Dispersive evaluation of the two-pion channel of HVP

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in collaboration with **Peter Stoffer**

Based on the **Colangelo-Hoferichter-Kubis-Leplumey-Stoffer** (CHKLS) framework

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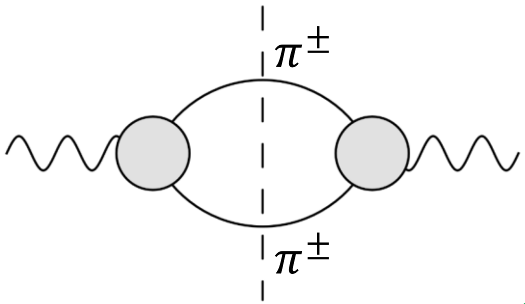
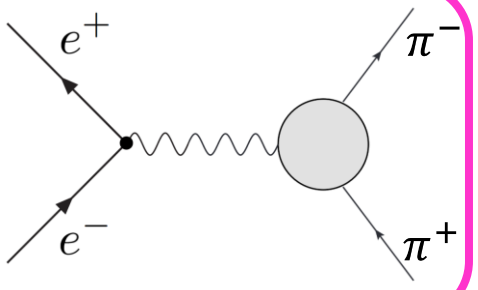
8^{th} plenary workshop of the muon $g-2$ theory initiative

IJCLab, Orsay, September 9, 2024

- $\pi\pi$ channel is the dominant source of uncertainty in HVP
- Many discrepancies remain between $e^+e^- \to \pi^+\pi^-$ experiments

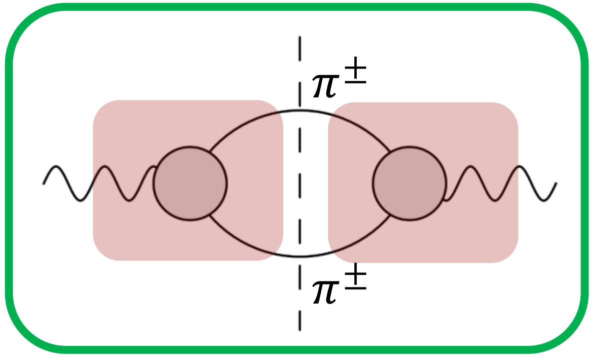


$$a_\mu^{\mathrm{HVP}(\pi\pi)} = \frac{m_\mu^2}{4\pi^2}\int_{s_{\mathrm{thr}}}^{\infty} \frac{\hat{K}(s)}{s}\sigma(e^+e^- \to \pi^+\pi^-(+\gamma))$$

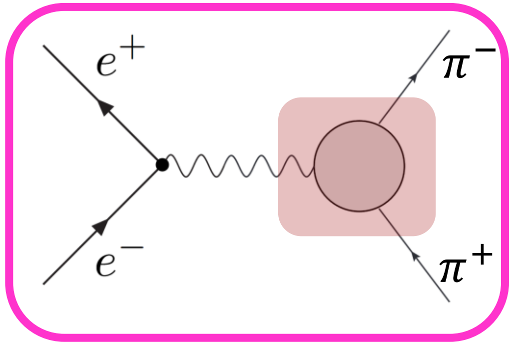


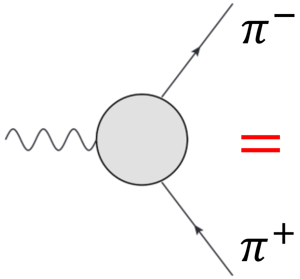
$\pi\pi$ CHANNEL OF HVP

- $\pi\pi$ channel is the dominant source of uncertainty in HVP
- Many discrepancies remain between $e^+e^- \to \pi^+\pi^-$ experiments
- How can we use theory inputs to shed light on these puzzles?



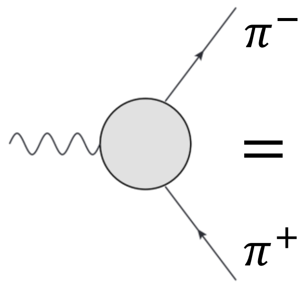
$$a_\mu^{\mathrm{HVP}(\pi\pi)} = \frac{m_\mu^2}{4\pi^2} \int_{s_{\mathrm{thr}}}^{\infty} \frac{\hat{K}(s)}{s} \sigma(e^+e^- \to \pi^+\pi^-(+\gamma))$$





$$= -ie(p' - p)^\mu F_\pi^V(s)$$

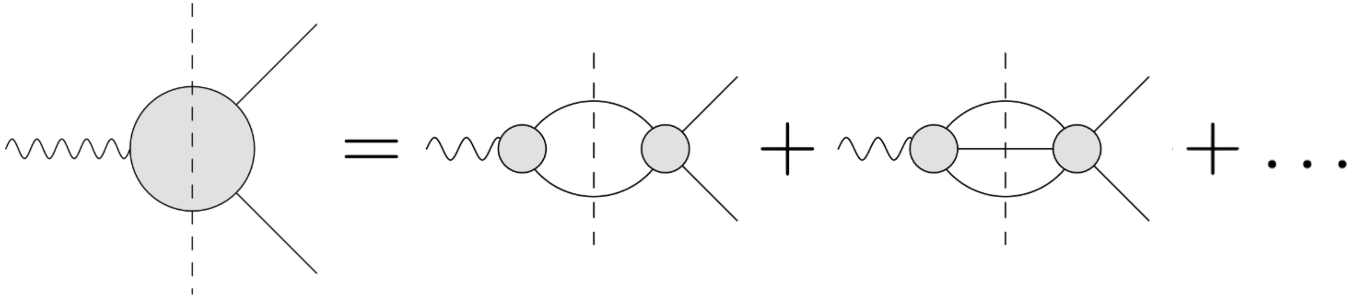
PION VECTOR FORM FACTOR



$$= -ie(p' - p)^\mu \boldsymbol{F_\pi^V(s)}$$

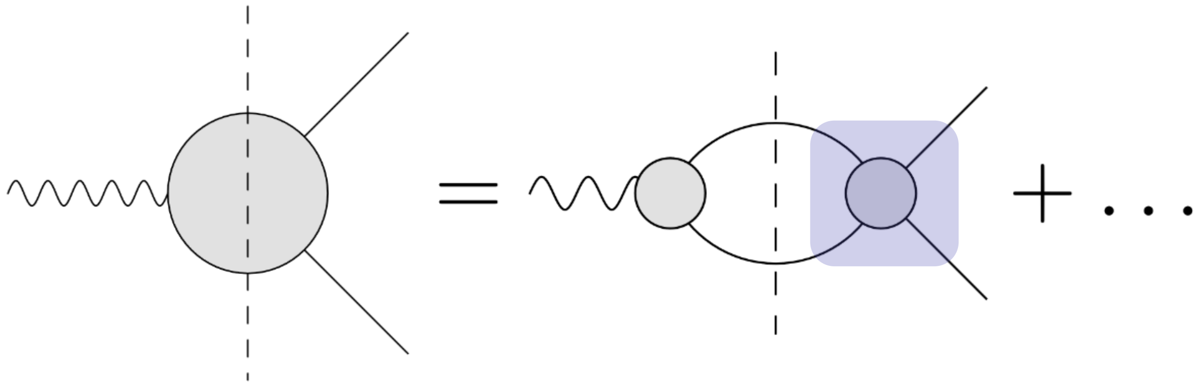
- **Unitarity:** The VFF can be decomposed into intermediate states contributions
- **Analiticity**: The knowledge of the VFF above threshold implies its knowledge everywhere

- With theory input on the channels ($\pi^+\pi^-, \pi^0\gamma, 3\pi[\omega, \dots], 4\pi[\pi^0\omega, \dots]$, etc.) we can write a **model-independent parameterized closed form** of the VFF that we can **fit to data**



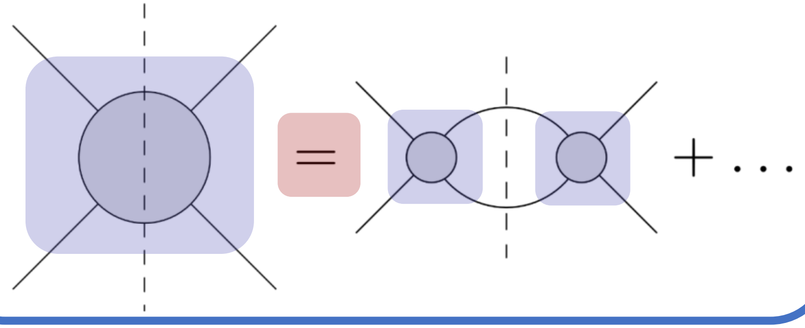
Model-independent description of the pion VFF

- Decomposition of the VFF in terms of intermediate states

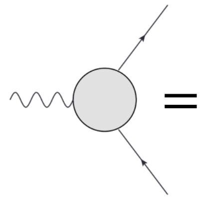


Caprini, Colangelo, Leutwyler, Eur.Phys.J.C 72 (2012) 1860

$\pi\pi$ RE-SCATTERING



Depends only on the *elastic phase shift* $\delta_1^1(s)$ of the P-wave in the isospin $I = 1$ channel

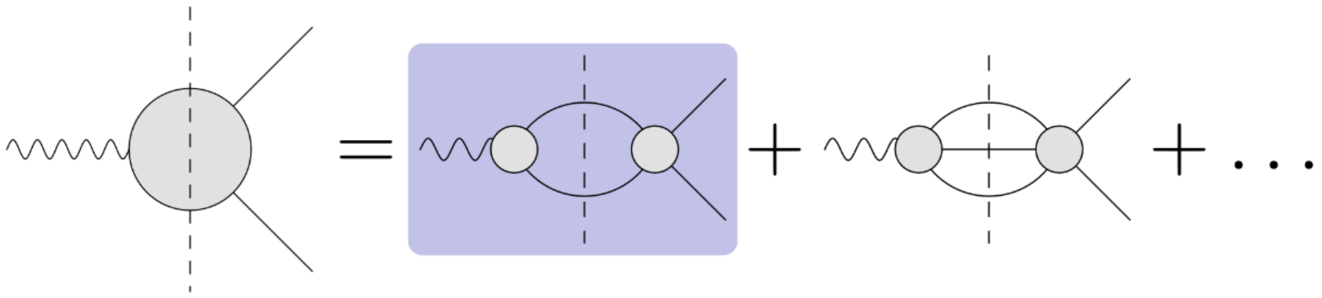


$= -ie(p' - p)^\mu F_\pi^V(s)$

$$t_1^1(s) = \frac{\sin \delta_1^1(s) e^{i\delta_1^1(s)}}{\sigma_\pi(s)} + \cdots$$

Roy equations

➡ Solution for $\delta_1^1(s)$ below ≈ 1.15 GeV

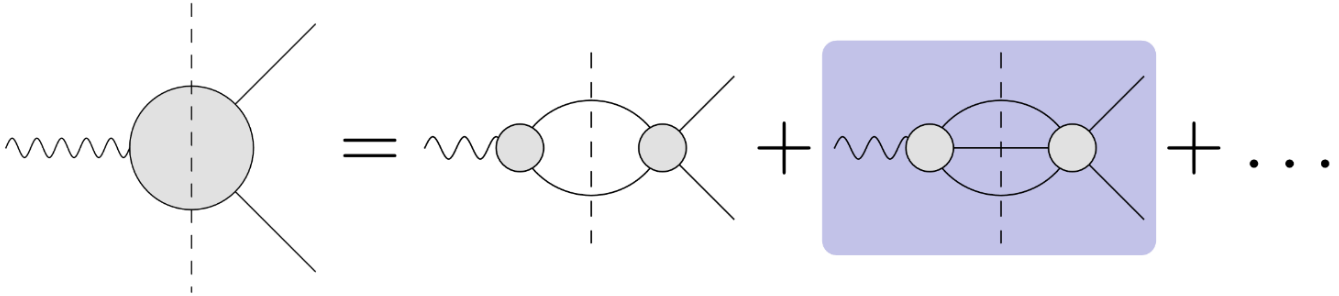


Colangelo, Hoferichter, Stoffer, JHEP 02 (2019) 006

$$F_\pi^V(s) = \Omega_1^1(s) \times G_{\omega(\phi)}(s) \times G_{\text{in}}(s)$$

- Omnès function with elastic $\pi\pi$-scattering P-wave phase shift $\delta_1^1(s)$ as input:

$$\Omega_1^1(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)}\right\}$$



Colangelo, Hoferichter, Stoffer, JHEP 02 (2019) 006

$$F_\pi^V(s) = \Omega_1^1(s) \times G_{\omega(\phi)}(s) \times G_{\text{in}}(s)$$

- Isospin-breaking 3π intermediate state: negligible apart from ω and ϕ resonances (mixing with the ρ resonance)

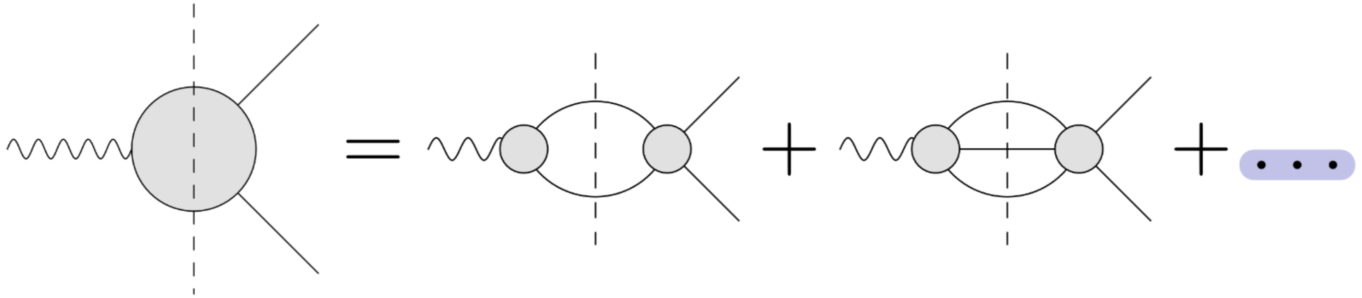
$$G_\omega(s) = 1 + \frac{s}{\pi}\int_{9M_\pi^2}^{\infty} ds' \frac{\text{Im} g_\omega(s')}{s'(s'-s)} \left(\frac{1-\frac{9M_\pi^2}{s'}}{1-\frac{9M_\pi^2}{M_\omega^2}}\right)^4$$

$$g_\omega(s) = 1 + \epsilon_\omega \frac{s}{(M_\omega - \frac{i}{2}\Gamma_\omega)^2 - s}$$

Colangelo, Hoferichter, Kubis, Stoffer, JHEP 10 (2022) 032

+ additional terms $\propto Im\epsilon_\omega$ to account for $\pi^0\gamma$ effects

+ additional terms for ϕ resonance



Colangelo, Hoferichter, Stoffer, JHEP 02 (2019) 006

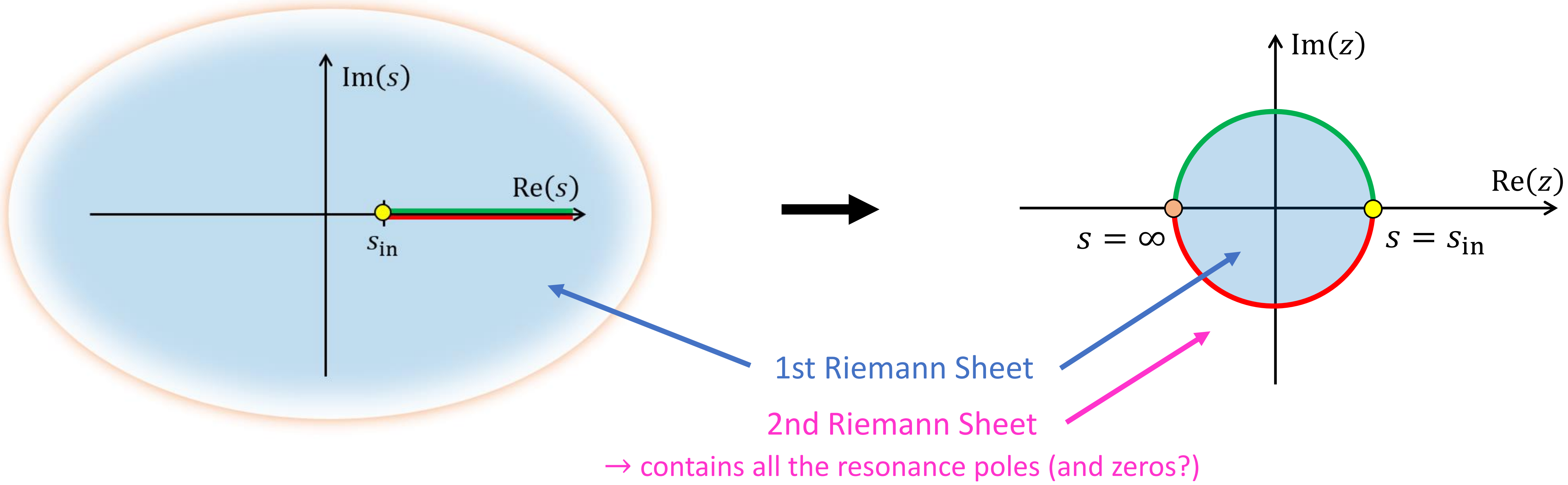
$$F_\pi^V(s) = \Omega_1^1(s) \times G_{\omega(\phi)}(s) \times G_{\text{in}}(s)$$

