Dispersive approach to isospin breaking in $\tau^- \to \pi^-\pi^0\nu_\tau$

in collaboration with M. Cottini, G. Colangelo and M. Hoferichter

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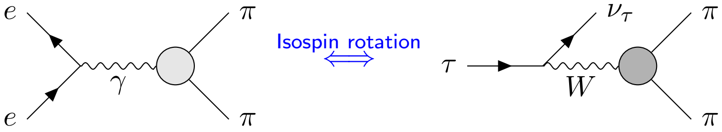
Breaking of the CVC relation

Make use of conserved-vector-current relation:



Alemany, Davier, Höcker 1997

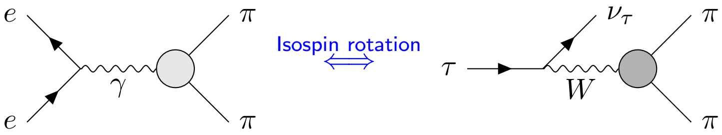
$$\sigma^{(0)}_{e^+e^-\to\pi^+\pi^-}(s) = \frac{K_\sigma(s)}{K_\Gamma(s)} \frac{\mathrm{d}\Gamma_{\tau\to\pi\pi\nu_\tau}}{\mathrm{d}s}$$

- $K_\sigma(s) = \pi\alpha^2/3s$, $K_\Gamma(s)$: decay couplings & τ phase space

$$a_\mu^{\mathrm{HVP,LO}}[\pi\pi,\tau] = \frac{1}{4\pi^3}\int_{4M_\pi^2}^{m_\tau^2} \mathrm{d}s\, K(s)\left[\frac{K_\sigma(s)}{K_\Gamma(s)} \frac{\mathrm{d}\Gamma_{\tau\to\pi\pi\nu_\tau}}{\mathrm{d}s}\right]$$

Breaking of the CVC relation

Make use of conserved-vector-current relation:



Alemany, Davier, Höcker 1997

with isospin breaking:

Cirigliano, Ecker, Neufeld 2001/2

$$\sigma^{(0)}_{e^+e^-\to\pi^+\pi^-}(s) = \frac{K_\sigma(s)}{K_\Gamma(s)} \frac{\mathrm{d}\Gamma_{\tau\to\pi\pi\nu_\tau[\gamma]}}{\mathrm{d}s} \times \frac{R_{\mathrm{IB}}(s)}{S_{\mathrm{EW}}^{\pi\pi}}$$

- $K_\sigma(s) = \pi\alpha^2/3s$, $K_\Gamma(s)$: decay couplings & τ phase space

$$a_\mu^{\mathrm{HVP,LO}}[\pi\pi,\tau] = \frac{1}{4\pi^3}\int_{4M_\pi^2}^{m_\tau^2} \mathrm{d}s\, K(s) \left[\frac{K_\sigma(s)}{K_\Gamma(s)} \frac{\mathrm{d}\Gamma_{\tau\to\pi\pi\nu_\tau[\gamma]}}{\mathrm{d}s}\right] \times \frac{R_{\mathrm{IB}}(s)}{S_{\mathrm{EW}}^{\pi\pi}}$$

IB corrections

- collected in

$$R_{\mathrm{IB}}(s) = \mathrm{FSR}(s) \frac{\beta^3_{\pi^+\pi^-}(s)}{\beta^3_{\pi^\pm\pi^0}(s)} \frac{|F^V_\pi(s)|^2}{|f_+(s)|^2} \frac{1}{G_{\mathrm{EM}}(s)}$$

- and $S^{\pi\pi}_{\mathrm{EW}}$: short-distance EM correction

focus on

- $G_{\mathrm{EM}}(s)$: long-distance EM corrections

▲ only combination of $S^{\pi\pi}_{\mathrm{EW}}$ and $G_{\mathrm{EM}}(s)$ is scheme independent

- $G_{\mathrm{EM}}(s)$ previously calculated in resonance chiral theory:

Cirigliano, Ecker, Neufeld 2001/2

Flores-Báez, Flores-Tlalpa, López Castro, Toledo Sánchez 2005/6

Davier et al. 2009-23

Miranda, Roig 2020; López Castro, Miranda, Roig 2024

→ need to care for effects of structure-dependent virtual photons

→ solid estimates of uncertainties

Outline of the calculation

- Structure-dependent virtual corrections $\rightarrow$ dispersive representation of f_+ to resolve pion-pole effects in box diagrams
- Matching to ChPT $\rightarrow$ make connection to short-distance physics contained in S_{EW}
- Threshold behavior $\rightarrow$ derive expansion of real and virtual contributions that allows stable evaluation down to threshold
- Fits to τ data using dispersive representation of f_+, including Roy-equation constraints and inelastic effects via ρ', ρ''
- Self-consistent iteration of f_+ in G_{EM}

More on G_{EM}

- for $\tau^-(l_1) \to \pi^-(q_1)\pi^0(q_2)\nu_\tau(l_2)[\gamma(k)]$:

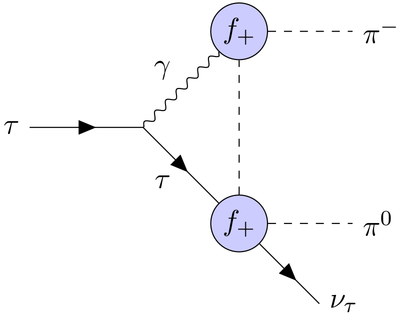
$$\frac{\mathrm{d}\Gamma_{\tau\to\pi\pi\nu_\tau[\gamma]}}{\mathrm{d}s} = S_{\mathrm{EW}}^{\pi\pi}\ K_\Gamma(s)\ \beta^3_{\pi^\pm\pi^0}(s)\ \big|f_+(s)\big|^2\ G_{\mathrm{EM}}(s)$$

$$G_{\mathrm{EM}}(s) = \frac{\int_{t_{\min}(s)}^{t_{\max}(s)} \mathrm{d}t\, D(s,t)\Big[1 + 2 f_{\mathrm{elm}}^{\mathrm{loop}}(s,t) + g_{\mathrm{rad}}(s,t)\Big]}{\int_{t_{\min}(s)}^{t_{\max}(s)} \mathrm{d}t\, D(s,t)}$$

- with Mandelstam $s = (q_1+q_2)^2$, $t = (l_1 - q_1)^2$ and tree-level kinematic function $D(s,t)$
- $G_{\mathrm{EM}} = 1 + \mathcal{O}(\alpha)$
- virtual photon contributions: $f_{\mathrm{elm}}^{\mathrm{loop}}(s,t)$
- real photon contributions: $g_{\mathrm{rad}}(s,t)$

Dispersive approach to virtual contributions – I

- sometimes called "FsQED"
- approximation: intermediate hadronic states up to two pions

