UF

15^{th} July 2025

Higgs Hunting 2025.

SMEFT Geometry, Amplitudes & a Shortcut to Riemann Normal Coordinates.

Soon to appear on arXiv.

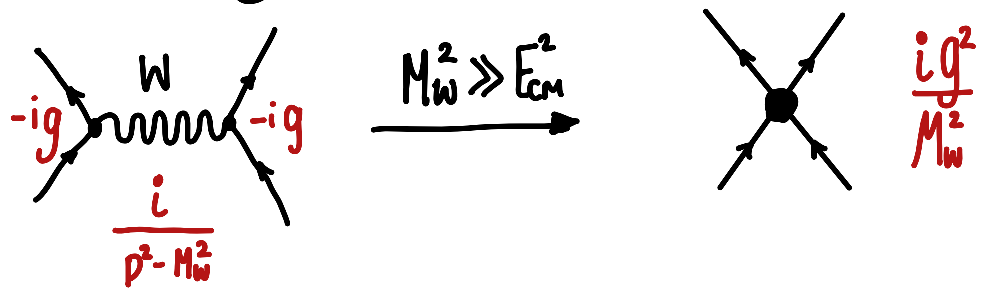
Mia West

in collaboration with Cris Caballeros & Rachel Houtz.

Effective Field Theory

"Top Down"

Think 4-fermi theory:



"Bottom Up"

Build a tower of higher dimensional operators supressed by the higher energy.

Standard Model EFT (SMEFT)

Wilson Coefficients

$$\mathcal{L}_{SMEFT} = \mathcal{L}_{SM} + \sum_{i=1}^{n_d} \frac{C_i^{(d)}}{\Lambda^d} Q_i^{(d)} \quad \text{for } d \geq 4.$$

Scale of new physics

Operators of SM fields

obey $SU(3)_c \times SU(2)_L \times U(1)_Y$

In This Talk...

* Focussing on the Scalar Sector
* Custodial Symmetry Limit $SU(2)_L \times SU(2)_R \sim O(4)$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_4 + i\phi_3 \end{pmatrix} \rightsquigarrow \vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

* Global Symmetry recover local symmetry following

[R. Alonso, E. Jenkins & A. Manohar: 1605.03602].

Scalar Lagrangian in Cartesians / 'Warsaw' to two-derivatives:

$$\mathcal{L} = \partial_\mu \phi^a \, G_{ab}(\phi) \, \partial^\mu \phi^b + V(\phi) + \text{(higher derivatives)}$$

where
$$G_{ab}(\phi) = \underbrace{A\left(\frac{\phi\cdot\phi}{\Lambda^2}\right)}_{1 + a_1 \frac{\phi\cdot\phi}{\Lambda^2} + a_2 \left(\frac{\phi\cdot\phi}{\Lambda^2}\right)^2 + \ldots} \delta_{ab} + \underbrace{B\left(\frac{\phi\cdot\phi}{\Lambda^2}\right)}_{b_1 + b_2 \left(\frac{\phi\cdot\phi}{\Lambda^2}\right) + b_3 \left(\frac{\phi\cdot\phi}{\Lambda^2}\right)^2 + \ldots} \frac{\phi_a \phi_b}{\Lambda^2}$$

Field Redefinitions which leave Amplitudes Invariant:

$$\underbrace{\phi^a \to F^{ab}(\tilde{\phi})}_{\text{Real analytic function of } \tilde{\phi}.}\tilde{\phi}^b \qquad \text{with} \qquad F^{ab}(0) = \delta^{ab}$$

* In general, $F^{ab}(X)$ MAY include derivative couplings.
  - ↳ But not in this talk.

[A. Manohar; hep-ph/1804.05863]

"Geometric SMEFT"

- $G_{ab}(\phi)$ is a metric tensor:

$$\tilde{G}_{ab}(\tilde{\phi}) = \frac{\partial \phi^c}{\partial \tilde{\phi}^a} G_{cd}(\phi(\tilde{\phi})) \frac{\partial \phi^d}{\partial \tilde{\phi}^b}$$

- $\partial_\mu \phi^a$ transforms as a tangent vector field:

$$\partial_\mu \tilde{\phi}^a = \frac{\partial \tilde{\phi}^a}{\partial \phi^b} \partial_\mu \phi^b$$

$$\mathcal{L}_{\text{kinetic}} = \frac{1}{2} \partial_\mu \phi^a G_{ab}(\phi) \partial^\mu \phi^b$$

$$= \frac{1}{2} \partial_\mu \tilde{\phi}^c \frac{\partial \phi^a}{\partial \tilde{\phi}^c} G_{ab}(\phi(\tilde{\phi})) \frac{\partial \phi^b}{\partial \tilde{\phi}^d} \partial^\mu \tilde{\phi}^d$$

$$= \frac{1}{2} \partial_\mu \tilde{\phi}^c \tilde{G}_{cd}(\tilde{\phi}) \partial^\mu \tilde{\phi}^d .$$

ie we can treat the fields *geometrically* living on the surface of a *field-space manifold.*

[L. Alvarez-Gaume, D.Z. Freedman and S. Mukhi; Annals. Phys. 148 (1981) 85].

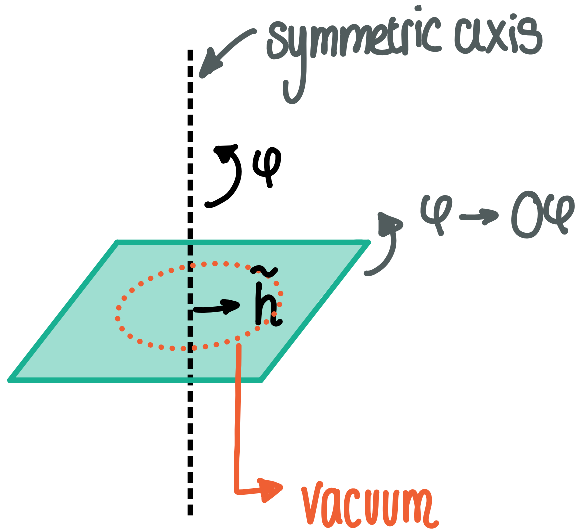
[R. Alonso, E. Jenkins and A. Manohar : 1605.03602] ← Geometry of scalars in HEFT

[A. Helset, A. Martin, M. Trott : 2001.01453] ← GeoSMEFT

...

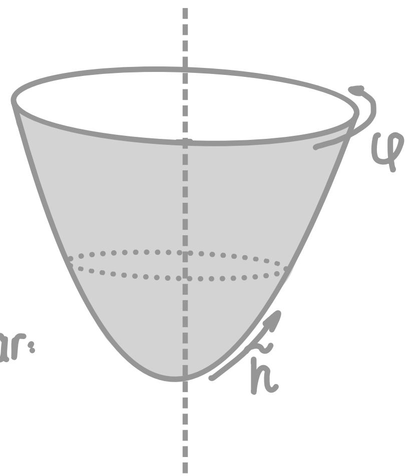
SMEFTs in pictures:

