Error mitigation and circuit division for early fault-tolerant quantum phase estimation

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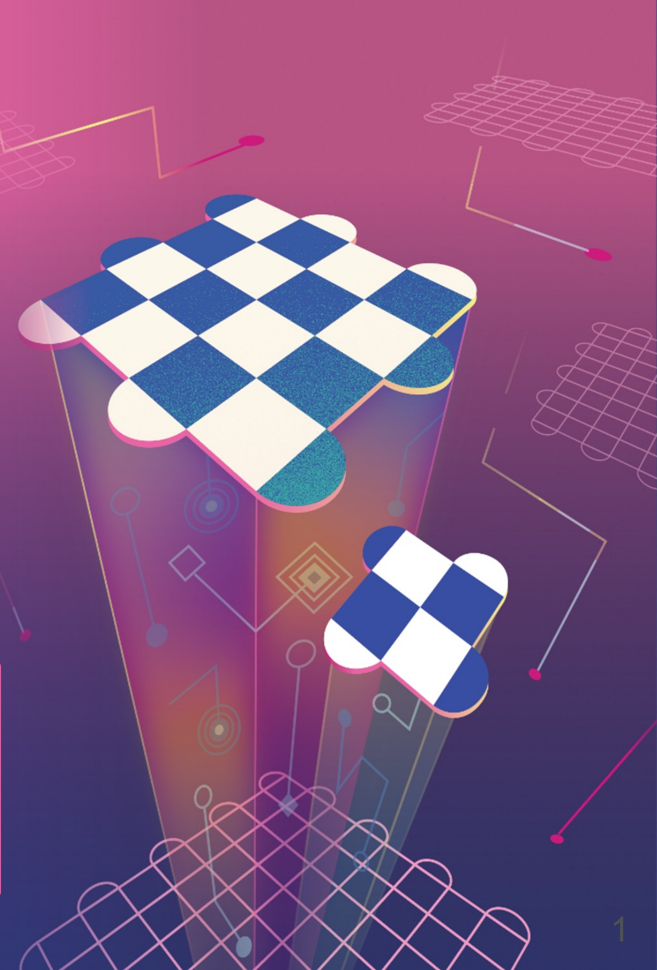
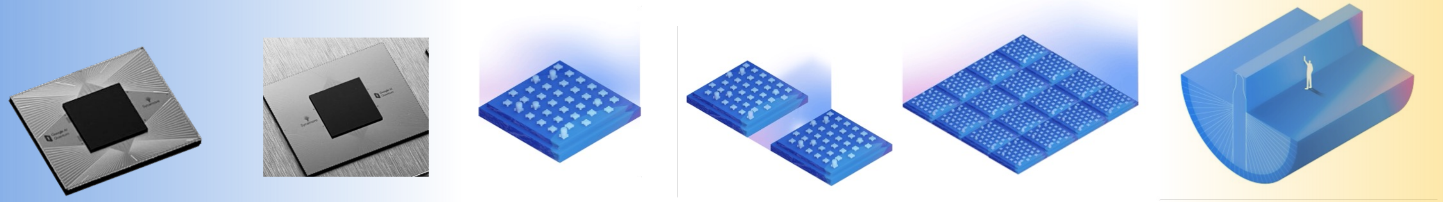




Logical error rate		[10^{-2}]	[10^{-6}]	[10^{-6}]	[10^{-8}]	[10^{-13}]
Number of physical qubits	53	10^2	10^3	10^4	10^5	10^6
	M1 (2019) Beyond classical benchmark ✓	**M2** (2023) Logical qubit prototype ✓	**M3** (2025+) 1 long-lived logical qubit	**M4** Tileable module (logical gate)	**M5** Engineering scale up	**M6** Error-corrected (EC) quantum computer

NISQ

Early fault-tolerant QC

Full FTQC

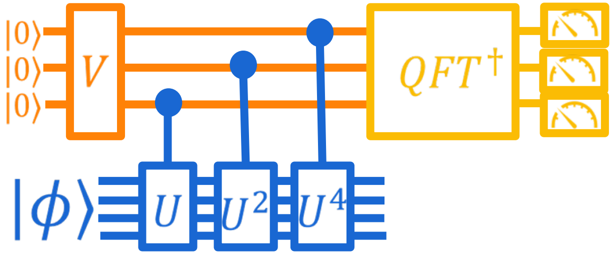
The era when we can ***correct all but $\Omega(1)$ errors****,*
and must design algorithms to be robust to the remainder

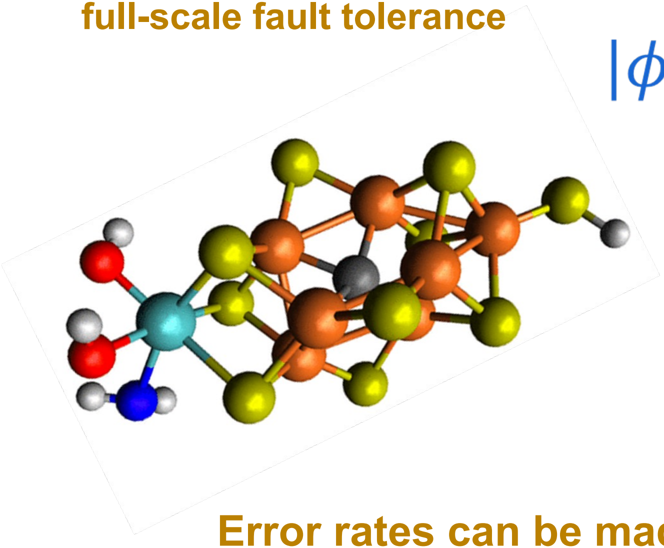
See also: Campbell ‘22, Wang *et al* ‘22, Wan *et al* ‘22, Dong *et al* ‘22, Lin, Tong ‘22, Katabarwa et al ‘23, Wang *et al* ‘23, Ding, Lin ‘23, Kshirsagar *et al* ‘24, Nelson, Baczewski ‘24, ...

Quantum AI

QPE benchmarks

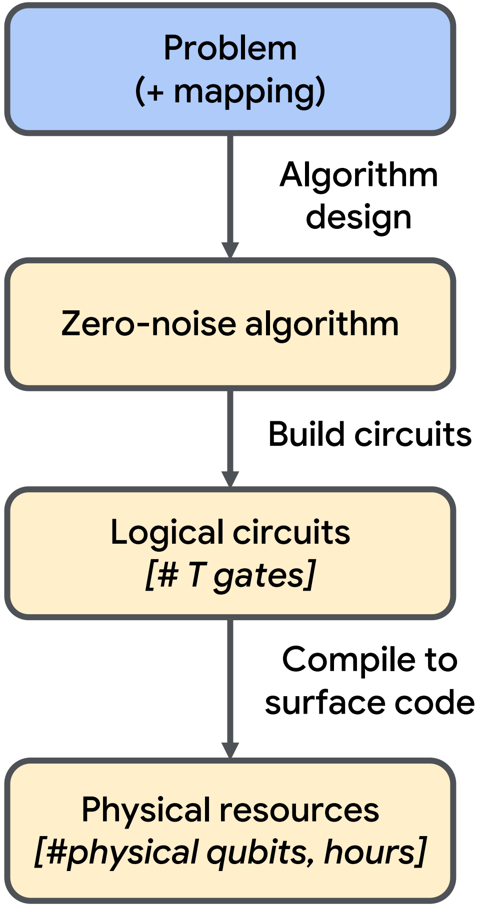
Most QPE benchmarks assume **full-scale fault tolerance**





