Perturbative QCD below charm threshold: theory and tensions with e^+e^- data

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USP FAPESP SÃO PAULO RESEARCH FOUNDATION

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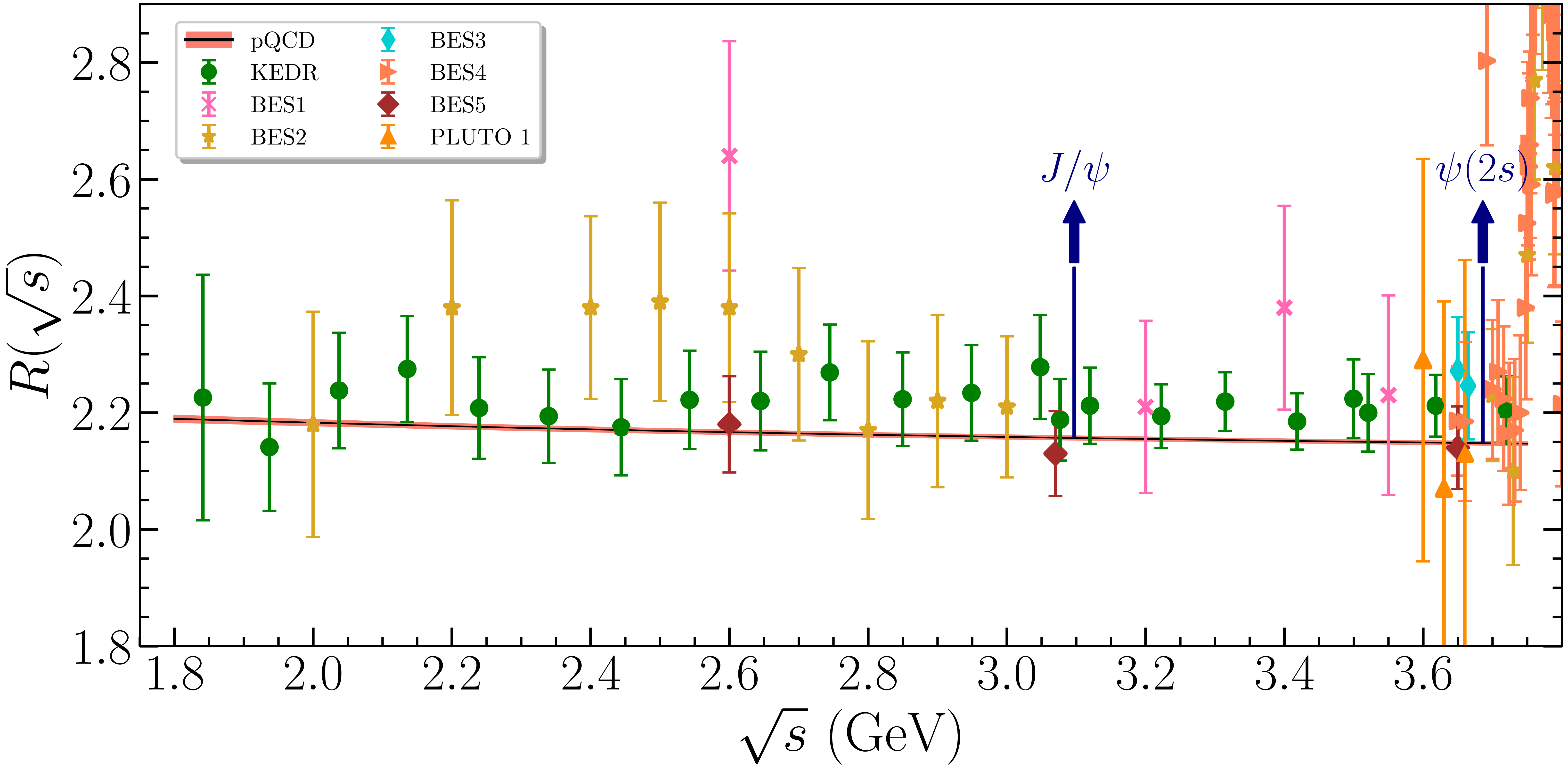
September 9, 2025

$R_{uds}(s)$ data and perturbative QCD

Inclusive data for *R(s)* below open charm (R_{uds})



