Eighth Plenary Workshop of the Muon g-2 Theory Initiative

FSR and ρ-width IB corrections beyond scalar QED and their impact on $\Delta a_\mu^{HVP}[\pi\pi, \tau]$

To be submitted to Arxiv

Genaro Toledo (IFUNAM)

In collaboration with
Francisco Flores Baéz (UANL)
Gabriel López Castro (CINVESTAV)

física teórica IFUNAM
UANL UNIVERSIDAD AUTÓNOMA DE NUEVO LEÓN
Cinvestav

Université Paris-Saclay, 8-12 sept 2025

Outline

- HVP evaluation using tau data, status of IB corrections
- ΔΓρ: radiative corrections, sQED and structure dependence
- Structure dependent effects on FSR
- Impact on $\Delta a_{\mu}^{HVP}[\pi\pi, \tau]$
- Conclusions

HVP evaluation using tau data

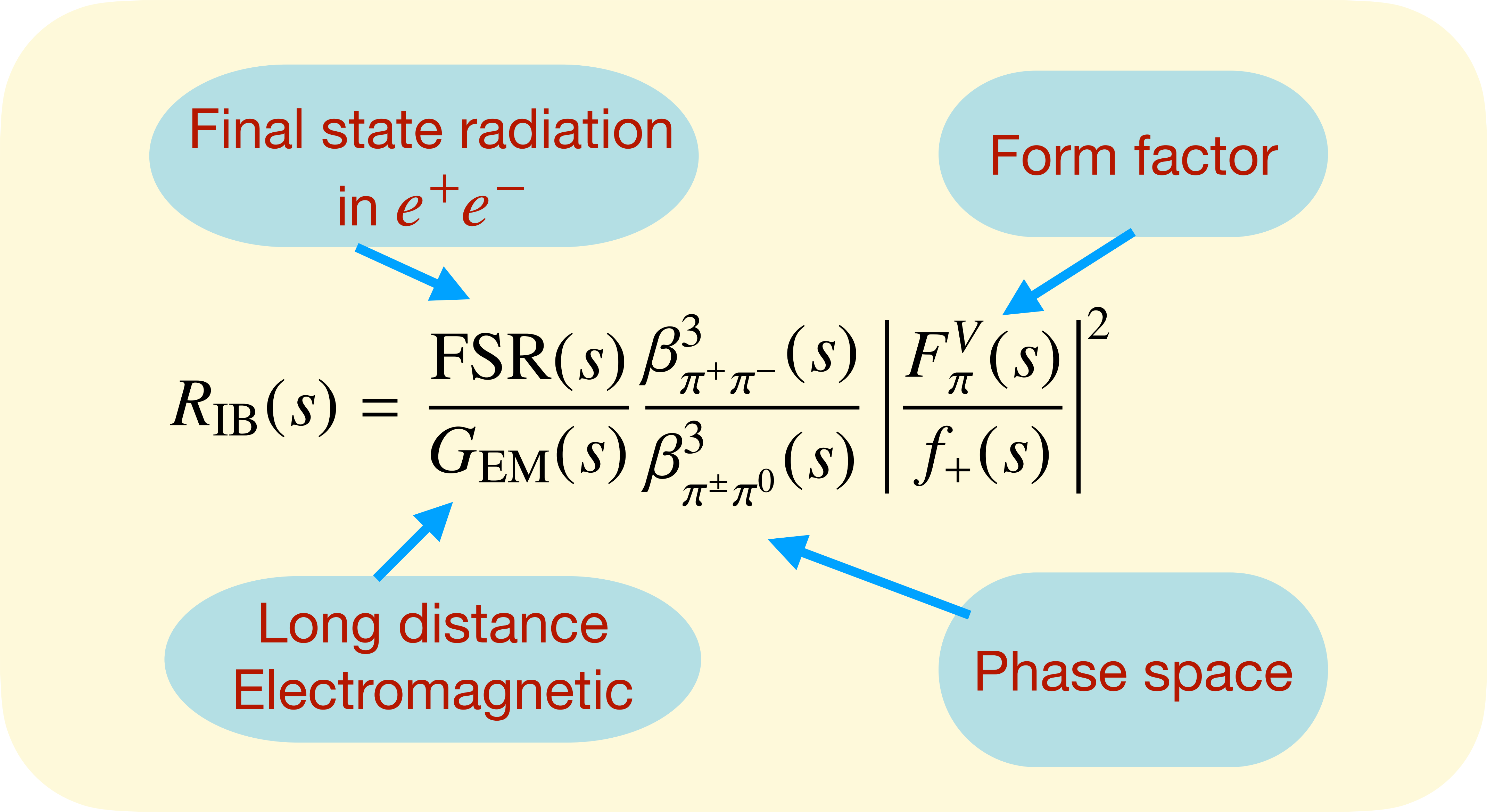
The leading order HVP contribution using tau data requires to account for the IB correction

$$\Delta a_\mu^{HVP}[\pi\pi,\tau] = \frac{\alpha^2 m_\tau^2}{6|V_{ud}|^2\pi^2}\frac{\mathcal{B}_{\pi\pi^0}}{\mathcal{B}_e}\int_{4m_\pi^2}^{m_\tau^2} ds\,\frac{K(s)}{s}\quad\frac{dN_{\pi\pi^0}}{N_{\pi\pi^0}\,ds}\left(1-\frac{s}{m_\tau^2}\right)^{-2}\left(1+\frac{2s}{m_\tau^2}\right)^{-1}\left[\frac{R_{\mathrm{IB}}(s)}{S_{\mathrm{EW}}}-1\right]$$

QED Kernel function

Described by data

The isospin breaking corrections are given by



Final state radiation in e^+e^-

Form factor

$$R_{\mathrm{IB}}(s) = \frac{\mathrm{FSR}(s)}{G_{\mathrm{EM}}(s)}\frac{\beta^3_{\pi^+\pi^-}(s)}{\beta^3_{\pi^\pm\pi^0}(s)}\left|\frac{F_\pi^V(s)}{f_+(s)}\right|^2$$

Long distance Electromagnetic

Phase space

S_{EW} Short distance EW correction

HVP contribution status

$\Delta a_\mu^{HVP}[\pi\pi,\tau]$ in 10^(-10) units

Phase space	-7.88	-7.52	–	$-7.7(2)$
S_{EW}	$-12.21(15)$	$-12.16(15)$	–	$-12.2(1.3)$
G_{EM}	$-1.92(90)$	$-1.67^{+0.60}_{-1.39}$	–	$-2.0(1.4)$
FSR	4.67(47)	4.62(46)	4.42(4)	4.5(3)
ρ–ω mixing	4.0(4)	2.87(8)	3.79(19)	3.9(3)

M. Davier, étal EPJC 84, 721 (2024),
M. Davier, étal , EPJC 66, 127 (2010)

Lopez Castro, Miranda, and Roig,
PRD 111, 073004 (2025)