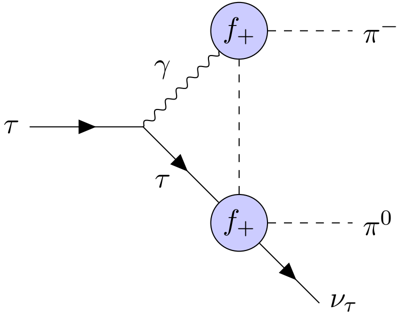


- for technical reasons: make use of unsubtracted dispersion relation

$$f_+(p^2) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}x \, \frac{\mathrm{Im}\, f_+(x)}{x - p^2 - i\epsilon}$$

- resulting amplitude $f_{\mathrm{elm}}^{\mathrm{disp}}(s,t)$: UV finite, IR divergent

Dispersive approach to virtual contributions – II



- amplitude:

$$f_{\mathrm{elm}}^{\mathrm{disp}}(s,t) = \alpha \int_{4M_\pi^2}^{\infty} \mathrm{d}s'\,\mathrm{d}s''\;\mathrm{Im}\,f_+(s')\;\mathrm{Im}\,f_+(s'') \sum_{k\in\{B,C,D\}} \mathcal{M}_k(s,t,s',s'')$$

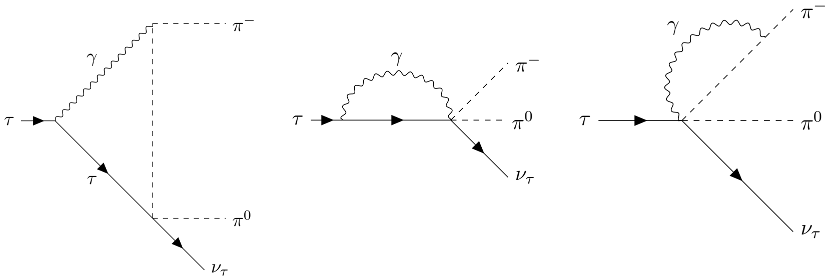
- $\mathcal{M}_k(s,t,s',s'')$: collection of Passarino–Veltman B_0, C_0, D_0 and kinematics

Matching to Chiral Perturbation Theory

- framework: ChPT with virtual photons [Urech 1995] and with virtual leptons [Knecht et al. 1999];
  amplitude $f_{\rm elm}^{\rm ChPT}(t)$ worked out in [Cirigliano, Ecker, Neufeld 2001]

contributions:

- photonic loops



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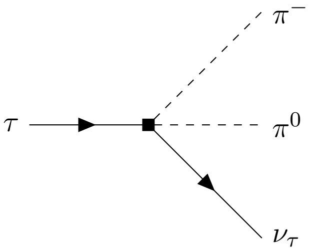
contributions:

- photonic loops, counter terms



Low-energy constants:

- $K_1,\ K_3,\ K_5,\ K_{12}$
- $X_1,\ X_2,\ X_3$

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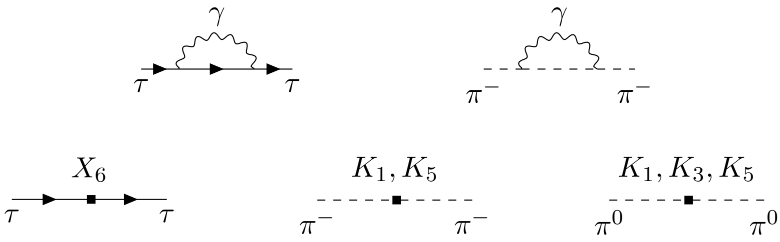
contributions:

- photonic loops, counter terms, self energies $\to \Sigma$: UV finite



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contributions:

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matching:

- dispersive "box": sensitive to high-energy behavior of FF f_+

$\to$ match to ChPT at $s, t = 0$ (outside of decay phase space)

$$f_{\mathrm{elm}}^{\mathrm{match}}(s,t) \coloneqq f_{\mathrm{elm}}^{\mathrm{disp}}(s,t) - f_{\mathrm{elm}}^{\mathrm{disp}}(0,0) + f_{\mathrm{elm}}^{\mathrm{ChPT}}(0)$$

- $f_{\mathrm{elm}}^{\mathrm{match}}$ has the correct chiral logarithm and IR structure
- in narrow resonance limit: $f_{\mathrm{elm}}^{\mathrm{disp}}(0,0) \to f_{\mathrm{elm}}^{\mathrm{ChPT}}(0)$

Note on low-energy constants

- after the dust settles:

$$f_{\text{elm}}^{\text{match}}(s,t) \supset -2\pi\alpha\left(\frac{4}{3}X_1 + X_6^r(\mu) - 4K_{12}^r(\mu)\right)$$

- in semileptonic decays X_6 and K_{12} always appear in same combination
  $X_6^{\text{phys}}(\mu) := X_6(\mu) - 4K_{12}(\mu)$ Descotes-Genon, Moussallam 2005
- it is here, where the short-distance part can be separated
  $X_6^{\text{phys}}(\mu) =: \tilde{X}_6^r(\mu) + X_{6,\text{SD}}^{\text{phys}}$

→ connection with S_{EW}:

Cirigliano, Ecker, Neufeld 2001

$$e^2 X_{6,\text{SD}}^{\text{phys}} = 1 - S_{\text{EW}} + \ldots$$

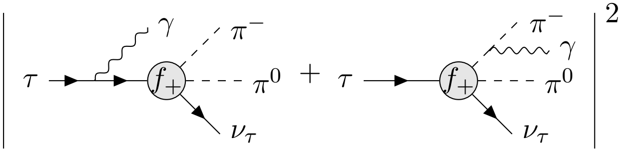
⇒ consistent choice of schemes for S_{EW} and G_{EM} necessary

- S_{EW}: known to NLL accuracy Erler 2002; Cirigliano et al. 2023
- G_{EM}: scheme-dependent nonlogarithmic term currently unknown

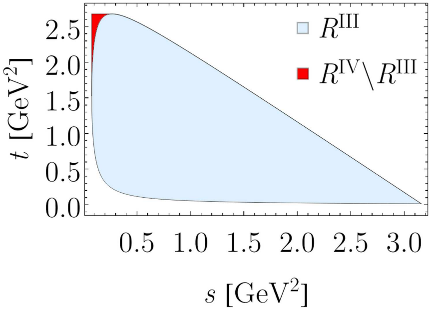
Cirigliano et al., in preparation

Radiative decay $\tau^- \to \pi^-\pi^0\nu_\tau\gamma$ – I

- IR divergent contribution follows from Low expansion of:



- further conts. finite in Low expansion Cirigliano, Ecker, Neufeld 2002
- due to nature of experimental data for $\mathrm{d}\Gamma_{\tau\to\pi\pi\nu_\tau[\gamma]}/\mathrm{d}s$: need to integrate over full photon phase space