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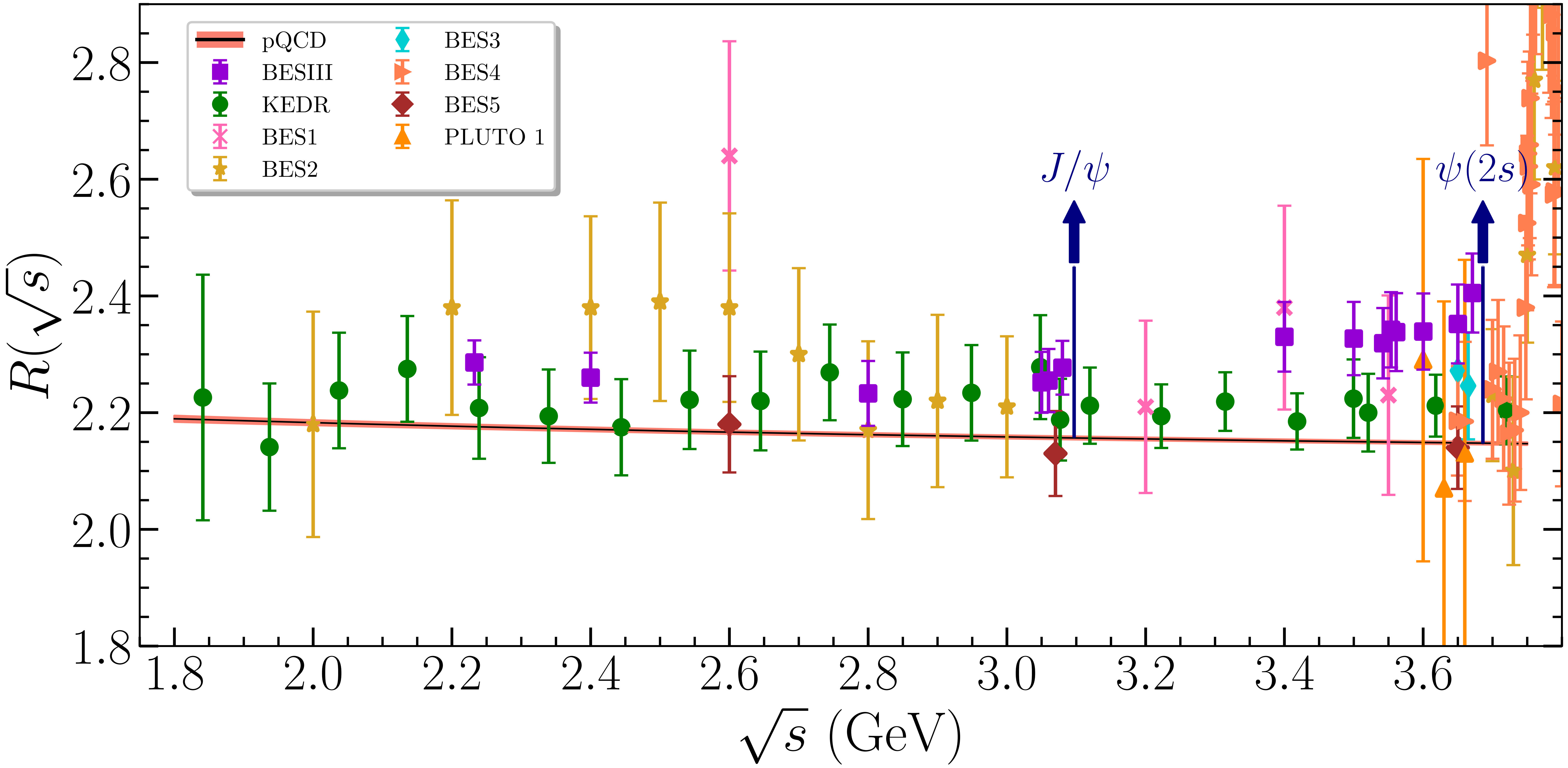

◉ 2021 **BES-III** results show a tension with pure pQCD. BES-III, 2112.11728, PRL (2022)

