Scattering amplitudes and the S-matrix bootstrap

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Introduction

Quantum Field Theory in d>2 space-time dimensions

Symmetries: Lorentz rotations + translations

We assume the presence of particles (asymptotic states)

Examples:

- Protons in QCD ✓
- Massless pions in QCD ✓
- Photons in QED ✓
- Gravitons
- Electrons in QED

IR divergences

Lagrangian → Correlation functions → Scattering amplitudes

LSZ

We can derive/postulate axioms

Axioms on scattering Amplitudes → Bounds on scattering Amplitudes

S-matrix bootstrap

This talk

2-to-2 scattering process: $p_1 p_2 \longrightarrow p_3 p_4, \qquad p_i^2 = -m^2$

Scattering amplitude:

$$S_I(p_1, p_2, p_3, p_4) = \delta^{d-1}(p_1 - p_3)\delta^{d-1}(p_2 - p_4) + i\, T_a(s,t,u)\; \mathbb{T}_I^a(p_1, p_2, p_3, p_4)$$

- S_I: Collective helicity label
- $\delta^{d-1}(p_1 - p_3)\delta^{d-1}(p_2 - p_4)$: free propagation
- $i\, T_a(s,t,u)\; \mathbb{T}_I^a(p_1, p_2, p_3, p_4)$: Interacting part
- $T_a(s,t,u)$: Scalar amplitudes
- $\mathbb{T}_I^a(p_1, p_2, p_3, p_4)$: Tensor structures

Mandelstam variables: $s \equiv -(p_1 + p_2)^2, \quad t \equiv -(p_1 - p_3)^2, \quad u \equiv -(p_1 - p_4)^2$

[A. Guerrieri; 2020]

Negative energy

$-p = \{-p^0, -\vec{p}\}$

Axioms

Crossing: $S_I(p_1, p_2, p_3, p_4) = M_{IJ} S_J(-p_1, p_2, -p_3, p_4)$

Analyticity: **Simple behaviour in the complex plane of energies**

Unitarity: $\langle \alpha | \alpha \rangle \geq 0$

Identical scalar particles

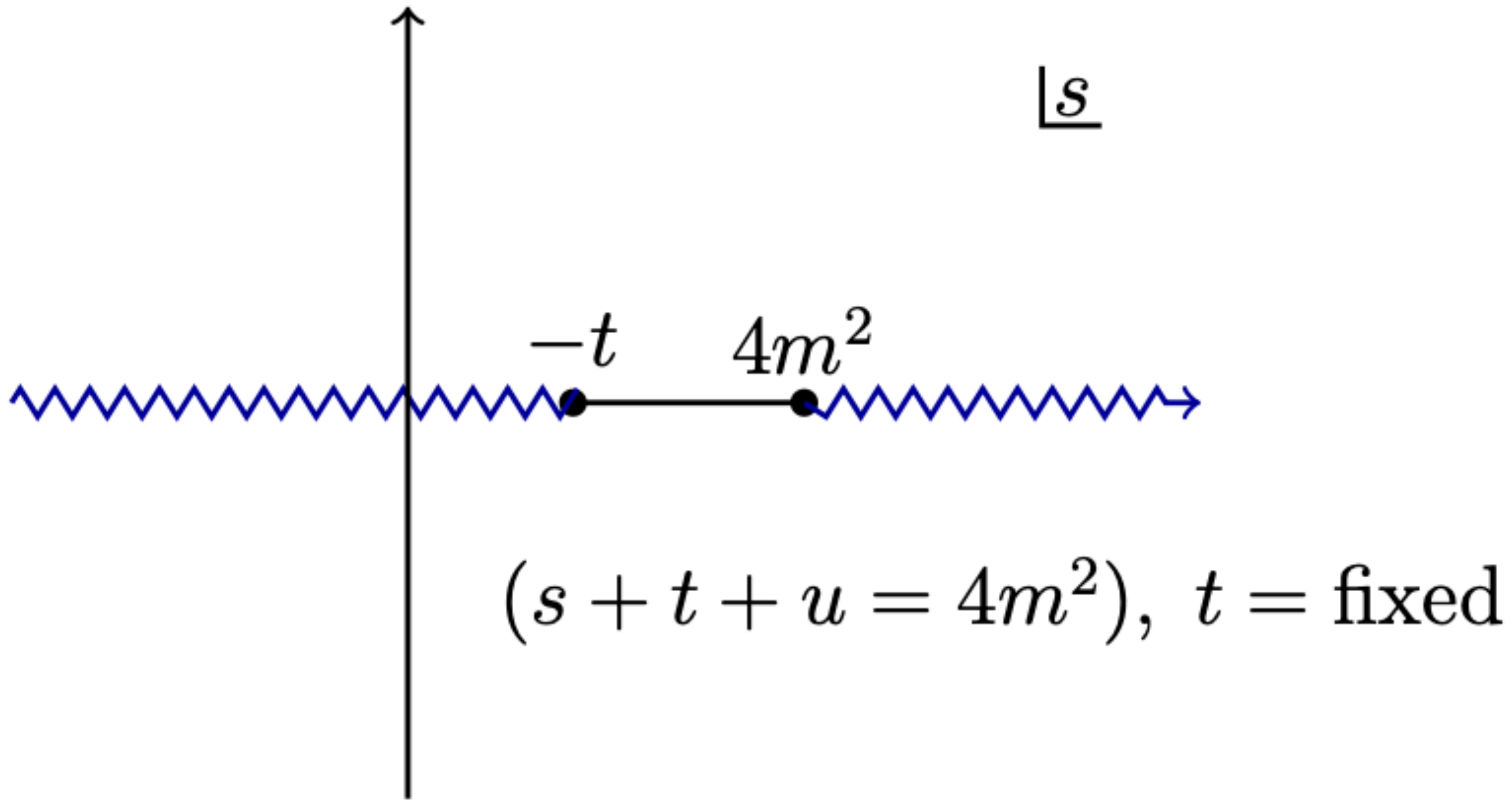
Identical scalar particles: $p_i^2 = -m^2$ → $T(s,t,u)$