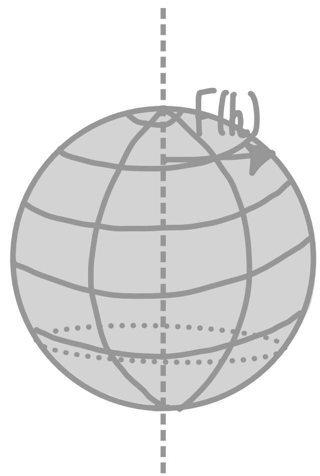
① SM



② SMEFTs

eg negative curvature models



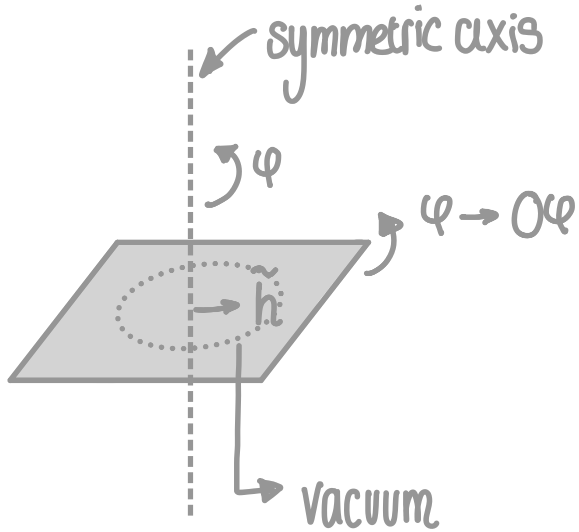


eg Composite Higgs models

[R. Alonso, E. Jenkins, A. Manohar: 1602.00706]

SMEFTs in pictures:

① SM



② SMEFTs

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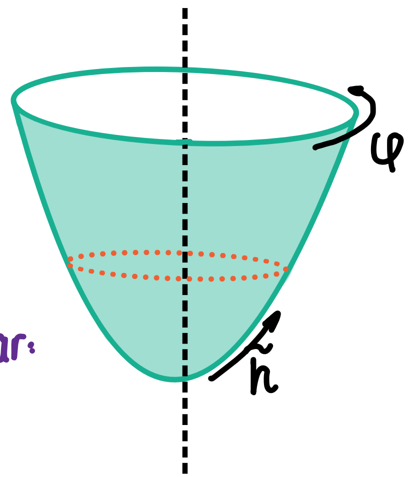
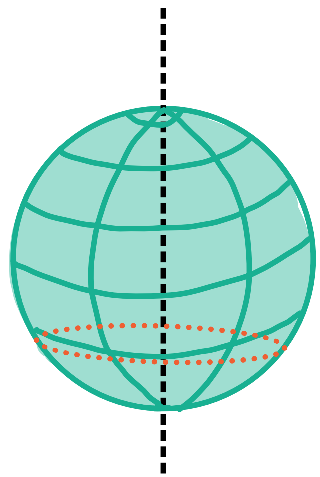
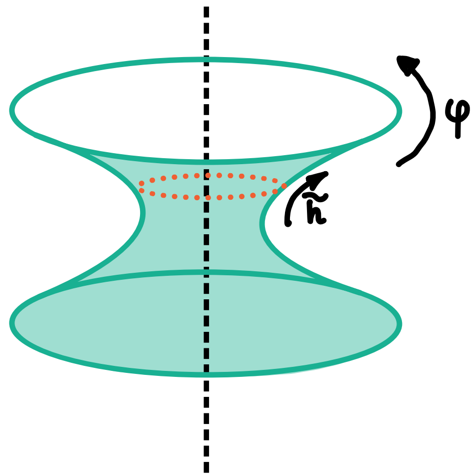


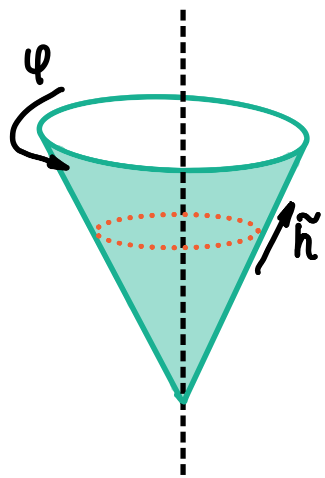


eg Composite Higgs models

③ HEFTs that aren't SMEFTs : Non-Linear Theories

(exclusive)





See eg:

[arxiv: 2109.13290; R. Alonso & MW]

[arXiv: 2008.08597; T. Cohen, N. Craig, X. Lu, D. Sutherland]

[A. Falkowski, R. Rattazzi; 1902.05936]

On-Shell Amplitudes in High Energy Limit ($E_{CM} \gg M_H, M_{W^\pm}, M_Z$).

$$A_3 = 0$$

$$A_4 = s\,\overline{R}_{1(34)2} + t\,\overline{R}_{1(24)3} + u\,\overline{R}_{1(23)4}$$

Evaluate at vacuum

[C. Cheung, A. Helset and J. Parra-Martinez: 2111.03045].

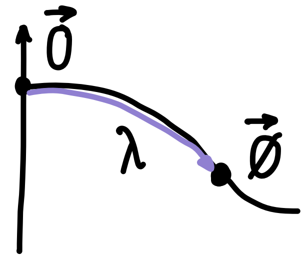
$$\begin{aligned}
A_4^{i_1 i_2 i_3 i_4} &= R^{i_1 i_3 i_2 i_4} s_{34} + R^{i_1 i_2 i_3 i_4} s_{24}, \\
A_5^{i_1 i_2 i_3 i_4 i_5} &= \nabla^{i_3} R^{i_1 i_4 i_2 i_5} s_{45} + \nabla^{i_4} R^{i_1 i_3 i_2 i_5} s_{35} + \nabla^{i_4} R^{i_1 i_2 i_3 i_5} s_{25} \\
&\quad + \nabla^{i_5} R^{i_1 i_3 i_2 i_4} s_{34} + \nabla^{i_5} R^{i_1 i_2 i_3 i_4} (s_{24} + s_{45}), \\
A_6^{i_1 i_2 i_3 i_4 i_5 i_6} &= -\tfrac{1}{72} \left(R^{i_1 i_3 i_2 j} s_{12} + R^{i_1 i_2 i_3 j} s_{13} \right) \frac{1}{s_{123}} \left(R_j{}^{i_6 i_5 i_4} s_{46} + R_j{}^{i_5 i_6 i_4} s_{45} \right) \\
&\quad + \tfrac{1}{108} \left(R^{i_1 i_3 i_2 j} (s_{12} - \tfrac{1}{6} s_{123}) + R^{i_1 i_2 i_3 j} (s_{13} - \tfrac{1}{6} s_{123}) \right) \left(R_j{}^{i_6 i_5 i_4} + R_j{}^{i_5 i_6 i_4} \right) \\
&\quad + \tfrac{1}{90} R^{i_1 i_6 i_5 j} R_j{}^{i_2 i_3 i_4} s_{13} + \tfrac{1}{80} \nabla^{i_6} \nabla^{i_5} R^{i_1 i_2 i_3 i_4} s_{13} + \text{perm}\,.
\end{aligned}$$

... "best basis" for computing amplitudes: local inertial "normal coordinate" frame.

[T. Cohen, N. Craig, X. Lu, D. Sutherland: 2108.03240]

Reach a generic point by following a *unique* geodesic from $\vec{0}$

$$\frac{d^2\phi^\alpha_{geo}}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}(\vec{\phi}_{geo}(\lambda))\,\frac{d\phi^\beta_{geo}}{d\lambda}\,\frac{d\phi^\gamma_{geo}}{d\lambda} = 0.$$



$\vec{\phi}_{geo}(0) = 0 \qquad \vec{\phi}_{geo}(1) = \vec{\phi}.$

Normal coordinate *tangent* to geodesic:

$$\eta^a \equiv \frac{d\phi^a_{geo}}{d\lambda}(0)$$

bar means evaluate at $\vec{0}$

OR $$\boxed{\phi^a = \phi^a_{geo}(1) = \eta^a - \sum_{n=2}^{\infty} \frac{1}{n!}\,\bar{\Gamma}^a_{(b_1\ldots b_n)}\,\eta^{b_1}\cdots\eta^{b_n}}$$

Why Riemann Normal Coordinates?