- Heavier intermediate states: 4π (mainly $\pi^0\omega$), $K\bar{K}$, …
- Description with a cut starting at the $\pi^0\omega$ threshold: $s_{\text{in}} = (M_{\pi^0} + M_\omega)^2$
- P-wave behavior imposed near the threshold

$$\operatorname{Im} G_{\text{in}}(s) \sim (s - s_{\text{in}})^{3/2}$$

→ Need an explicit parameterization of G_{in}

- Implements **branch cut** + **asymptotic behavior** $z(s) = \dfrac{\sqrt{s_{\rm in} - s_c} - \sqrt{s_{\rm in} - s}}{\sqrt{s_{\rm in} - s_c} + \sqrt{s_{\rm in} - s}}$



- Thus, the inelastic factor is conveniently written as a **function of $z(s)$**

- In our previous analyses, G_{in} was described as a **polynomial in z**:

$$G_{\text{in}}(z) = 1 + \sum_{k=1}^{N} \left(z^k - z_0^k \right)$$

Colangelo, Hoferichter, Stoffer, JHEP 02 (2019) 006

which is able to **fit the data up to $\approx$ 1GeV for degrees $N = 2 \ldots 6$**

- Our last analysis investigated the **impact of the zeros in the first Riemann Sheet**
  - Excluding these zeros resolves some instabilities and variability of the fits with N
  - Excluding zeros does not degrade the goodness of fit
  - The fitted zeros were already excluded by analyticity constraints
  - Zeros are excluded by χPT: all the zeros must be outside of the range of validity of χPT

Leplumey, Stoffer, arXiv:2501.09643

Leutwyler, Continuous advances in QCD (2002) 23-40

- In addition, we also implemented a **zero-free semi-model-dependent description of the inelasticities** (dispersively improved Gounaris-Sakurai functions) to fit the high-energy data (up to 3 GeV) and check consistency

Ruiz Arriola, Sanchez-Puertas, arXiv 2403.07121

Gounaris, Sakurai, Phys.Rev.Lett 21 (1968) 244-247

New implementation of the inelasticities

Alternative descriptions of the VFF use a conformal description from the elastic threshold $4M_\pi^2$ and incorporate an **outer function** (OF)

$$F_\pi^V(s) = \frac{1}{\phi(z_{\pi\pi})} \sum_{k=1}^{N} a_k \, z_{\pi\pi}^k$$

Buck, Lebed, Phys.Rev.D 58 (1998) 056001

Ananthanarayan, Caprini, Imsong, Phys.Rev.D 83 (2011) 096002

With the motivation of incorporating **dispersive bounds** written as

$$\frac{1}{2\pi i} \int_C \frac{dz}{z} \left|\phi(z) F_\pi^V(z)\right|^2 \leq 1 \quad \longrightarrow \quad \sum_{k=0}^{N} |a_k|^2 \leq 1$$

Such an OF can improve the convergence of the conformal expansion by setting this orthogonality constraint

One common choice of OF is the following

- $s_c = -1\ \mathrm{GeV}^2$: central point for conformal transformation
- s_0: branch cut threshold
- $Q^2 = -q^2 = -1\ \mathrm{GeV}^2$: point where the bound is evaluated

$$\phi(z) \propto \left[\sqrt{1-\frac{s_c}{s_0}}(1-z)+(1+z)\right]^{-1/2}\left[\sqrt{1+\frac{Q^2}{s_0}}(1+z)+\sqrt{1-\frac{s_c}{s_0}}(1-z)\right]^{-3}$$

- The initial motivation is a dispersive bound on the $\pi\pi$ channel of HVP

$$\frac{1}{\pi}\int_0^\infty ds\,\frac{\Pi_J^T(s)|_{\pi\pi}}{(s-q^2)^3} \leq \left[\frac{1}{2}\frac{\partial^2\Pi_J^T}{\partial^2(q^2)^2}\right]_{\mathrm{pQCD}}$$

Buck, Lebed, Phys.Rev.D 58 (1998) 056001

Kirk, Kubis, Reboud, van Dyk, Phys.Lett.B 861 (2025) 139266

- This OF mainly contains information on the **two-body kinematics**

- Supplemented by Blaschke factor corrections to remove sub-threshold singularities and correct the behavior at threshold and infinity

Adaptation of the Omnès parameterization

- The dispersive bound is saturated only at $\approx 45\%$, and therefore cannot be used to constrain any data fit

- Still, this OF is relevant to describe the **effective two-body kinematics** at $\pi^0\omega$ threshold

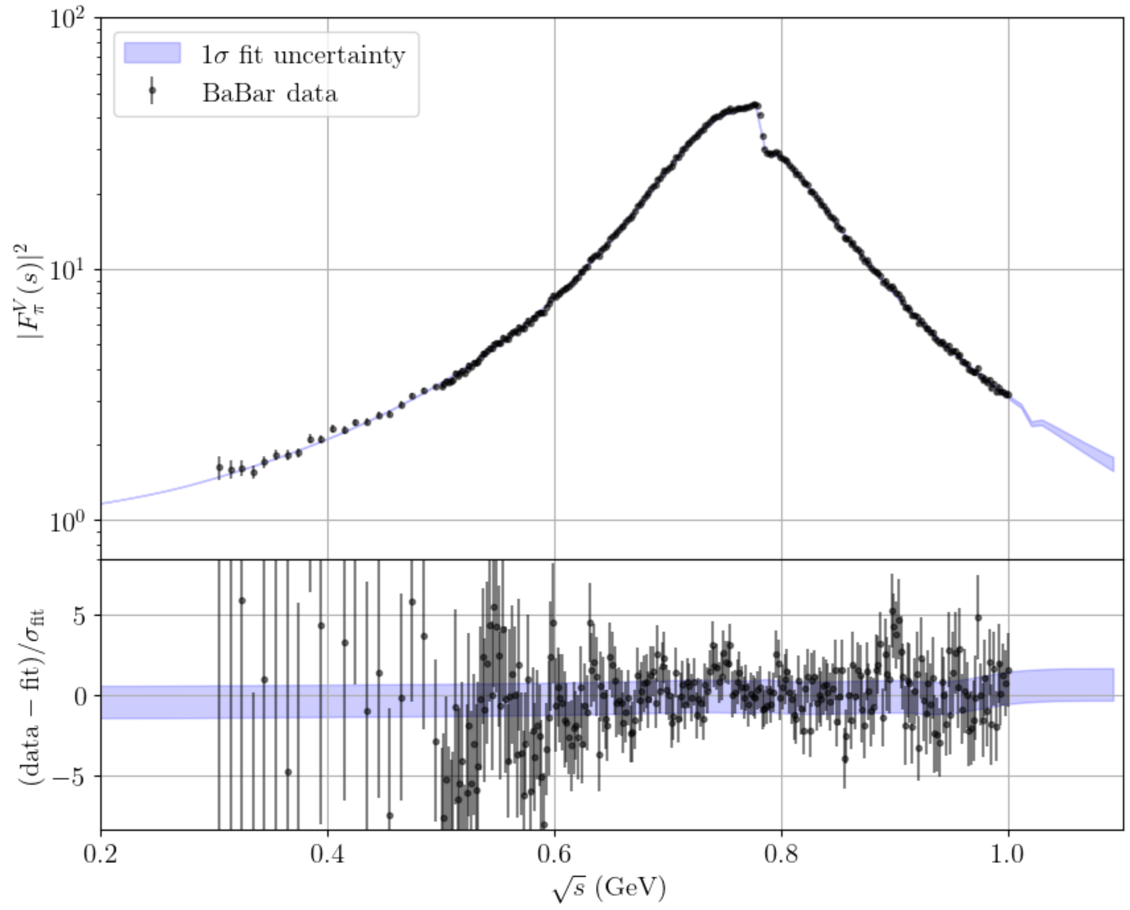
$$G_{\text{in}}(z) = \frac{1}{\phi(z)} P_N(z)$$

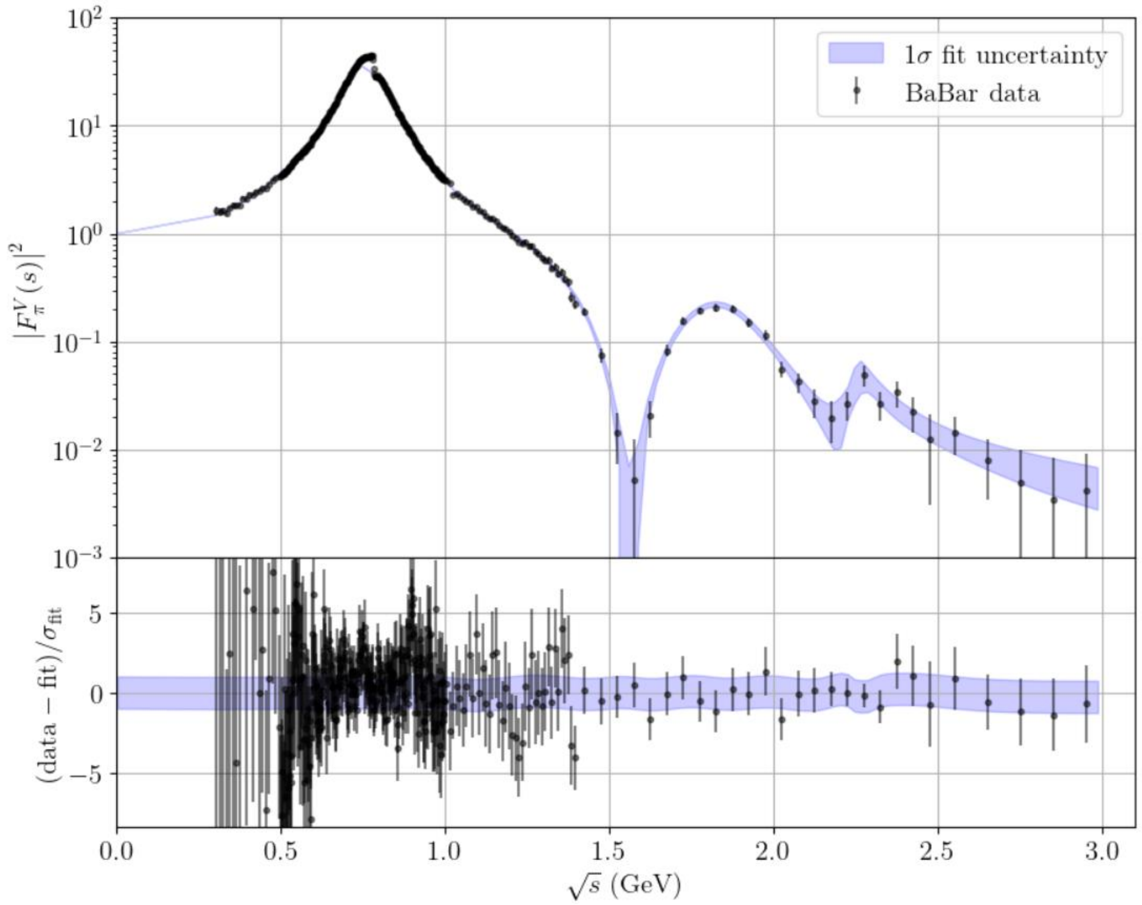
- It can be further complemented by the introduction of **explicit poles** in order to describe the resonances visible in multi-GeV data

Kirk, Kubis, Reboud, van Dyk, Phys.Lett.B 861 (2025) 139266

$$G_{\text{in}}(z) = \frac{1}{\phi(z)} \frac{P_N(z)}{\prod_j (z - z_j)(z - z_j^*)}$$

- Different fits are performed:
  - Fits to sub-GeV data only without explicit resonance poles in the parameterization (e.g. left)
  - Full-range fits with three explicit resonance poles in the parameterization (e.g. right)





Improved methodology for parameter inference

$$F_\pi^V(s) = \Omega_1^1(s) \times G_{\omega(\phi)}(s) \times G_{\text{in}}(s)$$

2 parameters:
Value of $\delta_1^1(s)$ at 2 particular points

3 parameters:
M_ω, $\mathrm{Re}\,\epsilon_\omega$, $\mathrm{Im}\,\epsilon_\omega$ (+ $\mathrm{Re}\,\epsilon_\phi$, $\mathrm{Im}\,\epsilon_\phi$ when ϕ visible in data)

$N-1$ parameters:
$c_2, \ldots, c_N$
(+ $2 \times n_{\text{poles}}$ for full-range fits)

- Multiple fits to individual experiments:
  - Direct-scan: SND06, CMD-2, SND20, CMD-3
  - Radiative-return: BaBar, KLOE, BESIII

- Use of unbiased fitting to avoid the d'Agostini bias

NNPDF Collaboration, JHEP 05 (2010) 075

D'Agostini, Nucl.Instrum.Meth.A 362 (1995) 487-498

The inference is based on a χ^2-like negative log-likelihood:

$$\chi^2 = \chi^2_{\mathrm{data}} + \chi^2_{\mathrm{EL}} + \chi^2_{\mathrm{zeros}} + \chi^2_{\mathrm{syst}}$$

- Data constraint: $\chi^2_{\mathrm{data}} = [\sigma(s_i,\theta) - \sigma_i]^{\mathrm{T}}\,\Sigma^{-1}\,[\sigma(s_i,\theta) - \sigma_i]$
- Eidelman-Łukaszuk bound: upper bound on the inelastic phase of the VFF close to the inelastic threshold, constrained by external experimental data

  Eidelman, Łukaszuk, Phys.Lett.B 582 (2004) 27-31

  Colangelo, Hoferichter, Kubis, Stoffer, JHEP 10 (2022) 032
- Sum-rule constraint for the absence of zeros $\dfrac{\sqrt{s_{\mathrm{in}}}}{\pi}\displaystyle\int_{s_{\mathrm{in}}}^{\infty}\frac{ds}{(s-s_{\mathrm{in}})^{3/2}}\log\left|\frac{G_{\mathrm{in}}(s)}{G_{\mathrm{in}}(s_{\mathrm{in}})}\right| = 0$

  Leplumey, Stoffer, arXiv:2501.09643
- Prior constraints on the model systematics parameters
  - Roy equation parameters, Γ_ω, M_ϕ, Γ_ϕ, s_c, asymptotic extrapolation of δ_1^1