Mandelstam variables: $s + t + u = 4m^2$

Physical range: $s \geq 4m^2, \qquad t \leq 0$

crossing

Interacting part of the scattering amplitude: $T(s,t,u) = T(t,s,u) = T(u,t,s)$



analyticity

Partial amplitudes:

angular momentum

scattering angle

$$S_j(s) = 1 + i\,T_j(s), \qquad j = 0,2,4,\ldots$$

$$t = -\frac{s-4m^2}{2}(1-x), \quad u = -\frac{s-4m^2}{2}(1+x), \quad x \equiv \cos\theta$$

Gegenbauer polynomial

Legendre polynomial in d=4

$$T_j(s) \equiv \# \int_{-1}^{+1} dx\,(1-x^2)^{\frac{d-4}{2}} C_j^{(d-3)/2}(x)\, T(s, t(s,x), u(s,x))$$

Non-linear unitarity

$$|S_j(s)| \leq 1$$

linearised unitarity

$$0 \leq ImT_j(s) \leq 2N$$

Positivity

$$0 \leq ImT_j(s)$$

What can we bound?

Set of observables 1:

$$\lambda_{0,0} \equiv T\left(s = t = u = 4m^2/3\right)$$

$$\lambda_{2,0} \equiv \partial_s^2 T\left(s = t = u = 4m^2/3\right)$$

$$\lambda_{2,1} \equiv \partial_s^2 \partial_t T\left(s = t = u = 4m^2/3\right)$$

Set of observables 2:

$$a_0^\ell \equiv T_\ell\left(s = 4m^2\right)$$

← **scattering lengths**

$$a_2^\ell \equiv \partial_s^2 T_\ell\left(s = 4m^2\right)$$

← **effective ranges**

Numerical method

[M. Paulos, J. Penedones, J. Toledo, B. van Rees, P. Vieira; 2017]

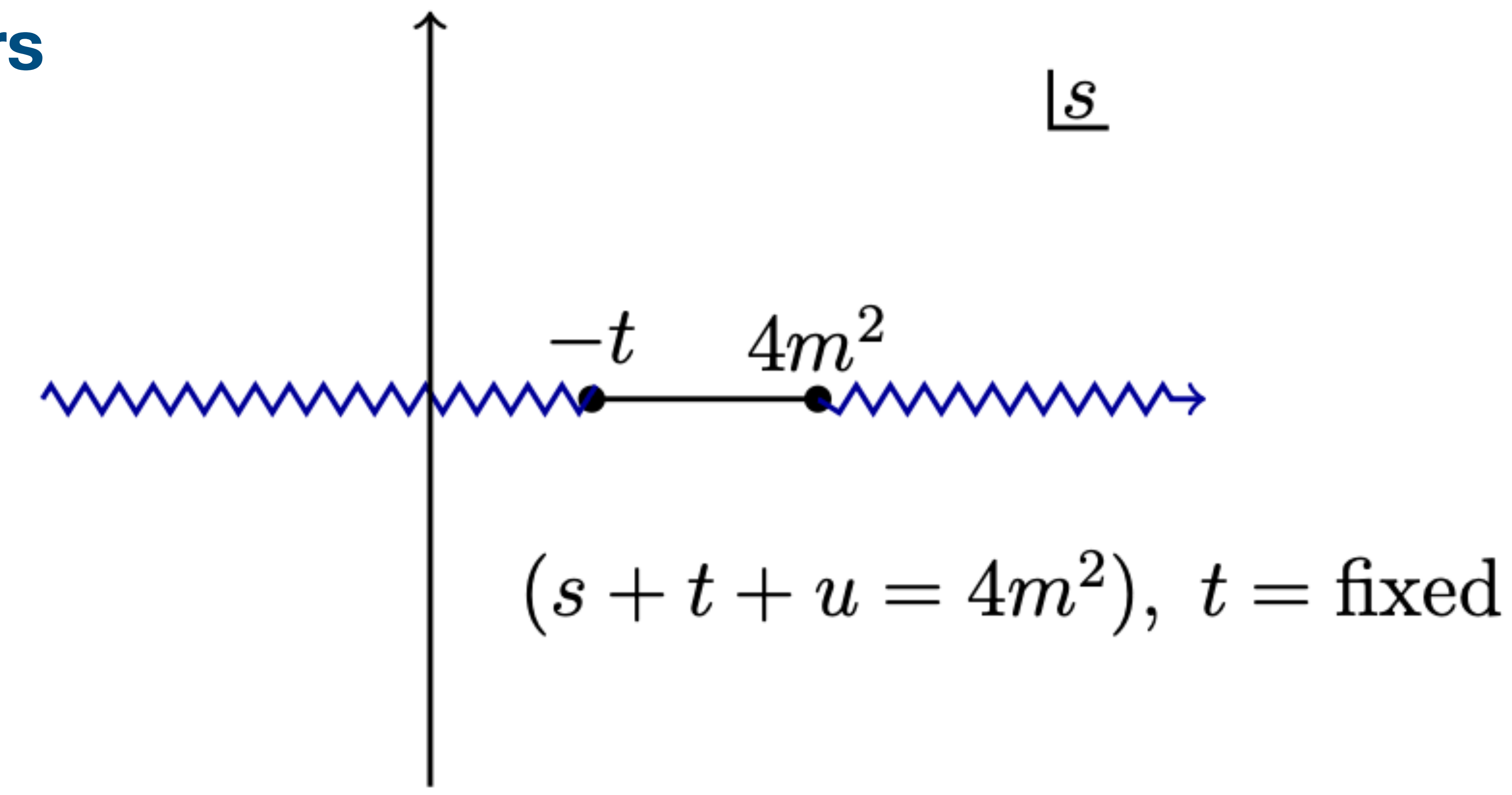
"Smart" variable: $$\rho(s) \equiv \frac{1-\sqrt{4m^2-s}}{1+\sqrt{4m^2-s}}$$