1// "Manifestly Field-Space Covarient":

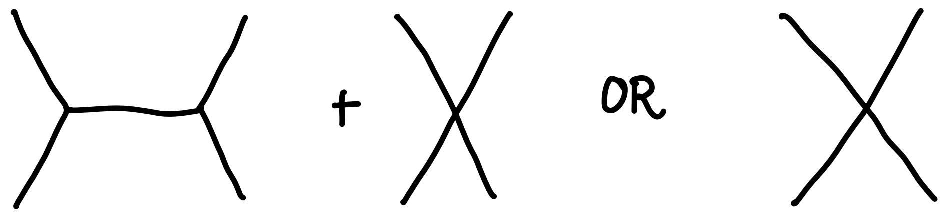
$$G_{ab}(\eta) = \delta_{ab} + \frac{1}{3}\bar{R}_{aijb}\eta^i\eta^j + \frac{1}{6}\nabla_k\bar{R}_{aijb}\eta^i\eta^j\eta^k$$

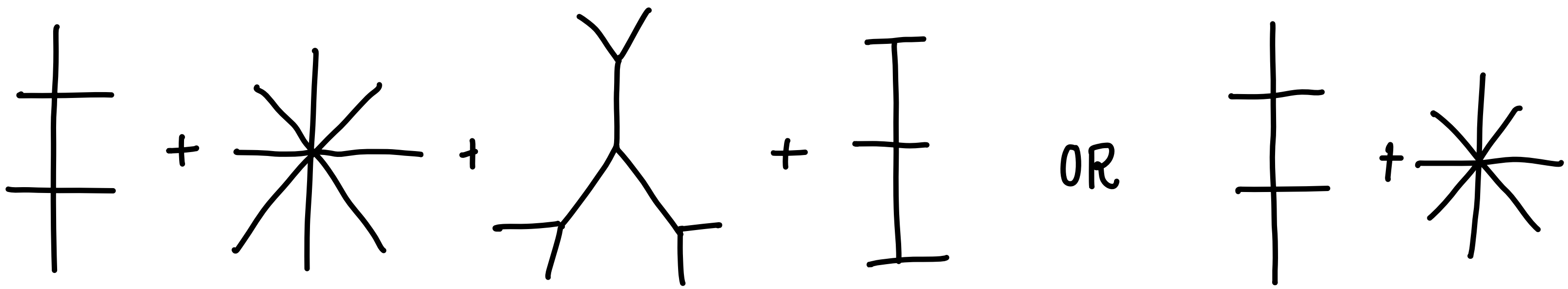
$$+ \frac{6}{5!}\left(\nabla_l\nabla_k\bar{R}_{aijb} + \frac{8}{9}\bar{R}_{aijc}\bar{R}^c{}_{klb}\right)\eta^i\eta^j\eta^k\eta^l + \ldots$$

[A. Hatzinikitas; hep-th/0001078].

$$\mathcal{L}_{kinetic} = \partial_\mu\eta^a G_{ab}(\eta)\partial^\mu\eta^b.$$

2// No 3-point vertex





3// Each Feynman graph has *distinct kinematics* with poles relating to each propagator denominator on-shell:

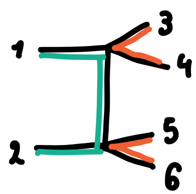
→ Vertices invariant under p_i deformations of external legs *off shell* & leaving Mandlestams unchanged.

[T. Cohen, N. Craig, X. Lu and D. Sutherland: 2108.03240].

For example:

type 1 particles

type 2 particles



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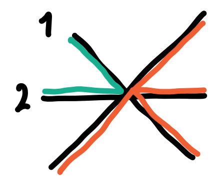

$$\frac{S_{34}S_{56}}{S_{134}} \qquad\qquad S_{12}$$

NB//
$S_{ij} = (p_i + p_j)^2$
$S_{ijk} = (p_i + p_j + p_k)^2$.

⟺ No "cancellations" between diagrams.

Generic RNC consruction method

$$\phi^a = \phi^a_{geo}(1) = \eta^a - \sum_{n=2}^{\infty} \frac{1}{n!} \overline{\Gamma}^a_{(b_1 \dots b_n)} \eta^{b_1} \dots \eta^{b_n}$$

RECURSIVE !!!

where

$$\Gamma^a_{b_1 \dots b_n c} = \partial_c \Gamma^a_{b_1 \dots b_n} - \sum_{j=1}^{n} \Gamma^\gamma_{c b_j} \Gamma^a_{b_1 \dots b_{j-1} \gamma b_{j+1} \dots b_n}$$

$$\Gamma^a_{bc}(\phi) = \frac{1}{2} G^{ai}(\phi) \left(\partial_c G_{ib}(\phi) + \partial_b G_{ic}(\phi) - \partial_i G_{bc}(\phi) \right)$$

[arXiv: 2008.08597; T. Cohen, N Craig X Lu, D. Sutherland]

Can we avoid this with Symmetry?

Metric transforms as a tensor around the symmetry point:

$$G_{ab}(\eta) = \delta_{ab} + \frac{1}{3} R_{aijb}(0)\,\eta^i \eta^j + \frac{1}{6} \nabla_k R_{aijb}(0)\,\eta^i \eta^j \eta^k + \dots$$

$$\rightarrow \delta_{cd} + \frac{1}{3} R_{aijb}(0)\, O^i{}_k O^j{}_l \,\tilde{\eta}^k \tilde{\eta}^l + \dots$$

$$= O_a{}^c \delta_{cd} O^{Td}{}_b + \frac{1}{3} R_{eijf}(0) (OO^T)^e_a (OO^T)^f_b \, O^i{}_k O^j{}_l \,\tilde{\eta}^k \tilde{\eta}^l$$

$$= O_a{}^c \left[\delta_{cd} + \frac{1}{3} \underbrace{\boxed{R_{eijf}(0)\; O^{T}{}_c{}^e O^f{}_d O^i{}_k O^j{}_l}}_{\text{Exactly an } O(4) \text{ isotropic tensor}} \tilde{\eta}^k \tilde{\eta}^l + \dots \right] O^{Td}{}_b$$

Isotropic Tensor Must be constructed from:

δ_{ij} AND ϵ_{ijk}

Kronecker Delta   Levi-Civita

[P.G Appleby, B.R. Duffy, R.W. Ogden: Glasgow Mathematical Journal 29(2) 185-196 (1987)]

Riemann Tensor at Fixed Point

$$\bar{R}_{a(ij)b} \overset{!}{=} \alpha \delta_{ai}\delta_{bj} + \beta \delta_{aj}\delta_{ib} + \gamma \delta_{ab}\delta_{ij}$$ ij symmerised

1// Skew Symmerry $R_{(ai)jb} = R_{ai(jb)} = 0 \Rightarrow \alpha = 0.$

2// Interchange Symmerry $R_{aijb} = R_{jbai} \Rightarrow \beta = -\gamma.$

$$\bar{R}_{a(ij)b} = \beta(\delta_{aj}\delta_{ib} - \delta_{ab}\delta_{ij})$$ ij symmerised