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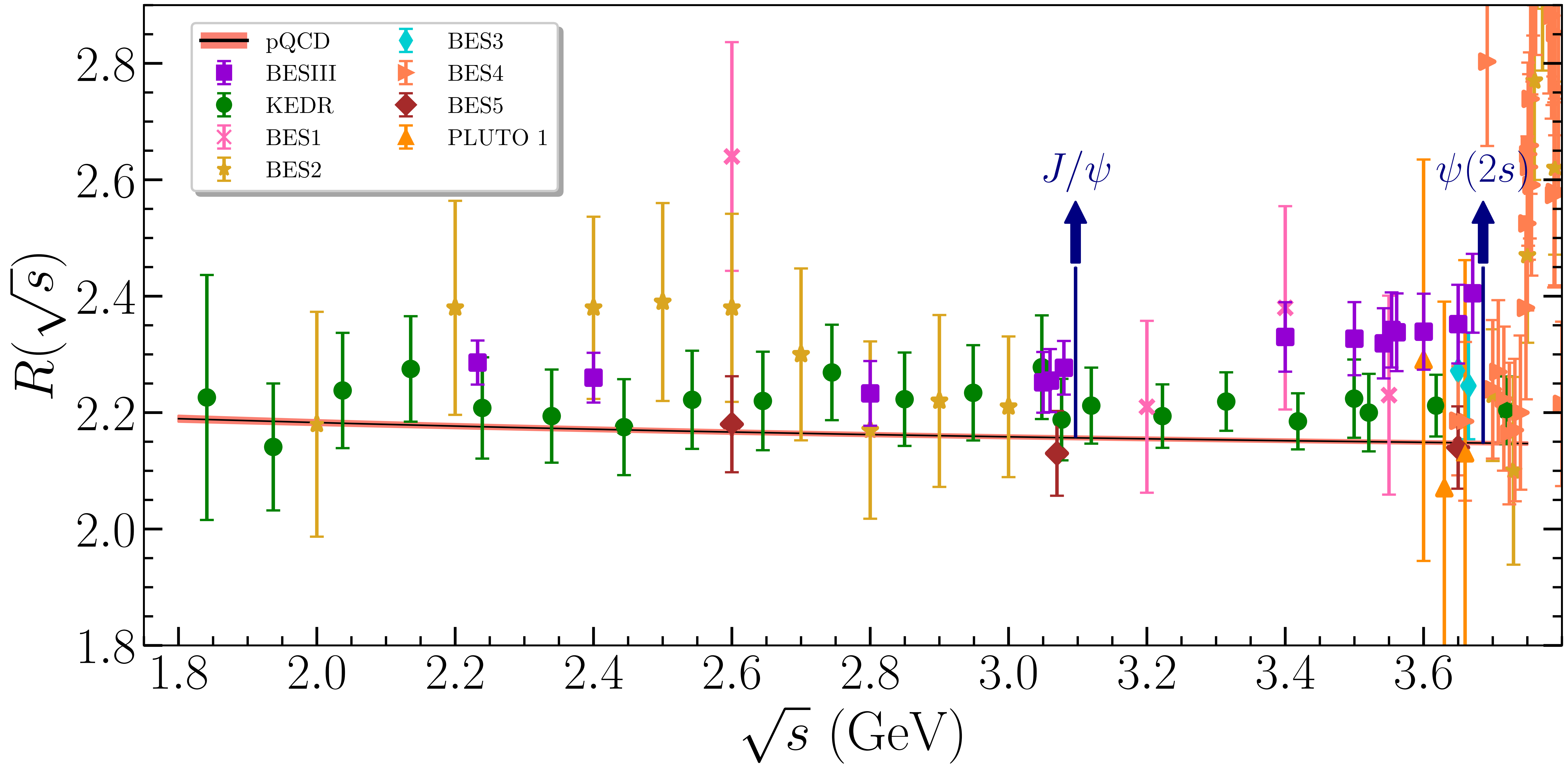

- 2021 **BES-III** results show a tension with pure pQCD. BES-III, 2112.11728, PRL (2022)
- Do the data agree with perturbative QCD (pQCD)? Possible duality violations (DV)?
- Are the data sets mutually compatible? Can the data be combined?
- Potential implications for g−2, strong coupling, and charm- and bottom-mass determinations.

R(s): theory

- Usual definition of *R*(*s*)

$$R(s) = \frac{3s}{4\pi\alpha_{\rm EM}^2}\sigma^{(0)}(e^+e^- \to \text{hadrons}(+\gamma))$$

- Correlator of two vector currents

$$\Pi^V_{\mu\nu}(q^2) = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2) = i\int d^4x\, e^{iq\cdot x}\langle 0|j^V_\mu(x)j^V_\nu(0)^\dagger|0\rangle$$

spectral function

$$\rho(s) = \frac{1}{\pi}{\rm Im}\Pi(s+i0)$$

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- We will work with the Adler function when discussing pQCD

$$D(q^2) = -q^2\frac{d}{dq^2}\Pi(q^2) = \frac{N_c}{12\pi^2}\left(1+\widehat{D}(q^2)\right)$$

hatted quantities contain only α_s corrections

- EM current for the *R*(*s*):

$$j^{\rm EM}_\mu = Q_u\,\bar{u}\gamma_\mu u + Q_d\,\bar{d}\gamma_\mu d + Q_s\,\bar{s}\gamma_\mu s = \frac{1}{2}(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d) + \frac{1}{6}(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d - 2\bar{s}\gamma_\mu s)$$

I=1 **I=0**

R(s): contributions

$$R_{uds}(s) = 12\pi^2 \rho_{\rm EM}(s) = N_c \sum_{q=u,d,s} Q_q^2 \left(1 + \delta^{(0)}_{\alpha_s} + \delta_{\rm DVs} + \delta_{m_q^2} + \delta_{\rm EM}\right)$$

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small OPE condensate contributions suppressed by α_s and 1/s not included

pQCD contribution

- pQCD computed with massless quarks.
- Singlet diagram contributions vanish because of the quark charges for R_{uds}.