G. Colangelo, etal JHEP 10, 032 (2022)
M. Hoferichter, étal PRL 131, 161905 (2023)

Theory initiative update
Arxiv: 2505.21476

FSR computed using sQED, first introduced in 2010 by M. Davier, étal EPJC 66, 127 (2010).
Uncertainty associated to the missing structure effects

HVP contribution status. Form Factors

$\Delta a_\mu^{HVP}[\pi\pi,\tau]$ in 10^(-10) units

$\frac{F_\pi^V}{f_+}$ (w/o ρ–ω)	ΔM_ρ	$0.20(^{+27}_{-19})(9)$	$1.95^{+1.56}_{-1.55}$	
	$\Delta\Gamma_\rho(\Delta M_\pi)$	$4.09(0)(7)$	3.37	
	$\Delta\Gamma_\rho(\pi\pi\gamma)$	$-5.91(59)(48)$	$-6.66(73)$	
	$\Delta\Gamma_\rho(g_{\rho\pi\pi})$	–	–	
	Total	$-1.62(65)(63)$	$(-1.34)^{+1.72}_{-1.71}$	$-1.5(4.7)$

M. Davier, étal EPJC 84, 721 (2024),
M. Davier, étal , EPJC 66, 127 (2010)

Lopez Castro, Miranda, and Roig,
PRD 111, 073004 (2025)

Theory initiative update
Arxiv: 2505.21476

Corrections computed using different FF parameterizations

$\rho \to \pi\pi(\gamma)$ decay, affecting the rho width difference, computed using sQED, structure dependent effects expected, 10% uncertainty assigned

The rho width difference gives the most important contribution to FF

The ρ width difference

$$\Delta\Gamma_\rho \equiv \Gamma_{\rho^0} - \Gamma_{\rho^+}$$

$$\Delta m_\rho \equiv m_{\rho^+} - m_{\rho^0}$$

Data based

$\Delta\Gamma_\rho = (0.3 \pm 1.3)$ MeV

$\Delta m_\rho = (+0.7 \pm 0.8)$ MeV

S. Navas et al. [Particle Data Group],
PRD 110 (2024)

$\Delta\Gamma_\rho = (-0.58 \pm 1.04)$ MeV

$\Delta m_\rho = (+0.30 \pm 0.53)$ MeV

Davier, Malaescu, and Zhang
ArXiv:2504.13789v1 (2025)

Theoretical, EM radiative corrections, sQED

$\Delta\Gamma_\rho = (+0.76 \pm 0.20)$ MeV

$+1.82\ (\pm 10\%)$

radiative correction

-1.06

Δm_π

structure

Flores-Baez, Castro and Toledo
PRD 76, 096010 (2007)

Approximations:

Structureless pion and rho photon interactions (sQED)

Only convection-convection terms in virtual corrections

The ρ width difference

The rho width difference can be split into $\Delta\Gamma_\rho = \Delta\Gamma_\rho[\pi\pi(\gamma)] + \Delta\Gamma_\rho(\text{rest})$

where $\Delta\Gamma_\rho(\text{rest})$ includes all the measured channels, except the two pions and the corresponding radiative channel

$\rho^0 : l^+l^-, \pi^0\gamma, \eta\gamma, 3\pi, 4\pi$ $\quad$ $\mathrm{BR}(\rho^0 \to \text{rest}) = (1.04 \pm 0.10) \times 10^{-3}$

$\rho^+ : \pi^+\gamma$ $\quad$ $\mathrm{BR}(\rho^+ \to \text{rest}) = (4.53 \pm 0.46) \times 10^{-4},$

S. Navas et al. [Particle Data Group], PRD 110 (2024)

Thus, for a common neutral and charged rho width (for example 150 MeV)

$$\begin{aligned}\Delta\Gamma_\rho(\text{rest}) &= \Gamma_{\rho^0} \times \mathrm{BR}(\rho^0 \to \text{rest}) - \Gamma_{\rho^+} \times \mathrm{BR}(\rho^+ \to \text{rest}) \\ &= (0.088 \pm 0.017)\ \mathrm{MeV}\ ,\end{aligned}$$

At the precision level of a few tenths of a percent,
the widths of rho mesons are driven by the $\rho \to \pi\pi(\gamma)$ decay

Contributions to $\Delta\Gamma_\rho[\pi\pi(\gamma)]$

The neutral and charged $\rho \to \pi\pi(\gamma)$ width are given by

$$\Gamma[\rho^+ \to \pi^+\pi^0(\gamma)] = \frac{g_+^2 m_{\rho^+}}{48\pi}\beta_+^3(1+\delta_+)$$

$$\Gamma[\rho^0 \to \pi^+\pi^-(\gamma)] = \frac{g_0^2 m_{\rho^0}}{48\pi}\beta_0^3(1+\delta_0)$$

where δ_+ and δ_0 account for the radiative correction

Thus, the rho width difference can be set, in terms of the IB parameters, as

$$\Delta\Gamma_\rho[\pi\pi(\gamma)] = \Gamma(\rho^0 \to \pi^+\pi^-)\left[\delta_0 - \delta_+ - \frac{\Delta m_\rho}{m_{\rho^0}} - \frac{6m_+^2}{m_{\rho^0}^2\beta_0^2}\left(\frac{\Delta}{m_+} + \frac{2\Delta m_\rho}{m_{\rho^0}}\right) - \frac{2\delta g}{g_0}\right]$$

where $\Delta \equiv m_+ - m_0$ $\qquad \Delta m_\rho \equiv m_{\rho^+} - m_{\rho^0}$ $\qquad \delta g \equiv g_+ - g_0$

Thus, the radiative corrections δ_+ and δ_0 for charged and neutral rho are needed

Previous analysis

Flores-Baez, Castro and Toledo
PRD 76, 096010 (2007)

$$\rho \to \pi\pi(\gamma)$$

Real photon emission: sQED + Model dependent

Virtual photon emission: $\rho^0 \to \pi^+\pi^-(\gamma)$ sQED

$\rho^+ \to \pi^+\pi^0(\gamma)$ sQED (convection terms only)

Meister and Yennie, PR 130, 1210 (1963)

Queijeiro and García, PRD 38, 2218 (1988)

$\rho\rho\gamma$ vertex

$$\Gamma^{\mu\nu\alpha} = (2P - K)^\alpha g^{\mu\nu} + \cancel{2(k^\mu g^{\nu\alpha} - k^\nu g^{\mu\alpha}) - P^\mu g^{\nu\alpha} - (P-k)^\nu g^{\mu\alpha}}$$

IR Finite
UV Finite with convection

Radiative correction for the neutral and charged processes and their difference, at m_ρ= 775 MeV