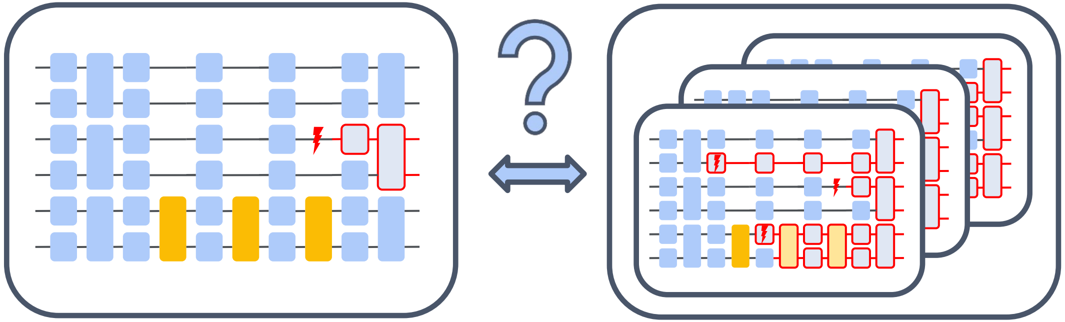
Reiher *et al.* [PNAS **114**, 2017]
Babbush *et al.* [PRX **8**, 2018]
Von Burg *et al.* [PRR **3**, 2021]
Lee *et al.* [PRQ **2**, 2021]

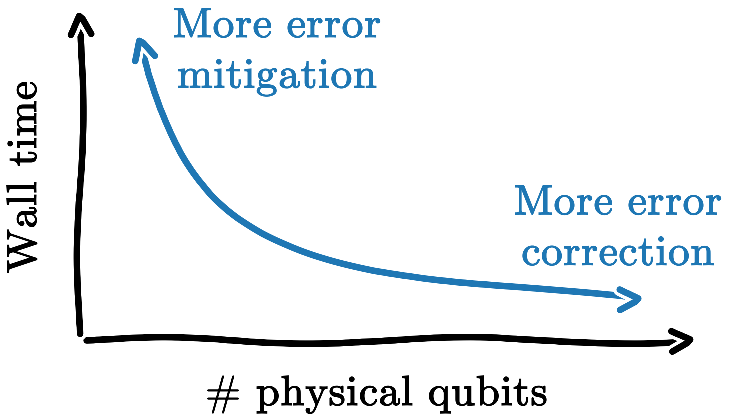
Error rates can be made arbitrarily small through error correction.