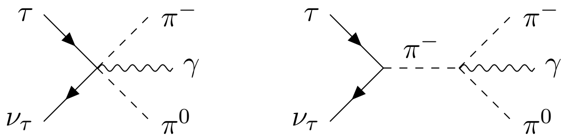


- R^{IV} full radiative phase space
- R^{III} accessible to non-radiative decay
- contribution in $R^{\mathrm{IV}}\backslash R^{\mathrm{III}}$ important for threshold behavior

Radiative decay $\tau^- \to \pi^-\pi^0\nu_\tau\gamma$ – II

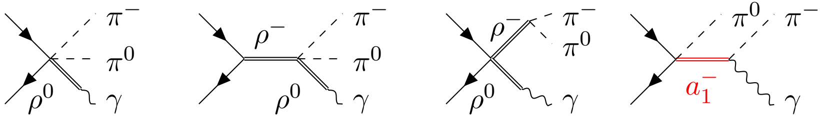
- further contributions at $\mathcal{O}(p^4)$ due to anomaly and resonance exchange
  Cirigliano, Ecker, Neufeld 2002; López Castro et al. 2005/6; Miranda, Roig 2020

Chiral anomaly (via Wess–Zumino–Witten action):



→ dependence on pion decay constant F_π

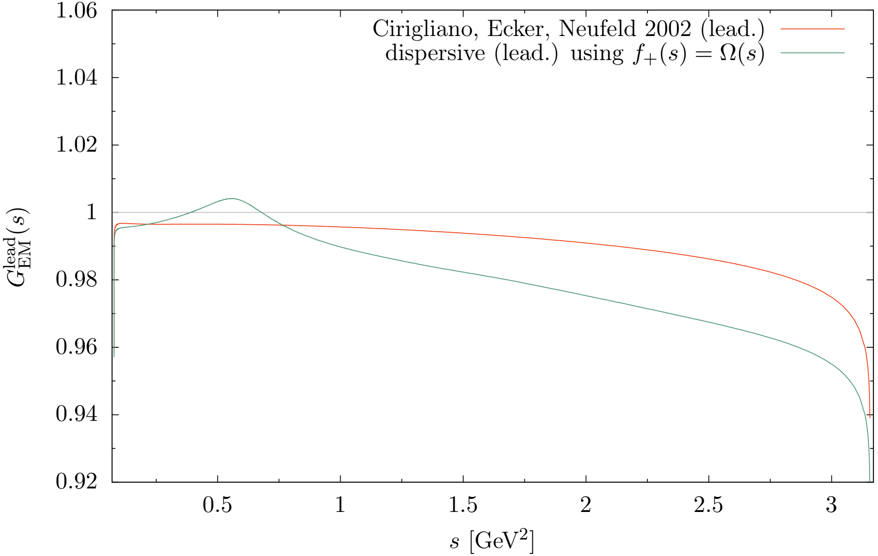
Resonance exchange (via Resonance Chiral Theory [Ecker et al. 1989])



→ dependence on couplings F_V, G_V, F_A

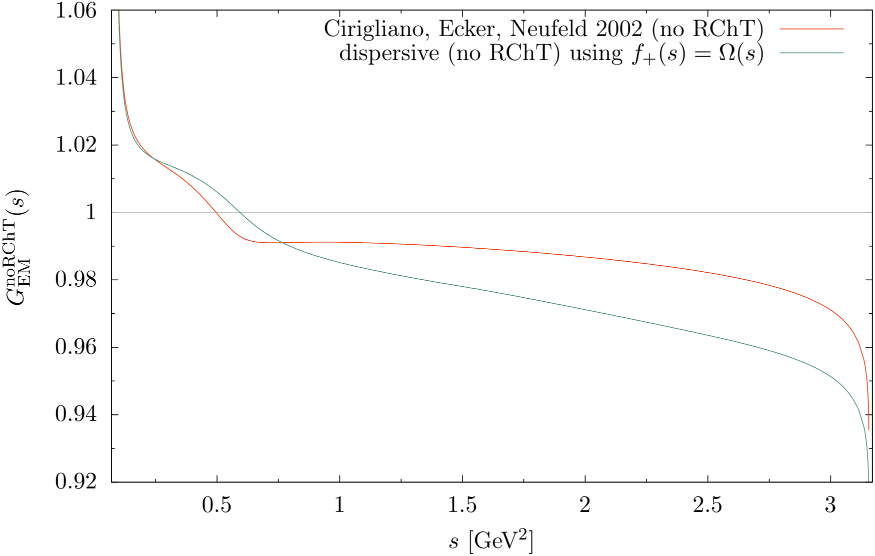
Putting the pieces together

$G_{\text{EM}}(s) = 1 + \text{virtual} + \text{soft brems.}$



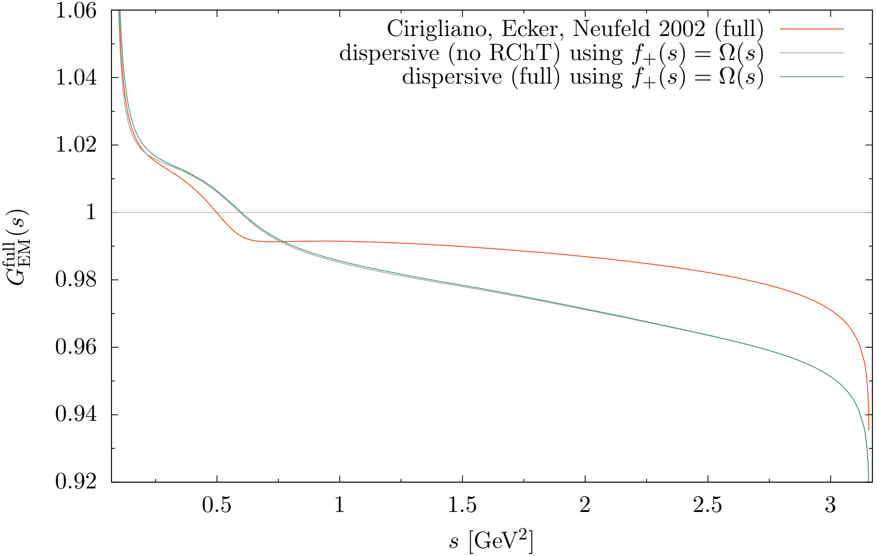
Putting the pieces together

$G_{\mathrm{EM}}(s) = 1 + \text{virtual} + \text{soft brems.} + \text{full real rad.}$



Putting the pieces together

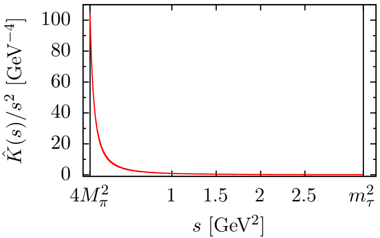
$$G_{\rm EM}(s) = 1 + \text{virtual} + \text{soft brems.} + \text{full real rad.} + (\text{anomaly \& RChT})$$