- Separate fits are performed for each value of N (degree of the conformal polynomial)
- Under Gaussianity, we derive the posterior for each fit separately against data D:
  - Studies have been performed to ensure Gaussianity hypothesis is nearly correct and conservative in our case

$$(\theta|D, N) \sim \mathcal{N}(\hat{\theta}_N, \hat{\Sigma}_N)$$

- The posteriors are then marginalized over N

$$p(\theta|D) = \sum_N p(\theta|D, N)p(N|D)$$

- If N has a flat (or exponential) prior, then one can show that

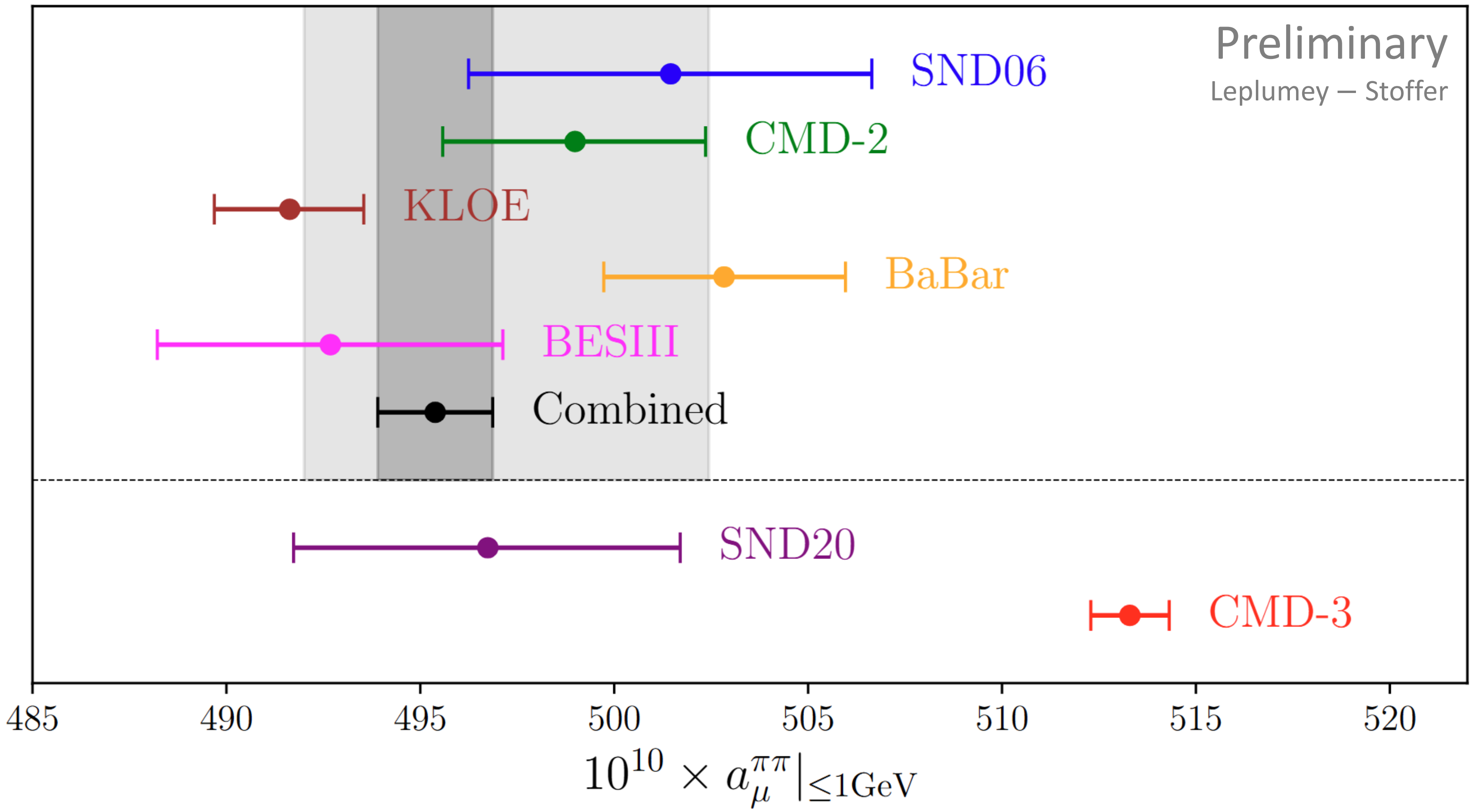
$$p(N|D) \simeq \exp\left(-\frac{1}{2}\left(\chi^2_N(\hat{\theta}_N) + \alpha N\right)\right)$$

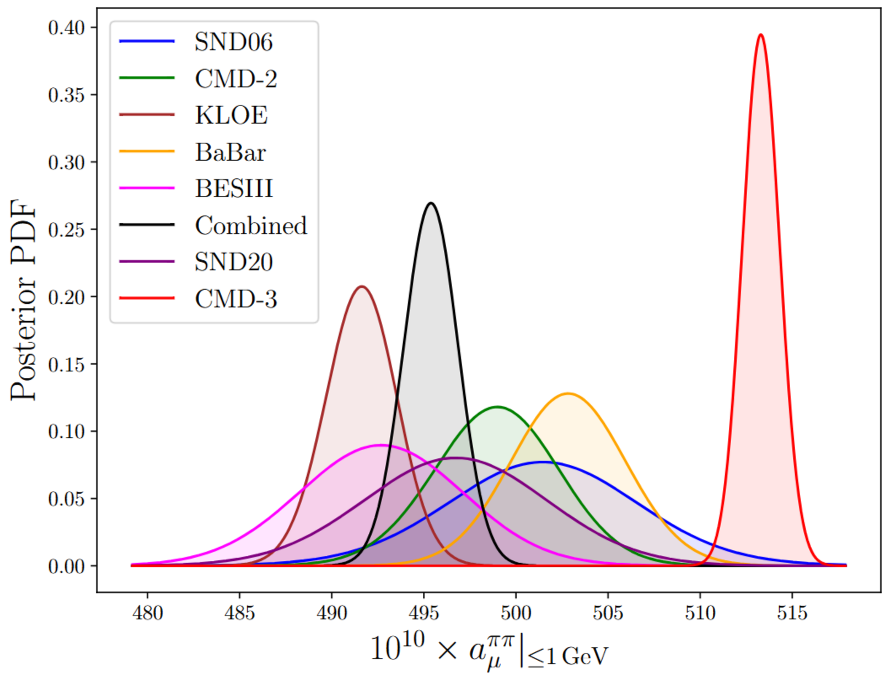
  - $\alpha \sim \log|D|$ would hold for very large dataset (Bayesian information)
  - Smaller α tend to be more conservative & accurate if none of the models is exact (e.g. $\alpha = 2$ for Akaike information)
  - We choose $\alpha = 1$ for our nominal inference, which penalizes the addition of one parameter by one χ^2 unit

RESULTS FOR $a_\mu^{\mathrm{HVP,LO}}[\pi\pi, e^+e^-]$

RESULT FOR $a_\mu^{\mathrm{HVP,LO}}[\pi\pi, e^+e^-]$ BELOW 1GEV

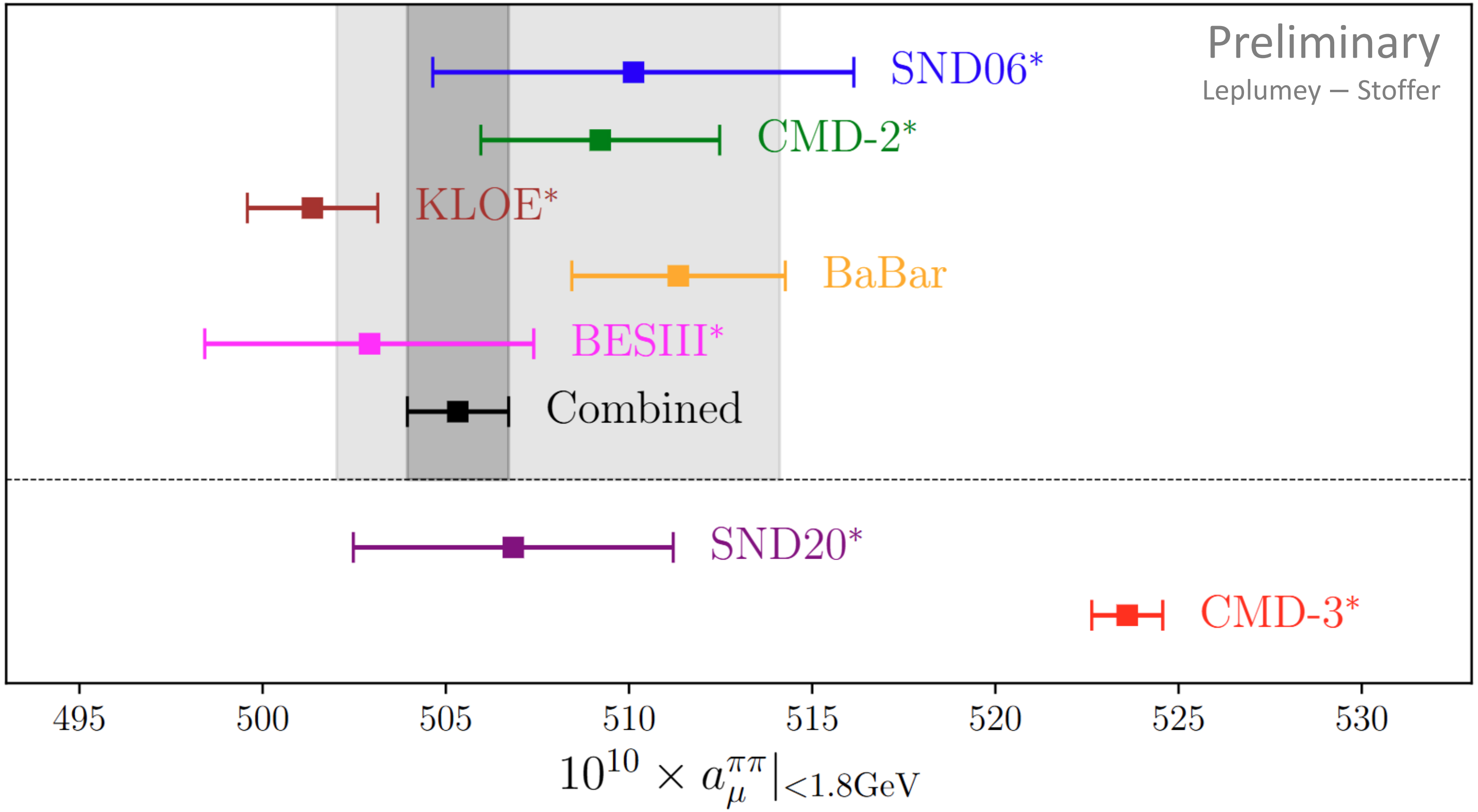
- All the datasets are truncated at 1 GeV
- No resonance poles are explicitly introduced in the inelastic factor

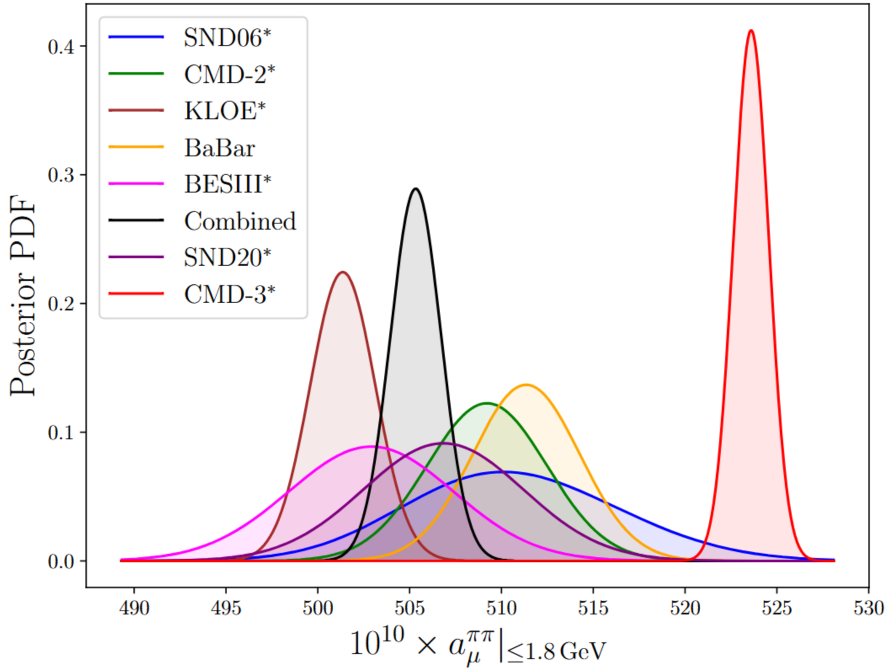




Result for $a_\mu^{\text{HVP,LO}}[\pi\pi, e^+e^-]$

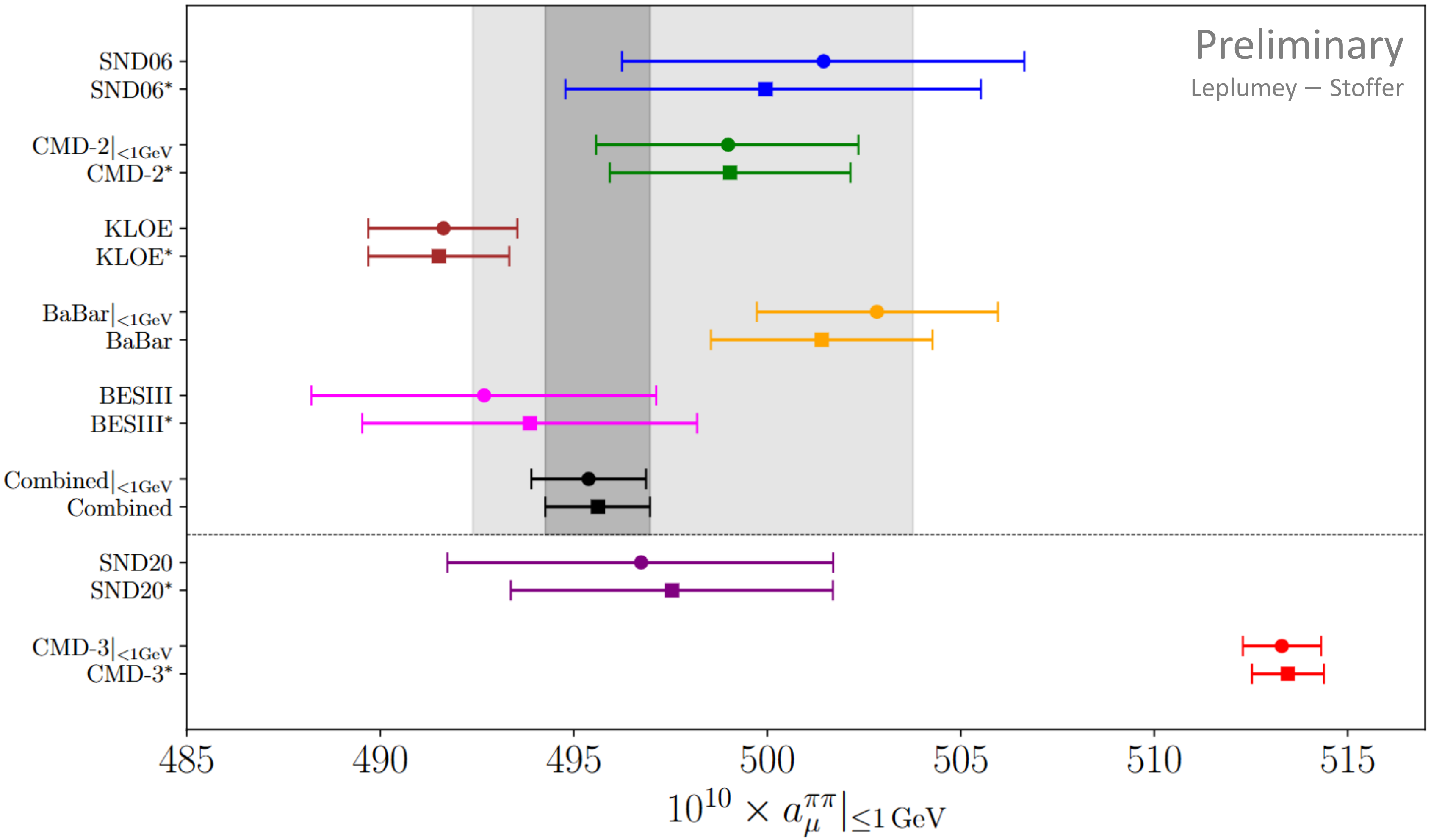
- All the datasets marked with a * are combined with BaBar data above 1.4 GeV
- 3 resonance poles are explicitly fitted in the parameterization (ρ', ρ'' and ρ''')