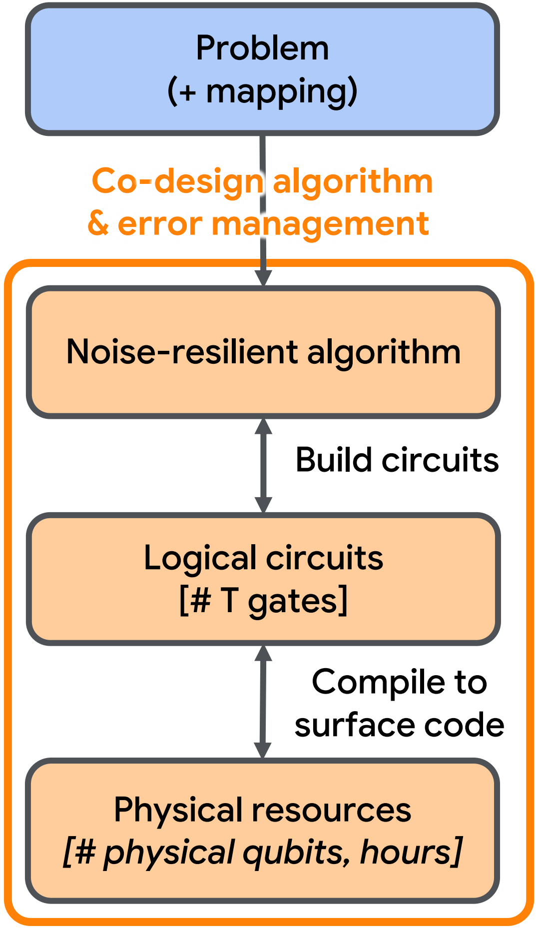


Circuit division

Circuit depth **Trade-off** **Repetitions**

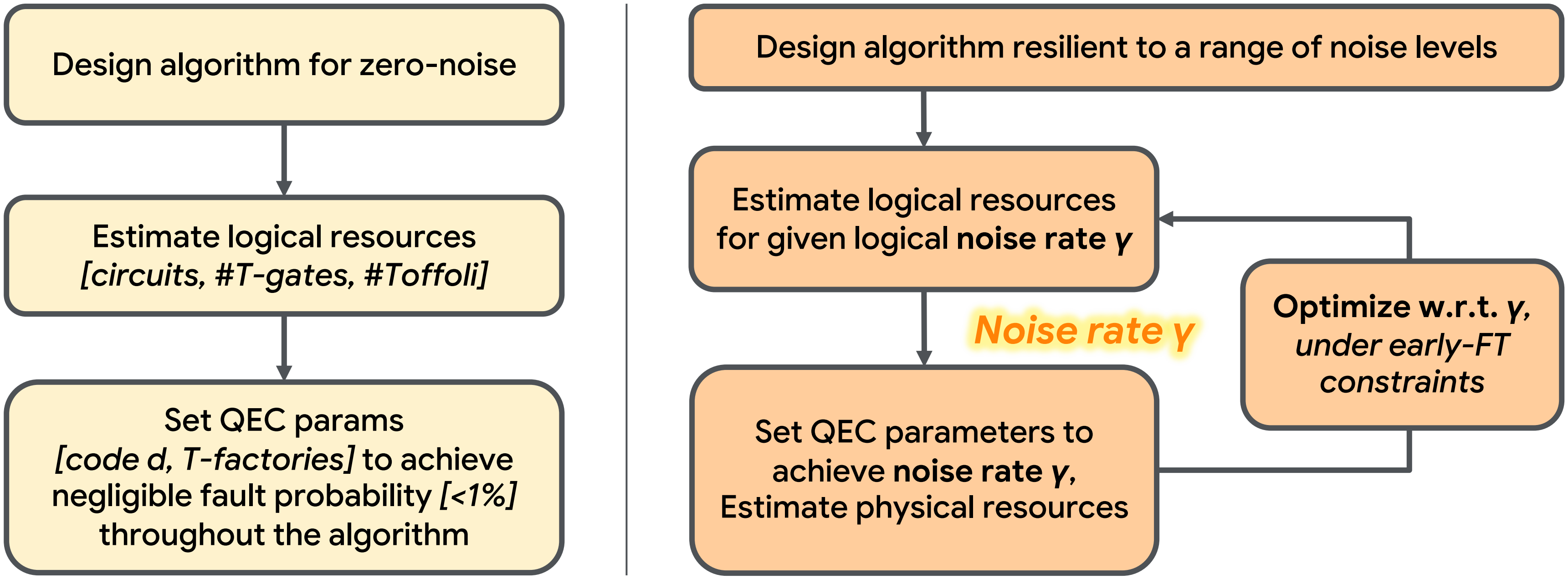






Algorithm design principles in FT

Full FTQC vs **Early FTQC**



Structure of the talk

QPE input:
ctrl-U circuit,
accuracy ε

Part 1:
Design QPE circuits & estimator given error rate γ

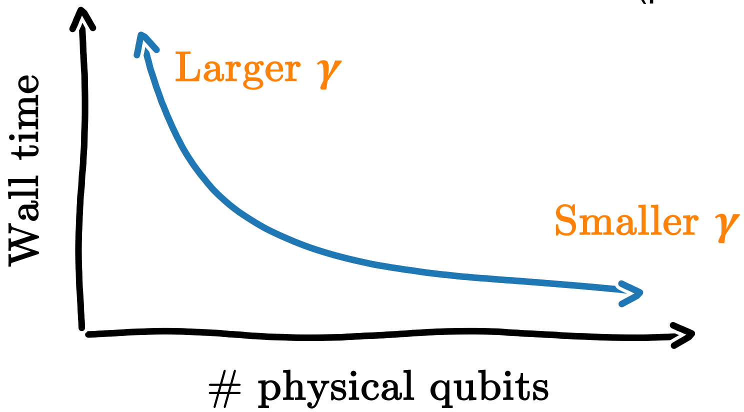
Physical resources

optimize over γ

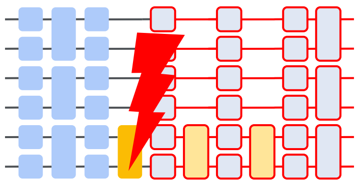
Circuits, #shots

Part 2:
Compiling and cost estimate

γ – Logical error rate (per ctrl-U)



Global depolarizing noise



Structure of the talk

QPE input:
ctrl-U circuit,
accuracy ε

Part 1:
Design QPE circuits & estimator given error rate γ

Physical resources

optimize over γ

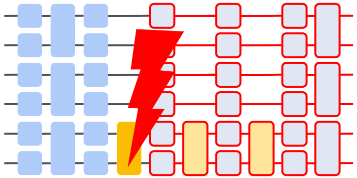
Circuits, #shots

Part 2:
Compiling and cost estimate

Overhead for removing bias

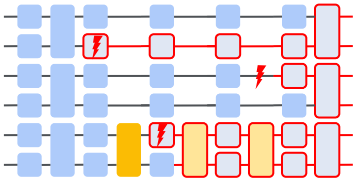
Part 3:
Define EM strategies to handle noise bias

Global depolarizing noise

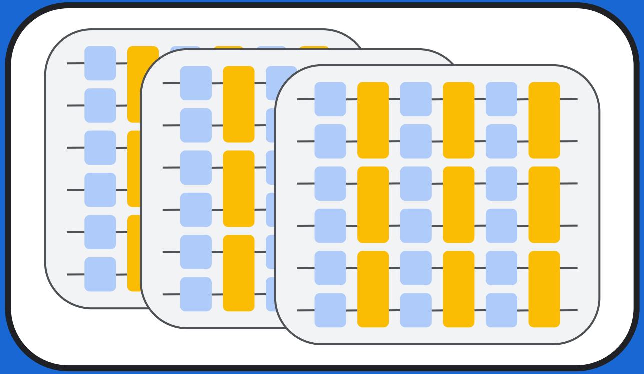


vs

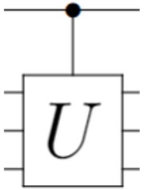
Realistic noise models



Quantum Phase Estimation with noise
(global depolarizing noise)



Quantum phase estimation



Noisy unitary
(global depolarising noise)

$$\Lambda[\rho] = F\mathcal{U}[\rho] + (1-F)\frac{\mathbb{I}}{2^n}$$

Cost = 1

Eigenstate

$$|\phi\rangle$$

Noise rate

$$\gamma$$

$$F = e^{-\gamma}$$

Target precision

$$\epsilon$$

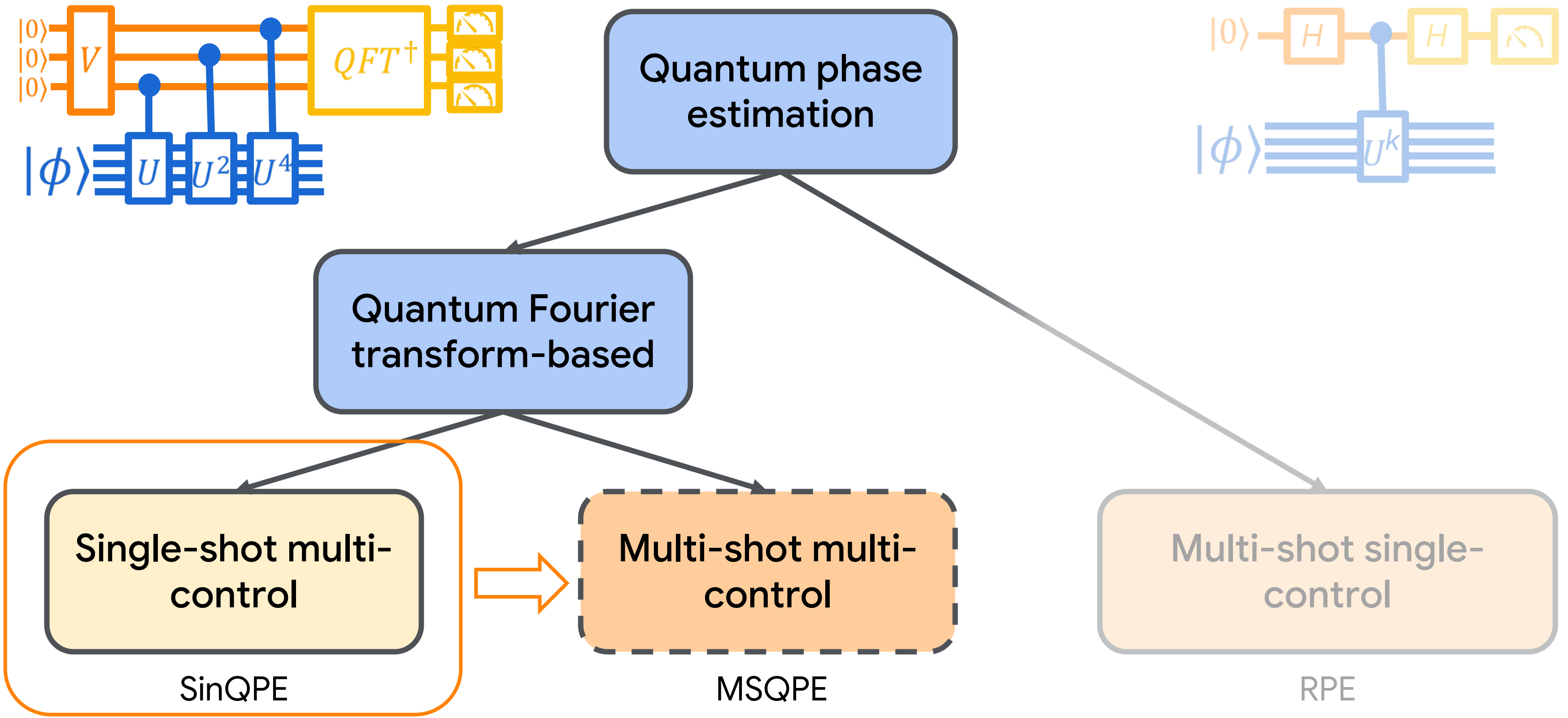
QPE algorithm = **Quantum circuits + data processing**