Threshold behavior

- in $a_\mu[\tau,\pi\pi]$ integrand: "QED kernel" $\hat{K}(s)/s^2$

$\rightarrow$ enhancement of threshold region



- subtle cancellations among Passarino–Veltman functions ensure regular behavior of (IR-finite) virtual contributions at threshold $\rightarrow$ maps onto ChPT; (soft radiation $\rightarrow$ logarithmic divergence)
- full G_{EM} behaves like $1/(s-4M_\pi^2)$ due to real emission $\rightarrow$ developed expansion for stable evaluation in threshold vicinity
- should be regulated by phase-space function $\beta^3_{\pi\pi^0}(s)$
- small changes in threshold region enhanced $\rightarrow$ certain e^4 terms matter!
- calculation was done in isospin limit $\rightarrow$ remap: $G_{\text{EM}}(s) \mapsto G_{\text{EM}}(\tilde{s}[s])$
  $\tilde{s}: \ [(M_\pi + M_{\pi^0})^2, m_\tau^2] \rightarrow [4M_\pi^2, m_\tau^2]$ linearly

$\rightarrow$ $\beta^3_{\pi\pi^0}(s) \times G_{\text{EM}}(\tilde{s}[s])$ has correct threshold behavior for $M_\pi \neq M_{\pi^0}$

Form factor f_+ input

- fix from fits to $\tau^- \to \pi^-\pi^0\nu_\tau$ spectrum for $s \in \left[(M_\pi + M_{\pi^0})^2,\, m_\tau^2\right]$
- ansatz: [analogous to dispersive $e^+e^- \to 3\pi$ fits Hoid et al. 2018-25]

$$f_+(s) = \Big(1 + G_{\rm in}^N(s) + \mathcal{A}_{\rho',\rho''}(s)\Big)\Omega(s),$$

Omnès function:

$$\Omega(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}x\, \frac{\delta(x)}{x(x-s-i\epsilon)}\right)$$

- $\pi\pi$ P-wave scattering phase shift $\delta(s)$ from solution of Roy equations
  Ananthanarayan et al. 2000; Caprini, Colangelo, Leutwyler 2011
- take $\delta(s_0)$, $\delta(s_1)$ as fit parameters, $s_0 = (0.8\,\mathrm{GeV})^2$, $s_1 = (1.15\,\mathrm{GeV})^2$
- analogous to what is done for dispersive $e^+e^- \to \pi^+\pi^-$ fits
  Colangelo, Hoferichter, Stoffer 2018; Leplumey, Stoffer 2025

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$$f_+(s) = \Big(1 + G_{\text{in}}^N(s) + \mathcal{A}_{\rho',\rho''}(s)\Big)\Omega(s),$$

Conformal polynomial:

$$G_{\text{in}}^N(s) = \sum_{k=1}^{N} p_k\big(z^k(s) - z^k(0)\big)$$

- description of inelastic effects beyond $\pi\pi$
- use of unsubtracted dispersion relation for virtual corrections $\to$ fix one coefficient to ensure normalization sum rule $\frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}s' \, \frac{\mathrm{Im}\, f_+(s')}{s'} = 1$
- another coefficient fixed to eliminate S-wave threshold behavior

$\to$ contains $N-2$ fit parameters

Form factor f_+ input

- fix from fits to $\tau^- \to \pi^-\pi^0\nu_\tau$ spectrum for $s \in \left[(M_\pi + M_{\pi^0})^2,\, m_\tau^2\right]$
- ansatz: [analogous to dispersive $e^+e^- \to 3\pi$ fits Hoid et al. 2018-25]

$$f_+(s) = \left(1 + G_{\text{in}}^N(s) + \mathcal{A}_{\rho',\rho''}(s)\right)\Omega(s),$$

Explicit contribution due to ρ', ρ'' resonances:

$$\mathcal{A}_{\rho',\rho''}(s) = \frac{s}{\pi}\int_{s_{\text{thr}}}^{\infty} \mathrm{d}x\, \frac{\operatorname{Im}\mathcal{A}(x)}{x(x-s-i\epsilon)},$$

$$\mathcal{A}(s) = \sum_{V\in\{\rho',\rho''\}} \frac{c_V}{M_V^2 - s - i\sqrt{s}\,\Gamma_V(s)}$$

- explicit inelastic contribution
- energy-dependent widths $\Gamma_V(s)$ constructed from $\omega\pi$ phase space

$\rightarrow$ 6 fit parameters: (real) couplings, mass parameters, on-shell widths

Fit strategy

- experiments provide spectra in the form of

$$\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}s} = S_{\rm EW}^{\pi\pi}\frac{\mathrm{Br}_e}{\mathrm{Br}_\pi}\frac{|V_{ud}|^2}{2m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+\frac{2s}{m_\tau^2}\right)\beta^3_{\pi\pi^0}(s)|f_+(s)|^2 G_{\rm EM}(s)$$

- input parameters (among others):
  - $S_{\rm EW}^{\pi\pi} = 1.0233(2)(24)$ WP25
  - $\mathrm{Br}(\tau^- \to e^-\bar{\nu}_e\nu_\tau) = 17.82(4)\,\%$, $\mathrm{Br}(\tau^- \to \pi^-\pi^0\nu_\tau) = 25.49(9)\,\%$

    HFLAV/PDG global fits
- further: bin averaging & procedure to avoid D'Agostini bias
- $G_{\rm EM}$ depends on f_+ input $\to$ iterative procedure:
  1. start with $f_+(s) = \Omega(s)$
  2. calculate $G_{\rm EM}$ from f_+
  3. insert $G_{\rm EM}$ into $\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}s}$
  4. fit f_+ in $\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}s}$ to data (arrow back to step 2)
- observation: tiny changes in fit parameters after three iteration steps

Sources of uncertainties

From data input:

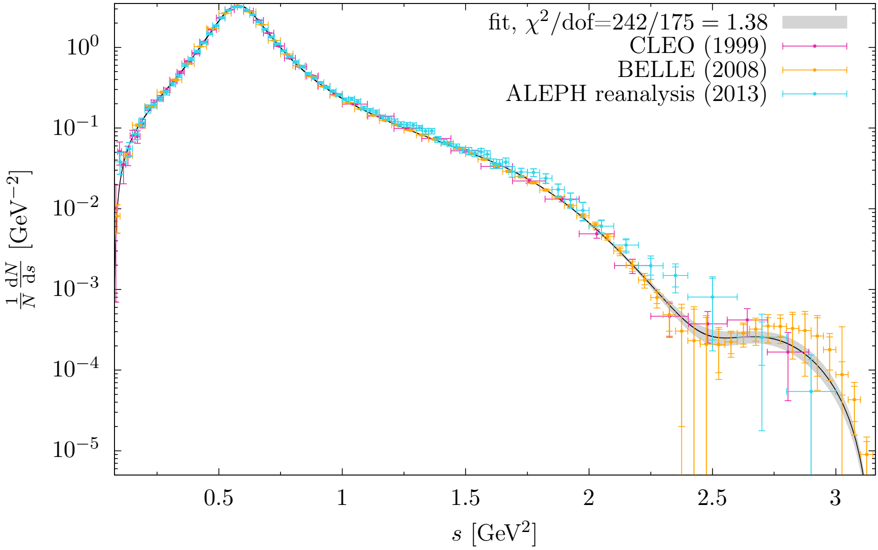
- statistical and systematic covariances on $\tau^- \to \pi^-\pi^0\nu_\tau$ data
  $\Rightarrow$ fit uncertainty on f_+, propagated to G_{EM}
- uncertainties on spectrum due to $\mathrm{Br}(\tau^- \to e^-\bar{\nu}_e\nu_\tau)$ and $\mathrm{Br}(\tau^- \to \pi^-\pi^0\nu_\tau)$ input