The spectral function can be written as an integral over the Adler function in the complex plane

$$\widehat{\rho}(s) = \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} \widehat{D}(sx)$$

Perturbative Adler function

$$\widehat{D}_{\mathrm{pert}}(q^2) = \sum_{n=1}^{\infty} a_s^n(\mu^2) \sum_{k=1}^{n+1} k\, c_{n,k} \left[\log\left(\frac{-q^2}{\mu^2}\right)\right]^{k-1}$$

$$a_s \equiv \alpha_s/\pi$$

- Coefficients $c_{n,1}$ exactly known up to five loops, $\mathcal{O}(\alpha_s^4)$.

  Baikov, Chetyrkin, and Kühn '08
- We will consider an estimate for the six-loop coefficient as well

$$c_{5,1} = 280 \pm 140$$

Beneke and Jamin '08
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This is a particular case of the integrated Adler function moments

$$\frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} W_m(x) \widehat{D}(sx)$$

- Etensively studied in the context of hadronic tau decays.
- There is a body of work about how these moments behave in perturbation theory, the main renormalons, potential problems and how to cure them etc.

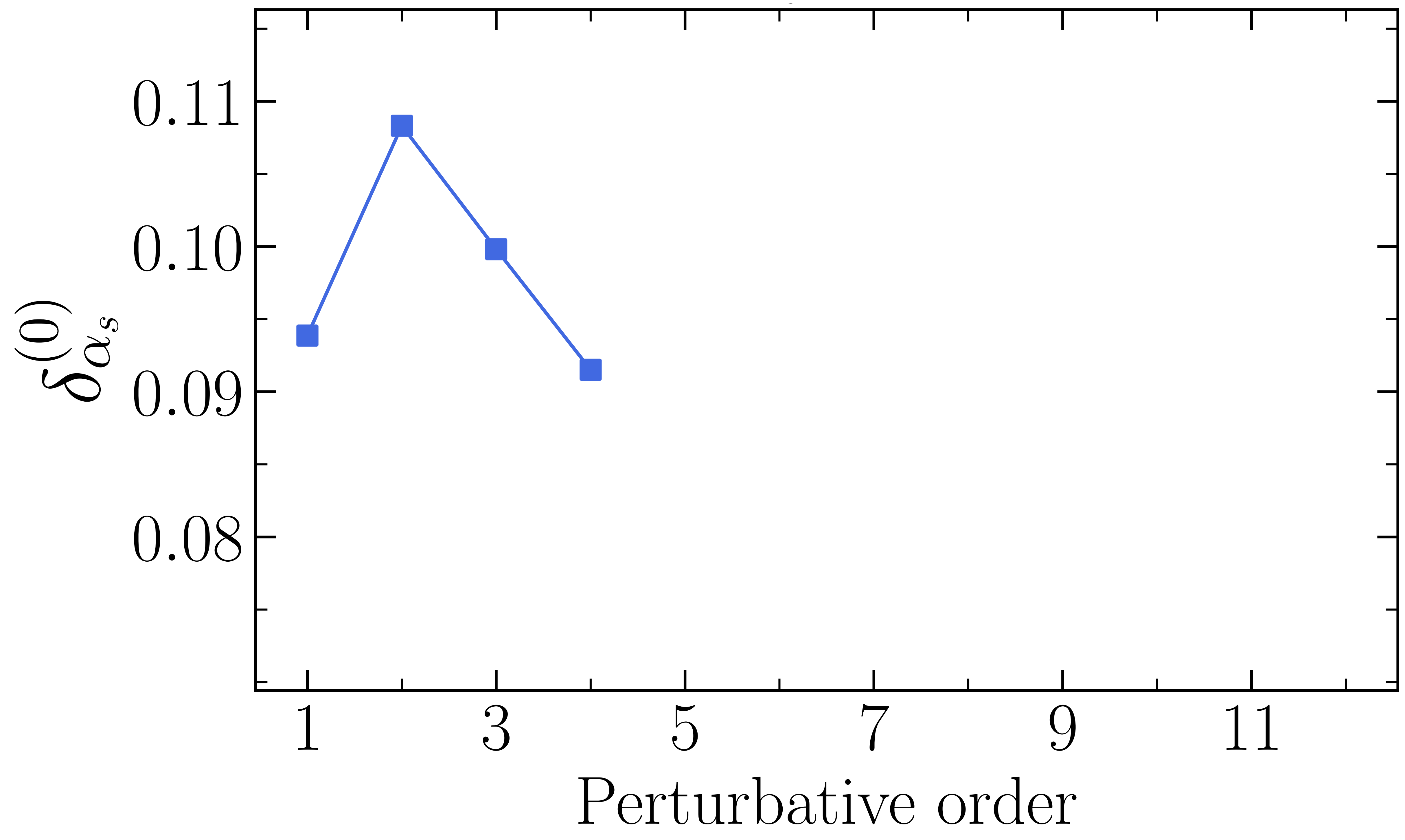
Beneke & Jamin ’08; Descotes-Genon & Malaescu ’10; DB, Beneke and Jamin ’12; DB and Oliani ’20; Hoang & Regner ’20; Benitez-Rathgeb, DB and Hoang ’22; Golterman, Maltman and Peris ’23, Gracia, Hoang and Mateu '23….