$\tilde{\phi}$ Phase estimate

T – Cost in terms of *total* #calls to noisy unitary

Types of quantum phase estimation

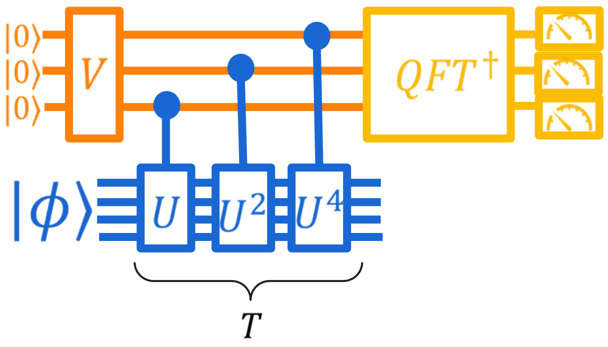
Part 1: QPE



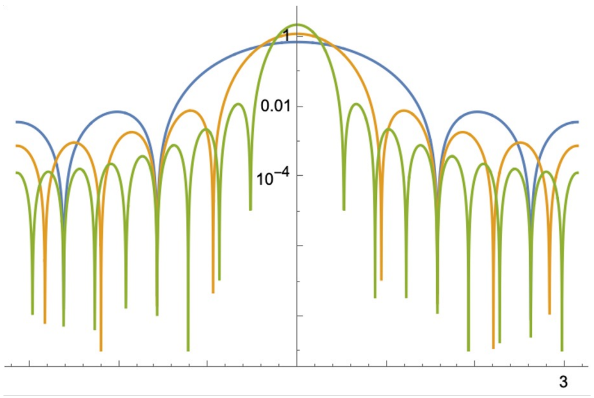
QFT-based quantum phase estimation

Part 1: QPE

Optimal strategy in the absence of noise.
Achieves minimal mean error $\epsilon = \pi/T$ [Luis & Perina, 1996].

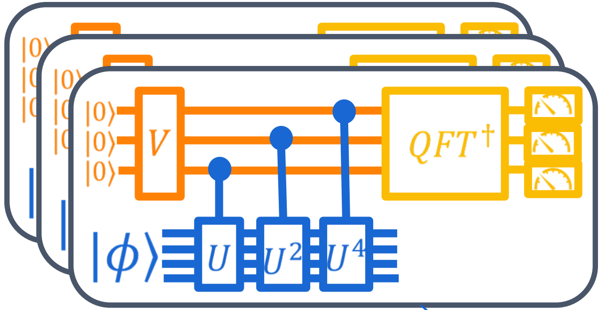


Known analytical expression for the output distribution:



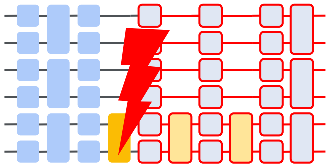
$$P(\tilde{\phi}|\phi) = \frac{\sin^2\frac{\pi}{T+2}}{(T+1)(T+2)} \frac{1+\cos\left((T+2)(\tilde{\phi}-\phi)\right)}{\left(\cos(\tilde{\phi}-\phi)-\cos\frac{\pi}{T+2}\right)^2}$$

Max-likelihood quantum phase estimator



+

Global depolarizing noise



$$P_\gamma(\tilde{\phi}|\phi) = e^{-\gamma T}\, P(\tilde{\phi}|\phi) + (1 - e^{-\gamma T})\, \frac{1}{2\pi}$$

Max-Likelihood estimation

$$\tilde{\phi} = \arg\max \log \mathcal{L}(\tilde{\phi}|\vec{x})$$

$$\log \mathcal{L}(\tilde{\phi}|\vec{x}) = \sum_{i=1}^{M} \log P(x_i|\tilde{\phi})$$

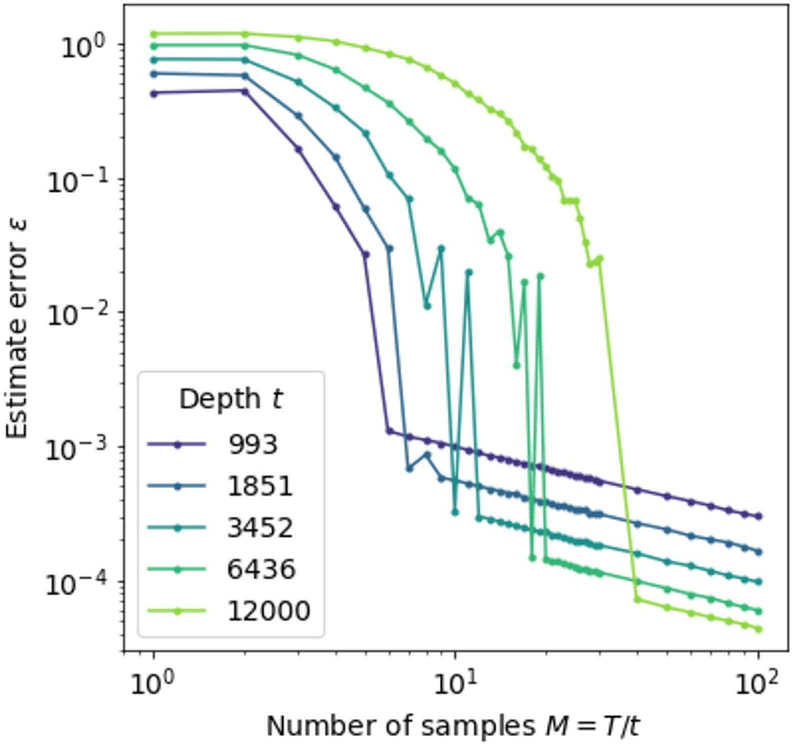
Max likelihood estimation

Asymptotically, the Cramer-Rao bound can be saturated using **max-likelihood estimation (MLE)**

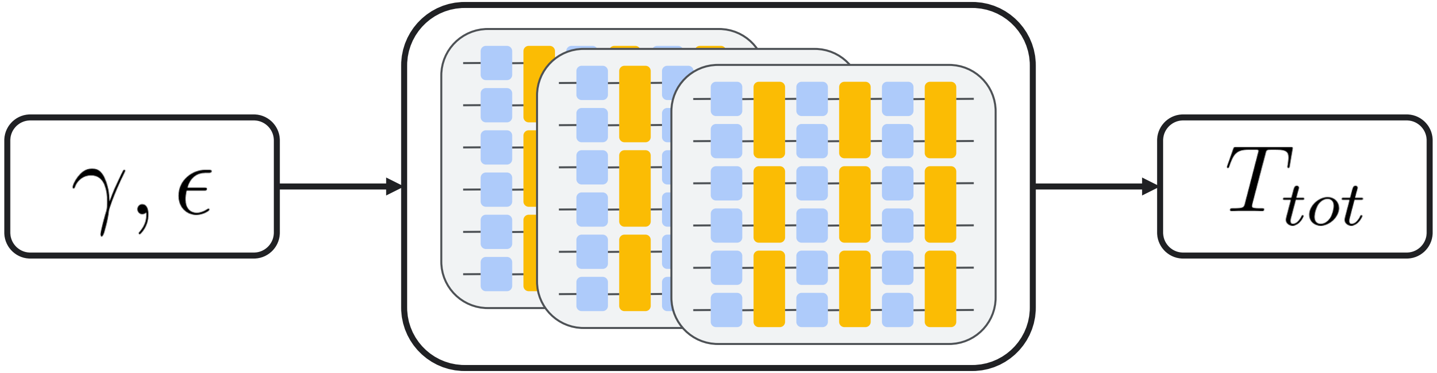
$$\tilde{\phi} = \arg\max \log \mathcal{L}(\tilde{\phi}|\vec{x})$$

$$\log \mathcal{L}(\tilde{\phi}|\vec{x}) = \sum_{i=1}^{M} \log P(x_i|\tilde{\phi})$$

However, with a **finite number of samples**, MLE is not guaranteed to converge.