Riemann Tensor at Fixed Point

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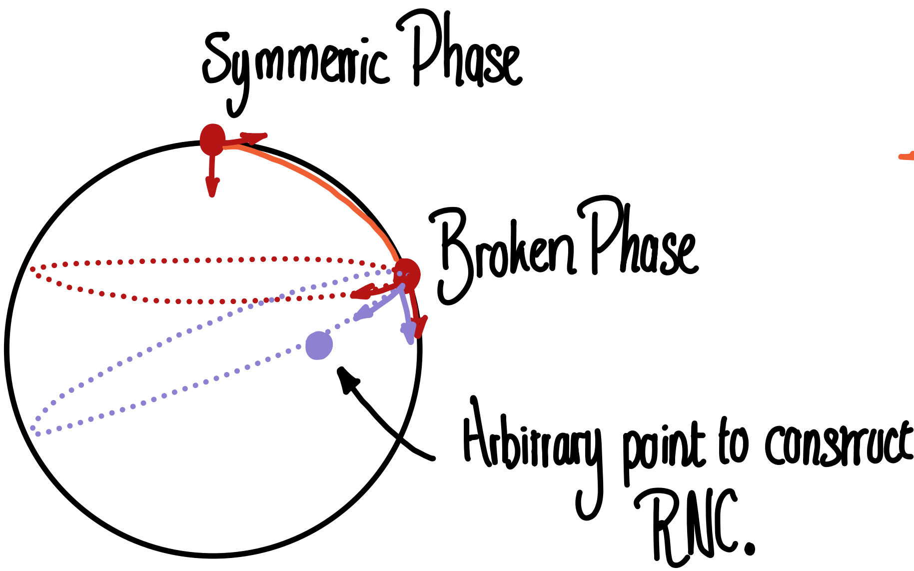
Symmetric Point Summary

$$G_{ab}(\eta) = \delta_{ab} + \frac{1}{3}\bar{R}_{aijb}\,\eta^i\eta^j + \frac{1}{6}\nabla_k\bar{R}_{aijb}\,\eta^i\eta^j\eta^k$$

$$+ \frac{6}{5!}\left(\nabla_l\nabla_k\bar{R}_{aijb} + \frac{8}{9}\bar{R}_{aijc}\bar{R}^c{}_{klb}\right)\eta^i\eta^j\eta^k\eta^l + \ldots$$

$$= \delta_{ab} + \sum_{n=2}^{\infty}\frac{C_{(n-1)/2}}{\Lambda^{(n-2)}}\tilde{R}_{a i_1 i_2 b}\,\delta_{i_3 i_4}\ldots\delta_{i_{(n-1)} i_n}\,\eta^{i_1}\ldots\eta^{i_n}.$$

Breaking the Symmetry (in Progress...)



— Parallel Transport the RNC to the vacuum

→ *The vector looks exactly the same*

— Re-establish the normal coordinate

→ *More Christoffels*

RNC in the Literature:

- Covariant effective actions [R. Alonso and MW: 2207.02050].

  Renormalising field-space geometry [P. Aigner, L. Bellafronte, E. Gendy, D. Haslehner and A. Weiler: 2503.09785]

  [B. Assi, A. Helset, A. Manohar, J. Pagés C-H. Shen: 2303.03187]

- Unitarity Bounds in HEFT [T. Cohen, N. Craig, X. Lu and D. Sutherland: 2108.03240].
- Geomeric soft theorems [C. Cheung, A. Helset and J. Parra-Martinez: 2111.03045].

Summary & Future Prospects

1// Geometry is a neat way to encapsulate field-redefinition invariance ON-SHELL.

2// Riemann normal coordinates motivated as "good" choice of field-space basis.

3// Riemann normal coordinates easier with symmetry! Avoiding the Christoffels...

In the future this maths could be used elsewhere:

a// Gauge symmetries?

b// Other groups may have similar trick?

Summary & Future Prospects

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Thank You!

Derivative Field Redefinitions

Jet Bundle Geometry [M. Alminawi, I. Brivio and J. Davighi; 2308.00017]

Geometry-Kinematics Duality [C. Cheung, A. Helset and J. Parra-Martinez; 2202.06972]

Functional Geometry [T. Cohen, X. Lu and Z. Zhang; 2410.21378]

- Non-local field redefinitions

[T. Cohen, M. Forslund, A. Helset; 2412.12247]

Equations of Motion:

The classical equation of motion $E[\phi] \equiv \frac{\delta S}{\delta \phi}$

eg $\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!}\lambda \phi^4 \Rightarrow E[\phi] = -\partial^2 \phi(x) - m^2 \phi(x) - \frac{1}{3!}\lambda \phi^3(x)$

consider a field redefinition with $\phi = \phi' + \epsilon F[\phi']$

$$\mathcal{L}[\phi] = \mathcal{L}[\phi'] + \epsilon F[\phi'] \frac{\delta S[\phi']}{\delta \phi} + \mathcal{O}(\epsilon^2)$$

$$= \mathcal{L}[\phi'] + \epsilon \,\Theta[\phi'] + \mathcal{O}(\epsilon^2)$$

where $\Theta[\phi] = F[\phi]\, E[\phi] = F[\phi] \frac{\delta S}{\delta \phi}$

ie a special case of a field redefinition [A. Manohar; hep-ph/1804.05863]

RNC in the broken phase...

$$\Gamma^h_{ij} = -v^2 F(h) F'(h)\, g_{ij}(\varphi) \qquad \Gamma^i_{hj} = \frac{F'(h)}{F(h)}\, \delta^i_j$$

$$\Gamma^i_{jk} = n_i\, g_{jk}(\varphi)$$

eg

$$\Gamma^h_{ijhk} = \partial_k \Gamma^h_{ijh} - \Gamma^a_{ki}\Gamma^h_{ajh} - \Gamma^a_{kj}\Gamma^h_{iah} - \Gamma^a_{kh}\Gamma^h_{ija}$$

$$\begin{aligned}
&= \partial_k \left[\partial_h \Gamma^h_{ij} - \Gamma^b_{hi}\Gamma^h_{bj} - \Gamma^b_{hj}\Gamma^h_{ib} \right] \\
&\quad - \Gamma^a_{ki}\left[\partial_h \Gamma^h_{aj} - \Gamma^b_{ha}\Gamma^h_{bj} - \Gamma^b_{hj}\Gamma^h_{ab} \right] \\
&\quad - \Gamma^a_{kj}\left[\partial_h \Gamma^h_{ia} - \Gamma^b_{hi}\Gamma^h_{ba} - \Gamma^b_{ha}\Gamma^h_{ib} \right] \\
&\quad - \Gamma^a_{kh}\left[\partial_a \Gamma^h_{ij} - \Gamma^b_{ai}\Gamma^h_{bj} - \Gamma^b_{aj}\Gamma^h_{ib} \right]
\end{aligned}$$

[arXiv: 2008.08597; T. Cohen, N Craig, X Lu, D. Sutherland]

