going with the classical flow

- Choose adapted coordinates (q^0, ζ^i) with $\dot{\zeta}^i = 0$ along all trajectories.
- Equation for quantum amplitude ψ

$$i\left(\underbrace{\partial^\mu S\,\partial_\mu\psi}_{\dot{q}^0\partial_0\psi = H_\star^{-1}\partial_t\psi} + \tfrac{1}{2}\,\Box\, S\cdot\psi\right) = \mathcal{O}(\alpha)$$

- In generic variables φ^i:

$$\partial^\mu S\partial_\mu\psi \sim H_\star^{-1}\partial_t\psi + \dot{\varphi}^i\partial_i\psi$$



Matching with perturbation theory

- **Perturbations at quadratic** $\varphi^i\varphi^j$

$$i\,\partial_t\psi_P = -\frac{\alpha}{2}\nabla_i^2\psi_P + \frac{1}{\alpha}\frac{M_{ij}(t)\varphi^i\varphi^j}{2}\psi_P$$

- **WdW WKB expansion in** α

$$i\left(\underbrace{\partial^0 S\partial_0}_{\frac{dq^0}{dt}\partial_0=\partial_t} + \partial^i S\partial_i + \frac{1}{2}\Box S\right)\psi = -\frac{\alpha}{2}\Box\psi$$

- KG-type current with leading order in WKB expansion probability density

$$dP(q^i|q^0) = |\psi(q)|^2\;|\partial^0 S|\sqrt{-G}\;d\vec{q}$$

- **Matching perturbative wavefunction** probability density $|\psi_P|^2 d\vec{q}$

$$\psi(q^i, q^0(t)) = \left(\sqrt{-G\partial^0 S}\;e^{\frac{i}{\alpha}f(q)}\right)^{-1}\;\psi_P(q^i, t) + \mathcal{O}(\alpha)$$

- $\sqrt{-G\partial^0 S}$: ensures correct relational time flow
- $e^{\frac{i}{\alpha}f(q)}$: accounts for classical potential term

Scalar field minimally coupled to FLRW spacetime

- Work in minisuperspace with metric of the form

$$ds^2 = -N^2 dt^2 + e^{2\rho(t)} d\vec{x}^{\,2} \,,$$

and model the matter content with a scalar field $\phi(t)$. The action reads

$$I = \frac{1}{\alpha H_\star} \int dt\, e^{3\rho} \left(-\frac{3\dot{\rho}^2}{N} + \frac{\dot{\phi}^2}{2N} - N H_\star^2 V(\phi) \right)$$

- Perturbations $\delta\rho = \zeta$, $\delta\phi = 0$ in unitary gauge satisfy

$$i H_\star^{-1} \partial_t \psi_P = -\alpha \frac{e^{-3\rho}}{6} \frac{H_\star^2 V(\phi)}{\dot{\phi}^2} \partial_\zeta^2 \psi_P$$

- Perturbations $\delta\phi = \varphi$, $\delta\rho = 0$ in spatially-flat gauge satisfy

$$i H_\star^{-1} \partial_t \psi_P = -\frac{\alpha}{6} \frac{H_\star^2 V(\phi)}{\dot{\rho}^2 e^{3\rho}} \partial_\varphi^2 \psi_P - \frac{3}{2\alpha} \frac{\dot{\rho}^2 e^{3\rho}}{H_\star^2 V(\phi)} \left(\frac{V'(\phi)^2}{V(\phi)} - V''(\phi) \right) \varphi^2 \psi_P$$

Slow-roll quasi-exponential potential $\epsilon, \eta \ll 1$

- Considering the first-two slow-roll parameters $\epsilon \equiv -\frac{\dot{H}}{H^2}, \eta \equiv \frac{\dot{\epsilon}}{\epsilon H} \ll 1$, allows to build the quasi-exponential potential ($\chi \equiv \frac{\phi}{\sqrt{2\epsilon}}$)

$$V(\chi) = (3-\epsilon)\exp\left[-2\epsilon\left(1+\frac{\eta}{6}\right)\chi - \frac{\epsilon\eta}{2}\chi^2 + \mathcal{O}(\epsilon^2, \eta^2)\right] .$$

- Unitary gauge: matter χ as clock $\{\rho, \chi\} \to \{\zeta(\rho, \chi) = \rho - \bar{\rho}(\chi), \chi\}$
- Schrödinger-like equation

$$i\left[\overbrace{e^{-\epsilon\chi}\left(1+\frac{(3-\epsilon)\eta}{6}\chi\right)\partial_\chi}^{\dot{\chi}\partial_\chi = \partial_t} + \frac{1}{2}\,\Box S\right]\psi =$$

$$-\,\alpha\frac{1}{\epsilon}\;e^{-3\bar{\rho}(\chi)}e^{-3\zeta}\;\left[\frac{3-\epsilon}{12}\,\partial_\zeta^2 + \underbrace{\frac{1}{4}\left((\bar{\rho}'^2-1)\,\partial_\zeta^2 - \bar{\rho}''\partial_\zeta - 2\bar{\rho}'\partial_\chi\partial_\zeta + \partial_\chi^2\right)}_{\mathcal{O}(\epsilon^0)}\right]\psi\,.$$

- α-corrections may dominate in eternal inflation $\frac{\alpha}{\epsilon} \gtrsim 1$
- WdW approach brings *higher "time" derivatives* ∂_χ, ∂_χ^2 and *non-Gaussian terms* $e^{-3\zeta}$ in this scalar field example.

Summary and outlook

- **Two complementary descriptions**:
  - Schrödinger evolution of perturbations on classical FLRW
  - Wheeler–DeWitt equation: quantize geometry + matter

- **Wavefunctions match at leading order:**

$$\psi_P = e^{\frac{i}{\alpha} f(q)} \sqrt{-G\partial^0 S}\, \psi + \mathcal{O}(\alpha)$$

- **Quantum gravity corrections** appear at higher orders in α
  - Higher "time" derivatives, cross terms, non-Gaussianities
  - Relevant when $\alpha/\epsilon \gtrsim 1$, e.g. in single scalar field eternal inflation

- **Beyond WKB: Bouncing solutions from relational Schrödinger dynamics** $\implies$ Time as test field, flat FLRW $x \sim a^{\frac{3}{2}}$ + radiation:

$$i\alpha H_\star^{-1} \partial_t \Psi(x,t) = -\frac{1}{2}\left(\alpha^2 \partial_x^2 + x^{-2/3}\right)\Psi$$

  - Interpreted as s-wave scattering off a potential
  - Suggests singularity resolution via quantum bounce
  - See recent work with F. Piazza `arXiv:2505.08703`