Isospin-breaking effects in inclusive hadronic τ data for the muon $(g-2)$

Mattia Bruno
work in collab. with T. Izubuchi, C. Lehner, A. Meyer, J. Parrino, X. Tuo
for the RBC/UKQCD collaborations

8th plenary workshop of the muon g-2 theory initiative
IJCLab, France, September 8th

The RBC & UKQCD collaborations

Boston University
Nobuyuki Matsumoto

BNL and BNL/RBRC
Peter Boyle
Taku Izubuchi
Christopher Kelly
Shigemi Ohta (KEK)
Amarji Soni
Masaaki Tomii
Xin-Yu Tuo
Shuhei Yamamoto

University of Cambridge
Nelson Lachini

CERN
Matteo Di Carlo
Felix Erben
Andreas Jüttner (Southampton)
Tobias Tsang

Columbia University
Norman Christ
Sarah Fields
Ceran Hu
Yikai Huo
Joseph Karpie (JLab)
Erik Lundstrum
Bob Mawhinney
Bigeng Wang (Kentucky)

University of Connecticut
Tom Blum
Jonas Hildebrand
Luchang Jin
Vaishakhi Moningi
Anton Shcherbakov
Douglas Stewart
Joshua Swaim

DESY Zeuthen
Raoul Hodgson

Edinburgh University
Luigi Del Debbio
Vera Gülpers
Maxwell T. Hansen
Nils Hermansson-Truedsson
Ryan Hill
Antonin Portelli
Azusa Yamaguchi

Johannes Gutenberg University of Mainz
Alessandro Barone

Liverpool Hope/Uni. of Liverpool
Nicolas Garron

LLNL
Aaron Meyer

Autonomous University of Madrid
Nikolai Husung

University of Milano Bicocca
Mattia Bruno

Nara Women's University
Hiroshi Ohki

Peking University
Xu Feng
Tian Lin

University of Regensburg
Andreas Hackl
Daniel Knüttel
Christoph Lehner
Sebastian Spiegel

RIKEN CCS
Yasumichi Aoki

University of Siegen
Matthew Black
Anastasia Boushmelev
Oliver Witzel

University of Southampton
Bipasha Chakraborty
Ahmed Elgaziari
Jonathan Flynn
Joe McKeon
Rajnandini Mukherjee
Callum Radley-Scott
Chris Sachrajda

Stony Brook University
Fangcheng He
Sergey Syritsyn (RBRC)

Hadronic τ decays for the HVP in the muon $(g-2)$ [Alemany et al '97]

- shed light given current tensions in e^+e^- panorama
- require isospin-breakin corrections
- for a complete introduction [talks by Zhang, Holz]

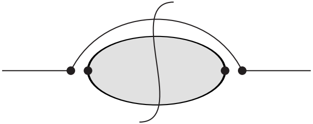
This talk = a strategy for IB corrections in hadronic τ decays for $(g-2)_\mu$

- based on Lattice QCD(+QED) simulations
  - first-principle systematically-improvable approach
- fully inclusive
- taking care of short-distance renormalization
- handling non-factorizable effects (aka G_{EM})

Hadronic τ decays. Short distance renormalization

$$\tau^-(P) \to \nu(q)\,\mathrm{had}_V^-(p)$$

$$\mathcal{M}_f(P,q,p_1\cdots p_{n_f}) = 2G_{\mathrm{F}}V_{\mathrm{ud}}\,\bar{u}(q)\gamma_\mu^L u(P)\;\langle \mathrm{out}, p_1\cdots p_{n_f}|\mathcal{J}_\mu^-(0)|0\rangle$$



$$d\Gamma = \frac{1}{4m} d\Phi_q \sum_f d\Phi_f \sum_{\mathrm{spin}} |\mathcal{M}_f|^2$$

Isovector spectral density isospin limit $= \rho$

$$\left[d\Phi_q = \frac{d^3q}{(2\pi)^3 2\omega_q} \right]$$

$$\frac{d\Gamma}{ds} = \frac{G_{\mathrm{F}}^2|V_{\mathrm{ud}}|^2 m_\tau^3}{8\pi}\,\kappa(\hat{s})\,\rho(s)\,, \qquad \kappa(\hat{s}) = (1+2\hat{s})\,(1-\hat{s})^2 \qquad \hat{s} = \frac{s}{m_\tau^2}$$