Theoretical uncertainties:

- for f_+:
  - variation of conformal polynomial degree N
  - variation of s_c parameter in conformal polynomial
- for G_{EM}:
  - variation of cutoff of dispersive integral
  - couplings in RChT part (F_V, G_V, F_A):
    $(\sqrt{2}F_\pi, F_\pi/\sqrt{2}, F_\pi)$ (via short-distance constraints) vs.
    $(0.16, 0.065, 0.12)\,\mathrm{GeV}$
    (via exp. decay widths $\rho \to e^+e^-$, $\rho \to \pi\pi$, $K^* \to K\pi$, $a_1 \to \pi\gamma$)
- additionally for Δa_μ (and spectrum):
  - uncertainty due to scheme dependence of $S_{\mathrm{EW}}^{\pi\pi}$

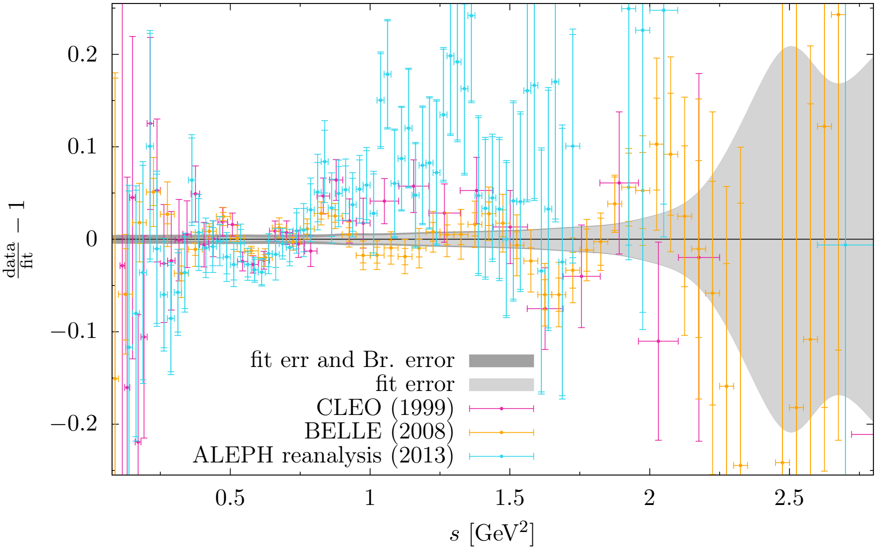
WP25

Fit results

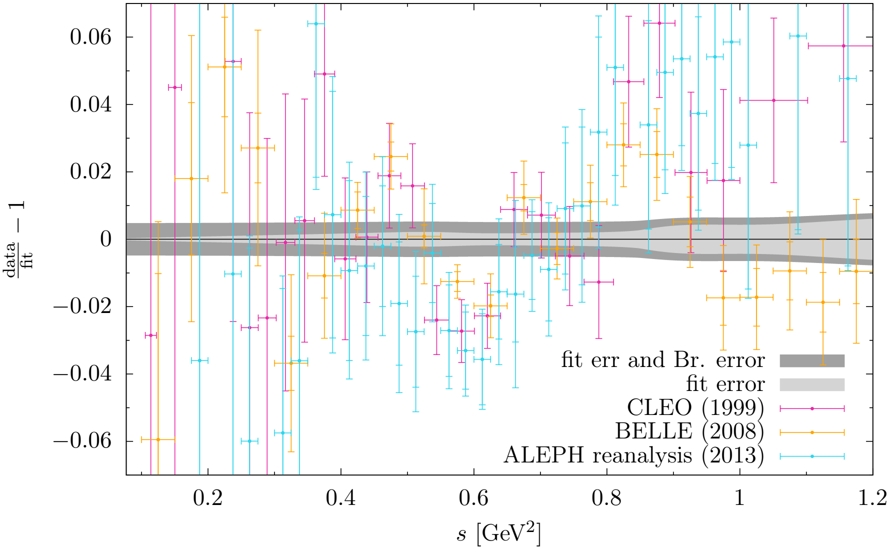


fit, χ^2/dof=242/175 = 1.38
CLEO (1999)
BELLE (2008)
ALEPH reanalysis (2013)
$\frac{1}{N}\frac{dN}{ds}$ [GeV^{-2}]
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
0.5
1
1.5
2
2.5
3
s [GeV^2]

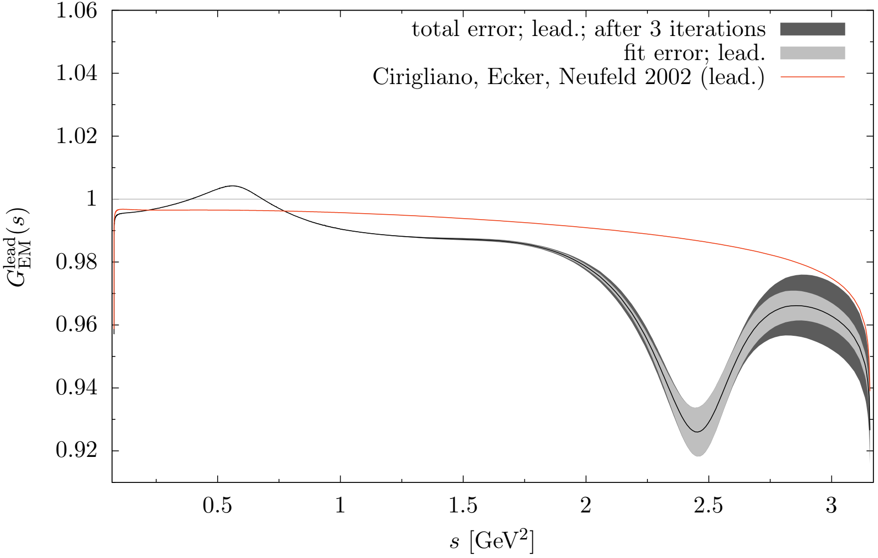
Fit results


0.2
0.1
0
−0.1
−0.2
$\frac{\text{data}}{\text{fit}} - 1$
0.5
1
1.5
2
2.5
s [GeV^2]
fit err and Br. error
fit error
CLEO (1999)
BELLE (2008)
ALEPH reanalysis (2013)