Ansatz: $$T(s,t,u) = \sum_{a,b,c=0}^{\infty} \alpha_{(abc)} (\rho(s))^a (\rho(t))^b (\rho(u))^c$$

unknown parameters

✓ analyticity

✓ crossing

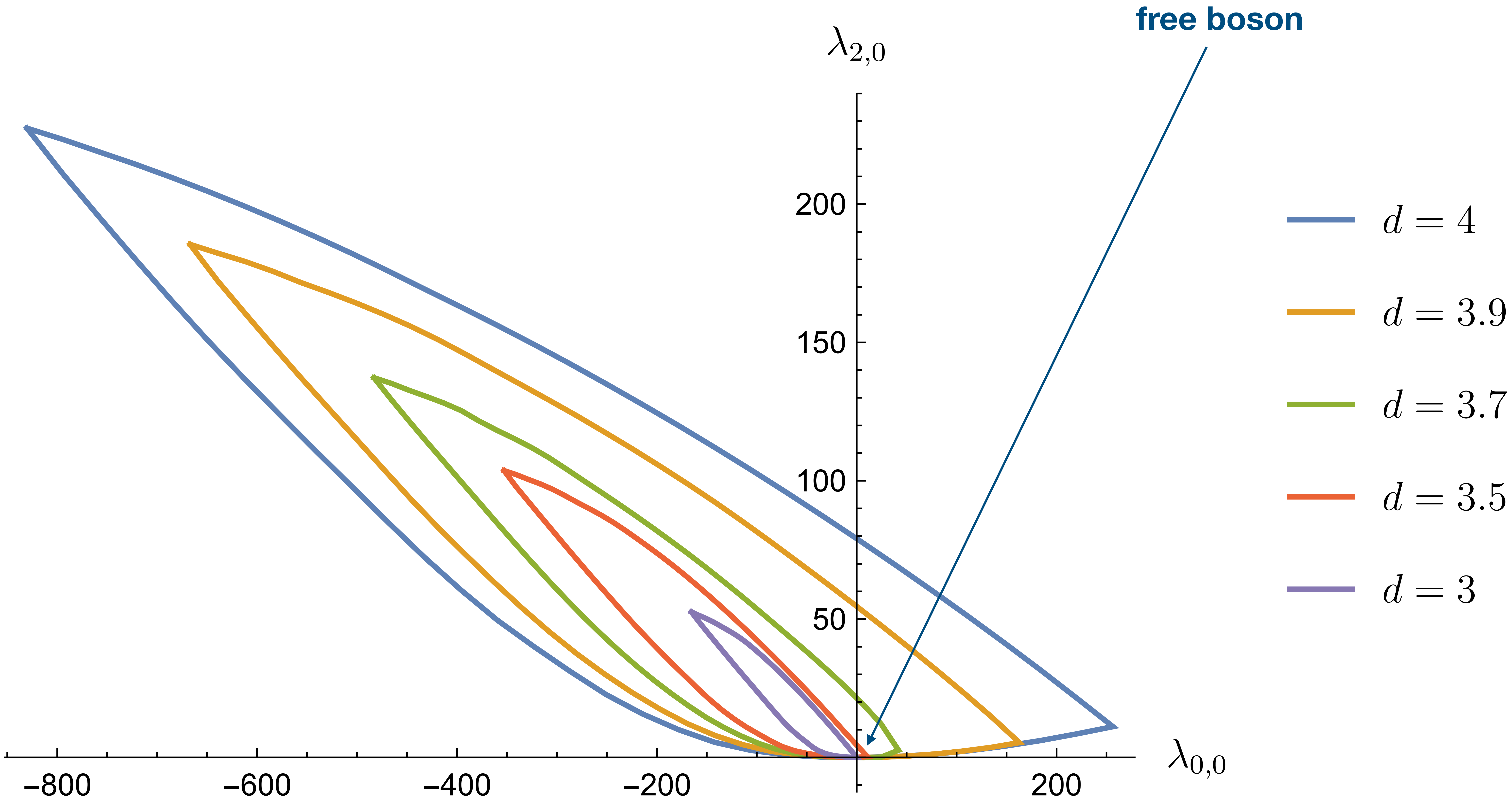


Unitarity: $$|S_j(s)| \leq 1$$ ← imposed numerically as a semi-definite problem

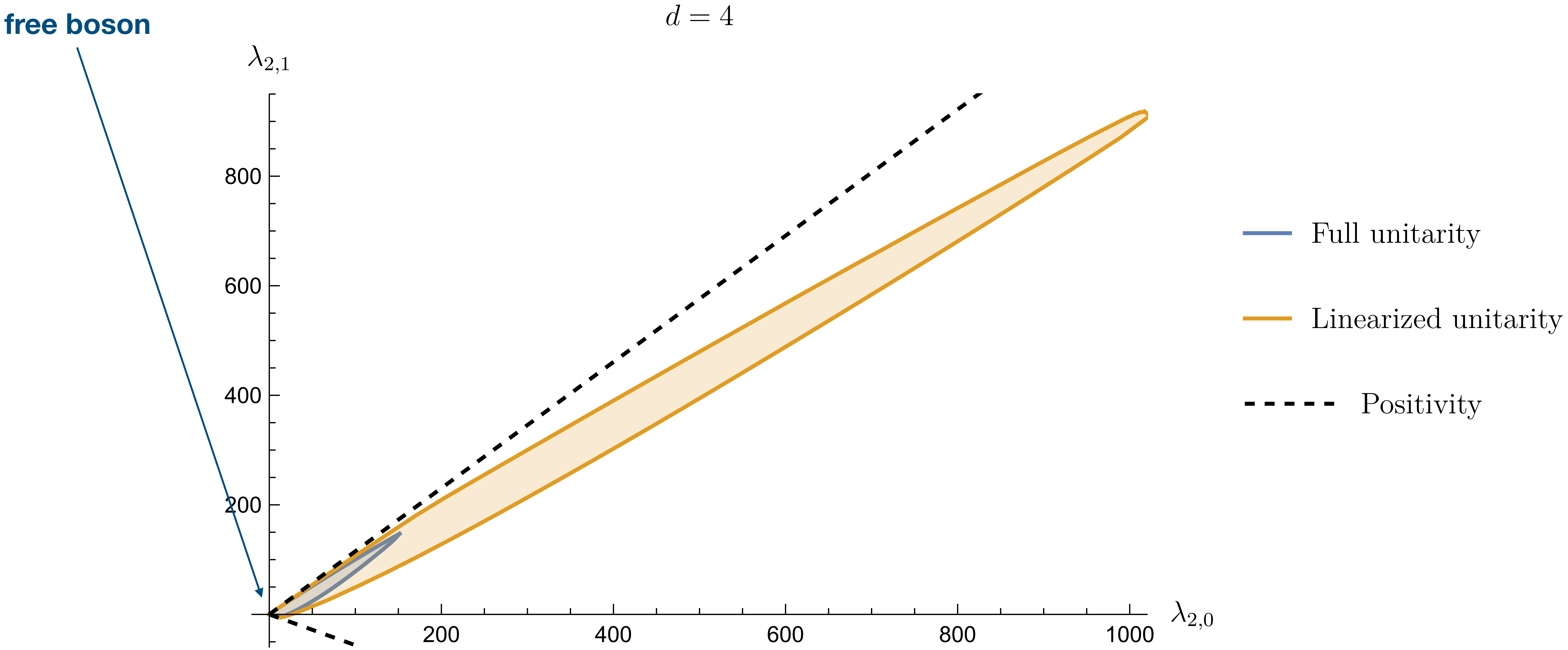
Observables: $$\textit{observable} = \textit{linear combination}(\alpha_{(a,b,c)})$$ ← minimize or maximize

Results: scalar amplitudes