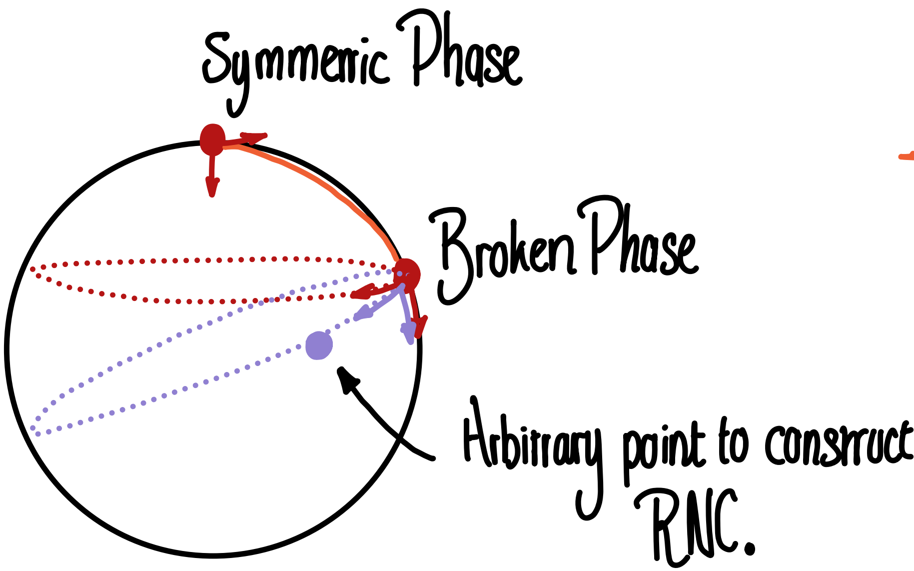
$$= -v^2\left(F'(h)^2 + F''(h)F(h)\right)\partial_k g_{ij}(\varphi) + F'(h)^2\left(\delta^l_i\, g_{lj}(\varphi) + \delta^l_j\, g_{il}(\varphi)\right)$$

$$- \varphi_l\, g_{ki}(\varphi)\left[-v^2\left(F'(h)^2 + F''(h)F(h)\right) g_{lk}(\varphi) - F'(h)^2\left(\delta^m_l\, g_{mj}(\varphi) + \delta^m_j\, g_{ml}(\varphi)\right)\right]$$

$$+ \ldots$$

Breaking the Symmerry

Breaking the Symmetry (in Progress...)



— Parallel Transport the RNC to the vacuum

→ The vector looks exactly the same

— Re-establish the normal coordinate

→ More Christoffels

The 'Wrong' way to do this: → loose all orders result!

① CANONICALISE:

Break the symmetry

$$\partial_\mu \eta^A G_{AB}(\eta)\, \partial^\mu \eta^B \to \partial_\mu \eta^A G_{AB}(\tilde{\eta}+v)\, \partial^\mu \eta^B$$

$$= \partial_\mu \tilde{\eta}^a e_a^A(v)\, G_{AB}(\tilde{\eta}+v)\, e_b^B(v)\, \partial^\mu \tilde{\eta}^b$$

vielbeins

where $\delta_{ab} = e_a^A(v)\, G_{AB}(v)\, e_b^B(v)$ does the canonicalisation.

[C. Cheung, A. Helset and J. Parra-Martinez: 2111.03045].

② Re-Establish The RNC

$$v^i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix}$$ for example!

$$\tilde{\eta}^a = \sigma^a + \sum_{i=2}^{\infty} \Gamma^a_{b_1 \dots b_n}\Big|_{\text{vacuum}} \sigma^{b_1} \dots \sigma^{b_n}$$

and

$$\Gamma^a_{b_1 \dots b_n}\Big|_{\text{vacuum}} = \Gamma^a_{b_1 \dots b_n}\Big|_{\text{restored}}$$

$$+ \Gamma^a_{b_1 \dots b_n, b_{n+1}}\Big|_{\text{restored}} v^{b_{n+1}}$$

$$+ \Gamma^a_{b_1 \dots b_n, b_{n+1} b_{n+2}}\Big|_{\text{restored}} v^{b_{n+1}} v^{b_{n+2}} \times \frac{1}{2}$$

$$+ \dots$$

$$G_{ab}(\eta) = \delta_{ab} + \frac{1}{3} c_1 \tilde{R}_{aijb}\, \eta^i \eta^j + \dots$$

$$\to \left(\frac{\partial \tilde{\eta}^c}{\partial \sigma^a}\right) e^C_c(v)\, G_{CD}(\tilde{\eta}+v)\, e^D_d(v) \left(\frac{\partial \tilde{\eta}^d}{\partial \sigma^b}\right)$$

$$= \Bigg[\delta^c_a - \left(\cancelto{0}{\tilde{\Gamma}^c_{(i_1 i_2)}}\, \sigma^{i_1} \delta^{i_2}_a + \frac{1}{2} \tilde{\Gamma}^c_{(i_1 i_2),v}\, \sigma^{i_1} \delta^{i_2}_a + \frac{1}{6} \tilde{\Gamma}^c_{(i_1 i_2),vv}\, \sigma^{i_1} \delta^{i_2}_a + \dots \right)$$
$$- \frac{1}{2} \left(\cancelto{0}{\tilde{\Gamma}^c_{(i_1 i_2 i_3)}}\, \sigma^{i_1} \sigma^{i_2} \delta^{i_3}_a + \frac{1}{2} \tilde{\Gamma}^c_{(i_1 i_2 i_3),v}\, \sigma^{i_1} \sigma^{i_2} \delta^{i_3}_a + \dots \right) + \dots \Bigg]$$

$$\times e^C_c(v) \left[\delta_{CD} + \frac{1}{3} c_1 \tilde{R}_{CijD} \left(v^i + \sigma^i - \frac{1}{2} \tilde{\Gamma}^i_{i_1 i_2, v}\, \sigma^{i_1} \sigma^{i_2} - \dots \right) \left(v^j + \sigma^j - \frac{1}{2} \tilde{\Gamma}^j_{j_1 j_2, v}\, \sigma^{j_1} \sigma^{j_2} + \dots \right) + \dots \right] e^D_d(v)$$

$$\times \Bigg[\delta^d_b - \left(\cancelto{0}{\tilde{\Gamma}^d_{(i_1 i_2)}}\, \sigma^{i_1} \delta^{i_2}_b + \frac{1}{2} \tilde{\Gamma}^d_{(i_1 i_2),v}\, \sigma^{i_1} \delta^{i_2}_b + \frac{1}{6} \tilde{\Gamma}^d_{(i_1 i_2),vv}\, \sigma^{i_1} \delta^{i_2}_b + \dots \right)$$
$$- \frac{1}{2} \left(\cancelto{0}{\tilde{\Gamma}^d_{(i_1 i_2 i_3)}}\, \sigma^{i_1} \sigma^{i_2} \delta^{i_3}_b + \frac{1}{2} \tilde{\Gamma}^d_{(i_1 i_2 i_3),v}\, \sigma^{i_1} \sigma^{i_2} \delta^{i_3}_b + \dots \right) + \dots \Bigg]$$

Which is Awful...

Below is a working idea to avoid this.

Fortunately we already know the answer:

$$G_{ab} = \delta_{ab} + \frac{1}{3}\bar{R}_{aijb}\,\sigma^i\sigma^j + \frac{1}{6}\nabla_k\bar{R}_{aijb}\,\sigma^k\sigma^i\sigma^j + \dots$$

new RNC at broken phase

So we can construct the new metric via a Taylor series:

$$\left.\frac{\partial^n G_{ab}}{\partial\sigma^{i_1}\dots\partial\sigma^{i_n}}\right|_{\text{Broken}} = e^{I_1}_{\ i_1}(\eta)\dots e^{I_n}_{\ i_n}(\eta)\,e^{A}_{\ a}(\eta)\,e^{B}_{\ b}(\eta)\left.\underbrace{\frac{\partial^n G_{AB}(\eta)}{\partial\eta^{I_1}\dots\partial\eta^{I_n}}}_{\text{We don't know this @ broken so take a Taylor expansion.}}\right|_{\text{Broken}}$$

$$= e^{i_1}_{\ I_1}(\eta)\dots e^{i_n}_{\ I_n}(\eta)\,e^{A}_{\ a}(\eta)\,e^{B}_{\ b}(\eta)\left[\frac{\partial^n G_{AB}(0)}{\partial\eta^{I_1}\dots\partial\eta^{I_n}} + \frac{\partial^{n+1} G_{AB}(0)}{\partial\eta^{I_1}\dots\partial\eta^{I_{n+1}}}\,\eta^{I_{n+1}} + \dots\right]_{\text{Broken}}$$