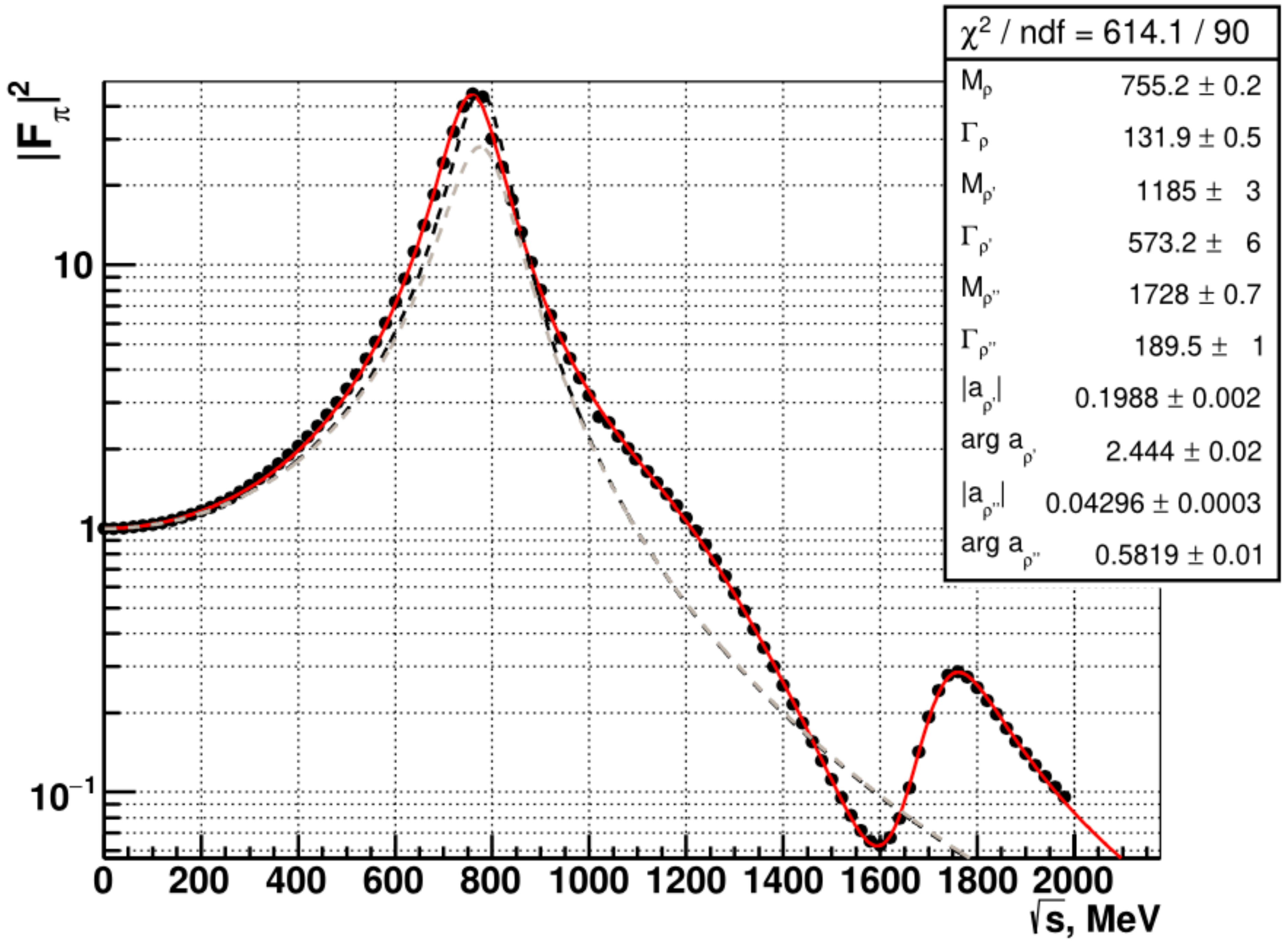
$$\delta_+ = -4.15 \times 10^{-3} \qquad \delta_0 = 8.05 \times 10^{-3} \qquad \delta_0 - \delta_+ = 12.2 \times 10^{-3}$$