[H. Chen, L. Fitzpatrick, DK; 2022]



free boson
$\lambda_{2,0}$
$\lambda_{0,0}$
200
150
100
50
-800
-600
-400
-200
200
$d = 4$
$d = 3.9$
$d = 3.7$
$d = 3.5$
$d = 3$



free boson
$d = 4$
$\lambda_{2,1}$
800
600
400
200
200
400
600
800
1000
$\lambda_{2,0}$
Full unitarity
Linearized unitarity
Positivity

Application: Majorana fermions in d=4

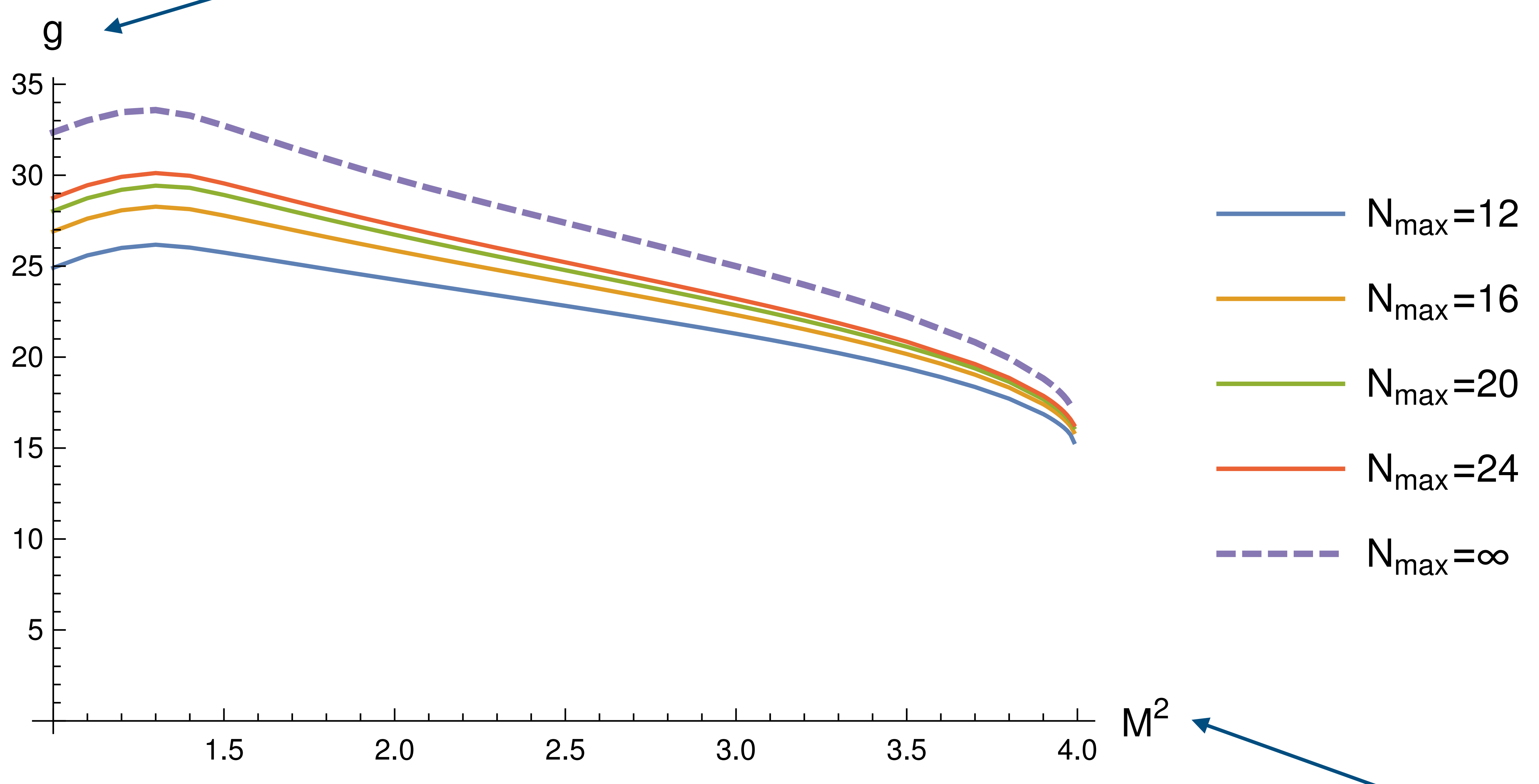
[A. Hebbar, DK, J. Penedones; 2020]

