Dispersive analysis of the isospin-breaking corrections to $\pi\pi$ scattering and pion vector form factor

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Introduction

2π contribution $\sim 72\,\%$ of $a_\mu^{\mathrm{HVP,LO}}$

$$\langle \pi(p')|j^\mu_{em}(0)|\pi(p)\rangle = \pm(p'+p)^\mu F_\pi^V\left[(p'-p)^2\right]$$

$$\sigma(e^+e^-\to 2\pi) = \frac{\pi\alpha^2}{3s}\sigma_\pi^3(s)|F_\pi^V(s)|^2\frac{s+2m_e^2}{s\sigma_e(s)}$$

$\to$ Radiative corrections in $e^+e^-\to\pi^+\pi^-$ must be under control

- Watson theorem: same phase of $e^+e^-\to\pi^+\pi^-$ and $\pi\pi\to\pi\pi$ if only $\pi\pi$ intermediate state
- Importance of rescattering effects

Goal: dispersive treatment of the isospin-breaking effects for the pion vector form factor

$\pi\pi$ scattering amplitude

Isospin and charge basis

$$\langle \pi^a \pi^b | T | \pi^c \pi^d \rangle = A(s,t,u)\delta^{ab}\delta^{cd} + A(t,s,u)\delta^{ac}\delta^{bd} + A(u,t,s)\delta^{ad}\delta^{bc}$$

Isospin basis:

$$\begin{aligned} T^0(s,t) &= 3A(s,t,u) + A(t,u,s) + A(u,s,t) \\ T^1(s,t) &= A(t,u,s) - A(u,s,t) \\ T^2(s,t) &= A(t,u,s) + A(u,s,t) \end{aligned}$$

Charge basis:

$$T^c(s,t,u) := T(\pi^+\pi^- \to \pi^+\pi^-) = \frac{1}{6}T^2 + \frac{1}{2}T^1 + \frac{1}{3}T^0 = A(s,t,u) + A(t,u,s)$$

$$T^x(s,t,u) := T(\pi^+\pi^- \to \pi^0\pi^0) = -\frac{1}{3}T^2 + \frac{1}{3}T^0 \; = A(s,t,u)$$

$$T^n(s,t,u) := T(\pi^0\pi^0 \to \pi^0\pi^0) = \frac{2}{3}T^2 + \frac{1}{3}T^0 \; = A(s,t,u) + A(t,u,s) + A(u,s,t)$$

$\pi\pi$ scattering amplitude

Dispersive representation in the isospin limit

Starting point: Roy-equations for $\pi\pi$-scattering below $s_1 \sim 1$ GeV

[Ananthanarayan et al, '01 & Garcia-Martin et al, '11]

- $\pi\pi$ invariant amplitude: $A(s,t,u) = A(s,t,u)_{SP} + A(s,t,u)_d$
- A_{SP} contribution of the S and P waves below s_1:

$$A(s,t,u)_{SP} = 32\pi \left\{ \frac{1}{3} W^0(s) + \frac{3}{2}(s-u) W^1(t) + \frac{3}{2}(s-t) W^1(u) + \frac{1}{2} W^2(t) + \frac{1}{2} W^2(u) - \frac{1}{3} W^2(s) \right\}$$

$\rightarrow$ $W^I(s)$ only RHC: $W^0(s) = \frac{a_0^0\, s}{4M_\pi^2} + \frac{s(s-4M_\pi^2)}{\pi} \int_{4M_\pi^2}^{s_1} ds' \, \frac{\operatorname{Im} t_0^0(s')}{s'(s'-4M_\pi^2)(s'-s)}$

- $A_d(s,t,u)$ = background amplitude: higher partial waves and higher energies
  $\rightarrow$ for $s < s_1$ small and smooth $\rightarrow$ polynomial

$\Rightarrow$ construct isospin amplitudes T^0, T^1 and T^2

$\pi\pi$ scattering amplitude

Dispersive representation of isospin-breaking corrections

- Strong isospin breaking effects proportional to $(m_u - m_d)$
- Effects proportional to $M_\pi - M_{\pi^0}$
- Further photon exchanges