Max-likelihood estimation (MLE)



3 regimes:			Total cost:
Low noise	$\gamma \ll \epsilon$	Single circuit (like noiseless)	$\pi\epsilon^{-1}$
High precision	$\gamma \gg \epsilon$	Fixed depth (max Fisher Info)	$\approx 20\gamma\epsilon^{-2}$
Intermediate	$\epsilon \sim \gamma$	Interpolate (empirical law, numerical validation)	

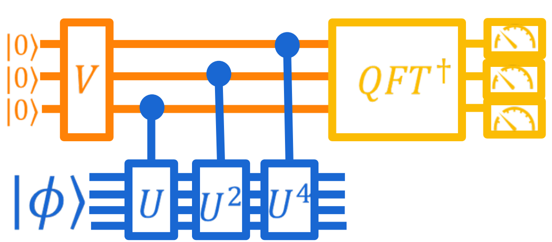
Summary MSQPE

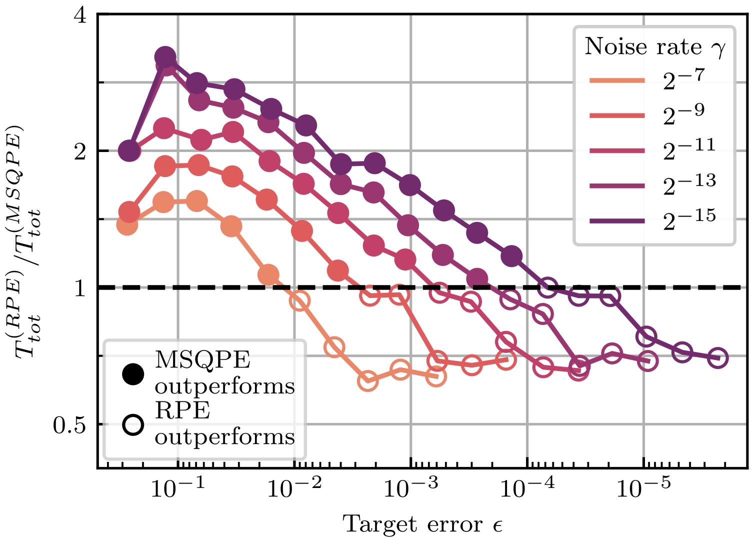
Part 1: QPE

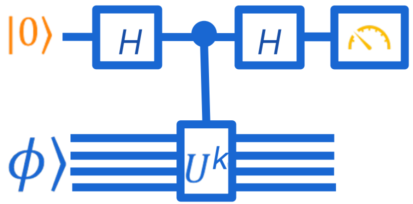
First complete multi-shot QFT-based QPE algorithm.

Optimised total runtime for global depolarising noise.

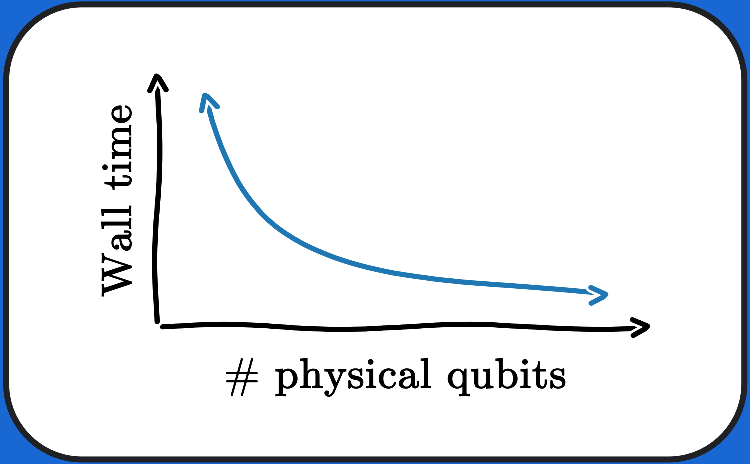
Advantage over Hadamard-test-based QPE for $\epsilon \sim \gamma$.







2. Compilation and cost estimate



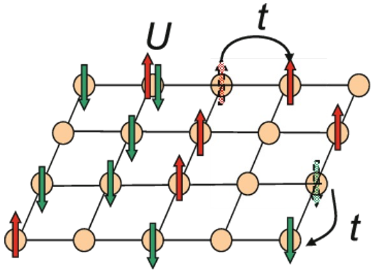
Choice of targets/benchmarks

Part 2: compilation

2D Fermi-Hubbard

(8–800 spin sites)

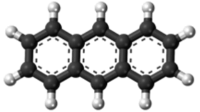
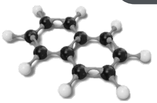
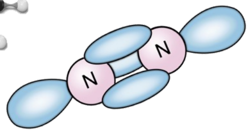
LCU block encoding

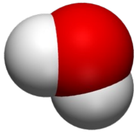
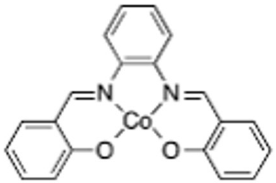


Small molecules

(12–52 spin orbitals)

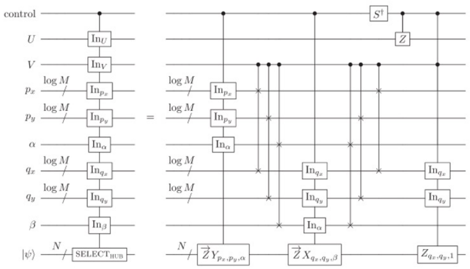
Tensor HyperContraction (THC)
+ Symmetry reduction of 1-norm

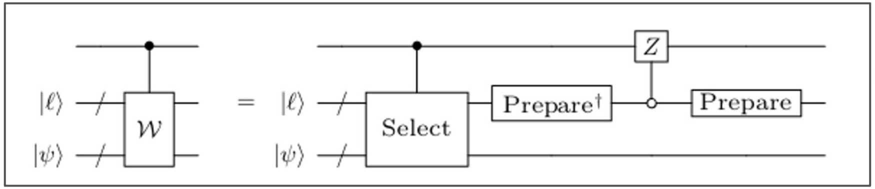




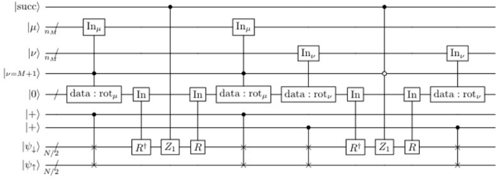



Co(salophen)