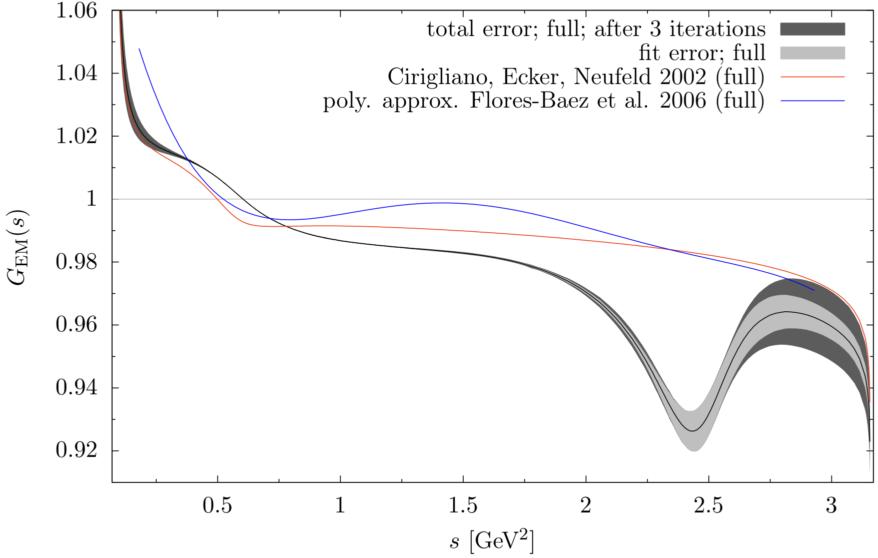
Fit results


0.06
0.04
0.02
0
−0.02
−0.04
−0.06
0.2
0.4
0.6
0.8
1
1.2
$\frac{\text{data}}{\text{fit}} - 1$
s [GeV^2]
fit err and Br. error
fit error
CLEO (1999)
BELLE (2008)
ALEPH reanalysis (2013)

G_{EM} results



G_{EM} results



IB corr. to $a_\mu^{\text{HVP, LO}}$ $(|F_\pi^V/f_+| = 1)$ [PRELIM.]

- τ spectral function: $v_\tau(s) = S_{\text{EW}}^{\pi\pi}\beta^3_{\pi\pi^0}(s)\,|f_+(s)|^2\,G_{\text{EM}}(s)$

$$\Delta a_\mu^{\text{HVP, LO}}[\pi\pi,\tau]\Big|_{r_{\text{IB}}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{4M_\pi^2}^{m_\tau^2} \mathrm{d}s\, \frac{\hat{K}(s)}{4s^2}\Big[r_{\text{IB}}(s) - 1\Big]v_\tau(s)$$

$r_{\text{IB}}(s)$	$\Delta a_\mu \times 10^{10}$

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$r_{\mathrm{IB}}(s)$	$\Delta a_\mu \times 10^{10}$
$1/G_{\mathrm{EM}}^{\mathrm{lead}}(s)$	$-0.221\,(6)_{\mathrm{exp}}\,(10)_{\mathrm{theo}}$
$1/G_{\mathrm{EM}}^{\mathrm{no\ RChT}}(s)$	$-3.163\,(17)_{\mathrm{exp}}\,(10)_{\mathrm{theo}}$

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$$\Delta a_\mu^{\mathrm{HVP,\,LO}}[\pi\pi,\tau]\Big|_{r_{\mathrm{IB}}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{4M_\pi^2}^{m_\tau^2} \mathrm{d}s\,\frac{\hat{K}(s)}{4s^2}\Big[r_{\mathrm{IB}}(s) - 1\Big]\,v_\tau(s)$$

$r_{\mathrm{IB}}(s)$	$\Delta a_\mu \times 10^{10}$
$1/G_{\mathrm{EM}}^{\mathrm{lead}}(s)$	$-0.221\,(6)_{\mathrm{exp}}\,(10)_{\mathrm{theo}}$
$1/G_{\mathrm{EM}}^{\mathrm{no\,RChT}}(s)$	$-3.163\,(17)_{\mathrm{exp}}\,(10)_{\mathrm{theo}}$
$1/G_{\mathrm{EM}}(s)$	$-3.39\,(2)_{\mathrm{exp}}\,(40)_{\mathrm{theo}}$

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$\beta^3_{\pi\pi}(s)/\beta^3_{\pi\pi^0}(s)$ [PS]	$-7.726\,(37)_{\mathrm{exp}}\,(4)_{\mathrm{theo}}$
$1/S_{\mathrm{EW}}^{\pi\pi}$	$-12.2\,(1)_{\mathrm{exp}}\,(1.3)_{\mathrm{scheme\,dep.}}$

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$\beta^3_{\pi\pi}(s)/\beta^3_{\pi\pi^0}(s)$ [PS]	$-7.726\,(37)_{\mathrm{exp}}\,(4)_{\mathrm{theo}}$
$1/S_{\mathrm{EW}}^{\pi\pi}$	$-12.2\,(1)_{\mathrm{exp}}\,(1.3)_{\mathrm{scheme\,dep.}}$
sum of above	$-23.3\,(1)_{\mathrm{exp}}\,(4)_{\mathrm{theo}}\,(1.3)_{\mathrm{scheme\,dep.}}$
$\beta^3_{\pi\pi}(s)/(\beta^3_{\pi\pi^0}(s)\,G_{\mathrm{EM}}(s)\,S_{\mathrm{EW}}^{\pi\pi})$	$-22.8\,(1)_{\mathrm{exp}}\,(4)_{\mathrm{theo}}\,(1.3)_{\mathrm{scheme\,dep.}}$

Comparison to previous works

Contributions to $\Delta a_\mu^{\mathrm{HVP,\,LO}}[\pi\pi,\tau]$ $[10^{-10}]$:

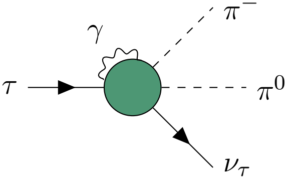
reference	PS	S_{EW}	G_{EM}	sum
this work [PRELIM.]	$-7.73\,(4)$	$-12.2\,(1.3)$	$-3.39(40)$	$-23.3\,(1.4)$
DHMLZ 2024	-7.88	$-12.21(15)$	$-1.92\,(90)$	$-22.01\,(91)$
LMR 2025	-7.52	$-12.16\,(15)$	$-1.67^{+0.60}_{-1.39}$	$-21.35^{+0.62}_{-1.40}$
WP 25	$-7.7\,(2)$	$-12.2\,(1.3)$	$-2.0\,(1.4)$	$-21.9\,(1.9)$

References:

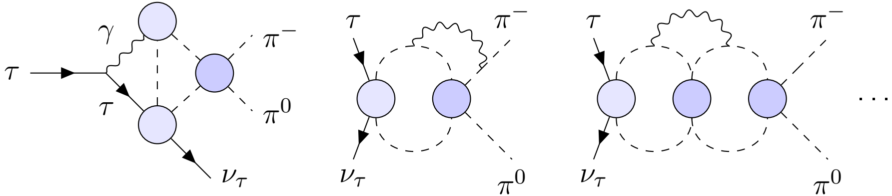
- DHMLZ 2024: Davier, Hoecker, Lutz, Malaescu, Zhang, Eur.Phys.J.**C**84, 721(2024)
- LMR 2025: López Castro, Miranda, Roig, Phys.Rev.**D**111, 073004 (2025)

Further contributions to dispersive calculation

general one-photon virtual corrections:



also include:



Work in progress $\rightarrow$ **Martina Cottini's talk tomorrow**

Conclusions and outlook

- dispersive calculation of leading effect in radiative corrections to $\tau^- \to \pi^- \pi^0 \nu_\tau$
  - $G_{\rm EM}$ correction: structure dependent effects prominent in resonance regions
  - correct low-energy behavior restored by ChPT matching procedure and connection to LECs
- reconstructed from charged current form factor f_+
  - representation fulfills analyticity constraints
  - beyond two-body unitarity included explicit inelasticities due to ρ', ρ''
  - fitted to experimental spectra
- for Δa_μ sum of contributions considered here: $-23.3\,(1.4) \times 10^{-10}$ vs. $-21.9\,(1.9) \times 10^{-10}$ [WP25]
  - however $\mathcal{O}(e^4)$ effects in full Δa_μ integrand give significant contribution ($\sim 0.5 \times 10^{-10}$) due to $G_{\rm EM}$ threshold behavior
- **in the future:** fully consistent matching between $G_{\rm EM}$ and $S_{\rm EW}$ $\to$ significant reduction of uncertainty
- in this talk $|F_\pi^V(s)/f_+(s)|$ in Δa_μ integrand ignored, more discussion $\to$ Martina Cottini's talk tomorrow

Backup

$\mathcal{O}(e^4)$ effects in linearization of Δa_μ integrand

Notation: $M_{\pi^0}^2 = M_\pi^2 - e^2\Delta_M$, $S_{\mathrm{EW}}^{\pi\pi} = 1 + e^2\Delta S_{\mathrm{EW}}^{\pi\pi}$, $G_{\mathrm{EM}} = 1 + e^2\Delta G_{\mathrm{EM}}$

Then difference of linearized and full integrand:

$$\left[\left(\frac{1}{G_{\mathrm{EM}}} - 1\right) + \left(\frac{\beta_{\pi\pi}^3}{\beta_{\pi\pi^0}^3} - 1\right) + \left(\frac{1}{S_{\mathrm{EW}}^{\pi\pi}} - 1\right) - \left(\frac{\beta_{\pi\pi}^3}{\beta_{\pi\pi^0}^3}\frac{1}{S_{\mathrm{EW}}^{\pi\pi} G_{\mathrm{EM}}} - 1\right)\right] v_\tau$$

$$= -\, e^4 \frac{\sqrt{s - 4M_\pi^2}}{s^{3/2}}\left[(s - 4M_\pi^2)\Delta G_{\mathrm{EM}}\, \Delta S_{\mathrm{EW}}^{\pi\pi} + 3\Delta_M\left(\Delta G_{\mathrm{EM}} + \Delta S_{\mathrm{EW}}^{\pi\pi}\right)\right]|f_+(s)|^2 + \mathcal{O}(e^6),$$

- ΔG_{EM} diverges at threshold $\to$ rather large effect on integral

Note on RChT contribution

In RChT: Ecker et al. 1989

$$\Gamma(a_1^\pm \to \pi^\pm\gamma) = \frac{e^2 F_A^2 m_{a_1}}{96\pi F_\pi^2}\left(1 - \frac{M_\pi^2}{m_{a_1}^2}\right)^3$$

- with $\Gamma(a_1) = 640\,(246)\,\mathrm{keV}$ [Zielinski et al. 1984] $\to F_A = 0.12\,(2)\,\mathrm{GeV}$
- on the other hand, short-distance constraints [Ecker et al. 1989]: $F_A = F_\pi \approx 0.92\,\mathrm{GeV}$

Alternative extraction via $a_1 \to \pi\rho \to \pi\gamma$

$$\Gamma(a_1^\pm \to \pi^\pm\gamma) = \frac{e^2 |\tilde{g}_{a_1\rho\pi}|^2 m_{a_1}^3}{96\pi g_{\rho\gamma}^2}\left(1 - \frac{M_\pi^2}{m_{a_1}^2}\right)^3$$

- assuming total width as $a_1 \to 3\pi$, get $\tilde{g}_{a_1\rho\pi}$
- $\Gamma_{a_1} = 0.25\ldots 0.6\,\mathrm{GeV}$ PDG 2024 $\to F_A = 0.083\ldots 0.13\,\mathrm{GeV}$

In RChT contribution to G_{EM}: addtly. include interference with ω exchange diagram [Fig. 2(g) in Flores-Tlapa et al. 2005] as suggested in [Davier et al. 2009]

Scheme dependence in S_{EW}: WP25 prescription

- at next-to-leading log (NLL), Wilson coefficients acquire a scheme dependence of $\mathcal{O}(\alpha/\pi)$

$\rightarrow$ to be absorbed by long-range correction, in this case $G_{\mathrm{EM}}(s)$

- current result (in the context of β decays) $S_{\mathrm{EW}}^{\pi\pi} = 1.0233\,(3)$ [Cirigliano et al. 2023] scheme dependence factored out, assumed to be absorbed

- at the moment, no control over scheme dependence in $G_{\mathrm{EM}}(s)$ (LECs)

$\rightarrow$ take ambiguity in the decay rate as $\alpha(m_\tau)/\pi \sim 0.24\,\%$
$\Rightarrow S_{\mathrm{EW}}^{\pi\pi} = 1.0233\,(3)\,(24)$

Numerical treatment of IR-divergent $D_0(m_\tau^2, M_\pi^2, M_\pi^2, 0, t, s, m_\tau^2, 0, M_\pi^2, s'')$

Singularity at $s'' = s$:

$$D_0(s,t) = \frac{1}{s''-s}\left[2d_0(t)\log\frac{s''}{s''-s} + D_0^{\text{rest}}(t,s'')\right]$$

$$f_+(s,t) \supset \int_{4M_\pi^2}^{\infty} \mathrm{d}s''\, \mathrm{Im}\, f_+(s'') \left(\frac{p_1(s,t)+p_2(s,t)s''}{s''-s}\right) \times \left[2d_0(t)\log\frac{s''}{s''-s} + D_0^{\text{rest}}(t,s'')\right] ,$$

$$I_{\ell 1}(s,\Lambda^2) = 2\int_{4M_\pi^2}^{\Lambda^2} \mathrm{d}s''\log\frac{s''}{s''-s}, \quad I_{\ell 2}(s) = 2\int_{4M_\pi^2}^{\infty} \mathrm{d}s''\frac{1}{s''-s-i\epsilon}\log\frac{s''}{s''-s-i\epsilon}$$