$\rightarrow$ each of them can be considered separately from the other two

$\pi\pi$ scattering amplitude

Dispersive representation of isospin-breaking corrections

- From isospin to charge basis: T^0, T^1, $T^2 \to T^c$, T^n, T^x

- Unitarity relation:

$$\begin{aligned}
\operatorname{Im} t_S^n(s) &= \sigma_0(s)|t_S^n(s)|^2 + 2\sigma(s)|t_S^x(s)|^2 \\
\operatorname{Im} t_S^x(s) &= \sigma_0(s)t_S^n(s)t_S^x(s)^* + 2\sigma(s)t_S^x(s)t_S^c(s)^* \\
\operatorname{Im} t_S^c(s) &= \sigma_0(s)|t_S^x(s)|^2 + 2\sigma(s)|t_S^c(s)|^2
\end{aligned}$$

with $\sigma(s) = \sqrt{1 - \frac{4M_\pi^2}{s}}$ and $\sigma_0(s) = \sqrt{1 - \frac{4M_{\pi^0}^2}{s}}$

$\to$ encode the effect of $\Delta_\pi = M_\pi^2 - M_{\pi^0}^2$

$\pi\pi$ scattering amplitude

Dispersive representation of isospin-breaking corrections

- Input above s_1 not changed for $\Delta_\pi \neq 0$
  $\rightarrow$ concentrate on T_{SP}: S and P waves below $s_1 \simeq 1$ GeV
- Express W^I in terms of imaginary parts of physical channels

$$T^n_{SP}(s,t,u) = 32\pi \left(W_S^{n,00}(s) + W_S^{n,+-}(s) + (s \leftrightarrow t) + (s \leftrightarrow u) \right)$$

where

$$W_S^{n,00}(s) = \frac{a_n^{00}\, s}{4M_{\pi^0}^2} + \frac{s(s-4M_{\pi^0}^2)}{\pi} \int_{4M_{\pi^0}^2}^{s_1} ds' \frac{\operatorname{Im} t_S^{n,00}(s')}{s'(s'-4M_{\pi^0}^2)(s'-s)}$$

$$W_S^{n,+-}(s) = \frac{s(s-4M_{\pi^0}^2)}{\pi} \int_{4M_{\pi}^2}^{s_1} ds' \frac{\operatorname{Im} t_S^{n,+-}(s')}{s'(s'-4M_{\pi^0}^2)(s'-s)}$$

with

$$\operatorname{Im} t_S^{n,00}(s) = \sigma_0(s) \left| t_S^n(s) \right|^2 , \qquad \operatorname{Im} t_S^{n,+-}(s) = 2\sigma(s) \left| t_S^x(s) \right|^2 \qquad (1)$$

$\pi\pi$ scattering amplitude

Strategy

- Analytical projection into S and P partial waves

$$t_S^n(s) = a_n^{00} + \int_{4M_{\pi^0}^2}^{s_1} \mathrm{d}s'\, K_n(s,s')\, \mathrm{Im} t_S^{n,00}(s') + \int_{4M_\pi^2}^{s_1} \mathrm{d}s'\, K_n(s,s') \mathrm{Im} t_S^{n,+-}(s') + d_S^n(s)$$

$\rightarrow K_n(s,s')$ analytically known
$\rightarrow d_S^n(s) =$ background integral contribution

- Scattering lengths: matching with χPT with Δ_π [Knecht and Nehme, '02]
- Partial wave parametrisation with $\delta_S^n(s)$:

$$t_S^n(s) = \frac{\eta_S(s)\, \mathrm{e}^{2i\delta_S^n(s)} - 1}{2i\sigma_0(s)}$$

$\pi\pi$ scattering amplitude

Strategy

- Isospin limit Roy-equation solutions of $\delta_S^{n,\mathsf{IL}}(s)$, $\delta_S^{c,\mathsf{IL}}(s)$, $\delta_P^{n,\mathsf{IL}}(s)$, ... and parametrize Δ_π effects:

$$\delta_S^n(s) = \delta_S^{n,\mathsf{IL}}(s)\left(1 + \Delta_\pi\left[\sum_{i=0}^{7} c_i^n \left(\frac{s-4M_{\pi^0}^2}{4M_{\pi^0}^2}\right)^i + \sigma(s)\sum_{i=0}^{1}\tilde{c}_i^n\left(\frac{s-4M_{\pi^0}^2}{4M_{\pi^0}^2}\right)^i\right]\right)$$