---

from experiment we get $\displaystyle \frac{1}{\Gamma}\frac{d\Gamma}{ds} \to \frac{\Gamma}{\Gamma_e}\frac{1}{\Gamma}\frac{d\Gamma}{ds} = \frac{1}{\Gamma_e}\frac{d\Gamma}{ds}$

$$\Gamma_e = \Gamma(\tau \to e\bar{\nu}\nu) = \frac{\mathcal{B}_e\Gamma}{\mathcal{B}} = \frac{G_{\mathrm{F}}^2 m_\tau^5}{192\pi^3}$$

At $O(\alpha)$ new divergences in EFT $\to$ need regulator, Z factors

W-regularization [Sirlin '82][Marciano, Sirlin '88][Braaten, Li '90]

$$\frac{1}{k^2} = \frac{1}{k^2 - m_W^2} - \frac{m_W^2}{k^2(k^2 - m_W^2)}$$ [Sirlin '78]

- universal UV divergences re-absorbed in G_{F}, i.e. drop out in ratio
- process-specific corrections in S_{EW}, like a Z factor

Effective Hamiltonian at $O(\alpha)$: $H_W \propto G_{\mathrm{F}} S_{EW}^{1/2} \mathcal{O}(x)$

- matching required as noted by [Carrasco et al '15][Di Carlo et al '19]

UNIVERSITA' DEGLI STUDI DI MILANO BICOCCA

[Brod, Gorbahn '08][Gorbahn et al '23][Bigi et al '23][Cirigliano et al '23]

$C^{\overline{\mathrm{MS}}}(\mu) = U(\mu, \mu_W) C^{\overline{\mathrm{MS}}}(\mu_W)$

initial conditions $C^{\overline{\mathrm{MS}}}(\mu_W) = 1 + \frac{\alpha}{2\pi}\left[\log\left(\frac{m_Z^2}{\mu_W^2}\right) - \frac{11}{6}\right]$

evolution operator $U(\mu, \mu_W)$, across quark thresholds

$$\gamma_W = \frac{\alpha}{4\pi}\gamma_W^{(0)} + \frac{\alpha\alpha_s}{(4\pi)^2}\gamma_W^{(1)} + \frac{\alpha\alpha_s^2}{(4\pi)^3}\gamma_W^{(2)} + O(\alpha\alpha_s^3, \alpha^2)$$

$\gamma_W^{(2)}$ not yet known but relevant in evolution at $O(\alpha\alpha_s)$

Dim.Reg. evidently not suitable for non-perturbative Lattice QCD

a different scheme is needed and a corresponding matching with $\overline{\mathrm{MS}}$

Regularization Invariant Momentum schemes [Martinelli et al '94]

suitable for Lattice QCD = off-shell (massless) quark states $p^2 < 0$

requires QCD gauge fixing (Landau)

[Gorbahn et al '23]

$$C^{\overline{\mathrm{MS}}\to\mathrm{RI}}(\mu) = 1 + \frac{\alpha}{4\pi} C_0^{\overline{\mathrm{MS}}\to\mathrm{RI}} + \frac{\alpha\alpha_s(\mu)}{(4\pi)^2} C_1^{\overline{\mathrm{MS}}\to\mathrm{RI}} + O(\alpha\alpha_s^2, \alpha^2)$$

available for generic QED and QCD gauges

and several kinematic configurations (RI-MOM, RI-SMOM)

$C^{\mathrm{RI}}(\mu) = C^{\overline{\mathrm{MS}}\to\mathrm{RI}}(\mu) U(\mu, \mu_W) C^{\overline{\mathrm{MS}}}(\mu_W)$

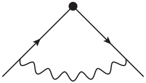
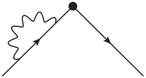
Physical effective Hamiltonian $H_W \propto C^{\mathrm{RI}}(\mu) Z^{\mathrm{RI}}(\mu, a) \mathcal{O}^{\mathrm{latt}}(a)$