Identical neutral massive spin 1/2 (Majorana) fermions

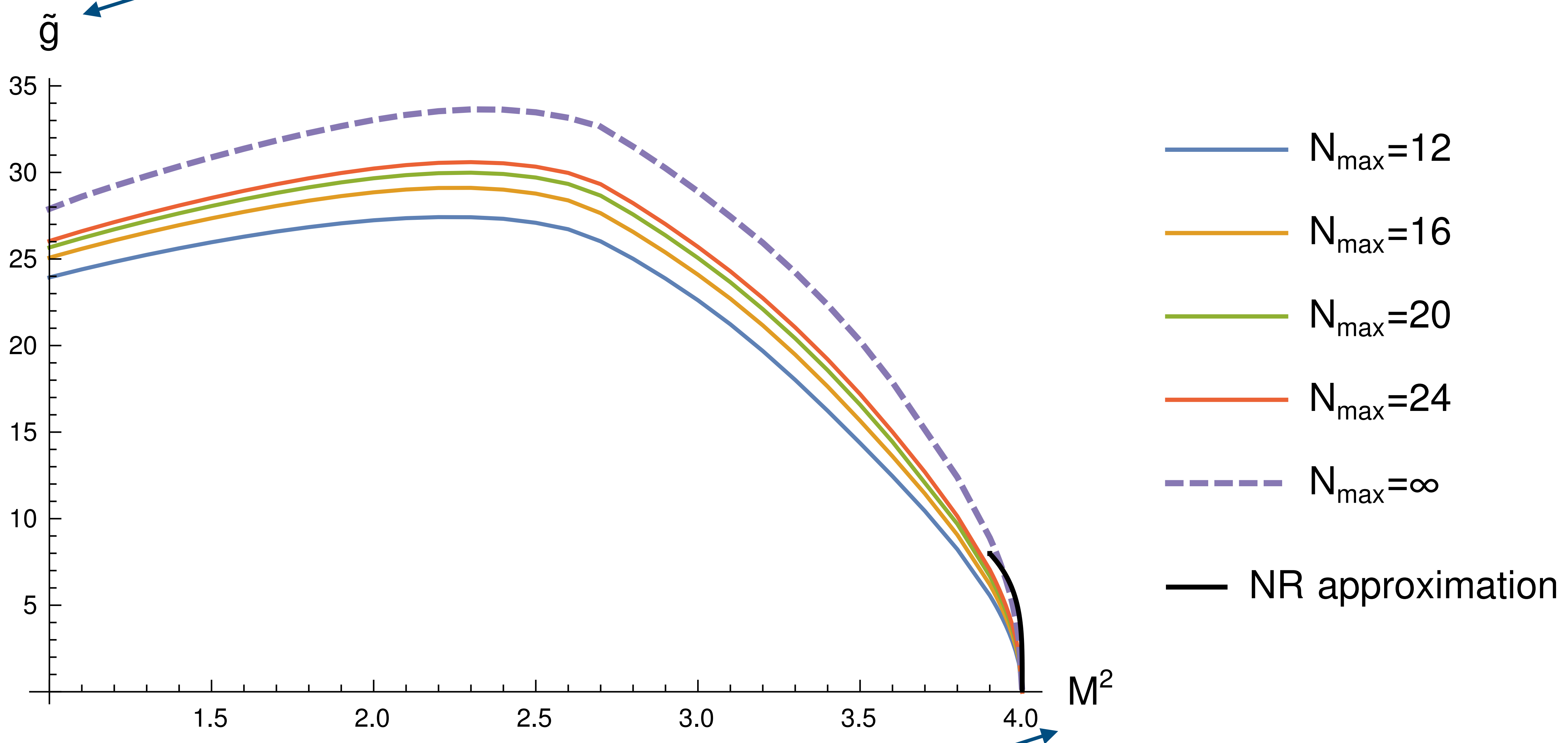
16 amplitudes → (Parity and identical particles) → **5 amplitudes** → (Crossing) → **3 amplitudes**

$$\frac{\lambda_{0,0}}{32\pi} \in [-3.25, \ +1.74]$$

Yukawa coupling



Pseudo-Yukawa coupling



Mass of the stable (pseudo-)scalar particle

Bounds on massless amplitudes

[F. Acanfora, K. Häring, A. Guerrieri, DK; 2023]