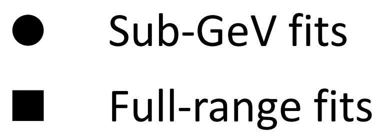




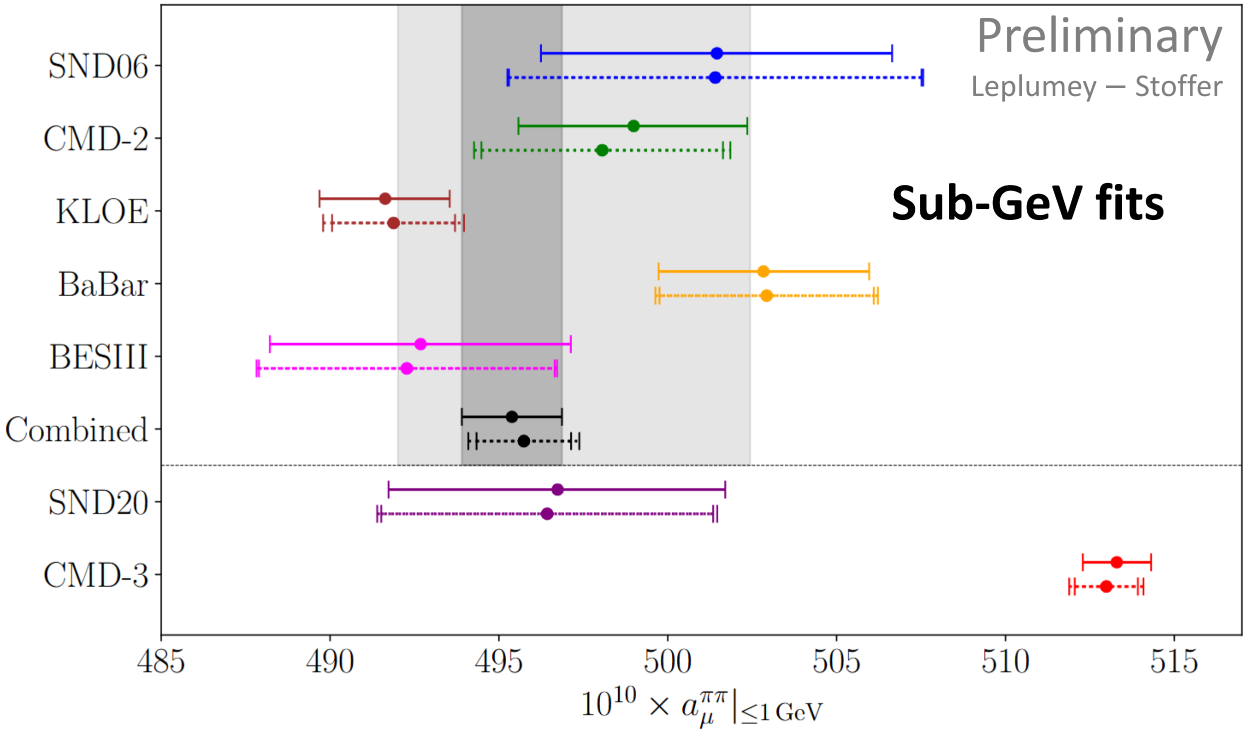
CONSISTENCY IN $a_\mu^{\text{HVP,LO}}[\pi\pi, e^+e^-]$ BELOW 1GEV

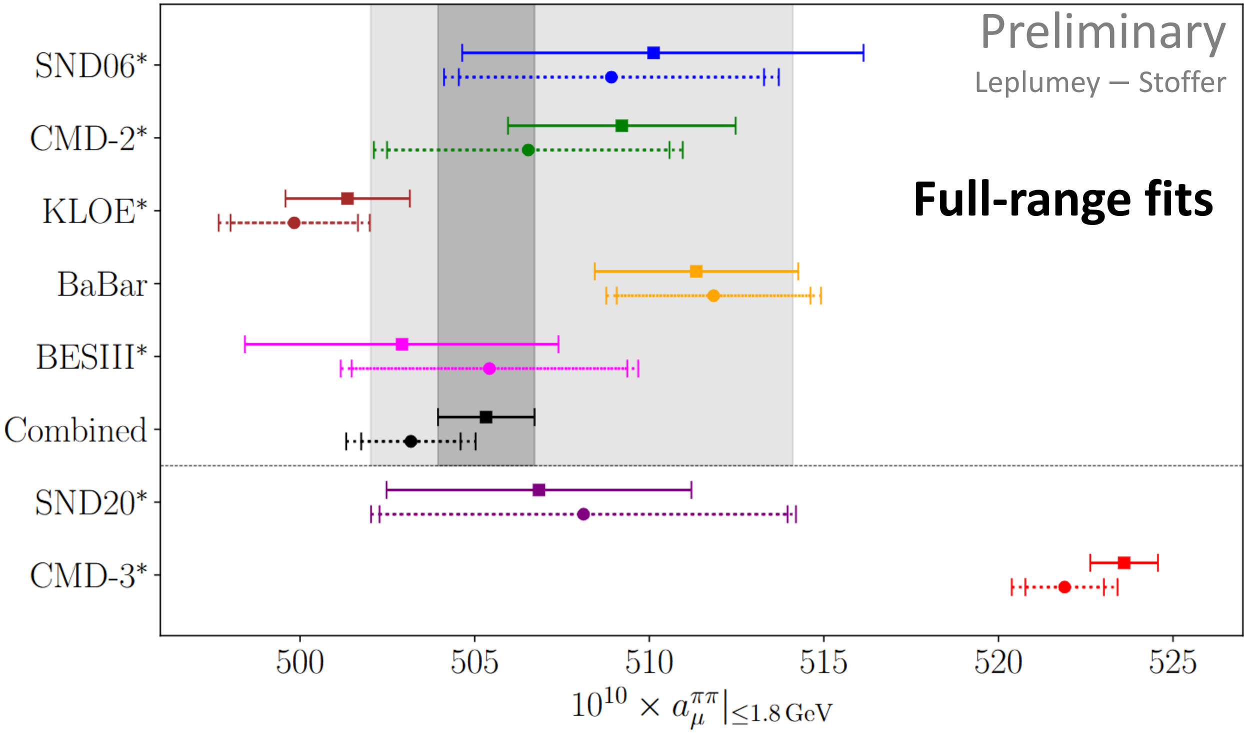
- Very good consistency is observed between sub-GeV and multi-GeV fits!





- Very good consistency is observed with our previous analysis
- The new analysis allows better interpretability of the credible intervals

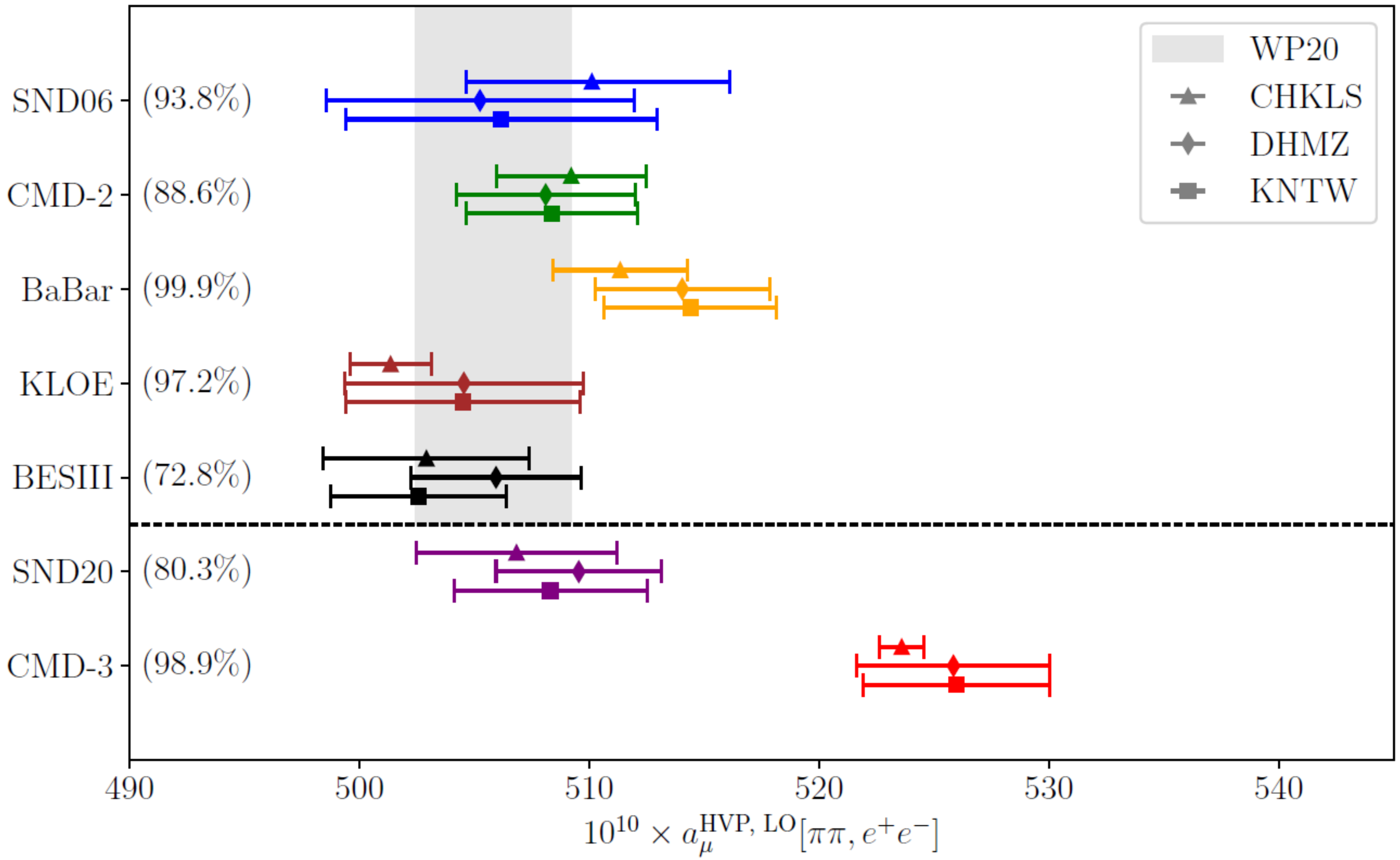


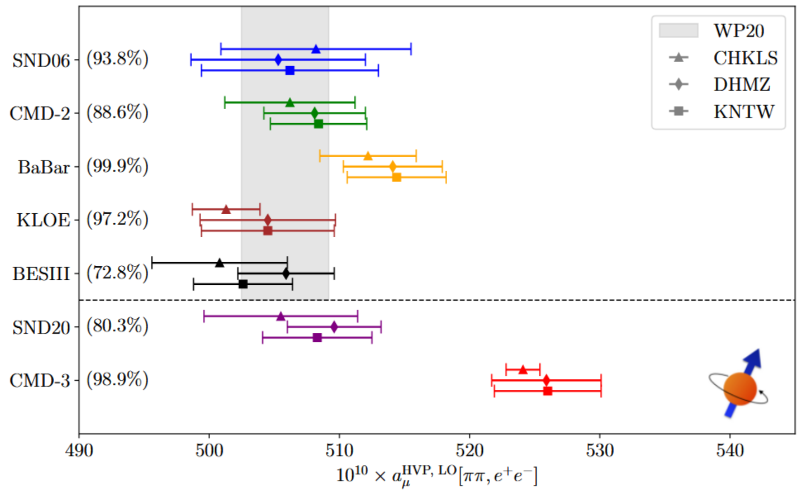


New analysis (preliminary)

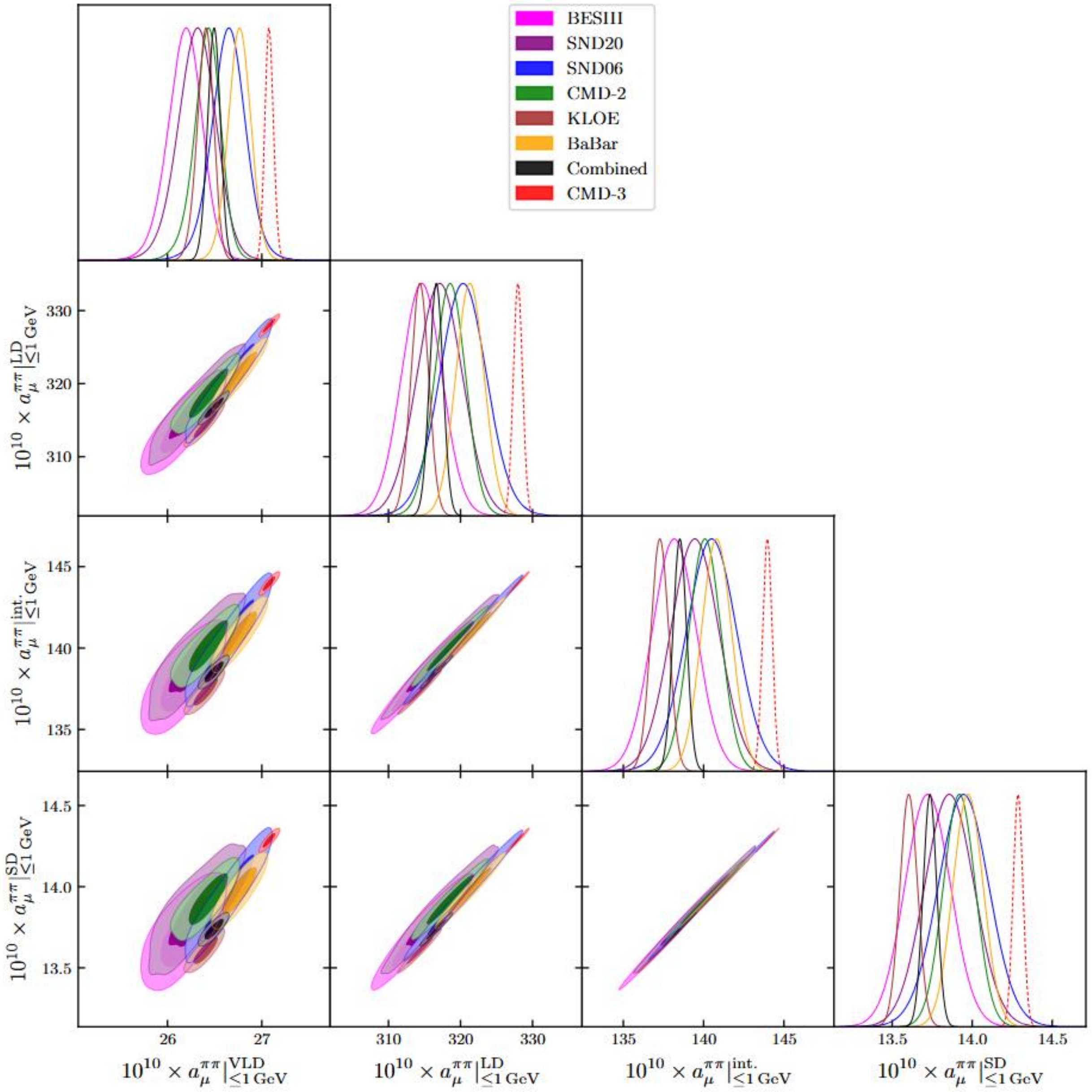
Previous analysis (arXiv:2501.09643)