$\to$ scheme + scale + gauge dependence cancels $C^{\mathrm{RI}} \leftrightarrow Z^{\mathrm{RI}}$

$\to$ errors $O(\alpha\alpha_s^2)$ from pert. truncation

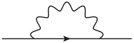
$$Z_{\mathcal{O}}^{\mathrm{RI}}(\mu) = \sqrt{Z_\tau(\mu)} \times Z_{\triangle}^{\mathrm{RI}}(\mu) \times Z_{\mathrm{ud}}^{\mathrm{RI}}(\mu)$$

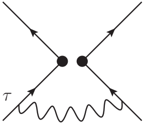
$Z_{\mathrm{ud}}^{\mathrm{RI}}(\mu)$ = QCD factorized radiative corrections

 $+$  $+\cdots$

Λ_μ = quark bilinear amputated GF

four-fermion operator $\Lambda_\mu \otimes \gamma_\mu^L$

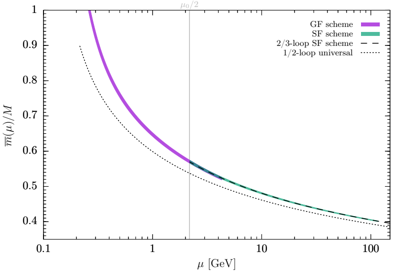
[corrections to quark propagator  $+$  $+\ldots$]



$Z_{\triangle}^{\mathrm{RI}}(\mu)$ = triangle short-distance radiative corrections

Uncertainties at $O(\alpha\alpha_s^2)$ can be improved w/ Lattice QCD, if needed

[ALPHA '15]



1. step-scaling on the lattice

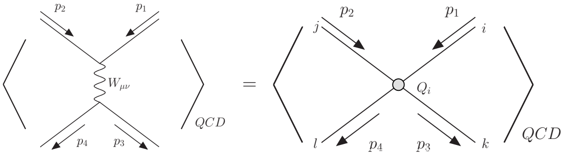
   [Lüscher et al '91,'94 ...]

   [Arthur,Boyle '10 ..]

2. matching across thresholds

   [Della Morte et al' 07 ...][Tomii et al '19]

3. matching initial conditions [MB, Lehner, Soni '17]



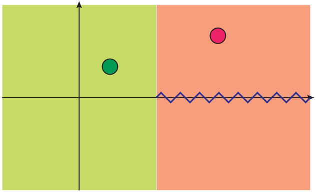
What are inverse problems and why do they matter here?

Lattice correlator
$\langle \widetilde{J}(t)\,\widetilde{J}(0)\rangle = \int d\omega\, e^{-\omega t}\,\rho(\omega)$

versus

Physical observable
$\rho_\kappa = \int d\omega\,\kappa(\omega)\,\rho(\omega)$

Can we find $f(t)$ such that $\rho_\kappa = \int dt\, f(t)\, C(t)$?



Example:

ρ has branch cut starting at multi-particle thresholds $E_{\rm thr}$

Kernel κ simple pole s_0 in complex plane

if $\mathrm{Re}s_0 \leq E_{\rm thr}$ $\rightarrow \forall t\ \exists M > 0 \mid f(t)C(t) < e^{-Mt}$
e.g. HVP contribution to $(g-2)_\mu$ [Blum '02][Bernecker-Meyer '11]

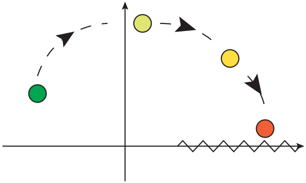
if $\mathrm{Re}s_0 > E_{\rm thr}$ $\rightarrow$ inverse problem, no "direct" analytic continuation

Increasingly high attention from the community on the subject

- formal and theoretical developments
  [Backus, Gilbert '68][Hansen, Meyer, Robaina '17][Hansen, Lupo, Tantalo '19]
  [Bailas et al '20][Del Debbio et al '24][Barone et al '24]
  [Jay et al '24] [MB, Giusti, Saccardi '25]

- analytic continuation via num approaches [ETMC '23][ETMC '24 ...]