Wilson coefficients

$$\mathscr{L}_{EFT} = -\frac{1}{2}(\partial\phi)^2 + c_4(\partial\phi)^4 + c_6(\partial_\mu\partial_\nu\partial_\rho\phi)(\partial_\mu\phi)(\partial_\nu\phi)(\partial_\rho\phi) + c_8(\partial_\mu\partial_\nu\phi)^4 + \dots$$

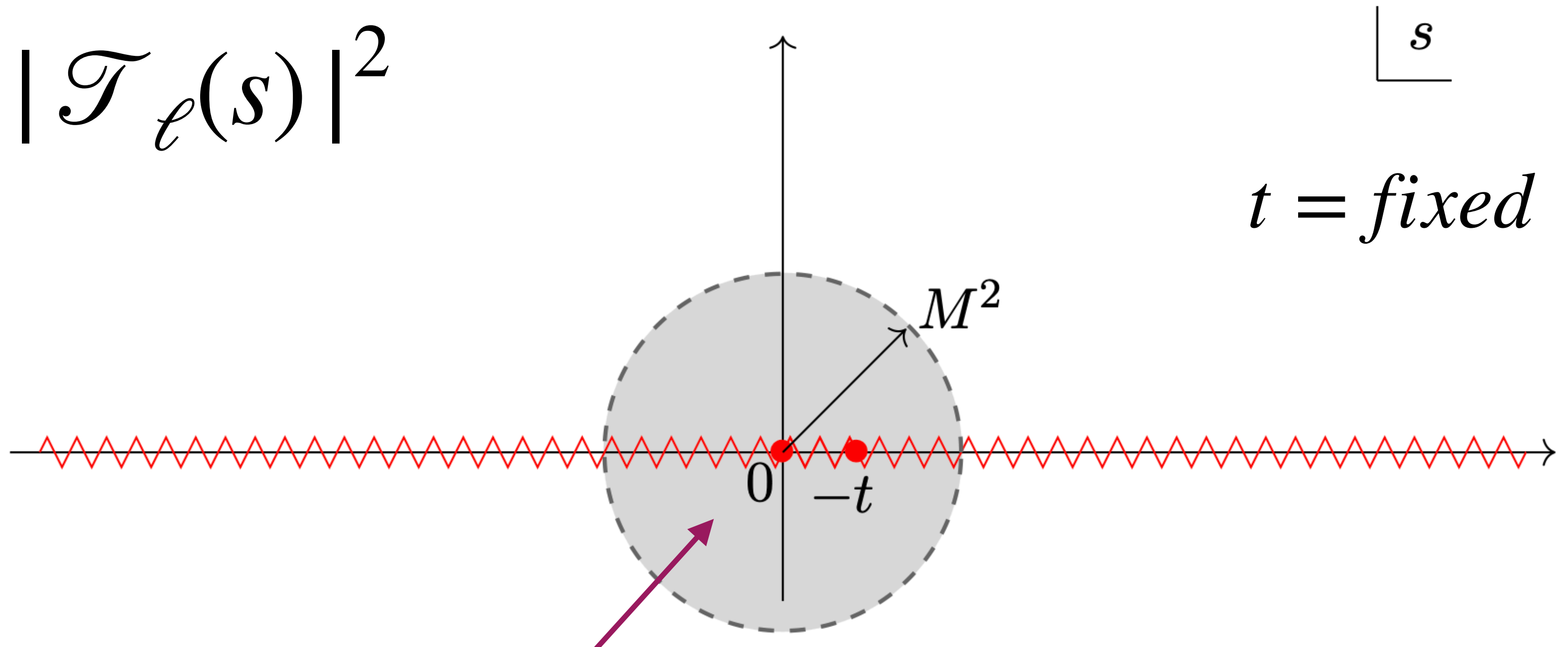
Shift symmetry: $\phi(x) \to \phi(x) + c$