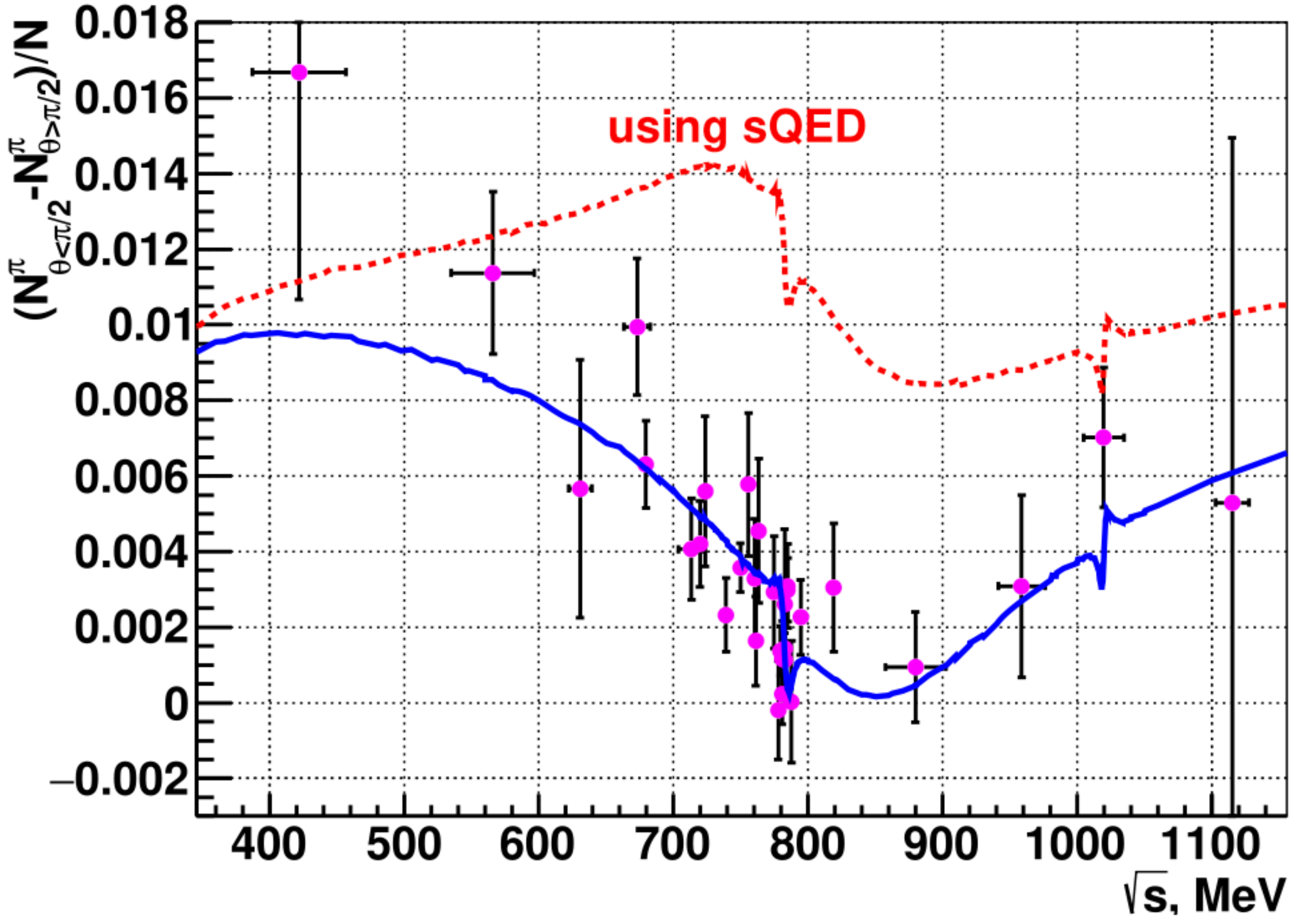
GVMD on pion FF

sQED insufficient to explain the charge asymmetry in $e^+e^- \to \pi^+\pi^-$. Pion structure required in virtual corrections [Ignatov, Lee PLB (2022)]

$$F_\pi(s) = \sum_{V=\rho,\rho',\rho''} a_v \frac{m_V^2 - im_V\Gamma_V}{m_V^2 - im_V\Gamma_V - s}$$

$$\sum a_V = 1$$





New analysis

F. V. Flores-Baez, G. L. Castro and G. Toledo
To be submitted to Arxiv

Improvements

- Structure effect via a form factor ✓
- Full EM vertex of the ρ, not only convection term ✓

Byproduct

- Implications for FSR ✓

Real photon emission not modified

Structure effect

✓ Beyond sQED by modifying the photon propagator in loops $\frac{1}{k^2} \to \frac{1}{k^2}\left[F_V(k^2)\right]^2$ makes virtual corrections finite

where $F_V(k^2) \equiv \dfrac{M_V^2}{M_V^2 - k^2}$ and $M_V^2 = m_\rho^2 - i m_\rho \Gamma_\rho$

Resembles the GVMD by Ignatov et al, where up to three resonances were used to fit the pion form factor and explain pi pi charge asymmetry

Ignatov and Lee, Phys.Lett.B833,137283(2022)

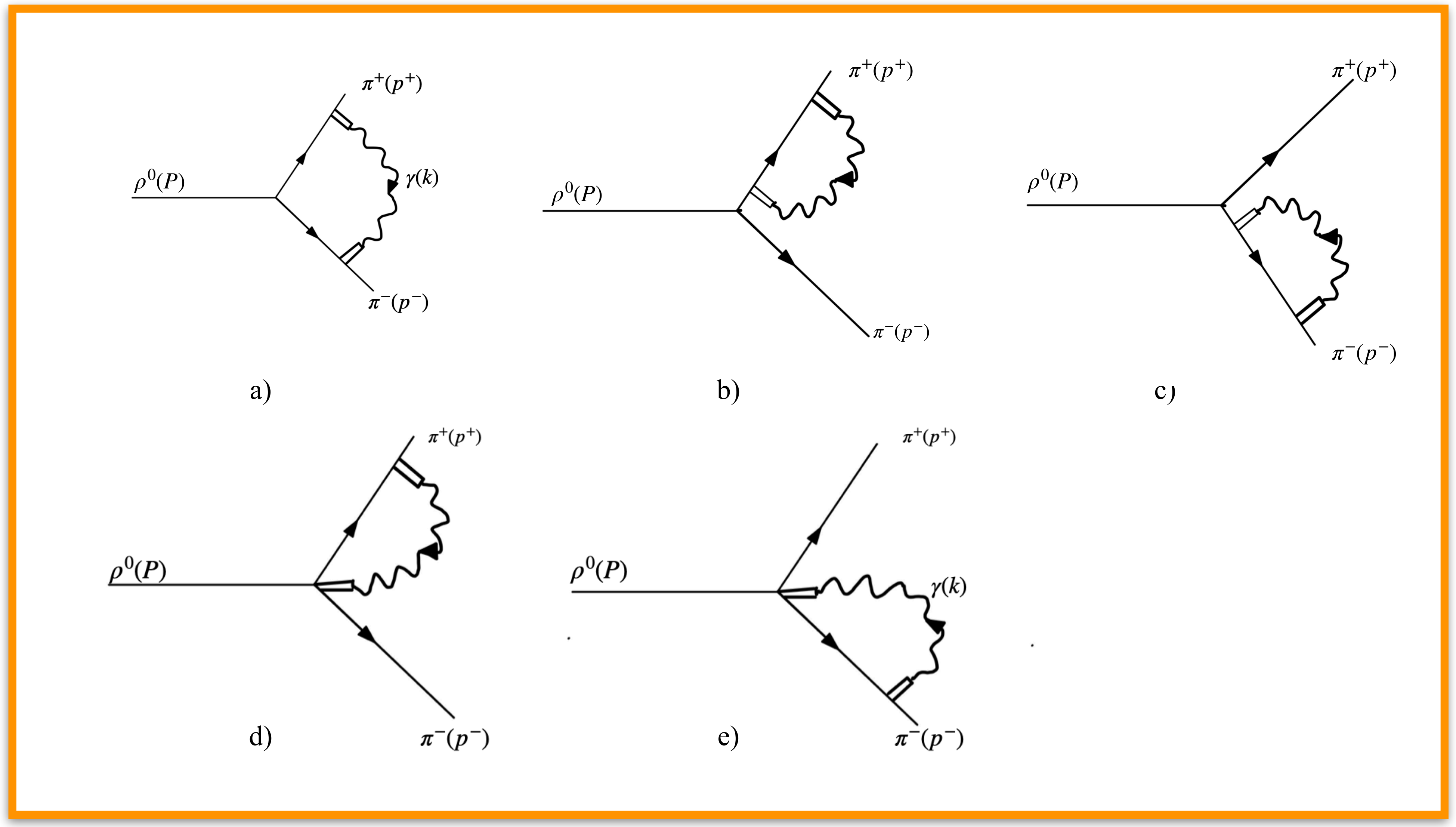
Colangelo, Hoferichter, Monnard, and Ruiz de Elvira, JHEP 08, 295 (2022)

We consider a single resonance, the ρ, with the mass and width as given in the PDG