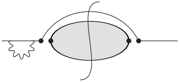
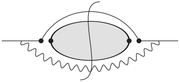
- direct solution via exclusive reconstruction [RBC/UKQCD '24][Mainz '24]
  [FNAL/HPQCD/MILC '24]



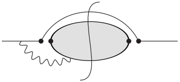
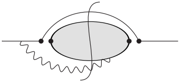
inverse problem solvable w/ modern tech
but closer to the cut more challenging
$\rightarrow$ noise, finite-volume effects ...

Hadronic τ decays. Long distance contributions

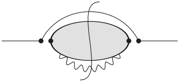
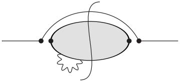
Three IR safe classes, each w/ separate UV counterterms

 $+$ 

$$+\cdots+\left(Z_\tau^{\mathrm{RI}}(\mu)-1\right) \equiv \frac{d\Gamma^{\mathrm{init}}}{ds}(\mu)$$

 $+$ 

$$+\cdots+2\left(Z_\triangle^{\mathrm{RI}}(\mu)-1\right) \equiv \frac{d\Gamma^{\mathrm{nonf}}}{ds}(\mu)$$

 $+$ 

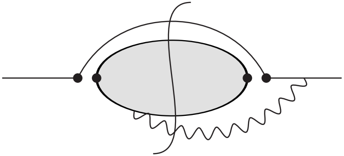
$$+\cdots+2\left(Z_{\mathrm{ud}}^{\mathrm{RI}}(\mu)-1\right) \equiv \frac{d\Gamma^{\mathrm{fin}}}{ds}(\mu)$$

Non-factorizable corrections

aka GEM

1. ampl $\mathcal{M}_f^{\rm real,had}$ from time-ordered prod
   $\int_x e^{ikx} \langle V | T\{\mathcal{J}^\gamma_\alpha(x) \mathcal{J}^{L,-}_\mu(0)\} | 0 \rangle$

2. interference $\mathcal{M}_f^{\rm real,had} \left[\mathcal{M}_f^{{\rm real},\tau}\right]^\dagger$
   $\int_x e^{ikx} \langle 0 | \mathcal{J}^+_\mu(0) | V \rangle \langle V | T\{\mathcal{J}^\gamma_\alpha(x) \mathcal{J}^{L,-}_\nu(0)\} | 0 \rangle$



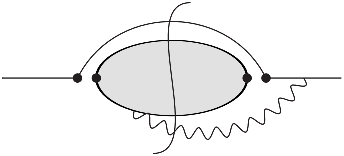
3. hadronic phase-space + sum over all vector states
   $$\int_x e^{ikx} \langle 0 | \mathcal{J}^+_\mu(0)\, \delta(\hat{H} - \sqrt{s})\, T\{\widetilde{\mathcal{J}}^\gamma_\alpha(x) \mathcal{J}^{L,-}_\nu(0)\} | 0 \rangle$$

4. calculable from Euclidean correlator $\langle \tilde{j}^+_\mu(t,\mathbf{0})\, \tilde{j}^\gamma_\alpha(\tau,\boldsymbol{k})\, j^{L,-}_\nu(0) \rangle$?
   there is δ-function $\to$ first inverse problem?
   there is T product $\to$ second inverse problem?

Notation: $\mathcal{J}^{L,-}$ charged weak, $\mathcal{J}^+$ charged vector, $\mathcal{J}$: Minkowski
$\tilde{j}$: Fourier trafo

1. ampl $\mathcal{M}_f^{\text{real,had}}$ from time-ordered prod

$$\int_x e^{ikx} \langle V| T\{\mathcal{J}_\alpha^\gamma(x)\mathcal{J}_\mu^{L,-}(0)\}|0\rangle$$



2. interference $\mathcal{M}_f^{\text{real,had}} \left[\mathcal{M}_f^{\text{real},\tau}\right]^\dagger$

$$\int_x e^{ikx} \langle 0|\mathcal{J}_\mu^+(0)|V\rangle\langle V|T\{\mathcal{J}_\alpha^\gamma(x)\mathcal{J}_\nu^{L,-}(0)\}|0\rangle$$