pQCD contribution

Using “standard" Fixed Order Perturbation Theory (FOPT) $(\mu^2 = s)$

$$\delta^{(0)}_{\alpha_s}(s) = a_s(s) + 1.6398\, a_s^2(s) - 10.284 a_s^3(s) - 106.88 a_s^4(s)$$

$$\delta^{(0)}_{\alpha_s}(4\ \mathrm{GeV}^2) = 0.09387 + 0.01445 - 0.008506 - 0.008298$$



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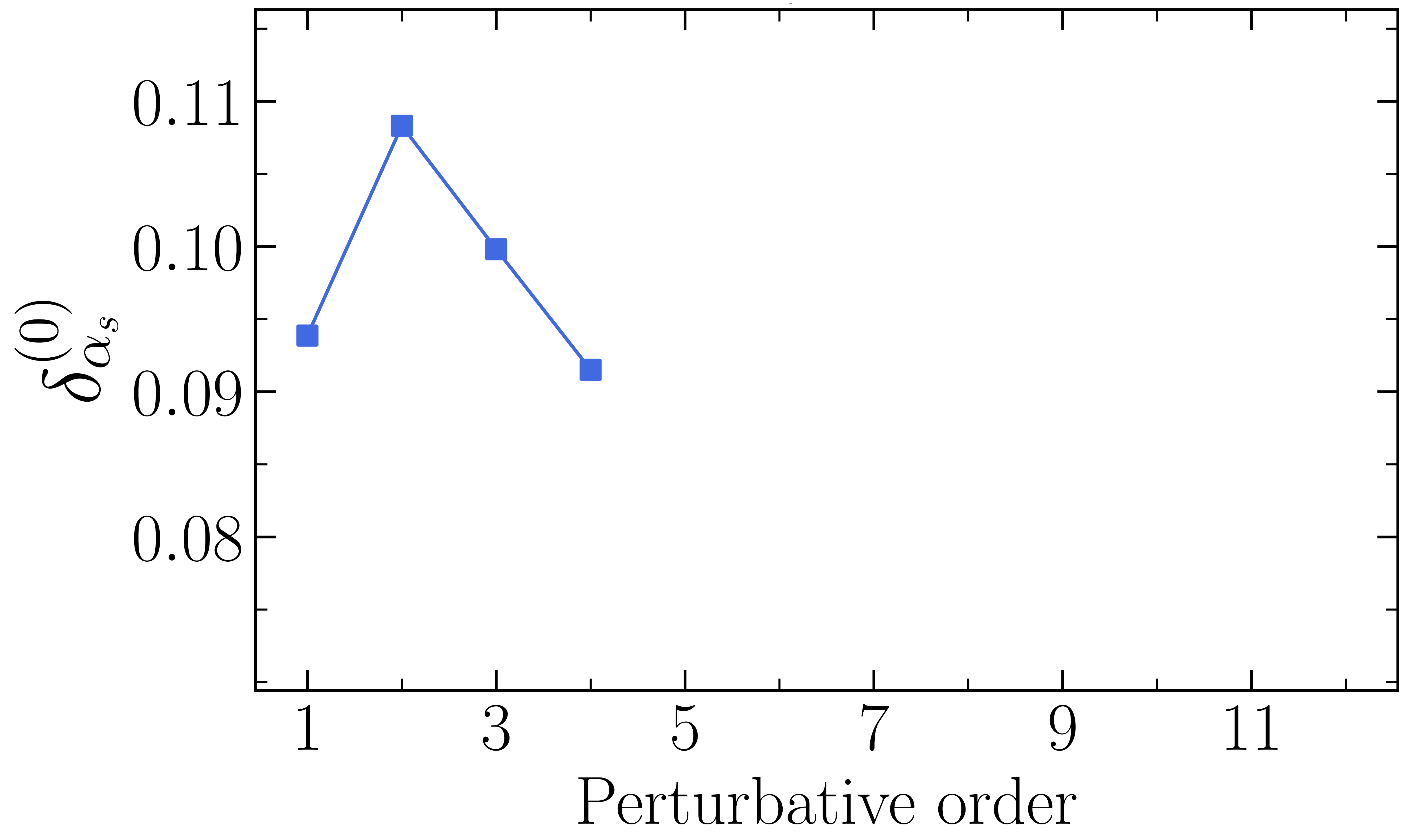