- Solve the coupled system of integral equations with c_i^n, $\tilde{c}_i^n$ as fitting parameters

Results

Scattering lengths

- Isospin limit [Colangelo et al, '01 & Caprini et al, '12]

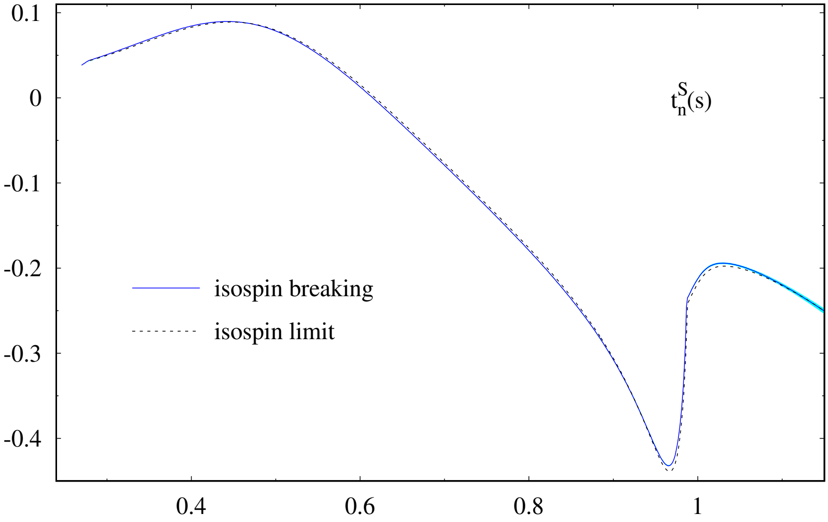
$$a^{00}_{n,\,\mathsf{IL}} = 0.0437\,, \quad a^{++}_{\mathsf{IL}} = -0.0444\,,$$
$$a^{+-}_{c,\,\mathsf{IL}} = 0.0659\,, \quad a^{+-}_{x,\,\mathsf{IL}} = 0.0881\,,$$
$$a^{+0}_{c,\,\mathsf{IL}} = -0.0222$$

- Isospin-breaking effect [Knecht and Nehme, '02]

$$\Delta a^{00}_{n} = -5.375 \times 10^{-3}\,, \quad \Delta a^{++} = 2.918 \times 10^{-3}\,,$$
$$\Delta a^{+-}_{c} = 4.076 \times 10^{-3}\,, \quad \Delta a^{+-}_{x} = 0.081 \times 10^{-3}\,,$$
$$\Delta a^{+0}_{c} = -1.474 \times 10^{-3}$$

Results

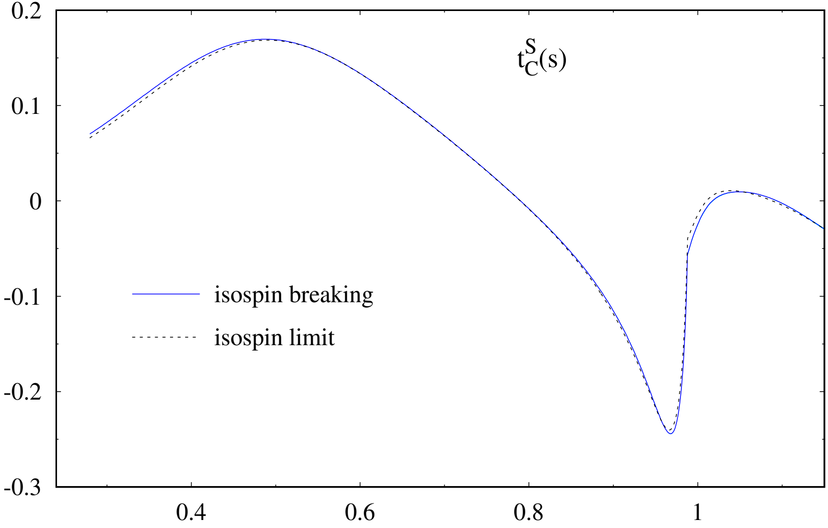
$t_{\mathrm{n}}^{\mathrm{S}}(s)$ [preliminary]