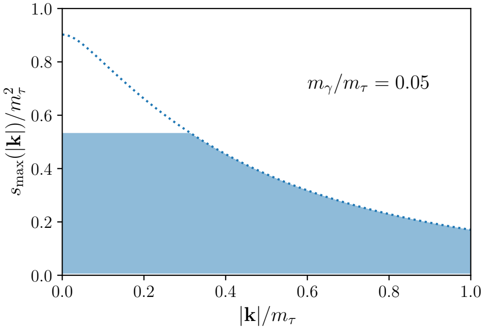
3. hadronic phase-space + sum over all vector states

$$\int_x e^{ikx} \langle 0|\mathcal{J}_\mu^+(0)\,\delta(\hat{H}-\sqrt{s})\,T\{\widetilde{\mathcal{J}}_\alpha^\gamma(x)\mathcal{J}_\nu^{L,-}(0)\}|0\rangle$$

4. calculable from Euclidean correlator $\langle \tilde{j}_\mu^+(t,\mathbf{0})\,\tilde{j}_\alpha^\gamma(\tau,\boldsymbol{k})\,j_\nu^{L,-}(0)\rangle$?
   there is δ-function → first inverse problem?
   there is T product → second inverse problem?

Notation: $\mathcal{J}^{L,-}$ charged weak, $\mathcal{J}^+$ charged vector, $\mathcal{J}$: Minkowski
$\tilde{j}$: Fourier trafo

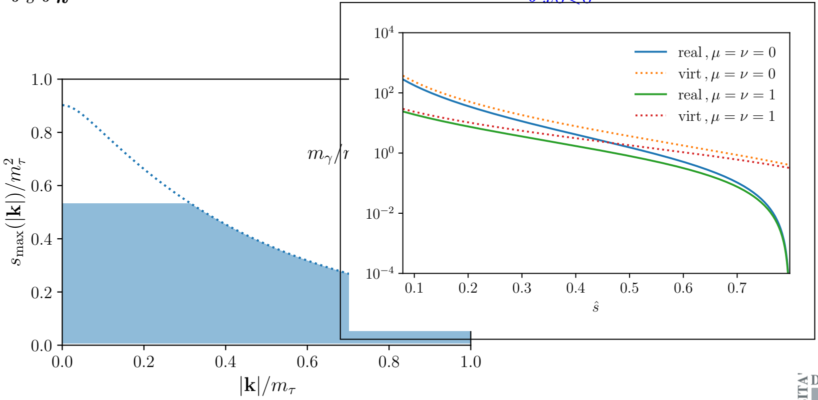
$$\int_s \int_{\boldsymbol{k}} \text{kernel}(\sqrt{s}, \boldsymbol{k}, m_\gamma) \langle 0 | \mathcal{J}^+_\mu(0) \delta(\hat{H} - \sqrt{s}) \times \int_{x0<0} \mathcal{J}^{L,-}_\nu(0) \widetilde{J}^\gamma_\alpha(x_0, \boldsymbol{k}) | 0 \rangle$$



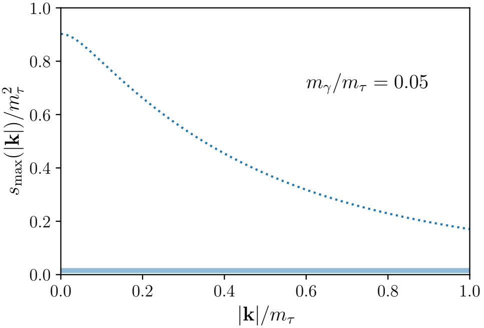
integral $x_0 < 0$
$\rightarrow$ directly calculable from Lattice QCD

integral ds
$\rightarrow$ calculable via exact reconstruction tail

$$\int_s \int_{\boldsymbol{k}} \text{kernel}(\sqrt{s}, \boldsymbol{k}, m_\gamma) \langle 0 | \mathcal{J}^+_\mu(0) \delta(\hat{H} - \sqrt{s}) \times \int_{x0<0} \mathcal{J}^{L,-}_\nu(0) \widetilde{J}^\gamma_\alpha(x_0, \boldsymbol{k}) | 0 \rangle$$