QPE of the qubitized walk operator







Logical costing using Qualtran

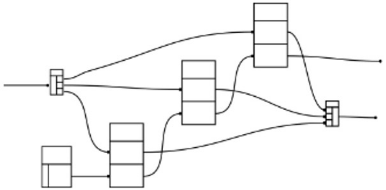
Part 2: compilation

github.com/quantumlib/Qualtran

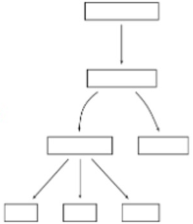
Release paper: arXiv:2409.04643

```
x = bb.add(IntState(val=1, bitsize=self.x_bitsize))
exponent = bb.split(exponent)

base = self.base
for j in range(self.exp_bitsize - 1, -1, -1):
    exponent[j], x = bb.add(
        CtrlModMul(k=base), ctrl=exponent[j], x=x)
    base = base * base % self.mod

return {
    'exponent': bb.join(exponent),
    'x': x
}
```

Code defines quantum subroutines



Use compute graph representation



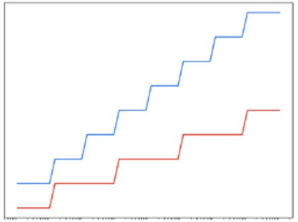
Call on a library of existing primitives

Simulate and test

Qubit counts

Gate counts

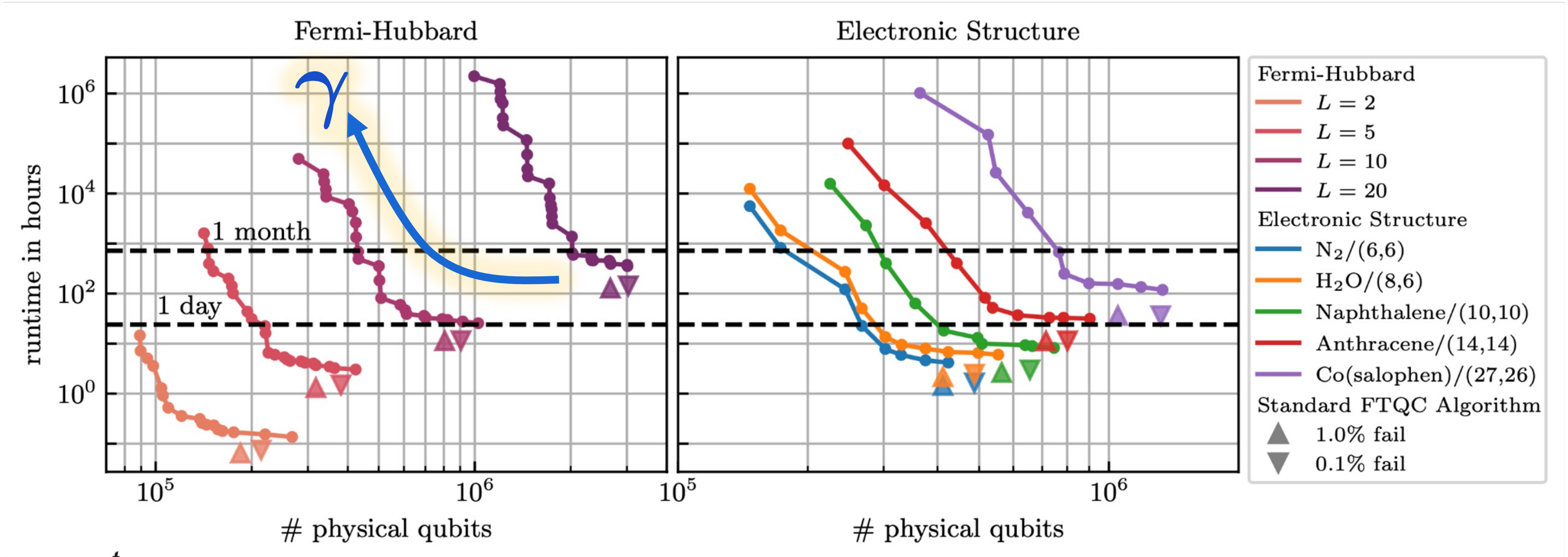
Analyze

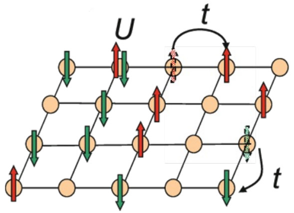


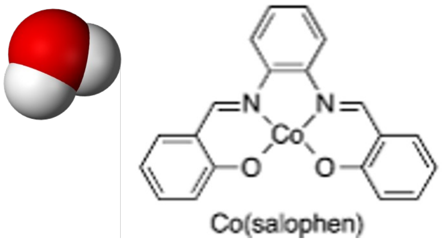
Model physical costs

```
walk = get_walk_operator_for_hubbard_model(x_dim, y_dim, t, mu).controlled(1)
complexity = walk.t_complexity()

ccz_from_rotations = (
    complexity.rotations * nbits_for_rotations
)  # CCZ cost for phase gradient rotations
ccz_from_logic = complexity.t / 4  # in this circuit all T gates come from CCZ gates
```

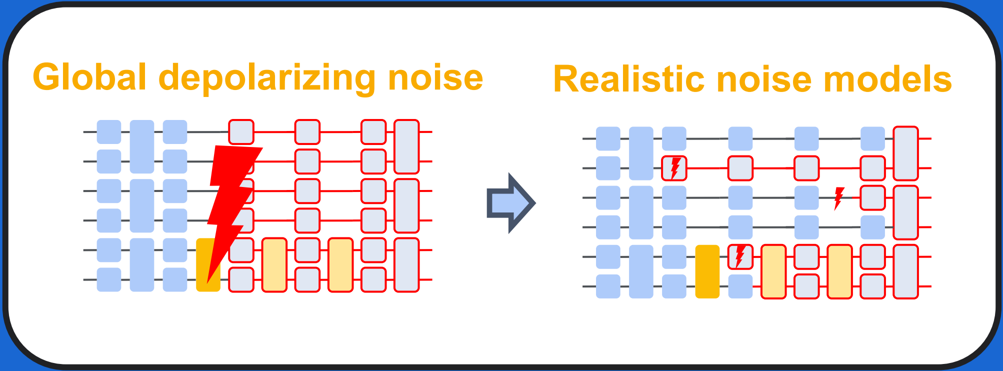
Trade off: physical qubits vs wall time





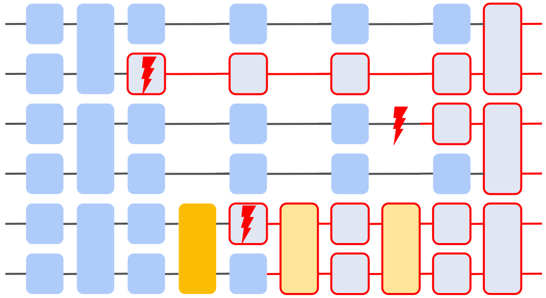


Error mitigation beyond expectation values



Logical noise (in surface code FTQC)

- Sources of logical noise
  - Data errors ← code distance [+ decoder limit]
  - T-state imperfections ← T-factory size