- Our results remain compatible with other approaches

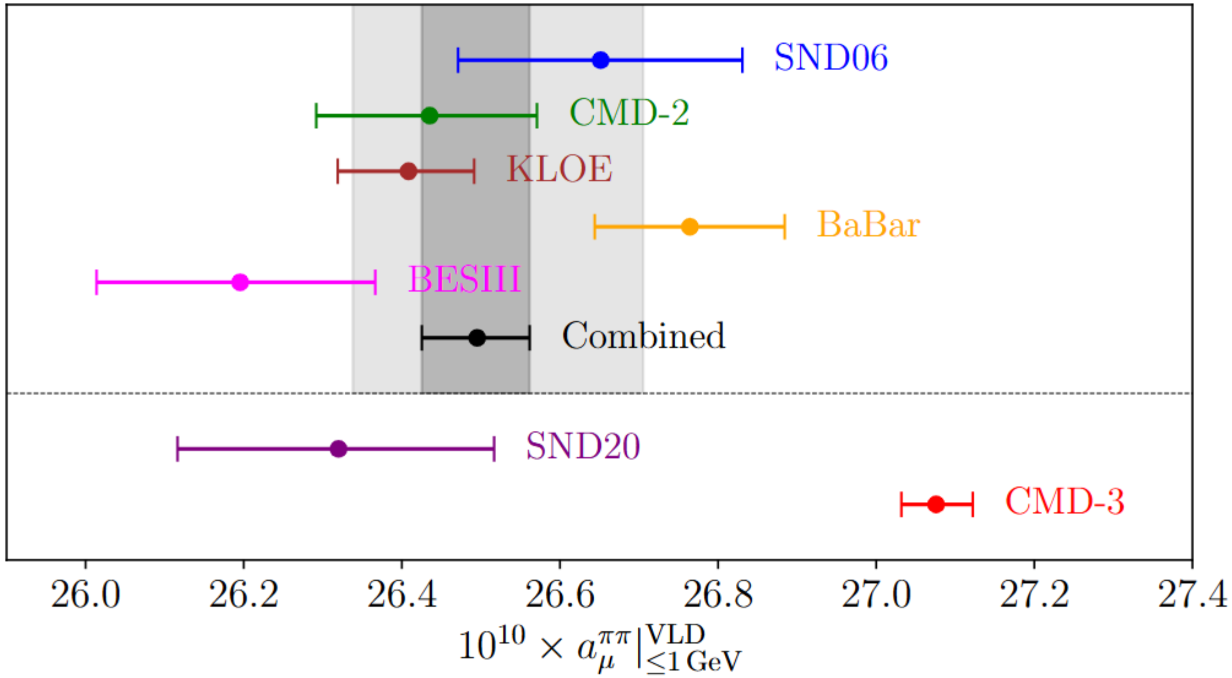




WP25: Fig. 26



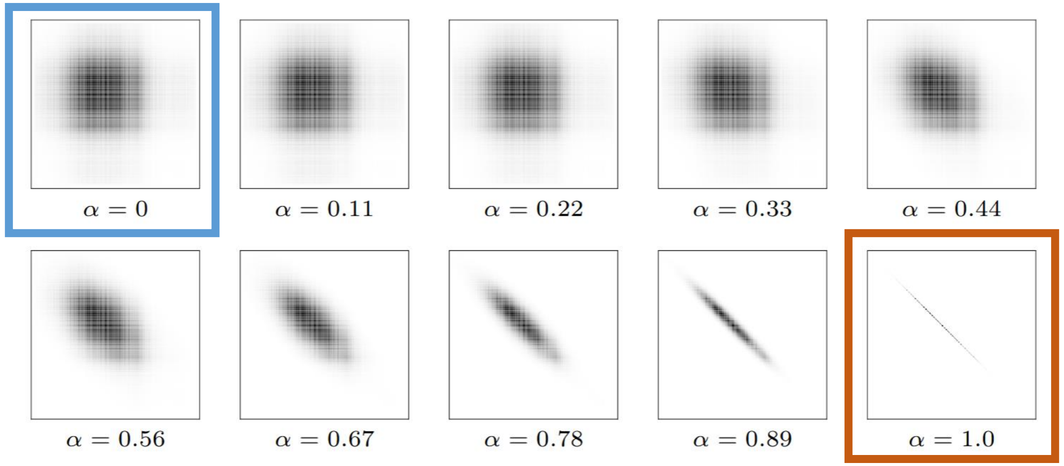
- The different euclidean windows are actually very **correlated by the data fit**
- Therefore, **the discrepancies propagate to all the windows**, even at very long distance!

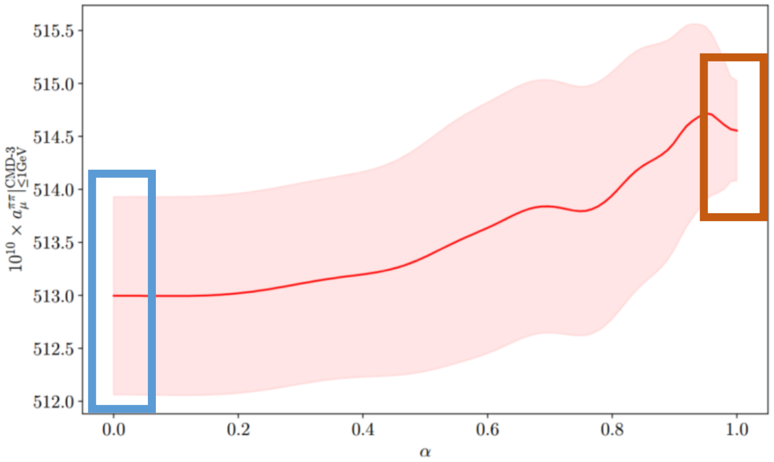


Very-long-distance window – sub-GeV description

- Some concerns have been raised about the impact of **correlations in CMD-3 data**
  - For **direct integration**, **fully correlated covariance** is clearly the **most conservative**
  - However, it is less clear *a priori* whether this choice is conservative or not in our framework

- In our last analysis, we implemented a “decorrelation scheme” to evaluate this:





Leplumey, Stoffer, arXiv:2501.09643

- Smaller correlations lead to **higher value of $a_\mu^{\pi\pi}$** and **smaller uncertainty**!
  - Full correlations allow global scale effects → analyticity constraints seem to pull the VFF down
  - Zero/negative correlations constrain the fit to be closer to the central values of the data points

Comment about the data correlations of CMD-3

- To assess this issue, we tried **tuning the covariance *a posteriori*** to get the largest posterior uncertainty in $a_\mu^{\pi\pi}$ (→ expected to be the most conservative choice)

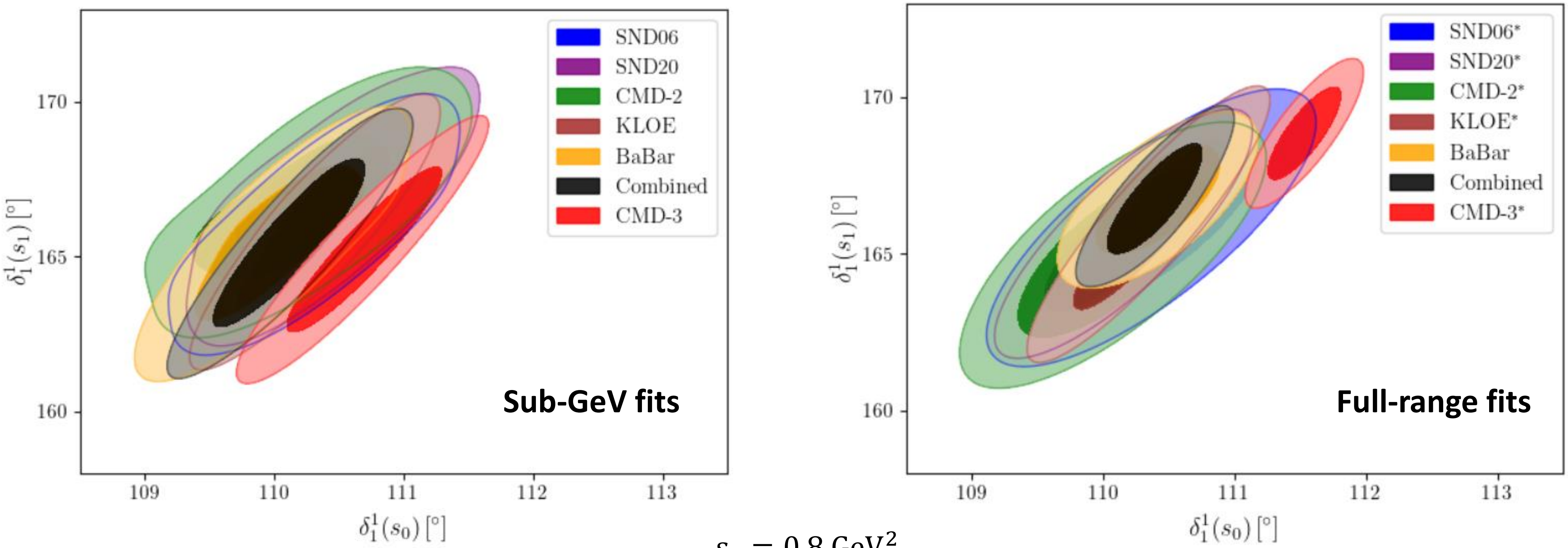
$$\mathrm{Corr}(\sigma_i, \sigma_j) = \mathrm{sign}\left(\frac{\partial a_\mu^{\pi\pi}}{\partial \sigma_i} \times \frac{\partial a_\mu^{\pi\pi}}{\partial \sigma_j}\right)$$

- However, the *a posteriori* conservative covariance **depends on the starting point**
  - Starting from the fully correlated best-fit point, the conservative option is consistently the fully correlated covariance matrix
  - Starting from the uncorrelated best-fit point, the conservative option contains **anti-correlations** between different energy regions, but **the overall uncertainty remains smaller**

- Without a clear prescription yet, we decided to **stick to the full corr. prescription**
  - This makes the interpretation and comparison with other results easier
  - This choice **does not overestimate the discrepancy** with other experiments

Results for other observables

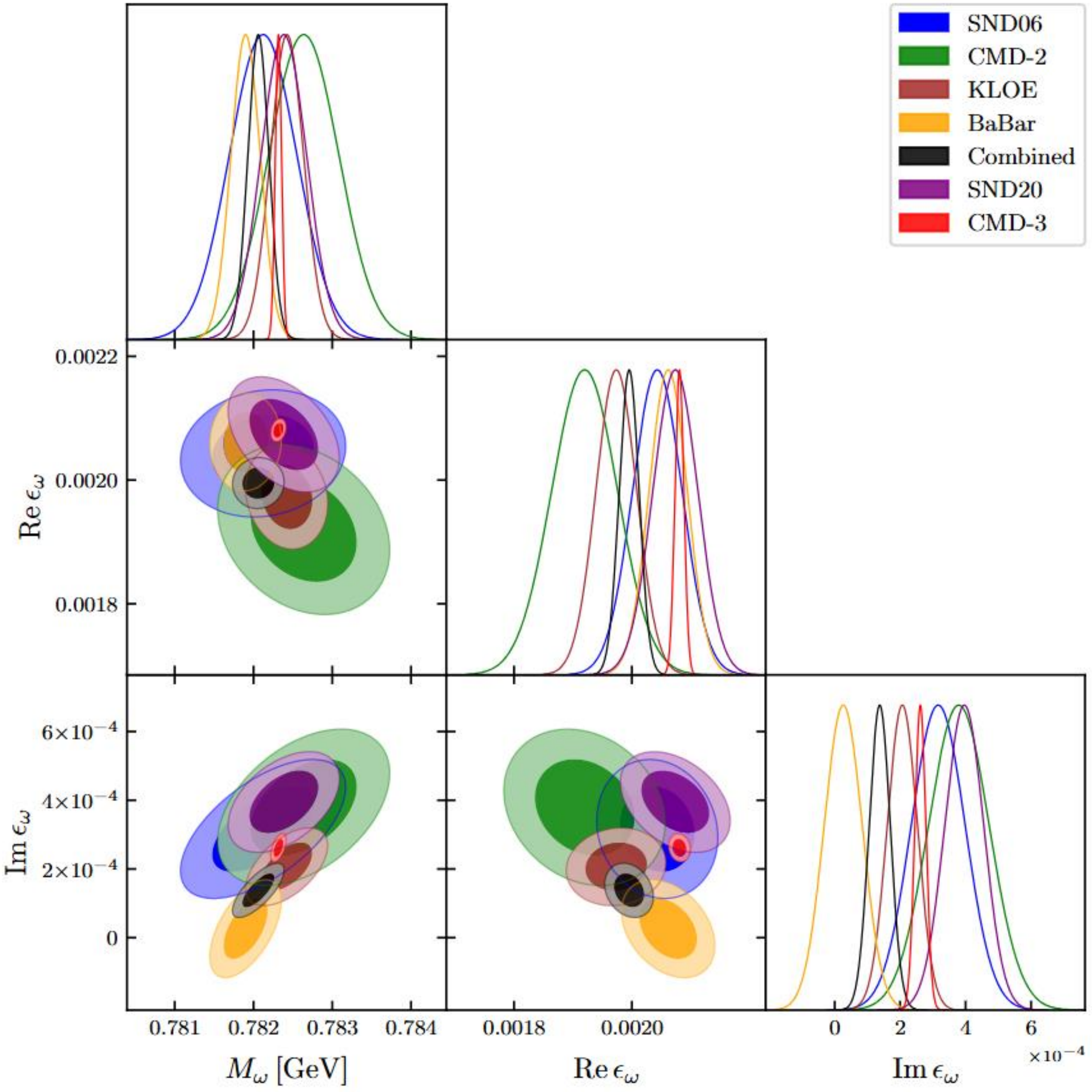
- The elastic contributions are described by the elastic phase shift δ_1^1
- Small disagreements are found between CMD-3 and other experiments