[Colangelo, MC, Ruiz de Elvira, to be submitted for publication]

Results

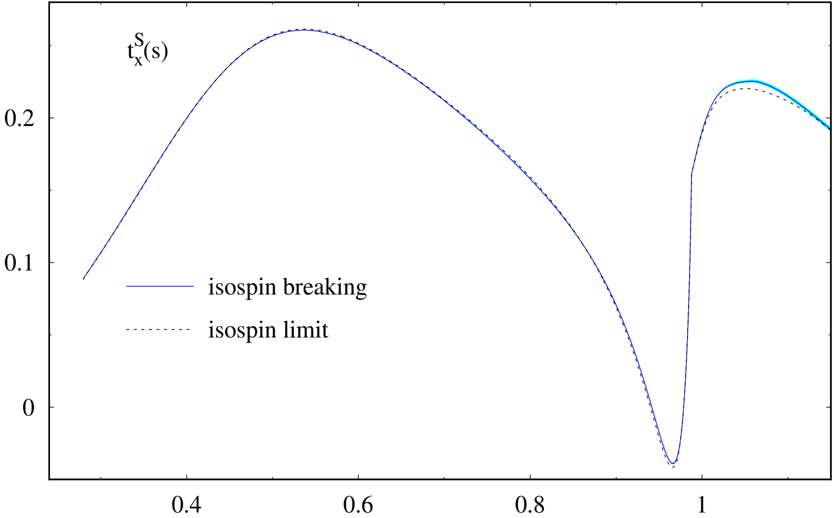
$t_c^S(s)$ [preliminary]



[Colangelo, MC, Ruiz de Elvira, to be submitted for publication]

Results

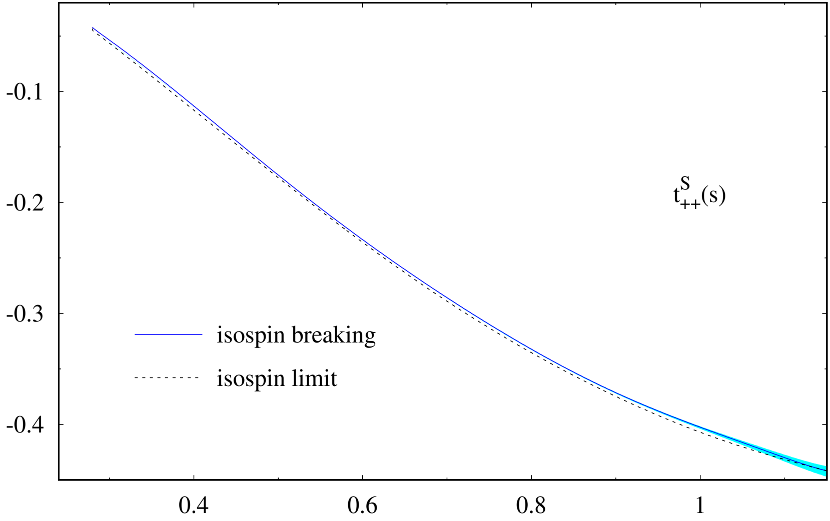
$t_x^S(s)$ [preliminary]



[Colangelo, MC, Ruiz de Elvira, to be submitted for publication]