$$\int_s \int_{\boldsymbol{k}} \text{kernel}(\sqrt{s}, \boldsymbol{k}, m_\gamma) \langle 0 | \mathcal{J}^+_\mu(0) \delta(\hat{H} - \sqrt{s}) \times \int_{x0>0} \widetilde{J}^\gamma_\alpha(x_0, \boldsymbol{k}) \mathcal{J}^{L,-}_\nu(0) | 0 \rangle$$



integral over $x_0 > 0$ and ds

$\rightarrow$ $\boldsymbol{k} = 0$ calculable via extract reconstruction tail

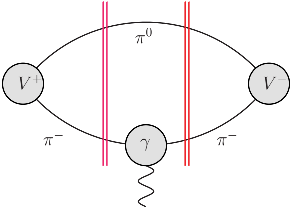
$\rightarrow$ $|\boldsymbol{k}| > 0$ double inverse problem

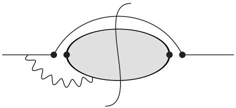
$$x_0 > 0 \qquad \langle 0|\mathcal{J}^+_\mu(0)\,\delta(\hat{H}-\sqrt{s})\widetilde{\mathcal{J}}^\gamma_\alpha(0,\boldsymbol{k})\,\frac{-i}{\hat{H}-\sqrt{s}-k_0-i\varepsilon}\,\mathcal{J}^{L,-}_\nu(0)|0\rangle$$

direct analytic continuation not possible

$\rightarrow$ inverse problem or ...

Restriction to 2π: need a representation of two-pion FF $\rightarrow$ a possible synergy with pheno/dispersive approach





once calculated, hadronic tensor must be integrated in loop integral $\rightarrow$ UV counter-term from $Z^{\mathrm{RI}}_\triangle(\mu)$

other schemes may be preferable

$\dfrac{Z^{\mathrm{RI}}_\triangle(\mu)}{Z^{\mathsf{s}}_\triangle(\mu)}$ from Lattice QCD

A strategy for the HVP prediction from τ decays

Charged vector current $j^-_\mu(x) = \frac{1}{\sqrt{2}}\bar{u}(x)\gamma_\mu d(x)$

$$G^W(t,\mu) = [Z^{\rm RI}_{\rm ud}(\mu)]^2 \langle j^+(t,\mathbf{0}) j^-(0)\rangle_{{\rm QCD}(1+O(\alpha))}$$

---

1. $\left[\frac{1}{\Gamma_e}\frac{d\Gamma}{ds}\right]^{\rm exp} - \frac{1}{\Gamma_e}\frac{d\Gamma^{\rm init}}{ds}(\mu) - \frac{1}{\Gamma_e}\frac{d\Gamma^{\rm nonf}}{ds}(\mu)$
2. Laplace transform result of point 1. w/ operator $\mathfrak{L}(t)$
3. multiply by $\frac{1+\delta\Gamma_e}{[C^{\rm RI}(\mu)]^2}$

$$G^{W,\rm exp}(t,\mu) \equiv \frac{1+\delta\Gamma_e}{[C^{\rm RI}(\mu)]^2}\mathfrak{L}(t)\cdot\left(\left[\frac{1}{\Gamma_e}\frac{d\Gamma}{ds}\right]^{\rm exp} - \frac{1}{\Gamma_e}\frac{d\Gamma^{\rm init}}{ds}(\mu) - \frac{1}{\Gamma_e}\frac{d\Gamma^{\rm nonf}}{ds}(\mu)\right)$$

$$a_\mu^{\mathrm{HVP}} = \int dt\, w(t, m_\mu)\, G^\gamma(t)$$

$G^\gamma(t) \equiv -\frac{1}{3} \sum_k \int d^3\boldsymbol{x}\ \langle j_k^\gamma(t, \boldsymbol{x}) j_k^\gamma(0) \rangle$