Assumptions

1. Crossing: $\mathcal{T}(s,t,u) = \mathcal{T}(t,s,u) = \mathcal{T}(u,t,s)$

2. Unitarity: $Im\mathcal{T}_\ell(s) \geq \frac{1}{16}|\mathcal{T}_\ell(s)|^2$

3. Maximal analyticity:



$$g_2 = 2c_4, \quad g_3 = 3c_6, \quad g_4 = \frac{1}{4}c_8$$

Wilson coefficients

4. Low energy expansion:

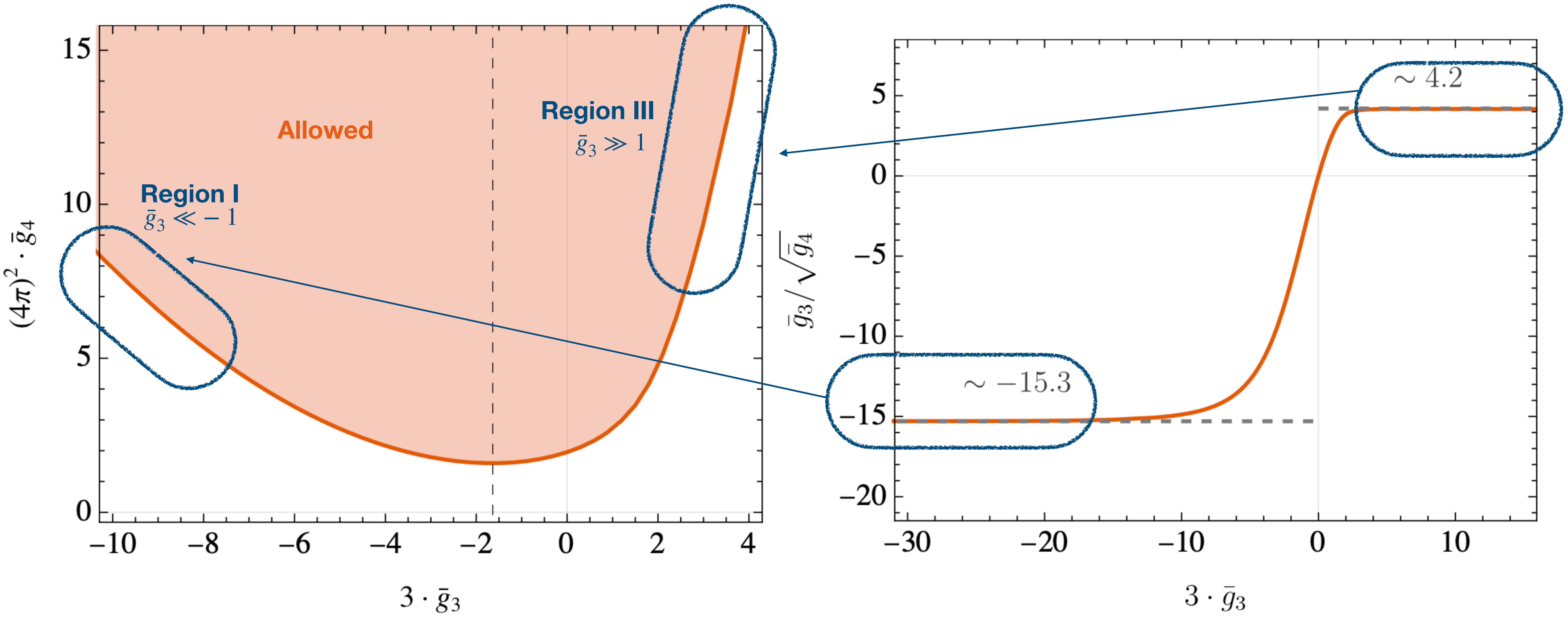
$$\mathcal{T}(s,t,u) = g_2(s^2+t^2+u^2) + g_3\, stu + g_4\left(s^2+t^2+u^2\right)^2 - \frac{g_2^2}{480\pi^2}\left(s^2(41s^2+t^2+u^2)\log\left(-s\sqrt{g_2}\right) + (s\leftrightarrow t) + (s\leftrightarrow u)\right) + O(s^5)$$

scale of the problem

$$\bar{g}_3 \equiv \frac{g_3}{g_2^{3/2}}, \quad \bar{g}_4 \equiv \frac{g_4}{g_2^2}$$