$$= e^{i_1}_{I_1}(\eta)\dots e^{i_n}_{I_n}(\eta)\, e^{A}_{a}(\eta) e^{B}_{b}(\eta) \left[\frac{\partial^n G_{AB}(0)}{\partial\eta^{I_1}\dots\partial\eta^{I_n}} + \frac{\partial^{n+1} G_{AB}(0)}{\partial\eta^{I_1}\dots\partial\eta^{I_{n+1}}}\, \phi^{I_{n+1}} + \dots \right]_{\text{Broken}}$$

$$= e^{i_1}_{I_1}(\eta)\dots e^{i_n}_{I_n}(\eta)\, e^{A}_{a}(\eta) e^{B}_{b}(\eta) \left[\delta_{I_1 I_2}\dots\delta_{I_{n-4} I_{n-3}}\, \tilde{R}_{A I_{n-1} I_n B} + \dots \right]_{\text{Broken}}$$

$$e^{I_1}_{i_1}(\eta)\, \delta_{I_1 I_2}\, e^{I_2}_{i_2}(\eta) \Big|_{\text{Broken}} \qquad \eta \to v \text{ and } \tilde{\eta} \to 0$$

$$= e^{I_1}_{i_1}(\eta)\, G_{I_1 I_2}(\tilde{\eta})\, e^{I_2}_{i_2}(\eta) \Big|_{\text{Broken}}$$

$$= e^{\tilde{I}_1}_{i_1}(\tilde{\eta})\, \frac{\partial\eta^{I_1}}{\partial\tilde{\eta}^{\tilde{I}_1}}\, G_{I_1 I_2}(\tilde{\eta})\, \frac{\partial\eta^{I_2}}{\partial\tilde{\eta}^{\tilde{I}_2}}\, e^{\tilde{I}_2}_{i_2}(\tilde{\eta}) \Big|_{\text{Broken}}$$

$$= e^{\tilde{I}_1}_{i_1}(\tilde{\eta})\, G_{\tilde{I}_1 \tilde{I}_2}(\eta)\, e^{\tilde{I}_2}_{i_2}(\tilde{\eta}) \Big|_{\text{Broken}} = \delta^{I_1}_{i_1}\, G_{I_1 I_2}(v)\, \delta^{I_2}_{i_2}.$$

So n-point Feynman rule is:

$$\frac{\partial^n G_{ab}}{\partial\sigma^{i_1}\cdots\partial\sigma^{i_n}}\Bigg|_{\text{Broken}} =$$

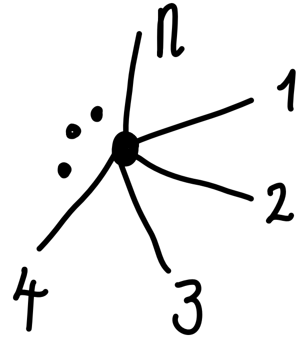
$$\frac{C_{(n-2)/2}}{\Lambda^{n-2}} G_{i_1 i_2}(v)\, G_{i_3 i_4}\cdots G_{i_{n-3}\, i_{n-2}}\Big(G_{a\, i_{n-1}}(v)\, G_{i_n b}(v) - G_{ab}(v)\, G_{i_{n-1} i_n}(v)\Big)$$

$$+ \frac{C_{n/2}}{\Lambda^{n}} G_{i_1 i_2}(v)\, G_{i_3 i_4}\cdots G_{i_{n-1}\, i_n}\Big(G_{a\, i_{n+1}}(v)\, G_{i_{n+2} b}(v) - G_{ab}(v)\, G_{i_{n+1} i_{n+2}}(v)\Big) \times v^{i_{n+1}} v^{i_{n+2}}$$

$$+ \cdots$$

where $v^i = \begin{pmatrix} 0 \\ \vdots \\ v \end{pmatrix}$

Relation to "Warsaw" / Generic Basis.



$$V^{RNC}_{n\text{-point}} = \frac{-i}{(n-2)!} \sum_{\substack{[1,\ldots,n] \\ \ni [a_1,\ldots,a_n]}} \delta_{\alpha_{a_1}\alpha_{a_2}} \delta_{\alpha_{a_3}\alpha_{a_4}} \ldots \delta_{\alpha_{a_{n-1}}\alpha_{a_n}} \frac{c_{(n-1)/2}}{\Lambda^{(n-2)}}$$

$$\times \left[2\left(p_{a_1}\cdot p_{a_2} + p_{a_3}\cdot p_{a_4} + \ldots + p_{a_{n-1}}\cdot p_{a_n}\right) + \left(p_1^2 + p_2^2 + \ldots + p_n^2\right) \right]$$

Matching to Generic Basis:

Via Matching Kinetics:

$$V^{\text{Generic}}_{n\text{-point}} = \frac{-i}{(n-2)!} \sum_{\substack{[1,\dots,n] \\ \ni [a_1,\dots,a_n]}} \delta_{\alpha_{a_1}\alpha_{a_2}} \delta_{\alpha_{a_3}\alpha_{a_4}} \dots \delta_{\alpha_{a_{n-1}}\alpha_{a_n}} \Bigg[\frac{1}{\Lambda^{(n-2)}}$$

$$a_{(n-2)/2} - b_{(n-2)/2} \Big[2(p_{a_1}\cdot p_{a_2} + p_{a_3}\cdot p_{a_4} + \dots + p_{a_{n-1}}\cdot p_{a_n}) + (p_1^2 + p_2^2 + \dots + p_n^2) \Big]$$

$$- a_{(n-2)/2} + b_{(n-2)/2} \Big[(p_1^2 + \dots + p_n^2) \Big] \Bigg]$$

$$V^{\text{RNC}}_{n\text{-point}} = \frac{-i}{(n-2)!} \sum_{\substack{[1,\dots,n] \\ \ni [a_1,\dots,a_n]}} \delta_{\alpha_{a_1}\alpha_{a_2}} \delta_{\alpha_{a_3}\alpha_{a_4}} \dots \delta_{\alpha_{a_{n-1}}\alpha_{a_n}} \frac{c_{(n-1)/2}}{\Lambda^{(n-2)}}$$

$$\times \Big[2(p_{a_1}\cdot p_{a_2} + p_{a_3}\cdot p_{a_4} + \dots + p_{a_{n-1}}\cdot p_{a_n}) + (p_1^2 + p_2^2 + \dots + p_n^2) \Big]$$

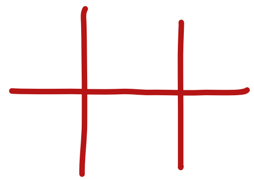
$$= 2(p_{a_1}\cdot p_{a_2} + p_{a_3}\cdot p_{a_4} + \ldots + p_{a_{n-1}}\cdot p_{a_n}) + (p_1^2 + p_2^2 + \ldots + p_n^2)$$

$$= p_1^2 + p_2^2 + \ldots + p_n^2$$

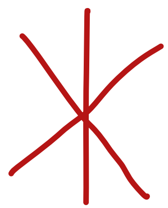
4pt: $c_1 \quad = (a_1 - b_1) \quad + (a_1 + b_1) \quad = 0.$