Isospin decomposition $G^\gamma(t) \equiv G_{00}^\gamma(t) + 2G_{01}^\gamma(t) + G_{11}^\gamma(t)$

Ward identity protects renormalization of G^γ and $G_{II'}^\gamma$ at $O(\alpha)$

$G^\gamma(t) =$	$G_{00}^\gamma(t) + 2G_{01}^\gamma(t)$		Lattice QCD+QED
	$+ G_{11}^\gamma(t) - G^W(t, \mu)$	$= \delta G_{11}(t, \mu)$	Lattice QCD+QED
	$+ G^W(t, \mu) - G^{W,\mathrm{exp}}(t, \mu)$	$= G^{W,>}(t, \mu)$	Exp. data or theory
	$+ G^{W,\mathrm{exp}}(t, \mu)$		Exp. data and theory

Calculate a_μ, windows etc... of any of the above

$G^{W,>}(t, \mu) \neq 0$ due to spectrum above τ mass

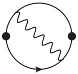
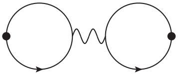
if taken from e^+e^- data risk of double counting IB effects

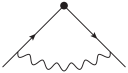
Neutral vs charged correlators

$\delta G_{11}(t,\mu) = G^{\gamma}_{11}(t) - G^{W}(t,\mu)$

calculable in Euclidean space from Lattice QCD

isosymmetric limit $\delta G_{11} = 0$

requires only  and 

renormalized only by (amputated) 

scheme + scale + gauge dependent

fully inclusive both in channels and energy

(expected to be) dominated by 2π

Strategy is now complete, paper towards finalization

- short-distance in RI-SMOM scheme
  - renormalization of charged current (beginning data production)
- initial-state corrections analytic
- final-state in Euclidean space
  - [talks by Julian, Christoph on progress w/ data generation]
- non-factorizable partly w/o inverse problem
  - partly w/ severe inverse problem

Thanks for your attention

UNIVERSITA' DEGLI STUDI DI MILANO BICOCCA

Initial state radiation

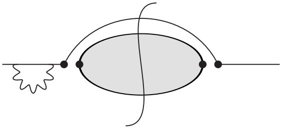
Wave-function renormalization

$$Z_\tau = 1 + \frac{\alpha}{2\pi}\left[\log\frac{m_\tau}{\mu} + 2\log\frac{m_\gamma}{m_\tau} + \cdots\right]$$

$$\frac{d\Gamma}{ds} \simeq 2\times\frac{1}{2}[Z_\tau - 1]|\mathcal{M}|^2$$

$$\delta Z_\tau \equiv \frac{\alpha}{2\pi}\log(m_W/m_\tau)$$

[Sirlin '82]



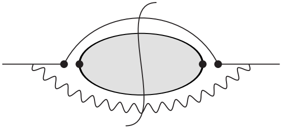
τ Bremsstrahlung

[Cirigliano et al '00, '01][MB et al, in prep]

$$\frac{d\Gamma}{ds}\ \frac{\alpha}{\pi}\left[B_{\log}(s,m_\gamma) + \ldots\right]$$

$$B_{\log}(s,m_\gamma) = \log\frac{m_\gamma}{m_\tau} + \cdots$$

$$\delta\kappa(s) \equiv B_{\log}(s,m_\tau) + B_1(s)\log(s) + B_2(s)$$



$$G_F^2|V_{ud}|^2\,\frac{m_\tau^3}{8\pi}\,\kappa(s)\,\rho(s)\left[\delta Z_\tau + \delta\kappa(s)\right]$$

UNIVERSITA' DEGLI STUDI DI MILANO BICOCCA

THE CHALLENGE = $\langle\pi\pi|j^\gamma|0\rangle$

[Maiani-Testa '90][MB-Hansen '20]

Starting point: euclidean correlator $\langle\widetilde{\pi}_{\boldsymbol{q}_1}(t_1)\widetilde{\pi}_{\boldsymbol{q}_2}(t_2)j^{\gamma,1}(0)\rangle$

$\widetilde{\pi}_{\boldsymbol{q}}$ pion interpolating operator projected to $\boldsymbol{q}$