Neutral rho meson

$$\rho^0 \to \pi^+\pi^-(\gamma)$$

Structured vertices



$$\delta_0 = 8.05 \times 10^{-3}$$

Flores-Baez, Castro and Toledo
PRD 76, 096010 (2007)

now

$$\delta_0 = 6.39 \times 10^{-3}$$

Real photon emission same as in the previous analysis, verified

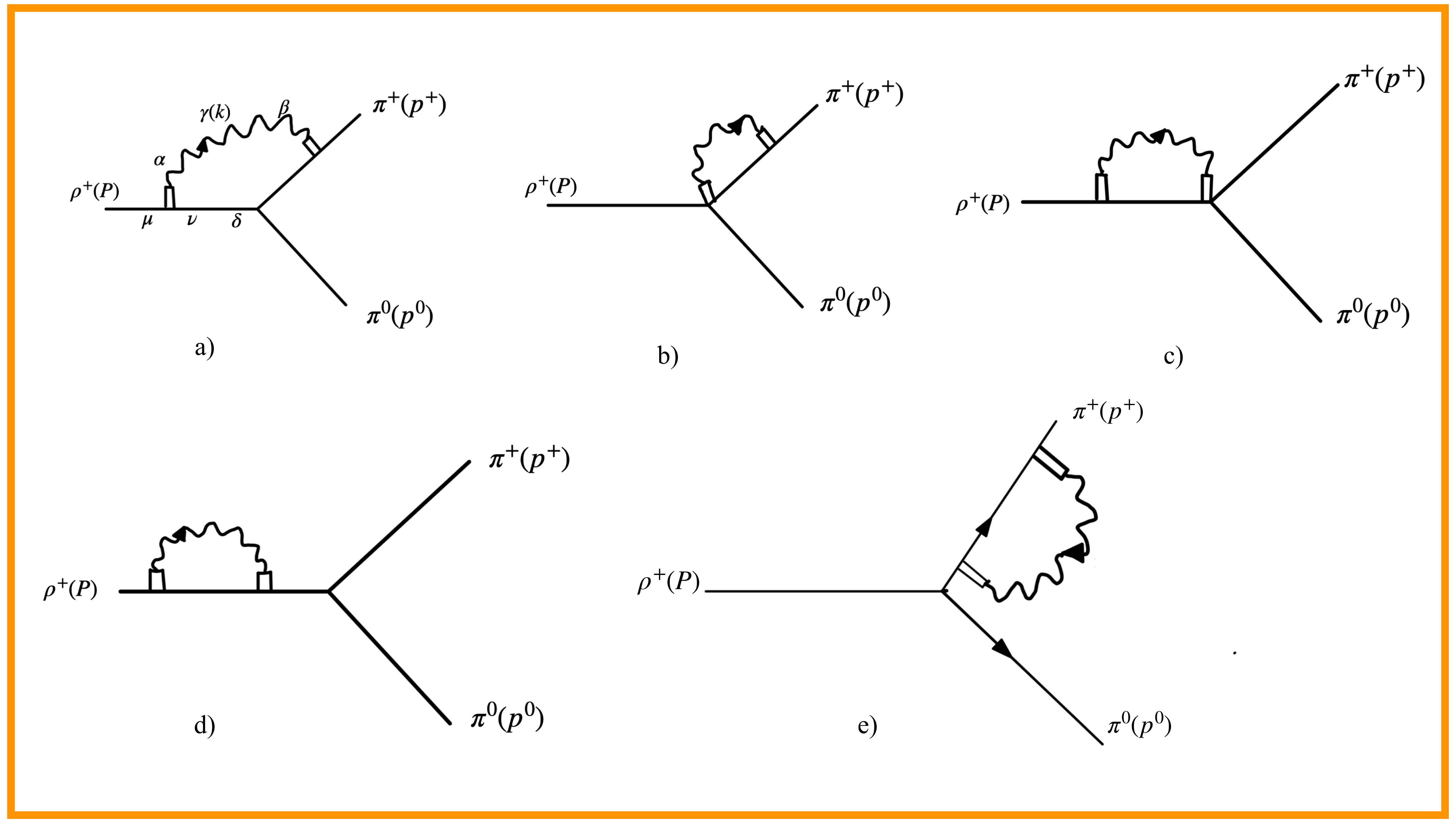
Infrared and UV finite upon inclusion of real photon emission contribution

Backup slides: Analytical expressions in terms of PaVe

Charged rho meson

$$\rho^+ \to \pi^+ \pi^0 (\gamma)$$

Structured vertices



$$\delta_+ = -4.15 \times 10^{-3}$$

Flores-Baez, Castro and Toledo
PRD 76, 096010 (2007)

now

$$\delta_+ = +0.85 \times 10^{-3}$$

$\rho\rho\gamma$ vertex

$$\Gamma^{\mu\nu\alpha} = (2P - K)^\alpha g^{\mu\nu} + 2(k^\mu g^{\nu\alpha} - k^\nu g^{\mu\alpha})$$
$$-P^\mu g^{\nu\alpha} - (P - k)^\nu g^{\mu\alpha}$$

✓

Real photon emission same as in the previous analysis, verified

Infrared and UV finite

Radiative correction

$m_{\rho^{+,0}}$ (MeV)	δ_0 $(\times 10^{-3})$	δ_+ $(\times 10^{-3})$	$\delta_0 - \delta_+$ $(\times 10^{-3})$
772	6.409	0.833	5.576
772.5	6.407	0.836	5.571
773	6.404	0.838	5.566
773.5	6.402	0.841	5.561
774	6.399	0.843	5.556
774.5	6.397	0.846	5.551
775	6.394	0.848	5.546
775.5	6.392	0.851	5.541
776	6.389	0.853	5.536
776.5	6.387	0.856	5.531
777	6.384	0.858	5.526
777.5	6.382	0.860	5.521
778	6.379	0.863	5.517

Radiative correction for the neutral and charged processes and their difference, ✓
as a function of the ρ mass, in the region around the physical value

Compared to the previous value at the same energy

$$\delta_0 = 8.05 \times 10^{-3} \qquad \delta_+ = -4.15 \times 10^{-3} \qquad \delta_0 - \delta_+ = 12.2 \times 10^{-3}$$

Flores-Baez, Castro and Toledo
PRD 76, 096010 (2007)

Contributions to $\Delta\Gamma_\rho[\pi\pi(\gamma)]$

The rho width difference, in terms of the IB parameters, is

$$\Delta\Gamma_\rho[\pi\pi(\gamma)] = \Gamma(\rho^0 \to \pi^+\pi^-)\left[\delta_0 - \delta_+ - \frac{2\delta g}{g_0} - \frac{\Delta m_\rho}{m_{\rho^0}} - \frac{6m_+^2}{m_{\rho^0}^2\beta_0^2}\left(\frac{\Delta}{m_+} + \frac{2\Delta m_\rho}{m_{\rho^0}}\right)\right]$$

Width difference contribution + 0.83 MeV + - 0.19 MeV + - 1.1 MeV ➡ $\Delta\Gamma_\rho[\pi\pi(\gamma)]$ = - 0.47 MeV

± 0.22 not included before

For $\Gamma(\rho^0 \to \pi^+\pi^-) = 150$ MeV

$\Delta = 4.5936$ MeV

$\Delta_\rho = (+0.7 \pm 0.8)$ MeV

S. Navas et al. [Particle Data Group], PRD 110 (2024)

vs previous $+1.82\ (\pm 10\%)$ + - 1.1 MeV

structure uncertainty now accounted

Flores-Baez, Castro and Toledo PRD 76, 096010 (2007)