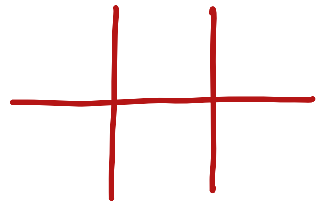
$$c_1 = (a_1 - b_1)$$

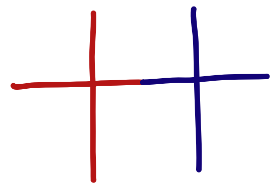
6pt:

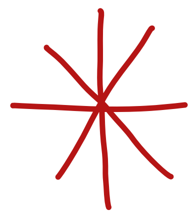
c_1^2 

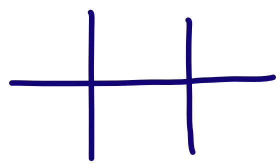
$+$

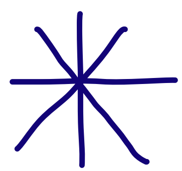
c_2 

$=$ 

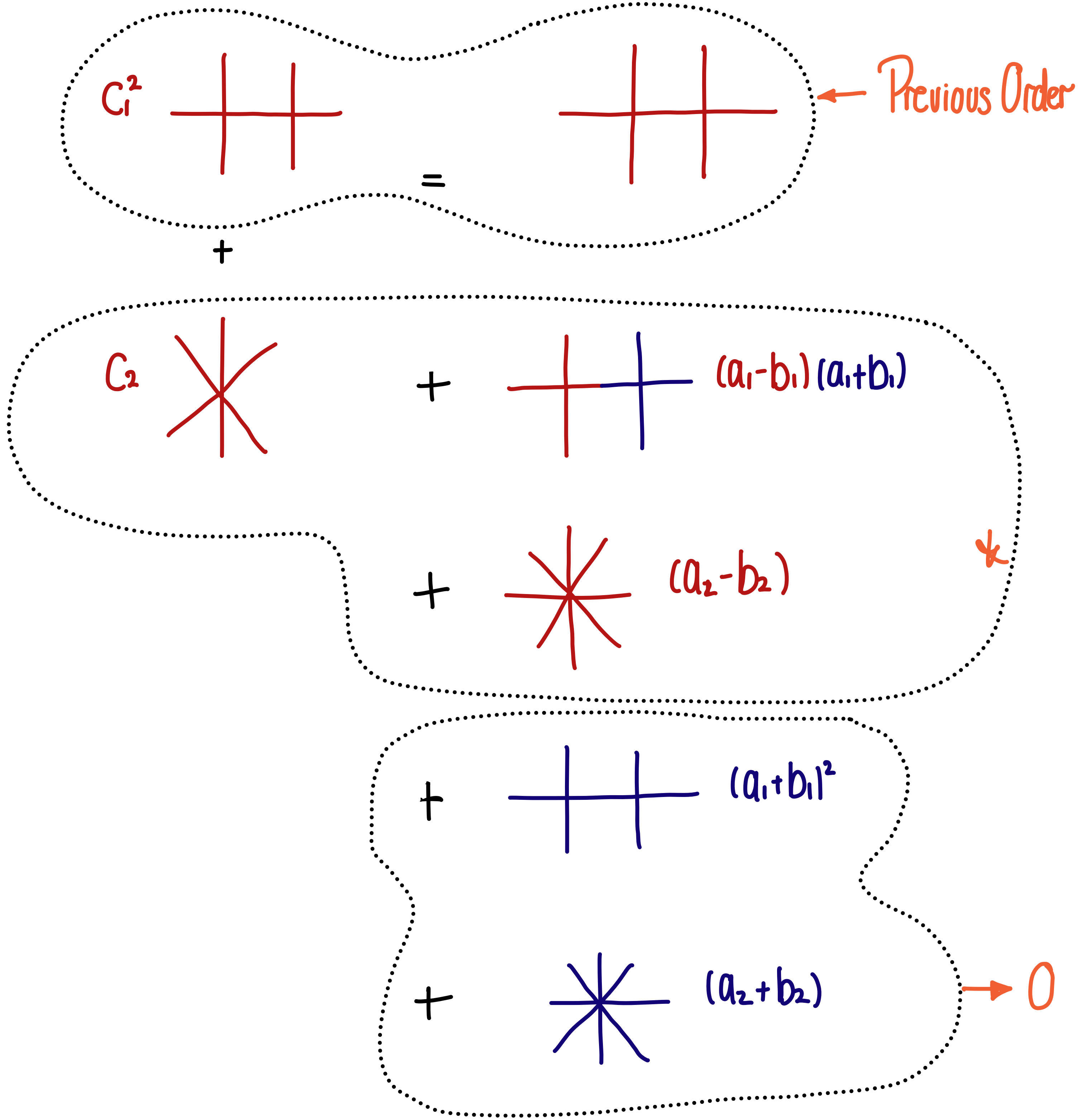
$+$  $(a_1-b_1)(a_1+b_1)$

$+$  (a_2-b_2)

$+$  $(a_1+b_1)^2$

$+$  (a_2+b_2)

6pt:



Symmetry point Riemann Tensor Rules

Odd Derivatives

eg $\nabla_{i_1} \bar{R}_{a i_2 i_3 b} \phi^{i_1} \phi^{i_2} \phi^{i_3}$

$\delta_{\cdot\cdot}$

even # of entries

Great but even × odd = even

$\epsilon_{\cdot\cdot\cdot}$

odd # of entries

$\epsilon_{ab i_1} = 0$ AND $\epsilon_{a i_1 i_2} = 0$

Nothing we can write down.

Odd Derivatives

eg $\nabla_{i_1} \bar{R}_{a i_2 i_3 b}\, \phi^{i_1} \phi^{i_2} \phi^{i_3}$

$\delta_{\cdot\cdot}$

even # of entries

Great but even × odd = even

$\epsilon_{\cdot\cdot\cdot}$

odd # of entries

$\epsilon_{a b i_1} = 0$ AND $\epsilon_{a i_1 i_2} = 0$

Nothing we can write down.

$\Rightarrow$ All $\nabla_{i_1} \ldots \nabla_{i_n} \bar{R}_{a\, i_{n+1}\, i_{n+2}\, b} = 0$ for odd n

$i_1 \ldots i_{n+2}$ Symmetrised

Even Derivatives

$i_1 \dots i_{n+2}$ Symmetrised

$$\nabla_{i_1}\nabla_{i_2}\dots\nabla_{i_n} R_{a\, i_{n+1}\, i_{n+2}}\Big|_{\phi=0} = \delta_{i_1 i_2}\,\delta_{i_3 i_4}\dots\delta_{i_{n-1} i_n}\,\tilde{R}_{a\, i_{n+1}\, i_{n+2}\, b}$$