$\omega_{\boldsymbol{q}_i} = \sqrt{M_\pi^2 + \boldsymbol{q}_i^2}$ pion energy

0. Limit of large t_1

$$\langle\widetilde{\pi}_{\boldsymbol{q}_1}(t_1)\widetilde{\pi}_{\boldsymbol{q}_2}(t_2)j^{\gamma,1}(0)\rangle \overset{t_1\to\infty}{\simeq} [2\omega_{\boldsymbol{q}_1}]^{-1}\sqrt{Z_\pi}e^{-\omega_{\boldsymbol{q}_1}t_1}\underbrace{\langle\pi,\boldsymbol{q}_1|\widetilde{\pi}_{\boldsymbol{q}_2}(t_2)\,j^{\gamma,1}(0)|0\rangle}$$

1. Use physical Hamiltonian $\hat{H}$ to remove t_2 dependence

$$e^{\omega_{\boldsymbol{q}_2}t_2} \times \langle\pi,\boldsymbol{q}_1|\widetilde{\pi}_{\boldsymbol{q}_2}(t_2)\,j^{\gamma,1}(0)|0\rangle = \langle\pi,\boldsymbol{q}_1|e^{+\hat{H}t_2}\widetilde{\pi}_{\boldsymbol{q}_2}(0)\,e^{-\hat{H}t_2} \times e^{\omega_{\boldsymbol{q}_2}t_2}\,j^{\gamma,1}(0)|0\rangle$$

2. set $\boldsymbol{q}_1 = -\boldsymbol{q}_2 = \boldsymbol{q} \to \langle\pi,\boldsymbol{q}|\widetilde{\pi}_{-\boldsymbol{q}}(0)\,e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2}\,j^{\gamma,1}(0)|0\rangle$

limit $t_2 \gg 0$ does not converge

THE CHALLENGE $= \langle \pi\pi | j^\gamma | 0 \rangle$

$\langle \pi, \boldsymbol{q} | \widetilde{\pi}_{-\boldsymbol{q}}(0)\, e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2}\, j^{\gamma,1}(0) | 0 \rangle$

1. Inverse problem formally defined by [Hansen-Bulava '19]

$$\int dt_2\, g(t_2|\varepsilon)\, e^{-x t_2} = \frac{1}{x + i\varepsilon}$$

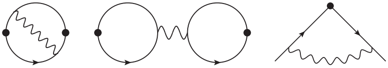
LSZ = amplitude from residue in limit $\varepsilon \to 0$

2. $e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2} \to \Theta(\hat{H} - 2\omega_{\boldsymbol{q}}) e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2}$ [MB-Hansen '20]

   amplitude (real and imaginary) from large t_2 limit

Discussion session on isospin-breaking for τ

Neutral vs charged correlators

$\delta G_{11}(t,\mu) = G^{\gamma}_{11}(t) - G^{W}(t,\mu) \rightarrow$ 

Pheno analysis of $\delta G_{11}(t,\mu)$: what's inside it?

- fully inclusive therefore 2π and 4π, and more ...
- intermediate/long distance windows preferable quantities
  - $\rightarrow$ suppression $4\pi, ..$

Neutral vs charged correlators

Pheno analysis of $\delta G_{11}(t,\mu)$: now focus on two pion channel

$$\int_E \int_{\text{ph.sp.}} |\langle 0|j^{\gamma,1}|\pi^+\pi^-(\gamma)\rangle|^2 e^{-E_{\pi^+\pi^-(\gamma)}t} - |\langle 0|j^-(\mu)|\pi^0\pi^-(\gamma)\rangle|^2 e^{-E_{\pi^0\pi^-(\gamma)}t}$$

1. difference of 2π form factors
   the charged FF is prescription dependent!
2. difference of phase space due to $m_{\pi^-} - m_{\pi^0}$
3. difference in final-state radiation

How to compare/mix/match Lattice and dispersive/pheno analysis?

1. dispersive adopts counter-terms from Lattice, i.e. RI-SMOM
2. lattice uses same prescription as disp

Same discussion applies to non-factorizable effects as well