Adding the difference from other channels, we obtain the total width difference

$$\Delta\Gamma_\rho = \Delta\Gamma_\rho[\pi\pi(\gamma)] + \Delta\Gamma_\rho(\text{rest}) = -0.38 \pm 0.22 \text{ MeV}$$

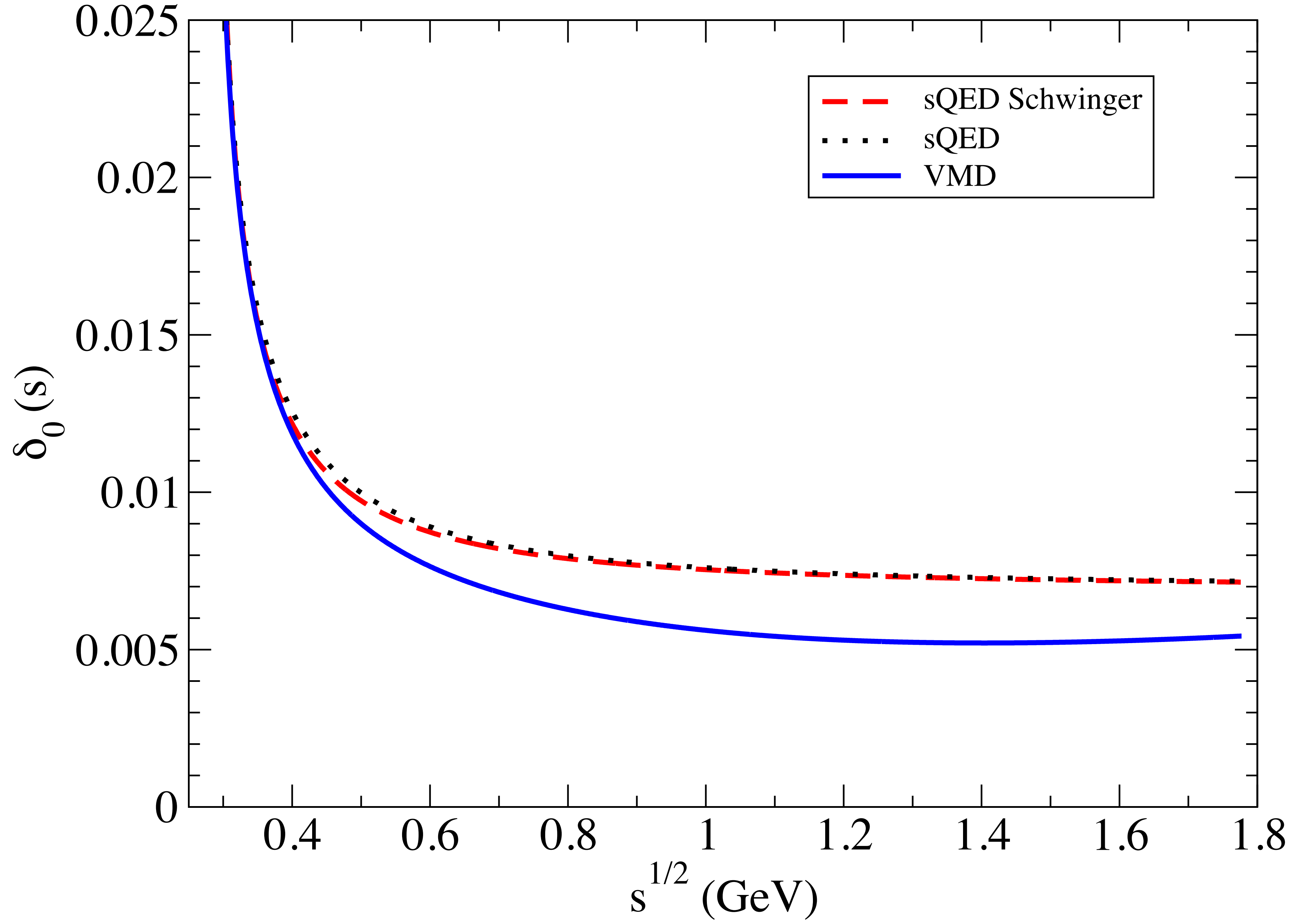
This can be compared with $\Delta\Gamma_\rho = (-0.58 \pm 1.04)$ MeV

Davier, Malaescu, and Zhang ArXiv:2504.13789v1 (2025)

FSR

The $\rho^0 \to \pi^+\pi^-(\gamma)$ decay is related to the FSR contribution in the $e^+e^- \to \pi^+\pi^-$ process.
Thus, our result for $\delta_0(s)$ incorporates the structure on the FSR in the VMD approach

$$FSR(s) = 1 + \delta_0(s)$$



Radiative correction δ_0 as a function of energy

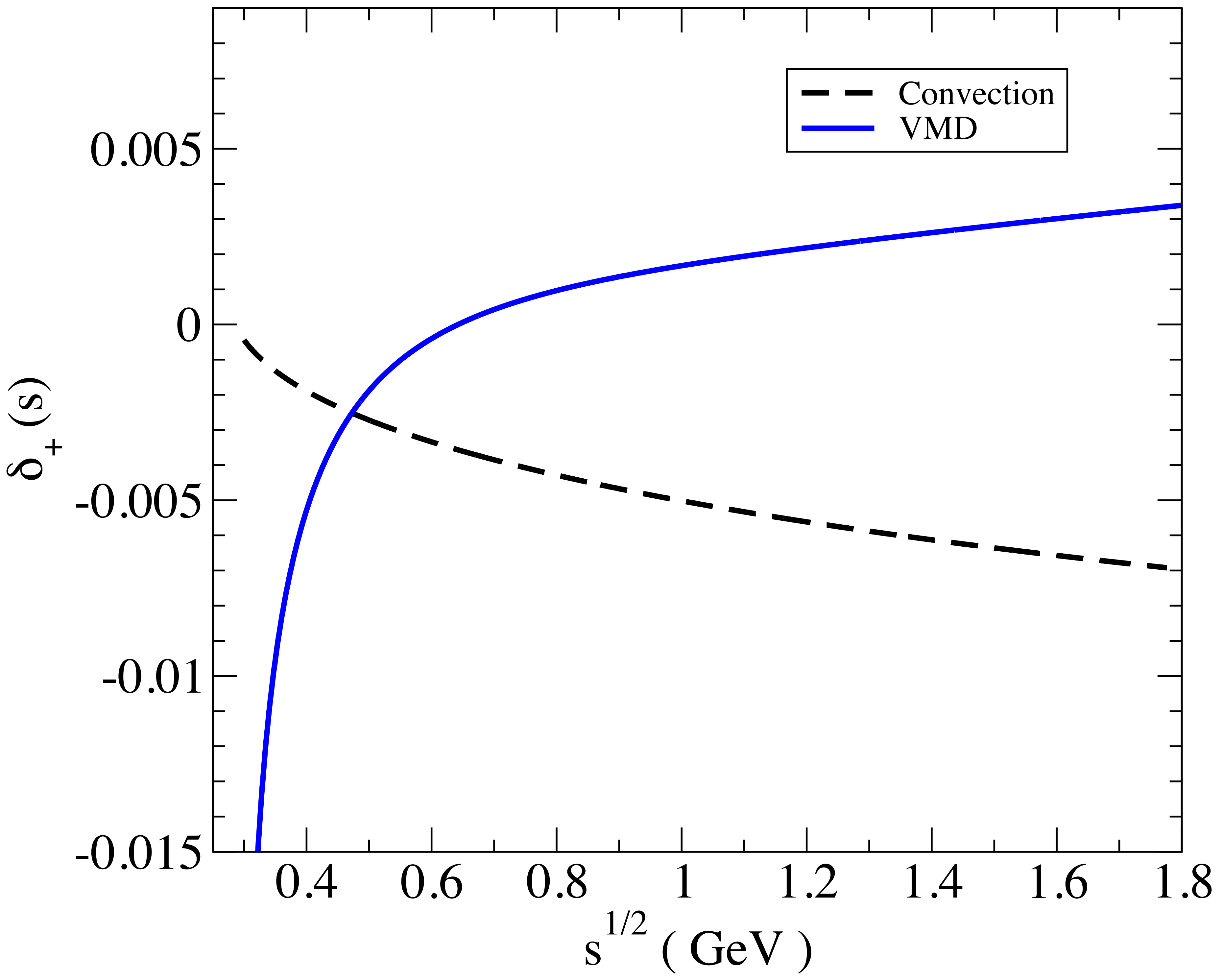
Cross-check: Our result matches the one by Schwinger (Drees-Hikasa) for sQED.
This allows to identify the additional structure effects on VMD