$s_0 = 0.8\ \mathrm{GeV}^2$

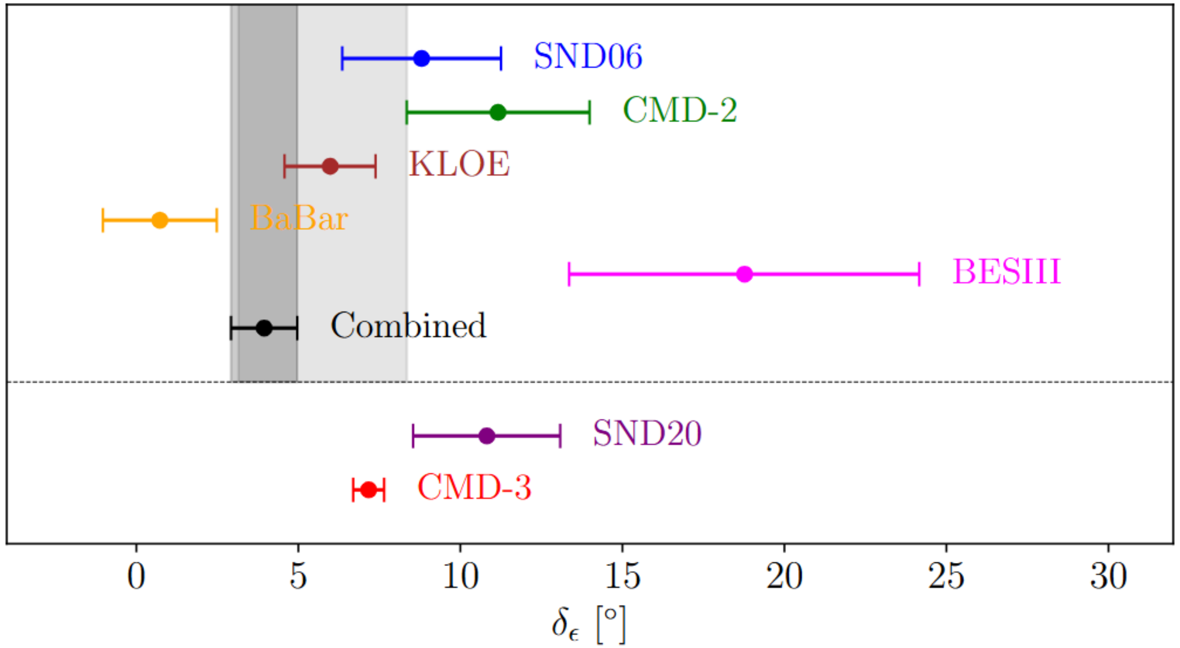
$s_1 = 1.15\ \mathrm{GeV}^2$

Result for the ω parameters

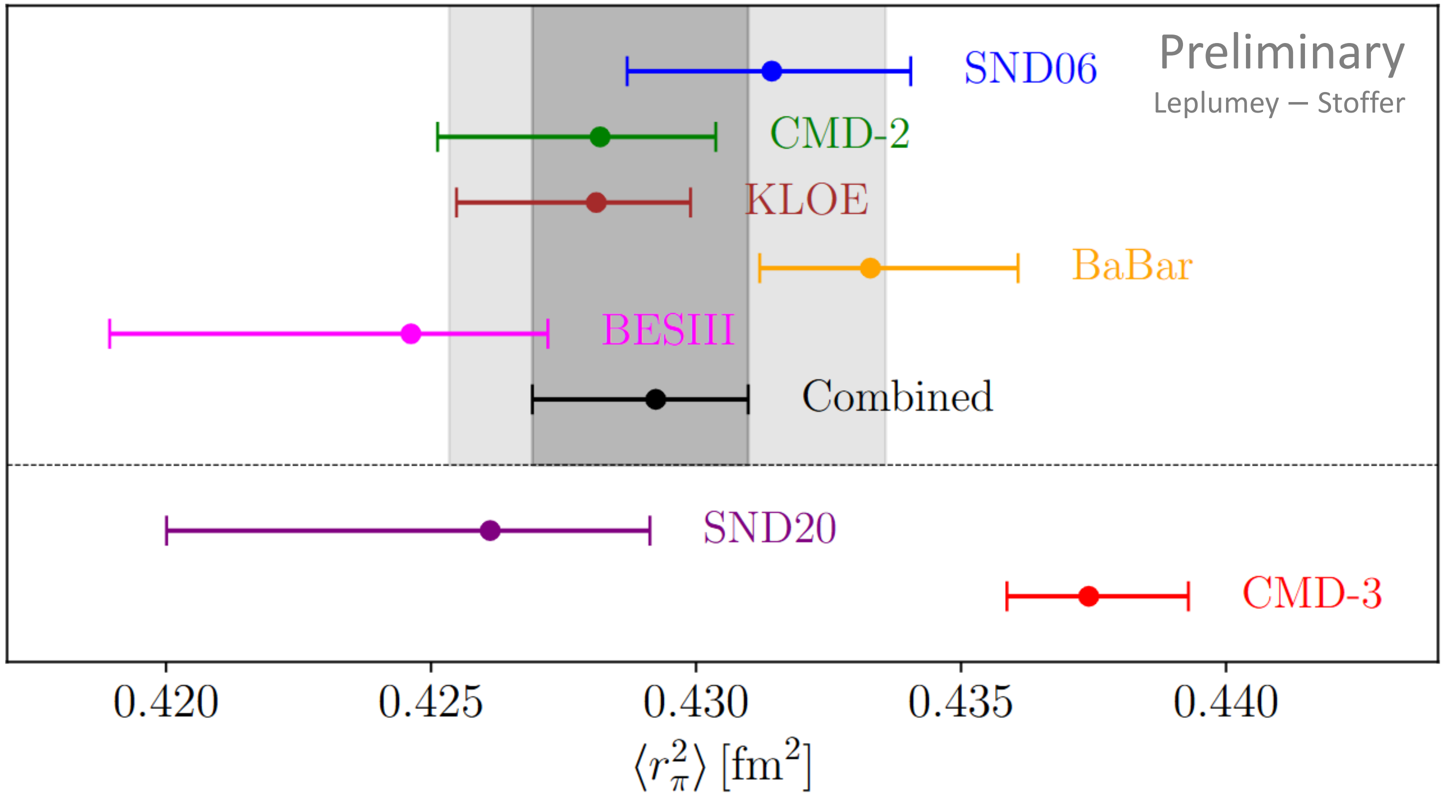


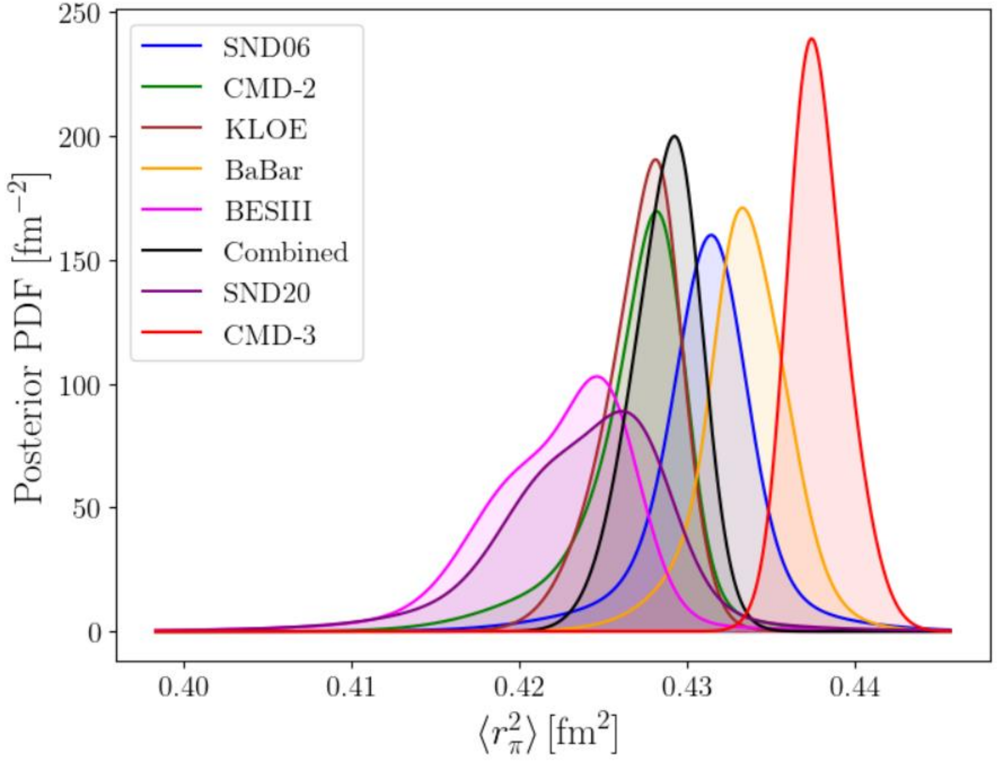
- The impact of the ω resonance is described with 3 parameters:
  - M_ω: The ω mass
  - $\mathrm{Re}(\epsilon_\omega)$: The scale of 3π channel effects
  - $\mathrm{Im}(\epsilon_\omega)$: The scale of $\pi^0\gamma$ channel effects

Colangelo, Hoferichter, Kubis, Stoffer, JHEP 10 (2022) 032

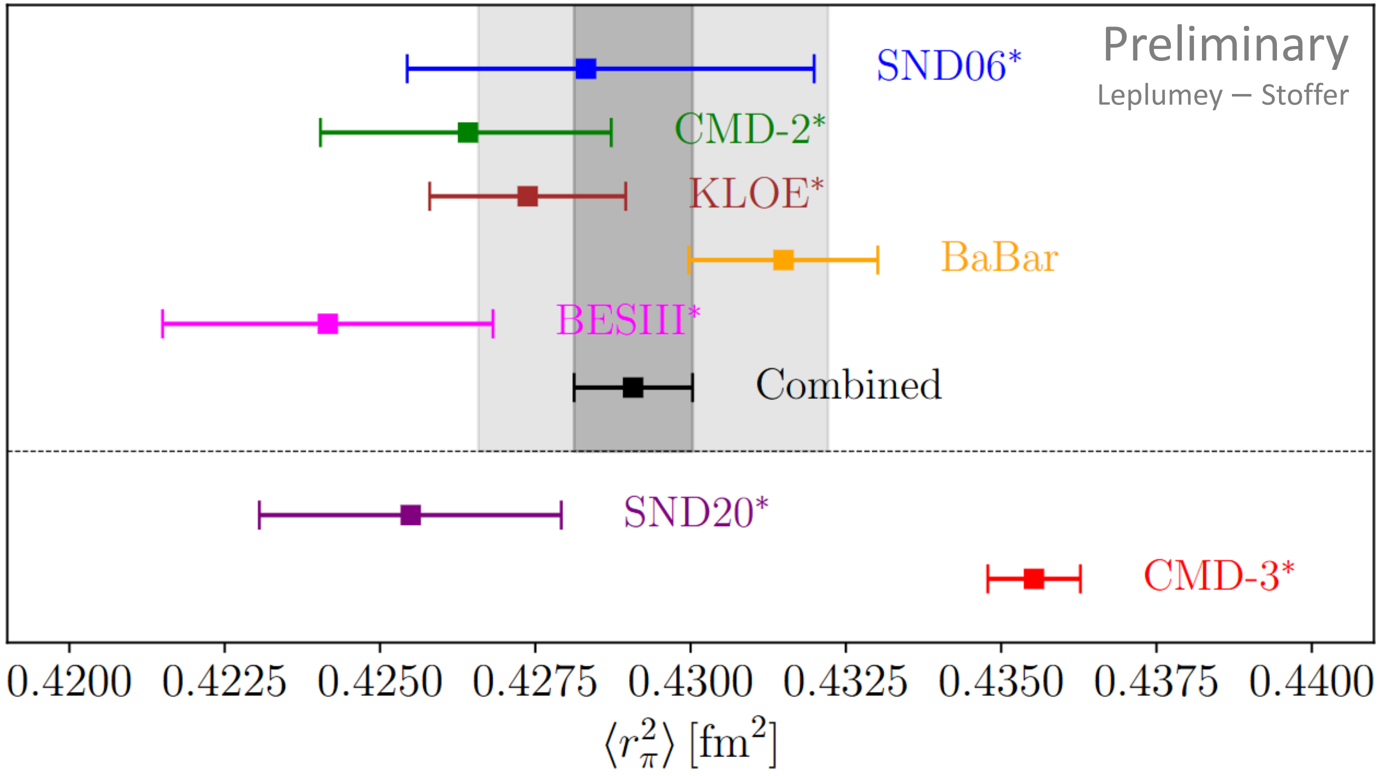


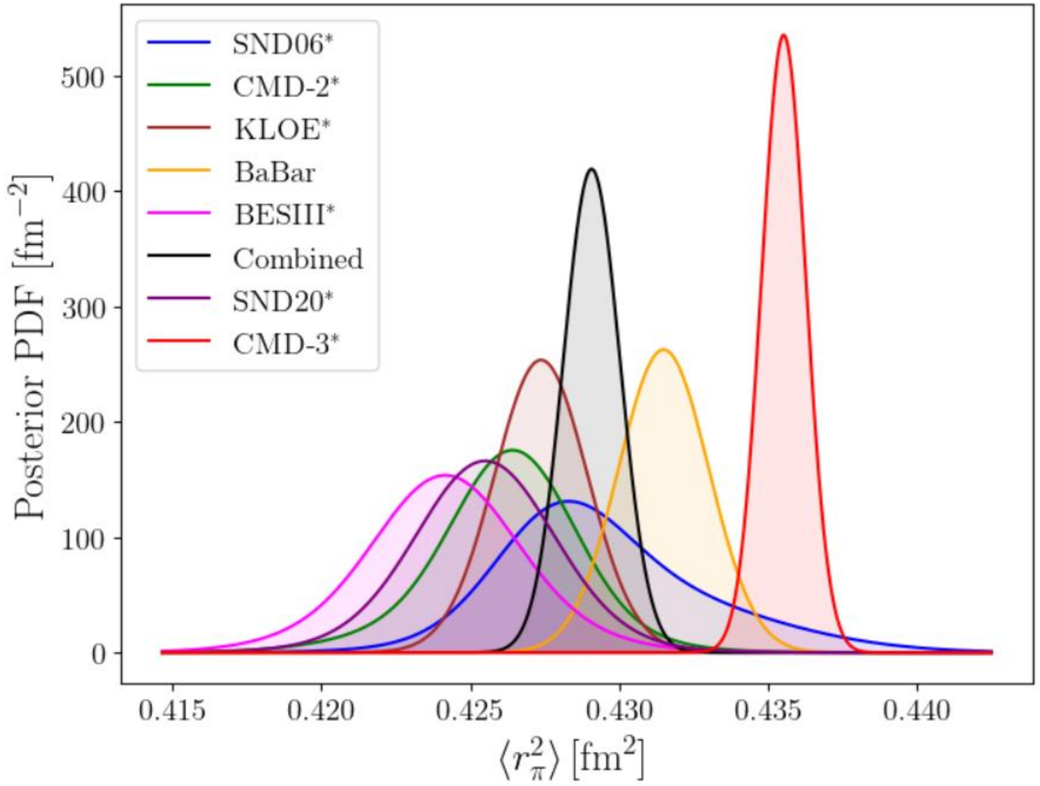
- The impact of marginalizing on N is much more visible in the charge radius
- This appears as large non-Gaussianities in the posterior distributions





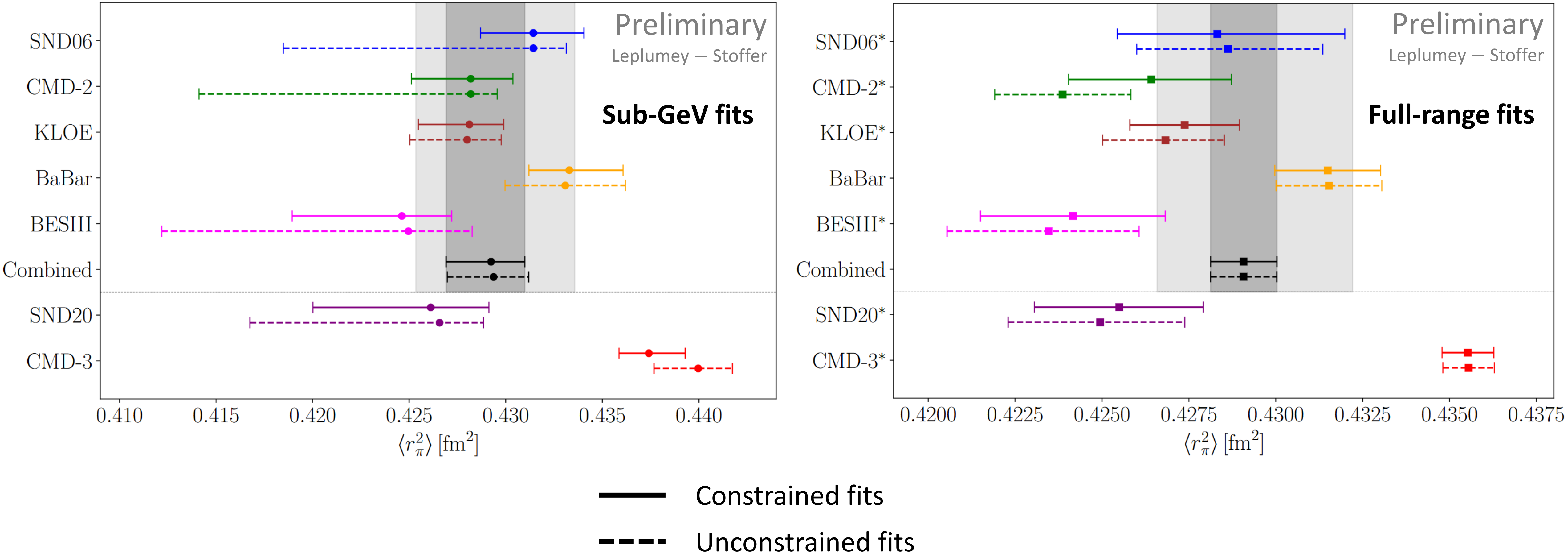
- However, better Gaussianity is restored in multi-GeV fits
- Multi-GeV data helps a lot in reducing the variability of the result with N



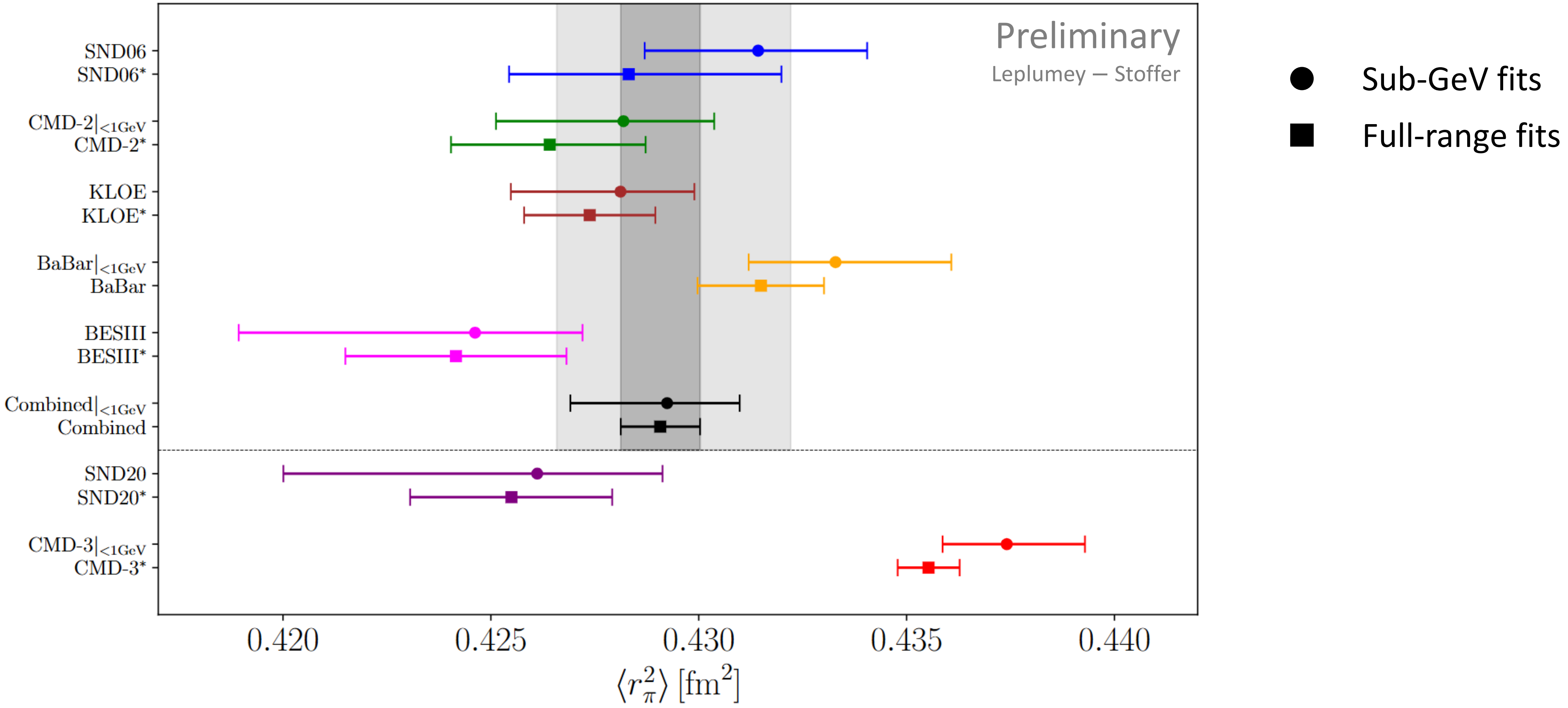


IMPACT OF ZEROS IN THE FIRST RIEMANN SHEET

- Excluding zeros reduces a lot the variability with N for sub-GeV fits
- However, the impact of zeros is almost invisible in multi-GeV fits even in $\langle r_\pi^2 \rangle$