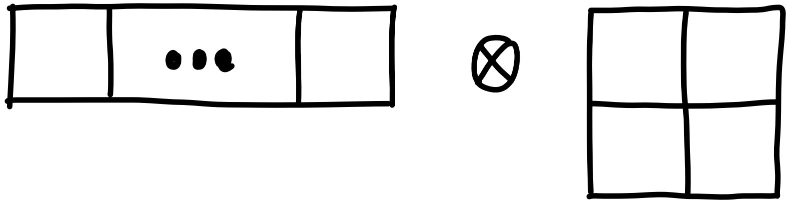
Proof

We want to be able to write LHS as

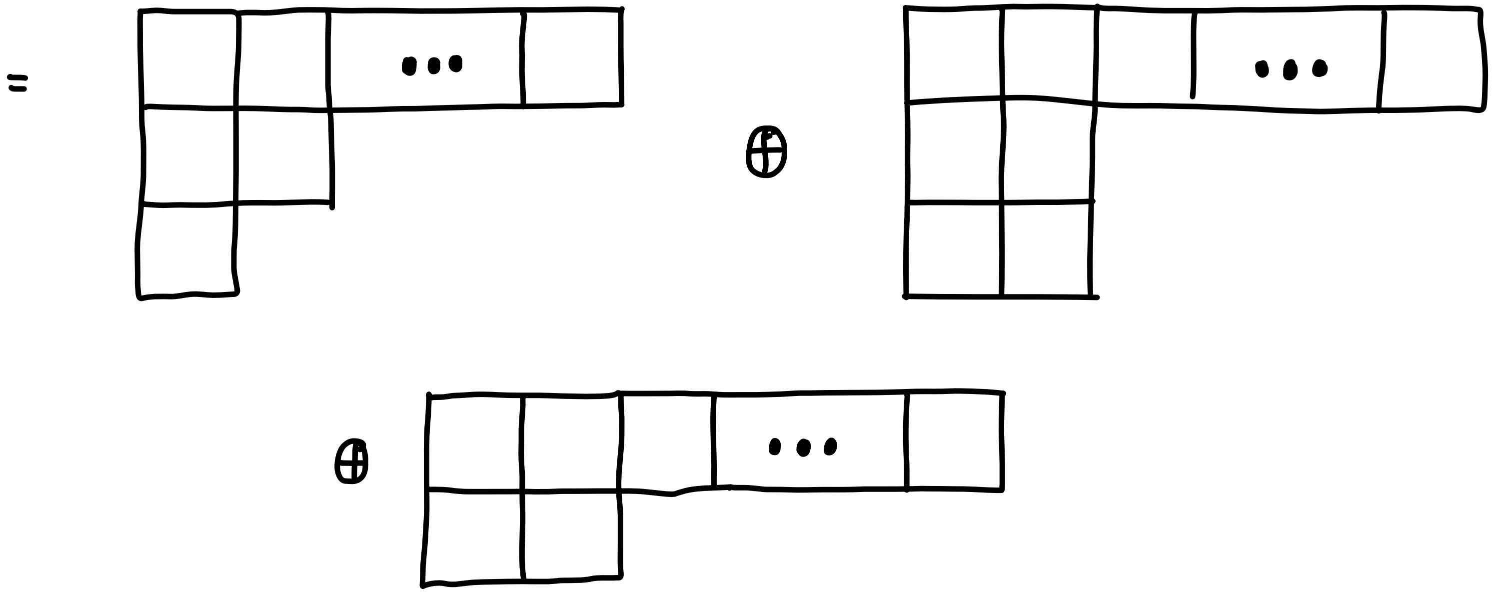
totally symmetric tensor

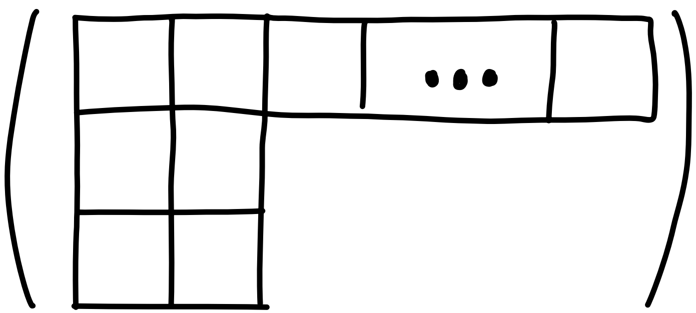
$$(S\otimes\tilde{R})_{i_1\dots i_n\, a\, i_{n+1}\, i_{n+2}\, b} \equiv S_{i_1\dots i_n}\,\tilde{R}_{a\, i_{n+1}\, i_{n+2}\, b}$$

which you can see by expressing ito the Young Tableau

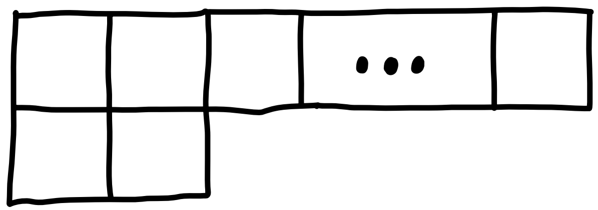


[H. Georgi; Front Phys 54 (1999) 1-320]



Then  $\eta^{i_1}\cdots\eta^{i_n} = 0$

etc.

$\Bigg($  $\Bigg)\, \eta^{i_1} \dots \eta^{i_n} \neq 0$ and so

$$\nabla_{i_1}\nabla_{i_2}\dots\nabla_{i_n} R_{a i_{n+1} i_{n+2}}\Big|_{\phi=0}\eta^{i_1}\dots\eta^{i_{n+2}} = (S\otimes\bar{R})_{i_1\dots i_n a i_{n+1} i_{n+2} b}\,\eta^{i_1}\dots\eta^{i_{n+2}}$$

$$= S_{i_1\dots i_n}\bar{R}_{a i_{n+1} i_{n+2} b}\,\eta^{i_1}\dots\eta^{i_{n+2}}$$

Isotropic properties

$$= \delta_{i_1 i_2}\delta_{i_3 i_4}\dots\delta_{i_{n-1} i_n}\tilde{R}_{a i_{n+1} i_{n+2} b}\,\eta^{i_1}\dots\eta^{i_{n+2}}$$

as we wanted!

"RR-reduction"

$$\tilde{R}_{aijc}\tilde{R}^{c}{}_{klb}\,\eta^i\eta^j\eta^k\eta^l = \left[(\delta_{aj}\delta_{ic}-\delta_{ij}\delta_{ac})(\delta^c{}_l\delta_{kb}-\delta^c{}_b\delta_{kl})\right]\eta^i\eta^j\eta^k\eta^l$$

$$= \Big[\underbrace{\delta_{aj}[\delta_{il}\delta_{kb}-\delta_{ib}\delta_{kl}]}_{\text{0 from symmerisation}} - \delta_{ij}[\delta_{al}\delta_{kb}-\delta_{ab}\delta_{kl}]\Big]\eta^i\eta^j\eta^k\eta^l$$

$$= -\delta_{ij}[\delta_{al}\delta_{kb}-\delta_{ab}\delta_{kl}]\,\eta^i\eta^j\eta^k\eta^l$$

$$= -\delta_{ij}\tilde{R}_{aklb}\,\eta^i\eta^j\eta^k\eta^l$$

Symmetric Point Metric

- $$\nabla_{i_1}\ldots\nabla_{i_n}\bar{R}_{a\,i_{n+1}\,i_{n+2}\,b}\,\eta^{i_1}\ldots\eta^{i_{n+2}} = \begin{cases} 0 & \text{for odd } n \\ & \text{for even } n \\ \delta_{i_1 i_2}\ldots\delta_{i_{n-1} i_n}\tilde{R}_{a\,i_{n+1}\,i_{n+2}\,b}\,\eta^{i_1}\ldots\eta^{i_{n+2}} & \end{cases}$$

- $$\nabla_{i_1}\ldots\nabla_{i_n}\bar{R}_{a\,i_{n+1}\,i_{n+2}\,b}\,\nabla_{i_{n+3}}\ldots\nabla_{i_m}\bar{R}^{b}{}_{i_{m+1}\,i_{m+2}\,c}\,\eta^{i_1}\ldots\eta^{i_{m+n+4}}$$
$$= \delta_{i_1 i_2}\ldots\delta_{i_{m+n+1}\,i_{m+n+2}}\tilde{R}_{a\,i_{m+n+3}\,i_{m+n+4}\,b}\,\eta^{i_1}\ldots\eta^{i_{m+n+4}}$$