J. S. Schwinger
Particles, Sources and Fields, vol. 3 (AW, 1989)

M. Drees and K. Hikasa
Phys.Lett.B252 127 (1990)

Flores-Baez, Castro and Toledo
PRD 76, 096010 (2007)

δ_+ energy dependence



$\rho\rho\gamma$ vertex

$$\Gamma^{\mu\nu\alpha} = (2P-K)^\alpha g^{\mu\nu} + 2(k^\mu g^{\nu\alpha} - k^\nu g^{\mu\alpha})$$
$$-P^\mu g^{\nu\alpha} - (P-k)^\nu g^{\mu\alpha}$$

Radiative correction δ_+ as a function of energy
Convection term considered to make UV finite (previous).
VMD (new) incorporates the full vertex structure

Estimated impact on $\Delta a_\mu^{HVP}[\pi\pi,\tau]$

$$\Delta a_\mu^{HVP}[\pi\pi,\tau] = \frac{\alpha^2 m_\tau^2}{6|V_{ud}|^2\pi^2}\frac{\mathcal{B}_{\pi\pi^0}}{\mathcal{B}_e}\int_{4m_\pi^2}^{m_\tau^2} ds\,\frac{K(s)}{s}\,\frac{dN_{\pi\pi^0}}{N_{\pi\pi^0}ds}\left(1-\frac{s}{m_\tau^2}\right)^{-2}\left(1+\frac{2s}{m_\tau^2}\right)^{-1}\left[FSR(s)\left|\frac{F_\pi^V(s)}{f_+(s)}\right|^2 - 1\right]$$

We compute the corrections using data (uncorrelated)

For the FF we use Gounaris-Sakurai parameterization. Radiative corrections are taken at m_ρ =775 MeV

$$\Delta a_\mu^{HVP}[\pi\pi,\tau]$$

	Source	$FSR \quad (\times 10^{-10})$	$\Delta\Gamma_\rho[\pi\pi(\gamma)] \quad (\times 10^{-10})$
	Davier et al. (10)	4.67(47)	-5.91(59)
This work	sQED	4.64(47)	-5.97(59)
	VMD	3.94	-2.68

For FSR we obtain around 15% reduction respect to sQED

For FF we obtain around 50% reduction respect to incomplete sQED

Conclusions

- We have computed the $\rho \to \pi\pi(\gamma)$ radiative corrections, including the structure dependence via a form factor, consistent with GVMD.

- Beyond the convection approximation was considered

- The rho width difference $\Delta\Gamma_\rho$ was obtained. This switched from

$(+0.76 \pm 0.20)$ MeV to (-0.27) MeV rad. corr. + Δ

(-0.47 ± 0.21) MeV + Δ_ρ

(-0.38 ± 0.22) MeV + $\Delta\Gamma_\rho(rest)$

- As a byproduct, the structure dependent effects on FSR was obtained

- We estimate the impact on $\Delta a_\mu^{HVP}[\pi\pi, \tau]$. FSR-> -15% and FF (radiative part)-> -50% are observed.

Thanks for your attention

Backup

Backup slides: Analytical expressions of delta0 in terms of PaVe

Scalar QED radiative correction

Neutral

$$\begin{aligned}\delta_0^{\mathrm{v}}(\mathrm{sQED}) \;=\; \frac{\alpha}{\pi}\Bigg[&\pi^2\left(\frac{1+\beta_0^2}{2\beta_0}\right) - 2\left(1+\ln\left[\frac{\lambda}{m_{\pi^+}}\right]\right)\left(1+\frac{1+\beta_0^2}{2\beta_0}\ln\left[\frac{1-\beta_0}{1+\beta_0}\right]\right)\\ &-\left(\frac{1+\beta_0^2}{\beta_0}\right)\left[\mathrm{Li}_2(\beta_0)-\mathrm{Li}_2(-\beta_0)\right] - \frac{1+\beta_0^2}{2\beta_0}\left(\mathrm{Li}_2\left[\frac{2}{1+\beta_0}\right]-\mathrm{Li}_2\left[\frac{2}{1-\beta_0}\right]\right)\Bigg]\end{aligned}$$

Charged

$$\begin{aligned}\delta_+^{\mathrm{v}}(\mathrm{sQED}) \;=\; \frac{\alpha}{\pi}\Bigg[&-1-2\ln\left(\frac{\lambda}{m_{\rho^+}}\right)\left[1+\frac{1}{2v_+}\ln\left(\frac{1-v_+}{1+v_+}\right)\right]+\frac{3}{4}\ln\left(\frac{1-v_+^2}{4}\right)\\ &+\frac{1}{1-v_+^2}\left[\ln\left(\frac{4}{1-v_+^2}\right)-v_+\ln\left(\frac{1+v_+}{1-v_+}\right)\right]\\ &+\frac{1}{2v_+}\Bigg\{\ln\left(\frac{1-v_+}{1+v_+}\right)\left[2\ln(v_+)+\frac{3}{4}\ln\left(\frac{1-v_+}{1+v_+}\right)\right]-\frac{\pi^2}{3}+\ln^2\left(\frac{1-v_+^2}{4}\right)\\ &+2\mathrm{Li}_2\left(\frac{1-v_+}{1+v_+}\right)+2\mathrm{Li}_2\left(-\frac{1-v_+}{1+v_+}\right)+2\mathrm{Li}_2\left(\frac{1+v_+}{2}\right)\Bigg\}\Bigg]\ ,\end{aligned}$$