Results

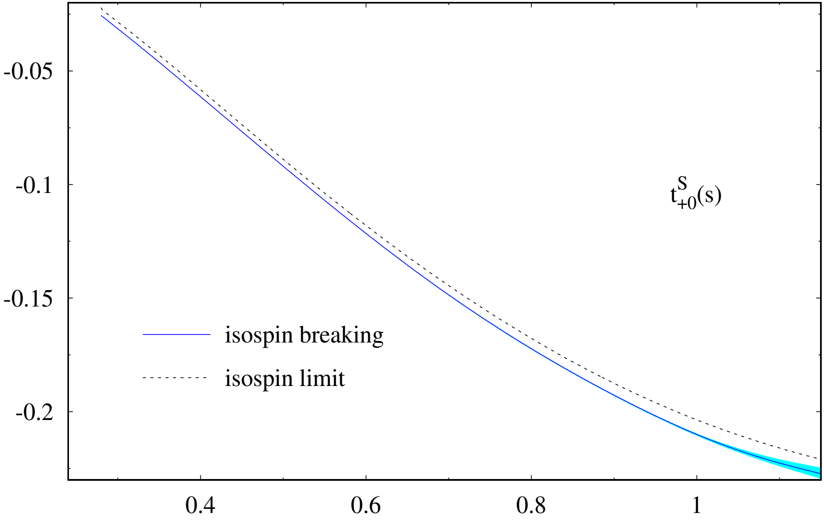
$t^{S}_{++}(s)$ [preliminary]



[Colangelo, MC, Ruiz de Elvira, to be submitted for publication]

Results

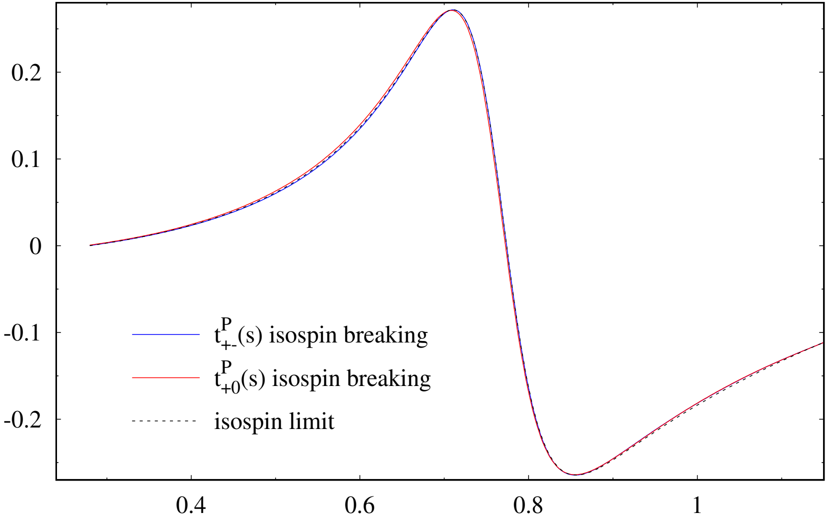
$t^S_{+0}(s)$ [preliminary]



[Colangelo, MC, Ruiz de Elvira, to be submitted for publication]

Results

$t^{\mathrm{P}}_{+-}(s)$ and $t^{\mathrm{P}}_{+0}(s)$ [preliminary]



[Colangelo, MC, Ruiz de Elvira, to be submitted for publication]

Pion vector form factor

$$F_\pi^V(s) = \Omega_1^1(s) G_\omega(s) G_{in}^N(s) \qquad \text{[Colangelo et al, '19]}$$

- $\Omega_1^1(s)$ is the Omnés function

$$\Omega_1^1(s) = \exp\left\{ \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)} \right\}$$

$\rightarrow$ Isospin breaking effects: $\delta_1^1(s) \rightarrow \delta^{\mathrm{IB}}(s)$

- G_ω accounts for $\rho - \omega$ mixing

$$G_\omega(s) = 1 + \frac{s}{\pi} \int_{9M_\pi^2}^{\infty} ds' \frac{\mathrm{Im} g_\omega(s')}{s'(s'-s)} \left(\frac{1 - \frac{9M_\pi^2}{s'}}{1 - \frac{9M_\pi^2}{M_\omega^2}} \right)^4$$