scale of the problem

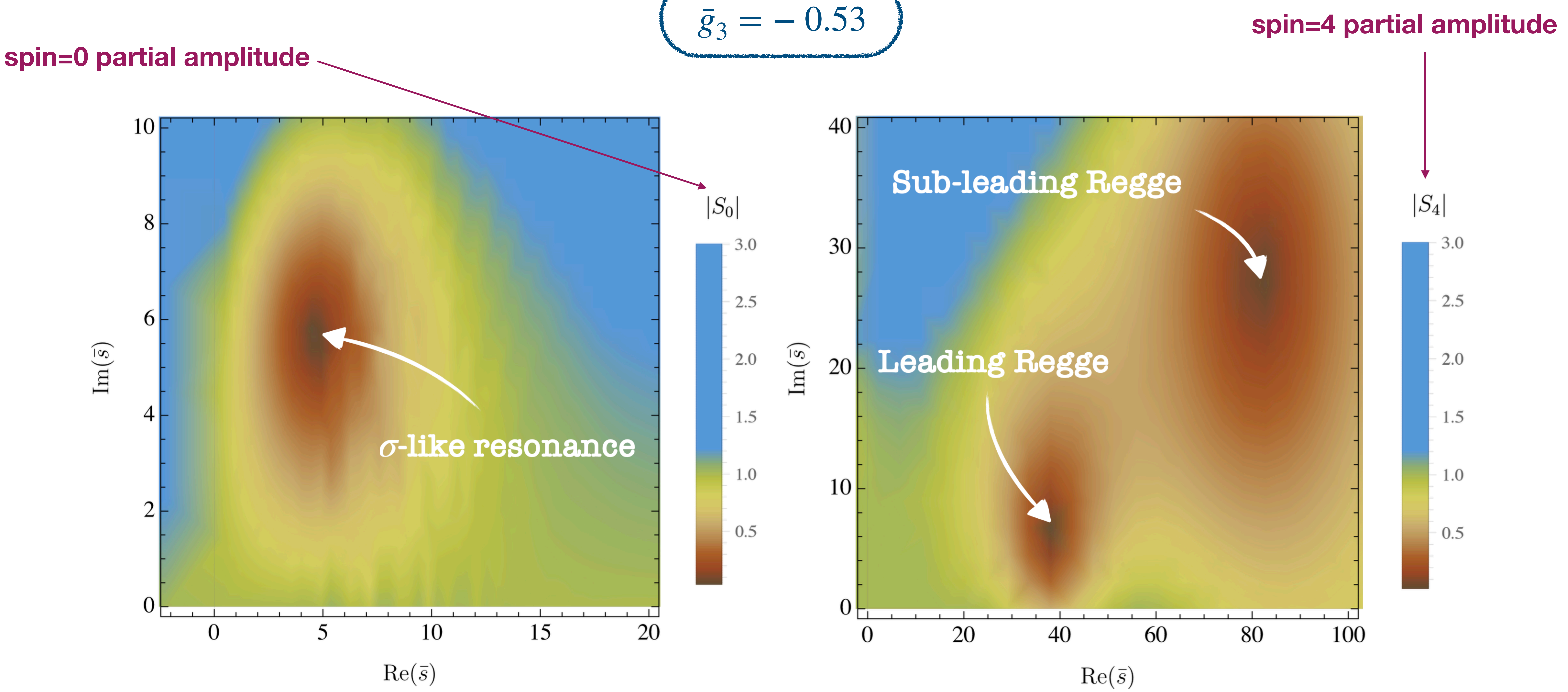
Bound



Allowed
Region III
$\bar{g}_3 \gg 1$
Region I
$\bar{g}_3 \ll -1$
$(4\pi)^2 \cdot \bar{g}_4$
$3 \cdot \bar{g}_3$
~ 4.2
~ -15.3
$\bar{g}_3 / \sqrt{\bar{g}_4}$
$3 \cdot \bar{g}_3$

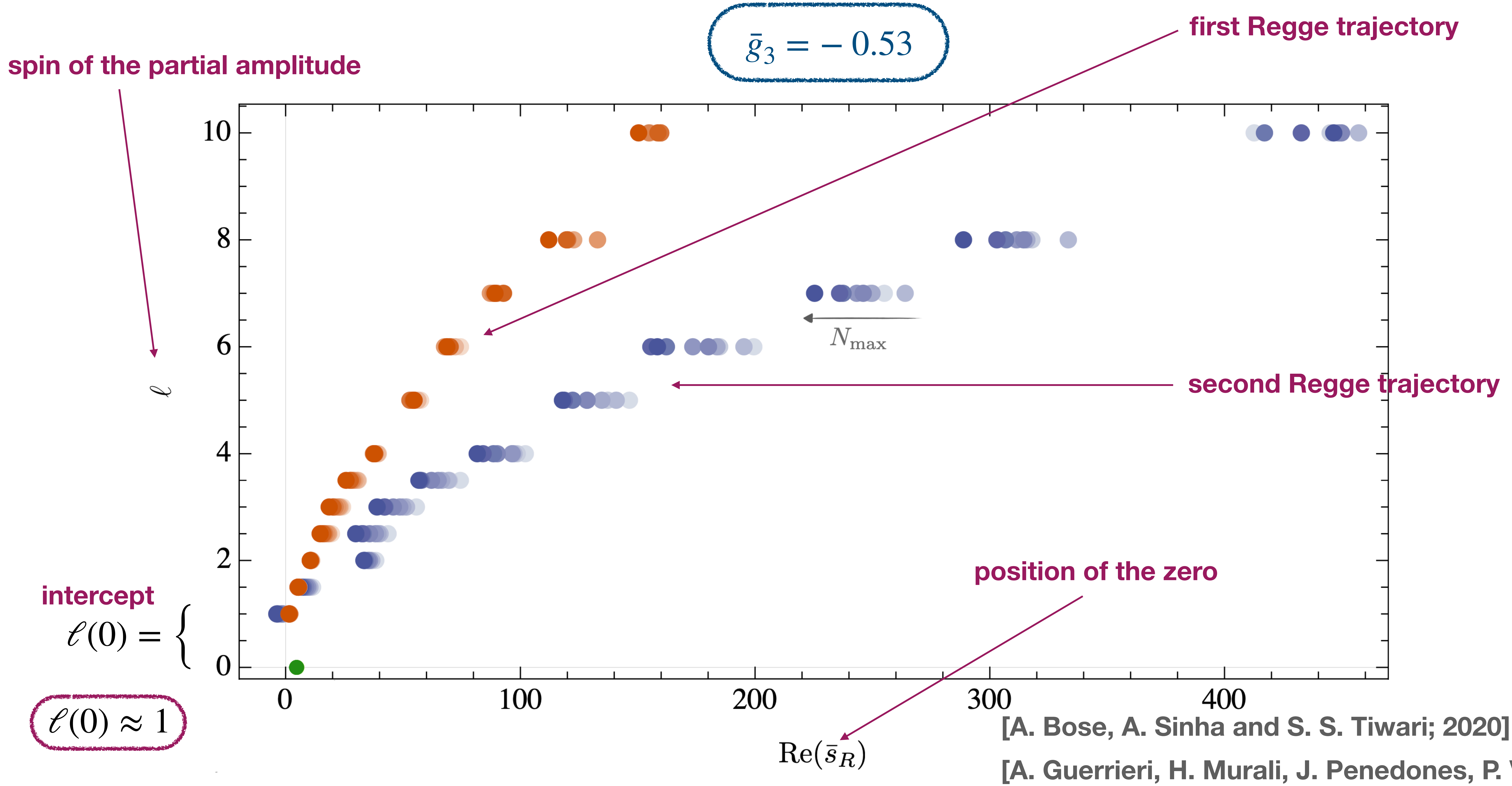
Resonances

$\bar{g}_3 = -0.53$



Zero on the first sheet = pole on the second sheet (for elastic amplitudes)

Regge trajectories



[A. Bose, A. Sinha and S. S. Tiwari; 2020]

[A. Guerrieri, H. Murali, J. Penedones, P. Vieira; 2022]

Conclusions

Axioms on scattering Amplitudes → (S-matrix bootstrap) → Bounds on scattering Amplitudes

This talk

Axioms on scattering Amplitudes + Extra data → (S-matrix bootstrap) → Scattering of pions, nucleons, ..

Present and future directions

[A. Guerrieri, J. Penedones, P. Vieira; 2018]

[Y. He, M. Kruczenski; 2025]

[Y. He, M. Kruczenski; 2024]

[Y. He, M. Kruczenski; 2023]

Thank you!