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$$\delta^{(0)}_{\alpha_s}(4\ \mathrm{GeV}^2) = 0.09387 + 0.01445 - 0.008506 - 0.008298 - 0.0036(10)_{c_{51}} + \cdots = 0.0879(21)$$

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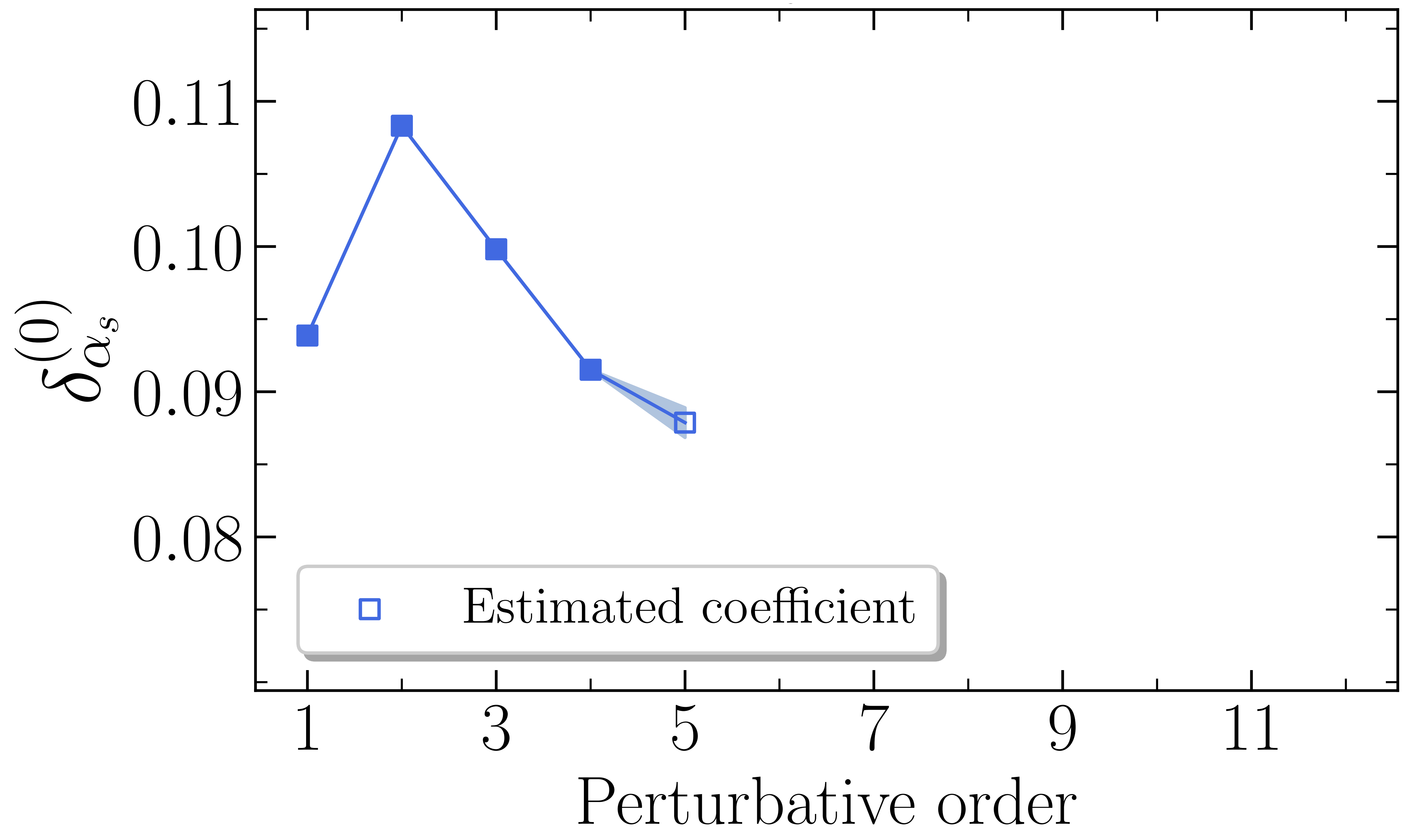