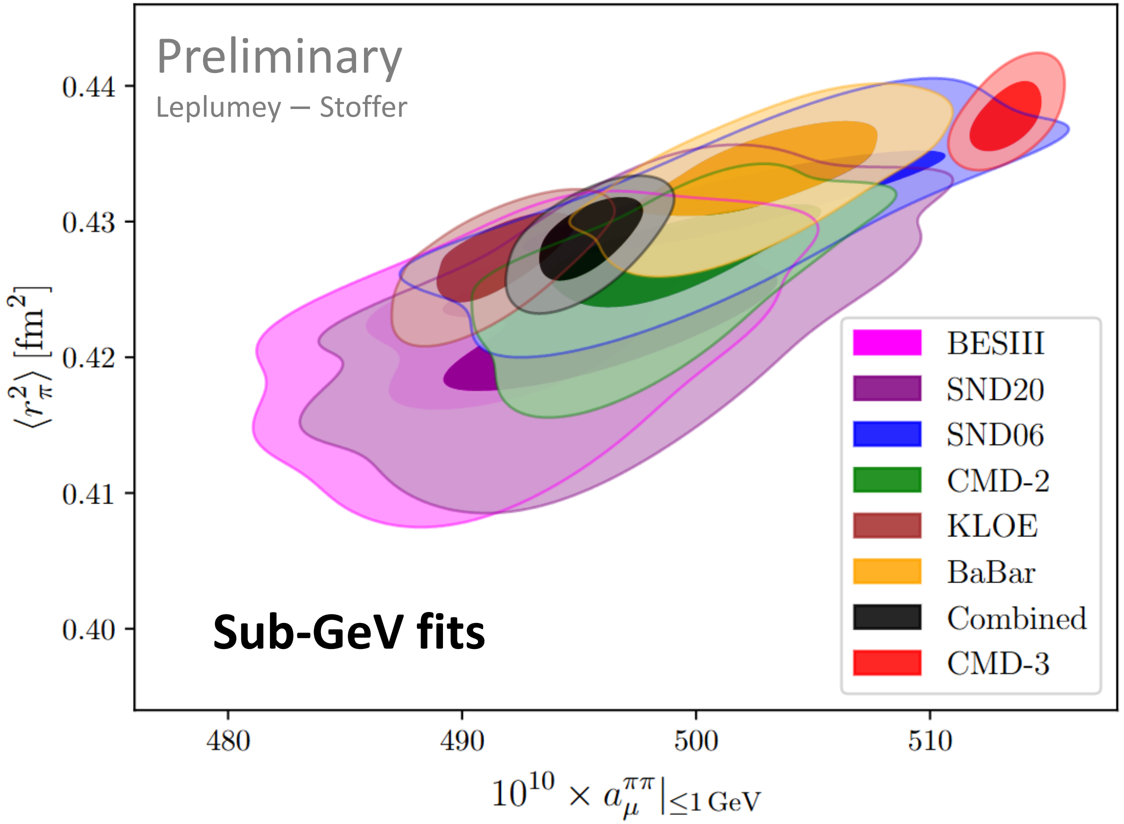


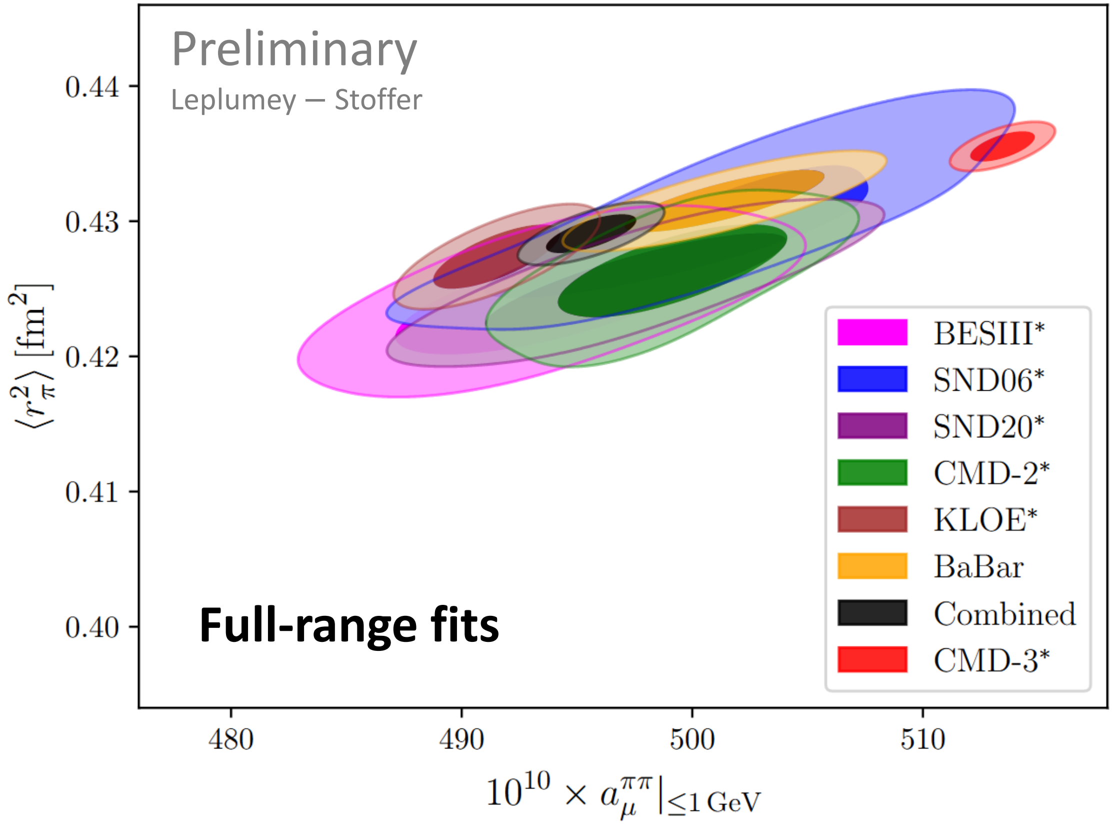
- Small variations are observed (although always less than $\sim 1\sigma$)
- The inclusion of multi-GeV data systematically reduces the value of $\langle r_\pi^2 \rangle$



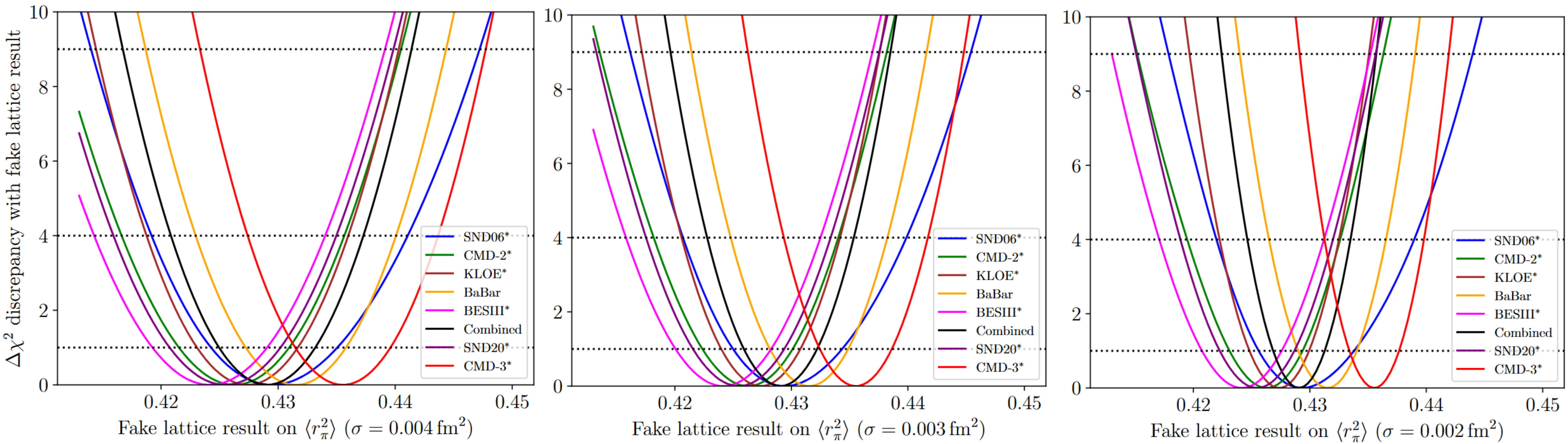
CORRELATION WITH $a_\mu^{\mathrm{HVP,LO}}[\pi\pi, e^+e^-]$

- **Strong correlations are observed between $\langle r_\pi^2 \rangle$ and $a_\mu^{\pi\pi}$**, both within each fit and between the fits
- Therefore, **an independent lattice calculation of $\langle r_\pi^2 \rangle$ could provide valuable insights on the observed discrepancies in $a_\mu^{\pi\pi}$**

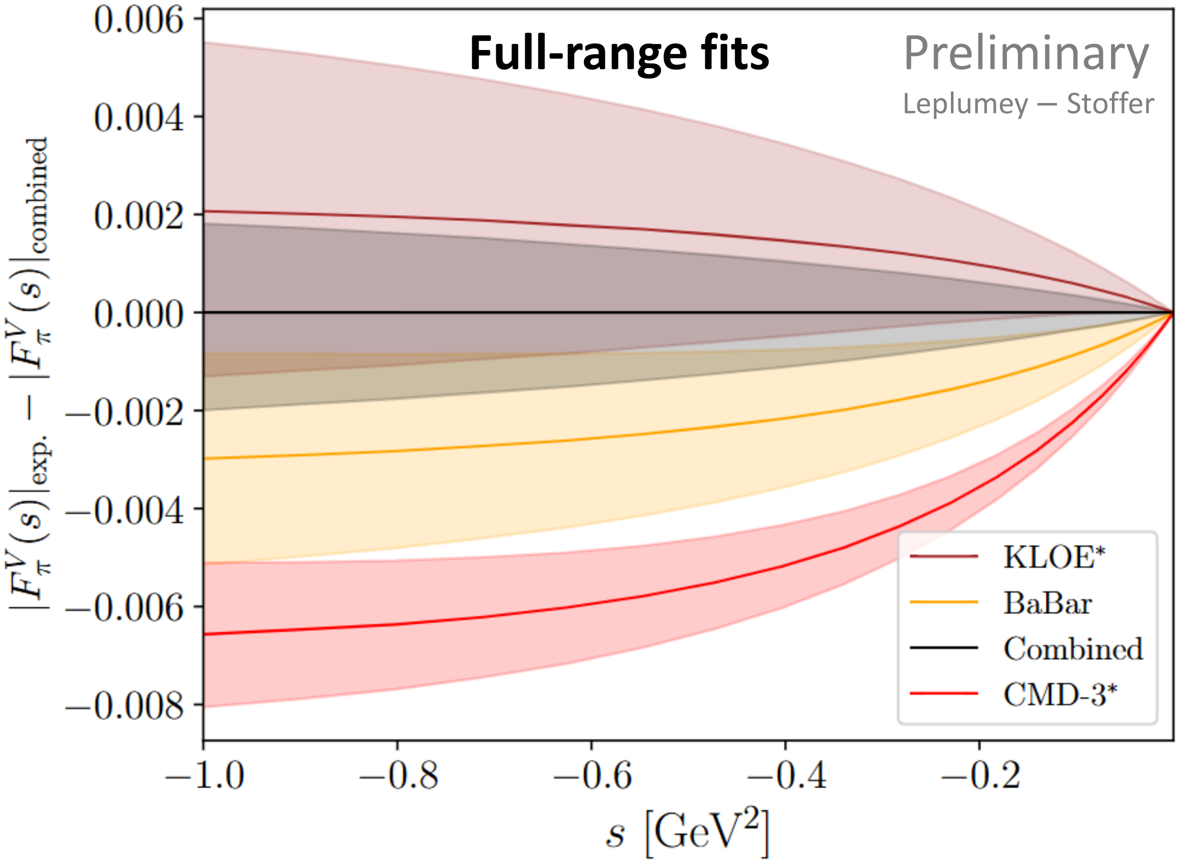


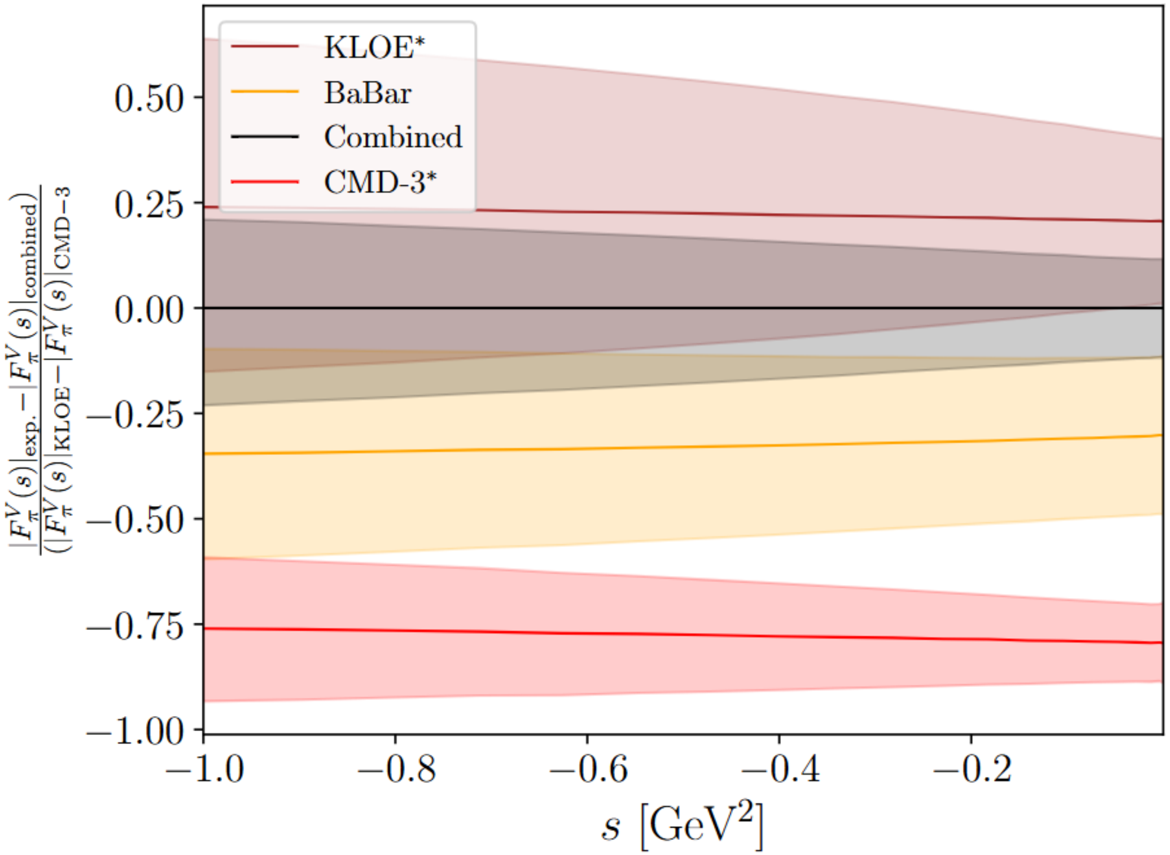


- The impact of a **future precise lattice calculation of $\langle r_\pi^2 \rangle$** can be assessed by the expected discrepancy with current results
  - The current world-leading χQCD result has $\sigma \sim 0.014\ \mathrm{fm}^2$: more precision would be needed
  - A precision of $0.003\ \mathrm{fm}^2$ (factor 4.6 reduction) would suffice to get $\sim 1.5\sigma$ tension with at least one experiment – up to 3σ in many cases