VMD radiative correction

Neutral

$$\begin{aligned}
\delta_0^{\rm v}(\mathrm{VMD}) = \frac{\alpha}{\pi}\Bigg\{ & -m_\rho^4 D0[0, m_{\pi^+}^2, m_{\rho^0}^2, m_{\pi^+}^2, m_{\pi^+}^2, m_{\pi^+}^2, m_{\rho^0}^2, m_{\rho^0}^2, m_{\pi^+}^2, m_{\pi^+}^2] \frac{[2+\beta_0^2]^2}{2\beta_0^2} \\
& +m_{\rho^0}^2 C0[m_{\pi^+}^2, m_{\pi^+}^2, m_{\rho^0}^2, m_{\pi^+}^2, m_{\rho^0}^2, m_{\pi^+}^2] \frac{(\beta_0^2-1)(\beta_0^2+2)}{2\beta_0^2} \\
& -\frac{2}{1-\beta_0^2}\left(\ln\left[\frac{4}{1-\beta_0^2}\right] + \frac{1+\beta_0^2}{2\beta_0}\ln\left[\frac{1-\beta_0}{1+\beta_0}\right]\right)\left(1+\frac{2}{\beta_0^2}\right) \\
& -\left[(3-\beta_0^2)\frac{\beta_0}{1-\beta_0^2}\ln\left[\frac{1+\beta_0}{1-\beta_0}\right] + \frac{2}{1-\beta_0^2}\ln[1-\beta_0^2] - \frac{4}{1-\beta_0^2}\ln[2]\right](1+\frac{1}{\beta_0^2}) \\
& -\left[\left(1-\frac{4}{1-\beta_0^2}\right)\ln\left[\frac{1-\beta_0^2}{4}\right] - 2 + \frac{1+3\beta_0^2}{\beta_0(1-\beta_0^2)}\ln\left[\frac{1-\beta_0}{1+\beta_0}\right]\right] \\
& \frac{4}{\beta_0(1-\beta_0^2)^2}\Bigg[\left(1-\frac{3}{2}(1-\beta_0^2)+\frac{3}{8}(1-\beta_0^2)^2\right)\ln\left[\frac{1+\beta_0}{1-\beta_0}\right] \\
& +\beta_0\left(\frac{1-\beta_0^2}{2} - \frac{1}{4}(1-\beta_0^2)^2 + \beta_0^2\ln\left[\frac{1-\beta_0^2}{4}\right]\right)\Bigg]\Bigg\}
\end{aligned}$$

sQED + VMD radiative correction

Charged

$$
\begin{aligned}
&\delta_{+}^{\mathrm{v}}(sQED+VMD)=\frac{\alpha}{\pi}\Bigg\{-\left[1+\frac{1}{2v_{+}}\ln\left[\frac{1-v_{+}}{1+v_{+}}\right]\right]2\ln\,[\frac{\lambda}{m_{\rho^{+}}}]\\
&+\frac{1}{2v_{+}}\ln\left[\frac{1-v_{+}}{1+v_{+}}\right]\left(\frac{1}{2}\ln\left[\frac{1-v_{+}^{2}}{4}\right]-\frac{1}{4}\ln\left[\frac{1-v_{+}}{1+v_{+}}\right]+2\ln\left[\frac{2v_{+}}{1+v_{+}}\right]\right)\\
&+\frac{1}{v_{+}}\left(-\frac{\pi^{2}}{6}+\mathrm{Li}_{2}[\frac{1-v_{+}}{1+v_{+}}]+\mathrm{Li}_{2}[\frac{1+v_{+}}{2}]+\mathrm{Li}_{2}[\frac{v_{+}-1}{1+v_{+}}]+\frac{1}{8}(\ln\,[\frac{1-v_{+}^{2}}{4}])^{2}\right)\\
&+\frac{1}{2}\ln\,[\frac{1-v_{+}^{2}}{4}]+m_{\rho^{+}}^{2}C0[m_{\pi^{+}}^{2},m_{\pi^{+}}^{2},m_{\rho^{+}}^{2},m_{\rho^{+}}^{2},m_{\pi^{+}}^{2},m_{\rho^{+}}^{2}]\left(\frac{7+4v_{+}^{2}}{4v_{+}^{2}}\right)\\
&+m_{\rho^{+}}^{4}D0[0,m_{\pi^{+}}^{2},m_{\pi^{+}}^{2},m_{\rho^{+}}^{2},m_{\pi^{+}}^{2},m_{\rho^{+}}^{2},m_{\rho^{+}}^{2},m_{\rho^{+}}^{2},m_{\pi^{+}}^{2},m_{\rho^{+}}^{2}]\frac{3(2v_{+}^{2}-1)}{4}\\
&-\frac{2}{1-v_{+}^{2}}\left(\ln\,[\frac{4}{1-v_{+}^{2}}]+\frac{1+v_{+}^{2}}{2v_{+}}\ln\,[\frac{1-v_{+}}{1+v_{+}}]\right)\left(\frac{3}{4v_{+}^{2}}+\frac{1}{2}\right)\\
&-\frac{1}{v_{+}(1-v_{+}^{2})}\ln\,[\frac{1+v_{+}}{1-v_{+}}]-\frac{1}{v_{+}^{2}(1-v_{+}^{2})}\ln\,[\frac{1-v_{+}^{2}}{4}]\\
&-\frac{1}{2}\left(1-\frac{4}{1-v_{+}^{2}}\right)\ln\,[\frac{1-v_{+}^{2}}{4}]-\frac{1+3v_{+}^{2}}{2v_{+}(1-v_{+}^{2})}\ln\,[\frac{1-v_{+}}{1+v_{+}}]\\
&+\frac{2}{(1-v_{+}^{2})^{2}}[\left(1-\frac{3}{2}(1-v_{+}^{2})+\frac{3}{8}(1-v_{+}^{2})^{2}\right)\ln\,[\frac{1+v_{+}}{1-v_{+}}]\\
&+v_{+}\left(\frac{1-v_{+}^{2}}{2}-\frac{(1-v_{+}^{2})^{2}}{4}+v_{+}^{2}\ln[\frac{1-v_{+}^{2}}{4}]\right)]+\frac{1287-119\sqrt{3}\pi}{1296}-\frac{\pi}{2\sqrt{3}v_{+}^{2}}\Bigg\}
\end{aligned}
$$