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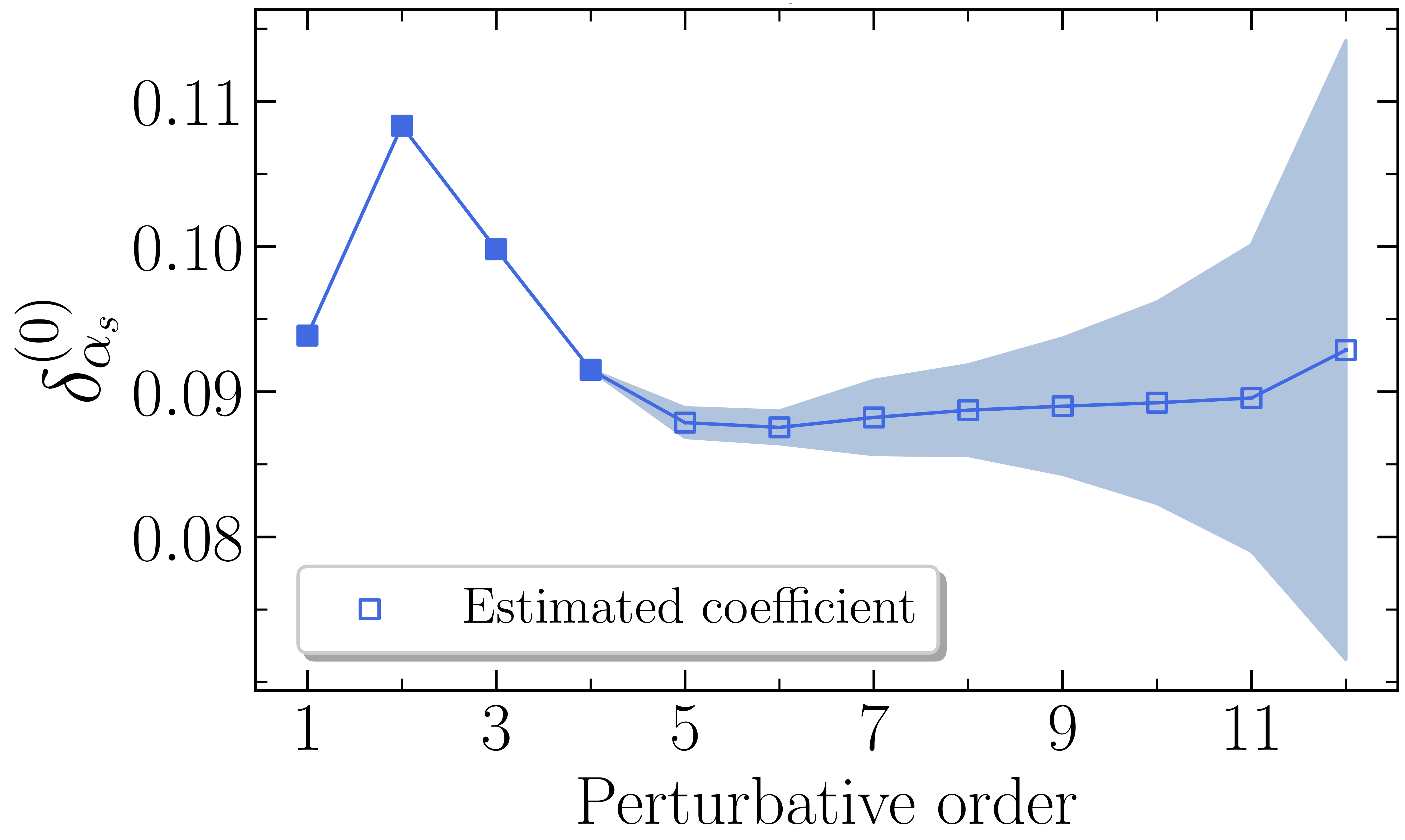
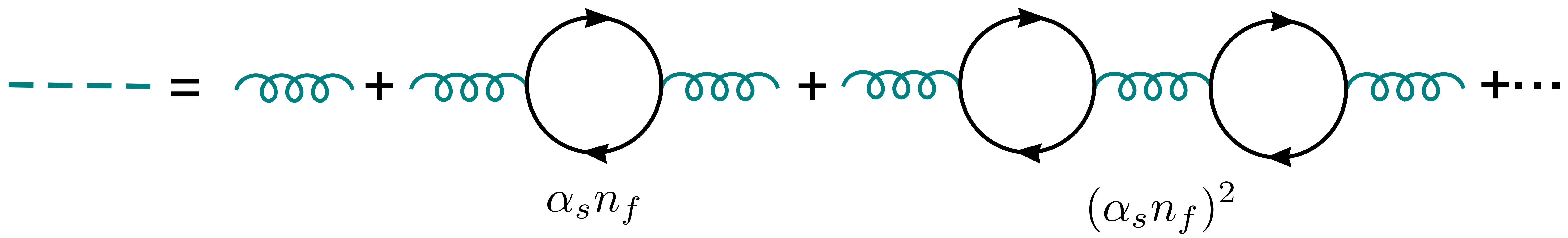