- Logical noise model
  - ***Local*** noise – random Pauli channel

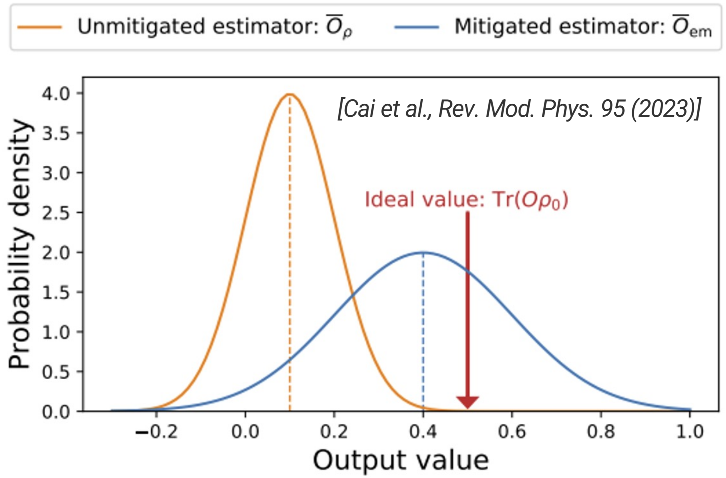
“Local noise biases phase kickback”

e.g. [T. E. O'Brien et al, New J. Phys. ***21*** *(2019)]*

Quantum error mitigation

Error mitigation is…

Trading bias for stochastic noise



Postselection	$C_{\text{em}} = \Lambda F^{-1}$	Symmetry verification, verified phase estimation*
Rescaling	$C_{\text{em}} = \Lambda F^{-2}$	Echo verification*, direct rescaling under GDN
Explicit unbiasing	$C_{\text{em}} = \Lambda F^{-4}$	Probabilistic error cancellation, virtual distillation

*depending if verification circuit is short

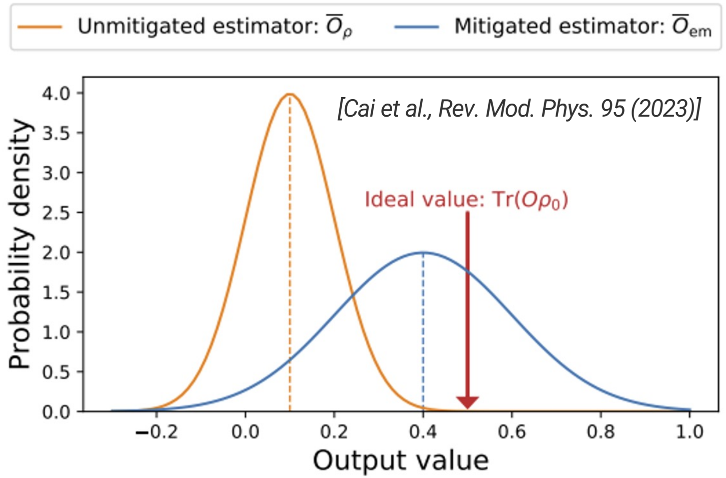
Quantum error mitigation

Error mitigation is…

Trading bias for stochastic noise

As we have control over circuit depth,

EM overhead can be reduced for QPE



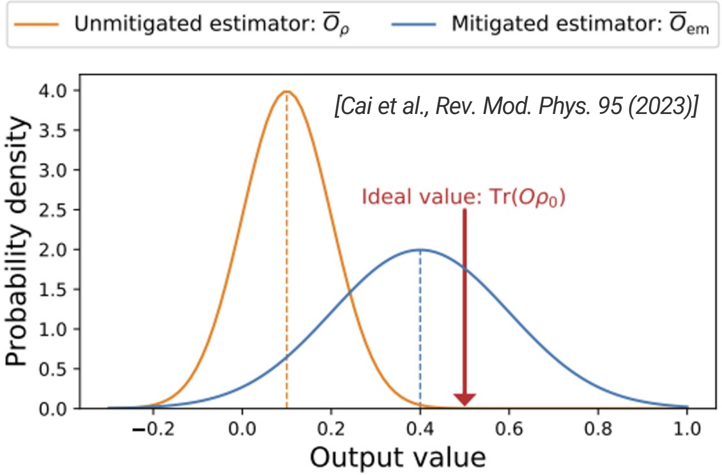
Postselection	$C_{\rm em} = \Lambda F^{-1}$	$C_{\rm em}^{\rm QPE}(\gamma,\epsilon) \leq \Lambda\kappa e/\pi \cdot \gamma/\epsilon$	Symmetry verification, verified phase estimation*
Rescaling	$C_{\rm em} = \Lambda F^{-2}$	$C_{\rm em}^{\rm QPE}(\gamma,\epsilon) \leq \Lambda\kappa e/\pi \cdot 2\gamma/\epsilon$	Echo verification*, direct rescaling under GDN
Explicit unbiasing	$C_{\rm em} = \Lambda F^{-4}$	$C_{\rm em}^{\rm QPE}(\gamma,\epsilon) \leq \Lambda\kappa e/\pi \cdot 4\gamma/\epsilon$	Probabilistic error cancellation, virtual distillation

*depending if verification circuit is short

Quantum error mitigation

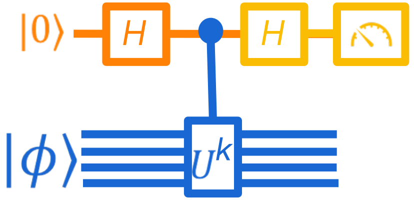
Error mitigation is…

Trading bias for stochastic noise



Current error mitigation schemes are designed **only for expectation values**

e.g.



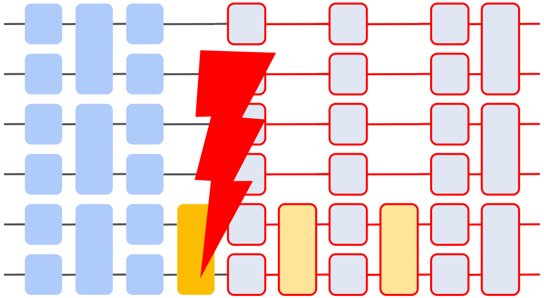
$$g(k) = \langle \phi | U^k | \phi \rangle$$

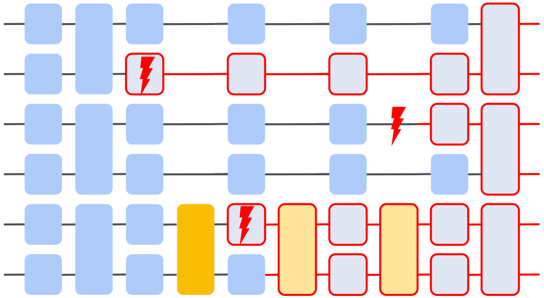
Global depolarising noise vs realistic noise

Max-Likelihood estimation

$$\tilde{\phi} = \arg\max \log \mathcal{L}(\tilde{\phi}|\vec{x})$$

$$\log \mathcal{L}(\tilde{\phi}|\vec{x}) = \sum_{i=1}^{M} \log P(x_i|\tilde{\phi})$$





$$P(\tilde{\phi}|\phi) = e^{-\gamma t} P_0(\tilde{\phi}|\phi) + (1 - e^{-\gamma t})\, \frac{1}{2\pi}$$

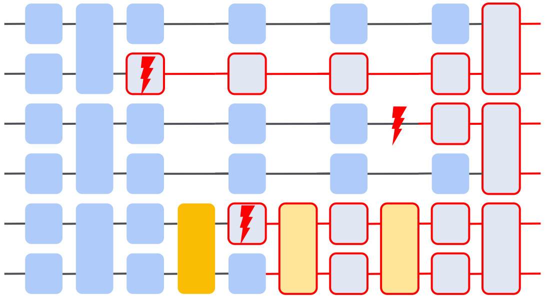
$$P(\tilde{\phi}|\phi) = e^{-\gamma t} P_0(\tilde{\phi}|\phi) + ???$$