- IR-divergences in dim-reg
- application: $e^+e^- \to \pi^+\pi^-$ asymmetry
  - simplified imaginary part in [Colangelo et al, '22], no scheme ambiguity
  - see new results in [Budassi et al, '24]

Endpoint singularities in the phase space

$$f_+^{\mathrm{box},F_\pi^V}(s,t) = f_+^{\mathrm{fin}}(s,t) + \frac{N(s,t)}{s(t-t_{\mathrm{min}})(t-t_{\mathrm{max}})}$$

$\rightarrow$ endpoint singularity in the t phase space integral BUT numerically showed that the two infinities cancel $\rightarrow$ finite result.
Analytically:

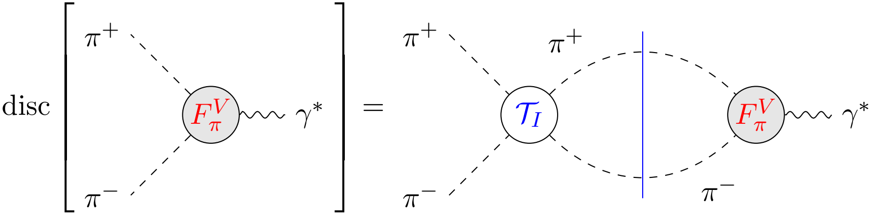
$$\begin{aligned} N(s,t) &= (t-t_{\mathrm{max}})N_+(s,t) \\ &= (t-t_{\mathrm{max}})(N_+(s,t) - N_+(s,t_{\mathrm{min}})) \\ &= (t-t_{\mathrm{max}})(t-t_{\mathrm{min}})\bar{N}(s,t) \ . \end{aligned}$$

$\rightarrow$ expand $\bar{N}(s,t)$ around $t = t_{\mathrm{max/min}}$ when the integration in t is close to the boundaries $t_{\mathrm{max/min}}$.

- can show that $\bar{N}(s,t_{\mathrm{max/min}}) = 0$ by relation between Passarino–Veltman C_0's and D_0's at that kinematic point Denner 1993

$\rightarrow$ intimately related to regular threshold behavior of (IR-finite) dispersive box contribution to G_{EM}

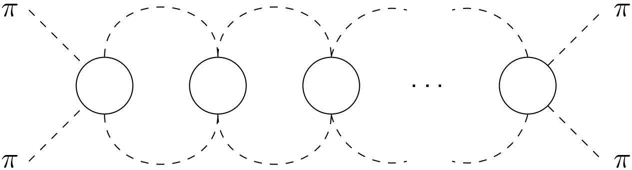
Pion Vector Form Factor



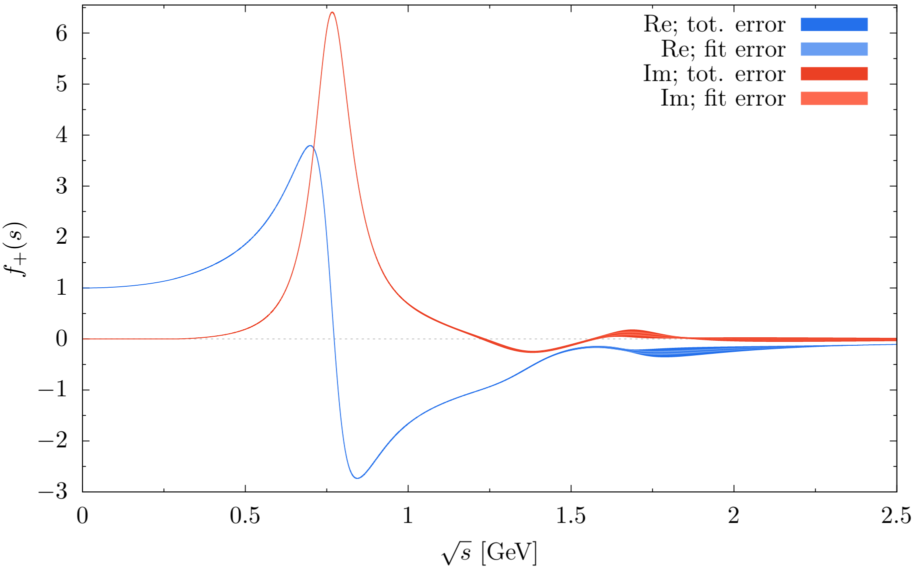
- solution of discontinuity equation: Omnès 1958

$$F_\pi^V(s) = R(s)\Omega(s)\ , \quad \Omega(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\delta(\omega)}{\omega(\omega-s)}\mathrm{d}\omega\right)$$

- δ: $I=1$ $\pi\pi$ P-wave phase shift

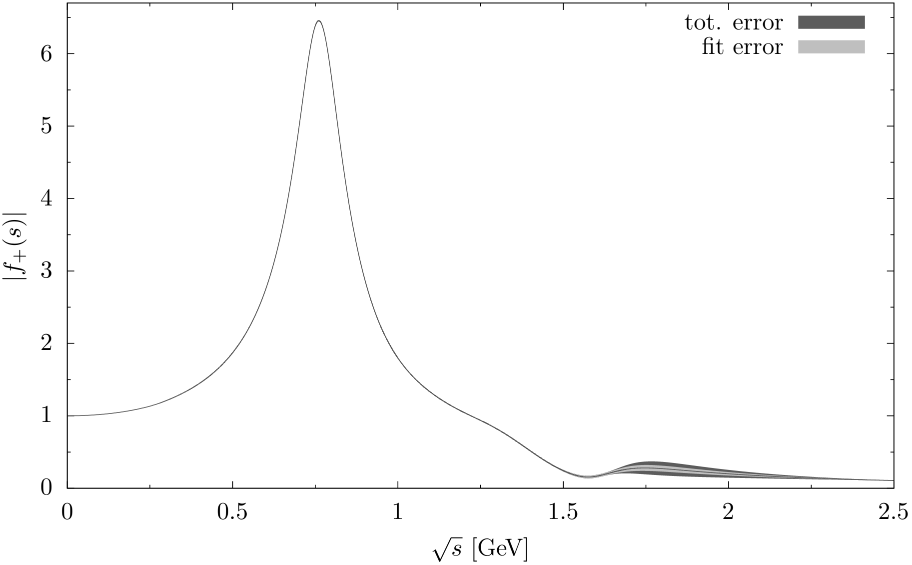


f_+ Plots



Re; tot. error
Re; fit error
Im; tot. error
Im; fit error
6
5
4
3
2
1
0
−1
−2
−3
$f_+(s)$
0
0.5
1
1.5
2
2.5
$\sqrt{s}$ [GeV]

f_+ Plots



f_+ Plots