- $G_{\mathrm{in}}^N(s)$ is a conformal polynomial taking into account inelastic channels

$$G_{\mathrm{in}}^N(s) = 1 + \sum_{k=1}^{N} c_k \left(z^k(s) - z^k(0) \right)$$

Pion vector form factor

Isospin-breaking corrections for the $\rho(770)$

- Only pole parameters provide a model-independent result

$$\sqrt{s_\rho} = M_\rho - i\frac{\Gamma_\rho}{2}$$

$\rightarrow$ Breit-Wigner or Gounaris-Sakurai parameters reaction-dependent

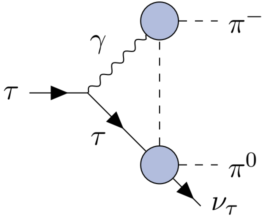
- Roy equations provide model-independent access to the complex plane

$$\sqrt{s_{\rho^0}} = 763.29 - i71.25\,\text{MeV}\,, \quad \sqrt{s_{\rho^\pm}} = 762.30 - i71.89\,\text{MeV}$$

$$\rightarrow M_{\rho^0} - M_{\rho^\pm}\Big|_{\Delta_\pi} \sim 1\,\text{MeV}$$

Pion vector form factor

τ-decay



$$f_+(s) = \left(1 + G^N_{\text{in}}(s) + \mathcal{A}_{\rho'\rho''}(s)\right)\Omega(s)$$

$\rightarrow$ see Simon's talk

$e^+e^- \rightarrow \pi^+\pi^-$

- ρ^0 resonance dominance
- $\rho - \omega$ mixing

$\rightarrow$P-wave phase in the $\pi^+\pi^-$ channel

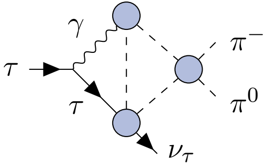
$\tau^\pm \rightarrow \pi^\pm\pi^0\nu_\tau$

- $\rho^\pm$ resonance dominance
- ρ', ρ'' contribution

$\rightarrow$P-wave phase in the $\pi^0\pi^\pm$ channel

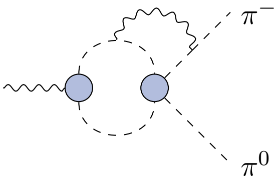
IB corrections due to Δ_π
for $\pi\pi$ scattering

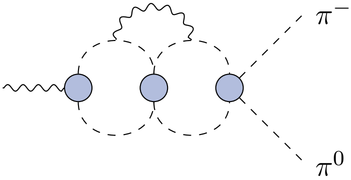
$\Rightarrow$



Higher order corrections

Work in progress: FSR

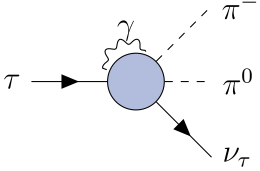




- Matching to χPT of all (sub)amplitudes
- improved estimate of uncertainties

$\rightarrow$ see Monnard PhD thesis and Jacobo's talk at KEK

$\Downarrow$



Conclusion and Outlook

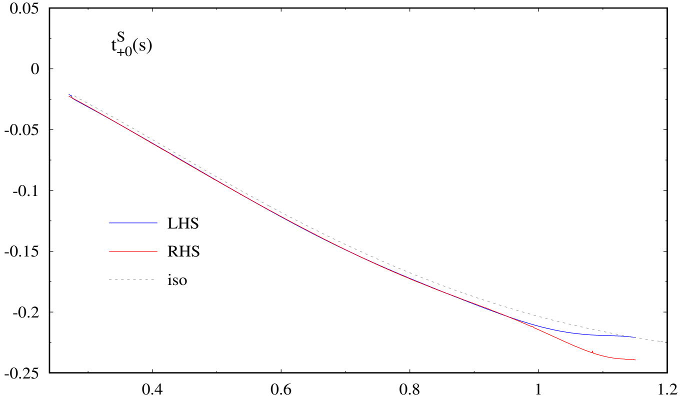
- Model-independent approach for the isospin breaking corrections due to Δ_π to the $\pi\pi$ scattering

- Isospin breaking effects for the pion vector form factor:
  - $F_\pi^V(s)$ in $e^+e^- \to \pi^+\pi^-$
  - $f_+(s)$ in $\tau^- \to \pi^-\pi^0\nu_\tau$

- Work in progress: dispersive approach for the higher order corrections (see Jacobo's talk at KEK)

- Final goal: provide a ready-to-use code to be implemented in MC and used in data analysis

$t^{S}_{+0}(s)$



- Unphysical bending enforced by the matching at s_1: Δ_π effects not neglected
- matching conditions at $s_2 = 4\,\text{GeV}^2$