- An alternative probe to the charge radius could be **values of the VFF at fixed spacelike Q^2**
  - Values further from $Q^2 = 0$ are much easier to access on the lattice
  - A compromise has to be found with the increasing fit uncertainty at larger Q^2





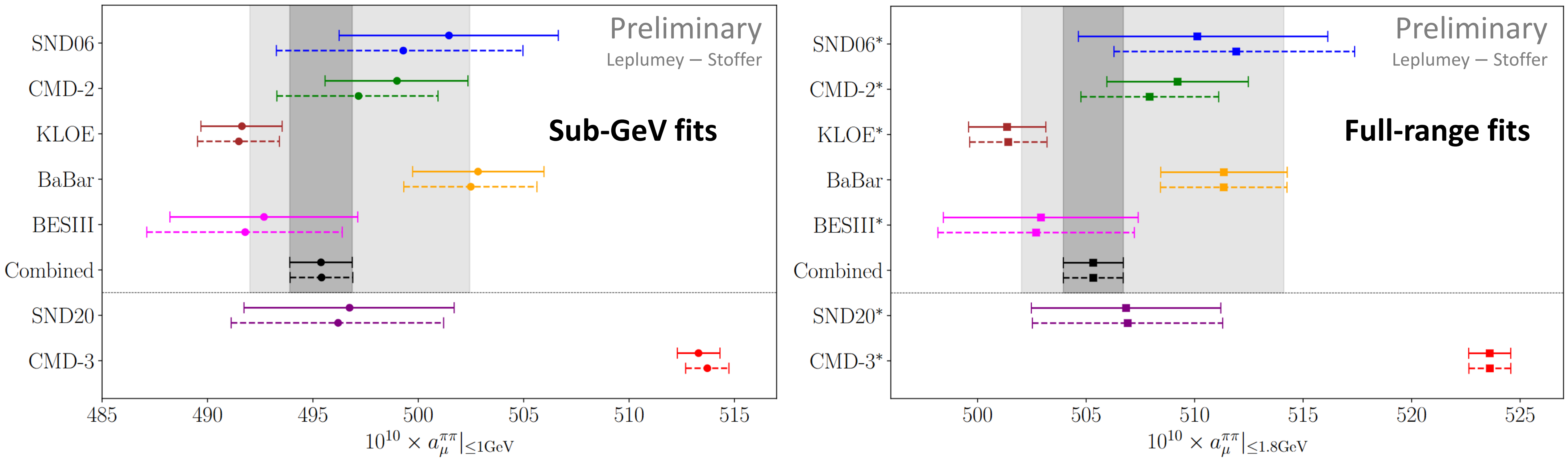
- Pion VFF representation incorporating dispersive constraints with **reduced model dependence** on full energy range
  - Determinations of $a_\mu^{\pi\pi}$ are **stable and robust under multiple parameterization changes**
  - Unitarity and analyticity constraints **propagate the discrepancies to the whole energy range**, including the very-long-distance window

- The **correlations with the pion charge radius** might be very helpful to probe the discrepancies if a **precise lattice calculation of the charge radius** arises
  - Removing zeros in the physical sheet helps a lot in stabilizing the result for sub-GeV fits
  - Nevertheless, full-range fits are nearly insensitive to the constraint and naturally exclude zeros

→ **Spacelike values of the VFF** might prove very valuable for **comparison with lattice calculations**!

BACKUP

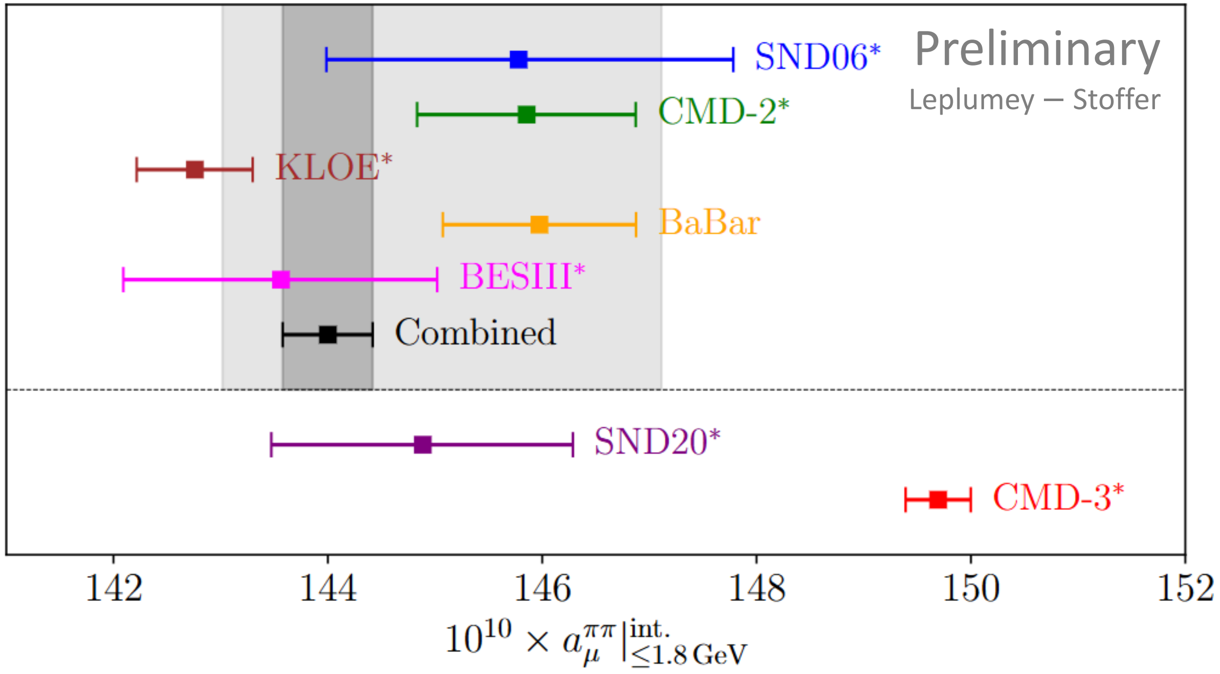
- The constraint of having no zeros is almost uneffective on $a_\mu^{\pi\pi}$ alone



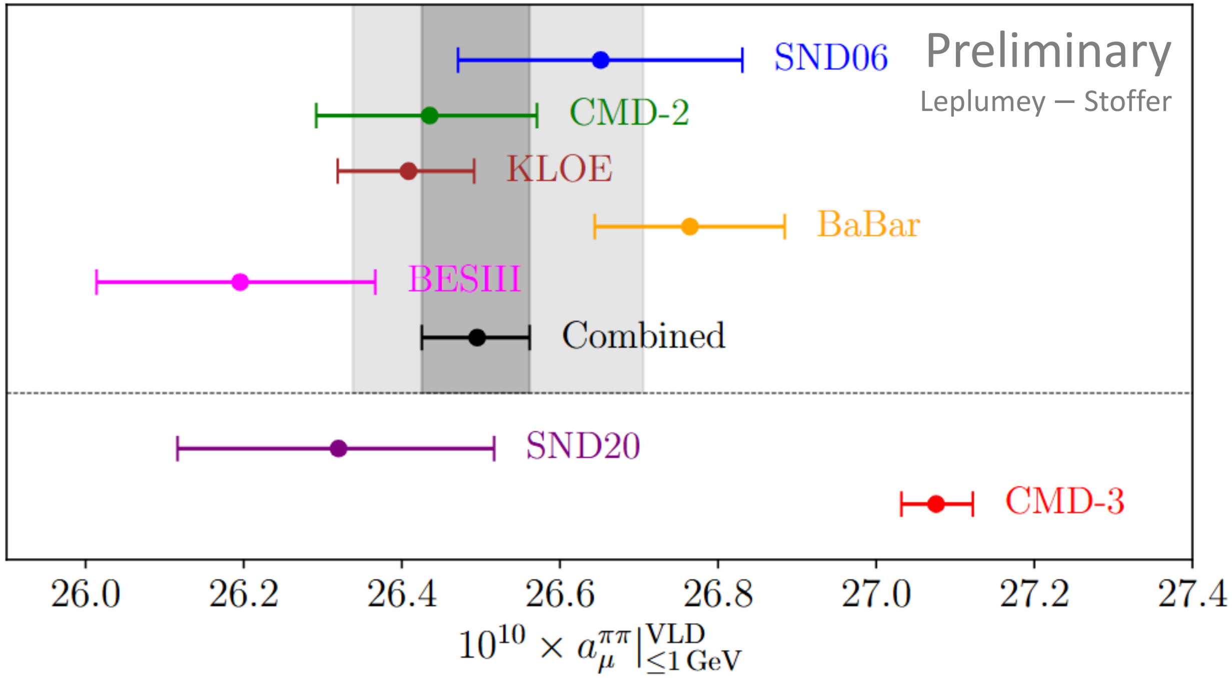
Constrained fits

Unconstrained fits

- The discrepancies are particularly visible in the intermediate window
- Still, very large discrepancies remain even at very long distance!

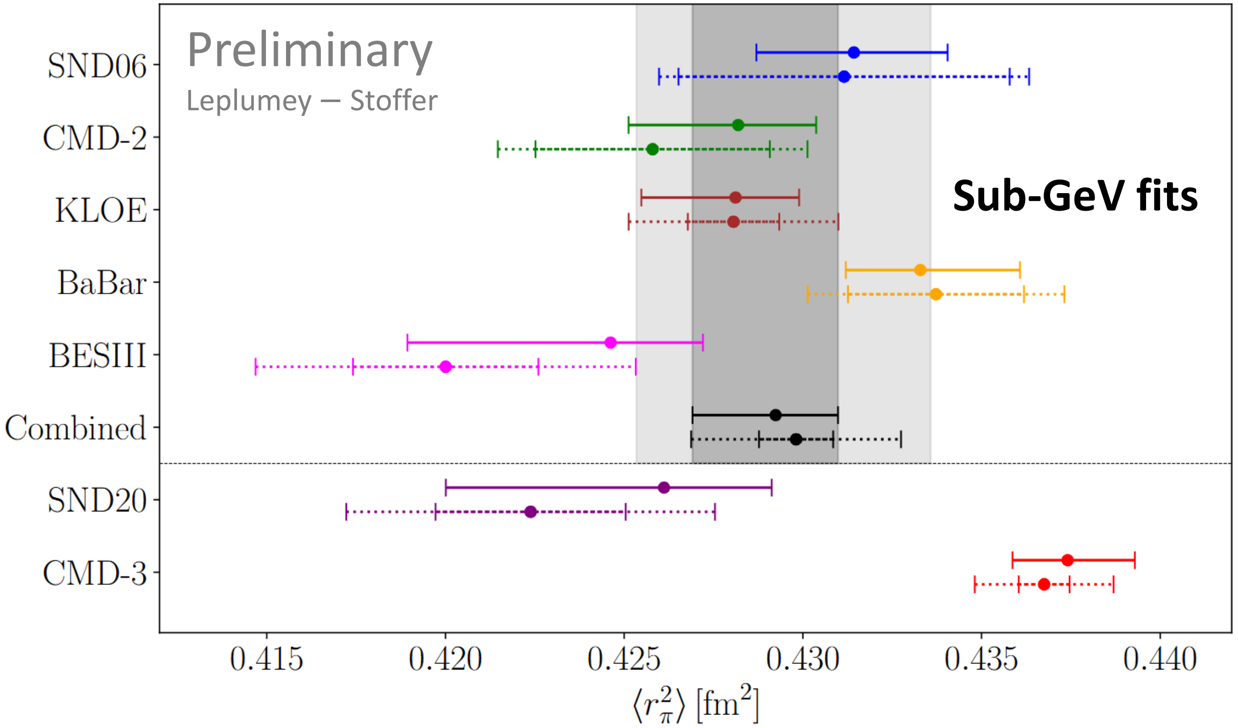


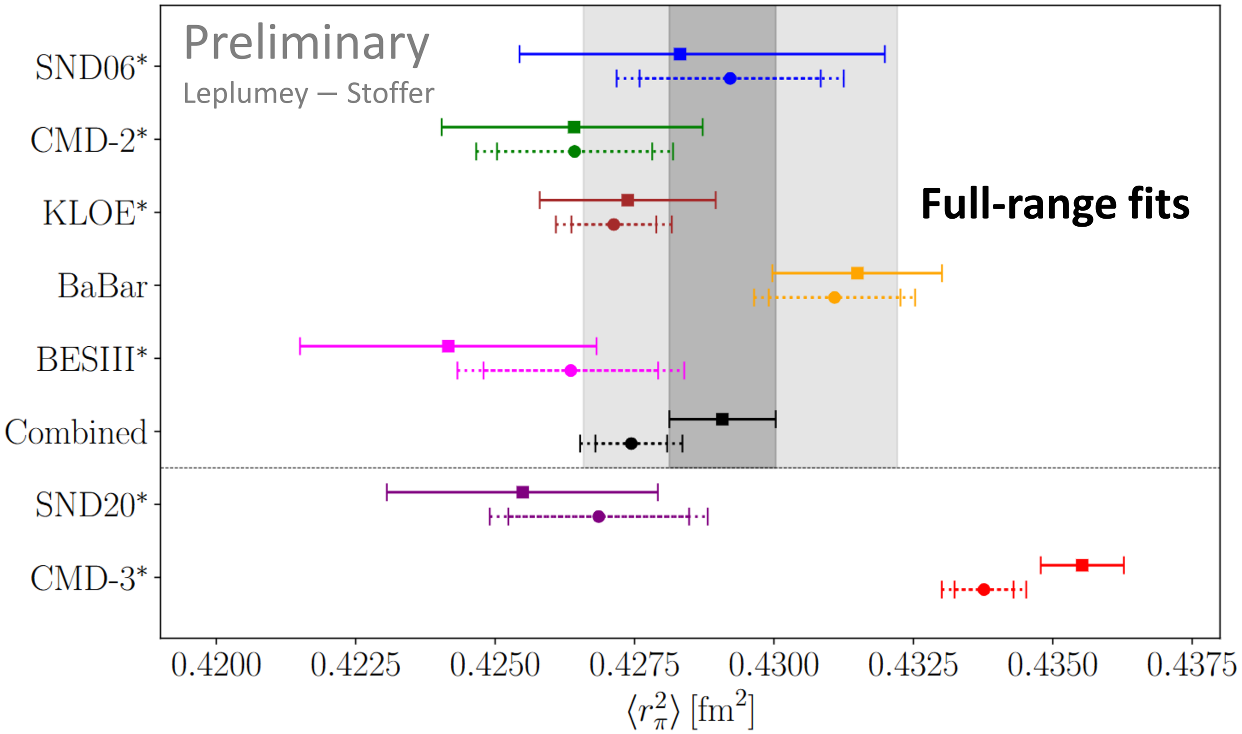
Intermediate window – full-range description



Very-long-distance window – sub-GeV description

- In sub-GeV fits, our new treatment of the systematics reduces the uncertainties
- In multi-GeV fits, switching to a less model-dependent parameterization slightly increased the uncertainties and shifted some of the results





New analysis (preliminary)

Previous analysis (arXiv:2501.09643)