Probabilistic error cancellation

Max-Likelihood estimation

$$\tilde{\phi} = \arg\max \log \mathcal{L}(\tilde{\phi}|\vec{x})$$

$$\log \mathcal{L}(\tilde{\phi}|\vec{x}) = \sum_{i=1}^{M} \log P(x_i|\tilde{\phi})$$



PEC decomposition

$$P_0(x) = \sum_j \alpha_j Q_j(x)$$

Noisy circuits we can actually sample from

$$P(\tilde{\phi}|\phi) = e^{-\gamma t} P_0(\tilde{\phi}|\phi) + ???$$

Probabilistic error cancellation

[Temme et al., 2017]

We decompose the ideal (noiseless) channel in implementable noisy circuits

$$\mathcal{U} = \sum_j \alpha_j \mathcal{B}_j$$

By linearity, the noiseless outcome distribution can be written as a nonconvex combination

$$P(x) = \sum_j \alpha_j Q_j(x)$$

$$Q_j(x) = \mathrm{Tr}\left[\, |x\rangle\langle x|\, \mathcal{B}_j(|0\rangle\langle 0|\,)\right]$$

PEC is defined for expectation values

$$\mathbb{E}[x|x \sim P(x)] = \sum_j \alpha_j\, \mathbb{E}[x|x \sim Q_j(x)].$$

We can estimate the sample mean by importance-sampling *j* with probability

$$p_j = \frac{|\alpha_j|}{\sum_j |\alpha_j|}$$

Overhead:

$$C_{\text{em}} = \left(\sum_j |\alpha_j| \frac{\sqrt{\mathrm{Var}[Q_j]}}{\sqrt{\mathrm{Var}[P]}} \right)^2 \stackrel{*}{=} F^{-4}$$

Probabilistic error cancellation for MLE?

$$P(x) = \sum_j \alpha_j Q_j(x)$$

PEC decomposition

Standard MLE:

1. Get M samples x_i from $P(x|\phi)$.
2. Calculate the log likelihood:

$$C(\tilde{\phi}) = \log \mathcal{L}(\tilde{\phi}|\vec{x}) = \frac{1}{M}\sum_{i=1}^{M} \log P(x_i|\tilde{\phi})$$

3. Maximize $C(\tilde{\phi})$.

$$C(\tilde{\phi}) \xrightarrow{M \to \infty} \mathbb{E}\left[\log P(x|\tilde{\phi}) \ |x \sim P(x|\phi)\right]$$

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Sample average

3. Maximize $C(\tilde{\phi})$.

$$C(\tilde{\phi}) \xrightarrow{M \to \infty} \mathbb{E}\left[\log P(x|\tilde{\phi}) \;|x \sim P(x|\phi)\right]$$

Expectation value

Explicitly-unbiased MLE

$$P(x) = \sum_j \alpha_j Q_j(x)$$

PEC decomposition

Standard MLE:

1. Get M samples x_i from $P(x|\phi)$.
2. Calculate the log likelihood:

$$C(\tilde{\phi}) = \log \mathcal{L}(\tilde{\phi}|\vec{x}) = \frac{1}{M}\sum_{i=1}^{M} \log P(x_i|\tilde{\phi})$$

3. Maximize $C(\tilde{\phi})$.

EUMLE:

1. Get M samples (j_i, x_i) from $\frac{|\alpha_j|}{\|\alpha\|} Q_j(x)$.
2. Calculate the log likelihood:

$$C(\tilde{\phi}) = \frac{\|\alpha\|}{M}\sum_{i=1}^{M} \mathrm{sgn}(\alpha_{j_i}) \log P(x_i|\tilde{\phi})$$

3. Maximize $C(\tilde{\phi})$.

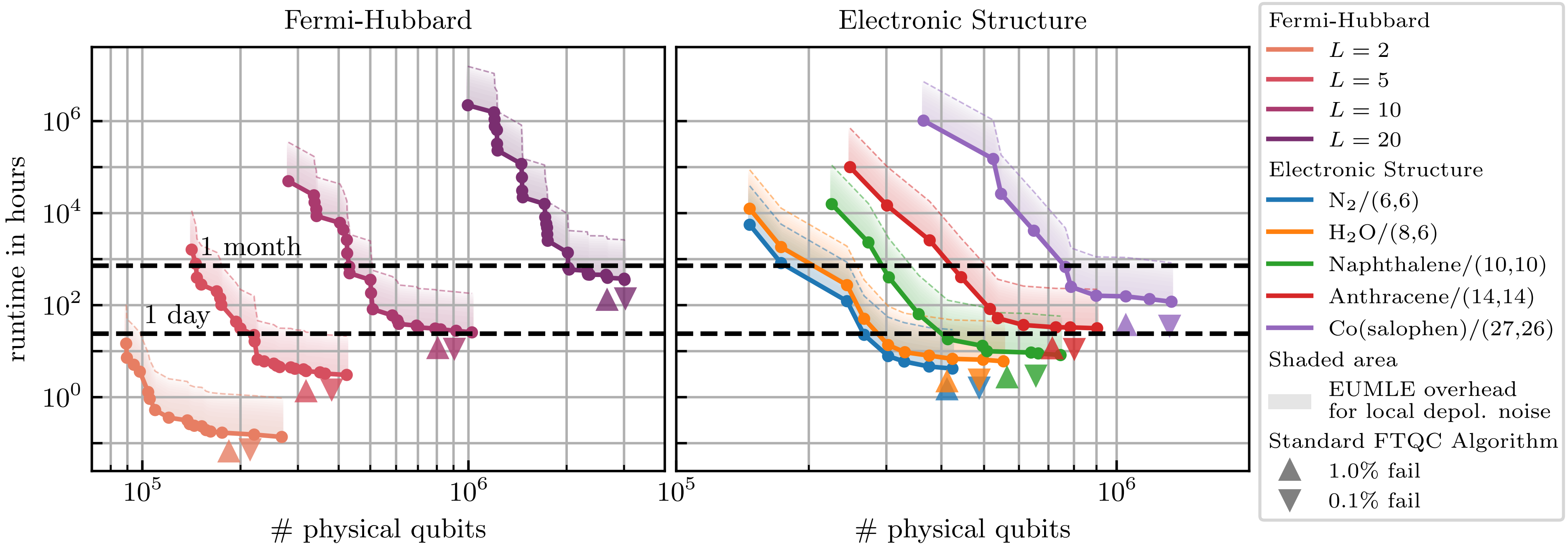
In both cases:

$$C(\tilde{\phi}) \xrightarrow{M \to \infty} \mathbb{E}[\log P(x|\tilde{\phi}) \; |x \sim P(x|\phi)]$$

Explicitly-unbiased max-likelihood estimator

- First error mitigation technique beyond expectation values
  (note: MLE is very general)
- Proven convergence and **asymptotic normality**
- We further introduce **regularization** and **filtering**
- Under *local Pauli noise,* regularized EUMLE incurs in **≈ 7x overhead** compared to *global depolarizing noise* in max-likelihood QPE.
  (Filtering could reduce this)

Resource estimates with EUMLE overhead



Take-home summary

- We can trade-off **qubits vs time** by combining error mitigation with error correction

- Designing algorithms to be noise-resilient is useful, even in fault-toletant QC

- Error mitigation can be extended to general estimation tasks

Error mitigation and circuit division for early fault-tolerant quantum phase estimation

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Stefano Polla

Maximillian Scheurer

Christian Gogolin

William J. Huggins

Thomas E. O’Brien

https://github.com/StefanoPolla/EarlyFT_QPE

arXiv:2410.05369