- Higher-order coefficients from a reconstruction using Padé approximants DB, Masjuan, Oliani '18
- Other higher-order models give similar results Beneke & Jamin '08
- Indication that the series stabilizes not too far from the result at $\mathcal{O}(\alpha_s^4)$

pQCD contribution: large-β_0

Gluon propagator with insertions of $q\bar{q}$ loops



$\alpha_s n_f \sim \mathcal{O}(1)$ $\quad \beta_{0,f} = \frac{n_f}{6\pi}$

"Non-abelianization" of the result

$$\beta_{0,f} \to \beta_0 = \beta_{0,f} + \beta_{0,\text{nA}}$$

$$n_f \to 6\pi\beta_0$$

A set of non-abelian diagrams is included (running coupling)

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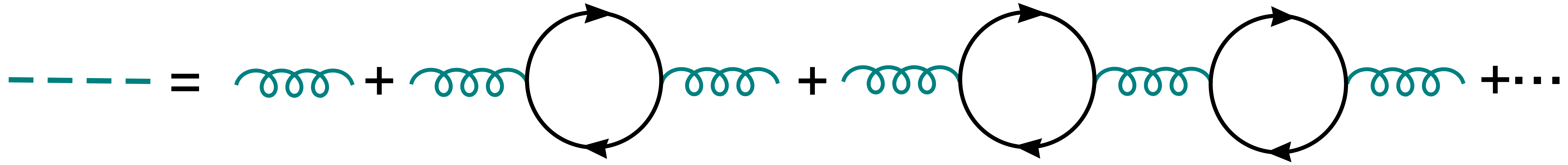

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Model for the series to all orders in pQCD

$$\widehat{D}(Q^2) = \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}(Q^2) \quad \Rightarrow \quad B[\widehat{D}](t) \equiv \sum_{n=0}^{\infty} r_n \frac{t^n}{n!}$$

Borel transform

pQCD contribution: large-β_0

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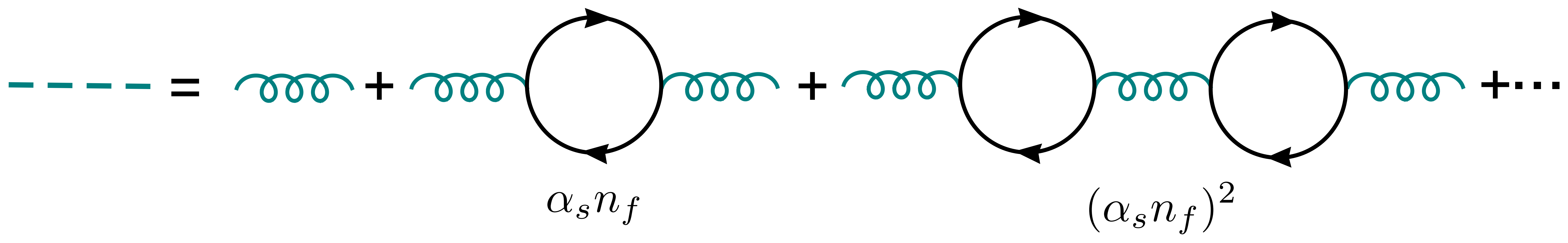

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Borel transform

Borel transform exactly known (here we know everything)

$$B[\widehat{D}_{L\beta_0}](u) = \frac{32}{3\pi}\left(\frac{Q^2}{\mu^2}\right)^{-u} \frac{\mathrm{e}^{-Cu}}{(2-u)} \sum_{k=2}^{\infty} \frac{(-1)^k k}{[k^2-(1-u)^2]^2}$$

Beneke '93
Broadhurst '93

$$\widehat{D}(\alpha) = \int_0^\infty dt\, e^{-t/\alpha} B[\widehat{D}](t)$$

True value of the series can be calculated from the Borel integral

pQCD contribution: large-β_0

large-β_0: $$\delta^{(0)}_{\alpha_s,L\beta_0} = a_s(s) + 1.56\,a_s^2(s) - 0.944a_s^3(s) - 52.9a_s^4(s) - 283a_s^5(s) + \cdots$$

QCD: $$\delta^{(0)}_{\alpha_s}(s) = a_s(s) + 1.6398\,a_s^2(s) - 10.284a_s^3(s) - 106.88a_s^4(s) - (499.6 \pm 140)a_s^5(s) + \cdots$$

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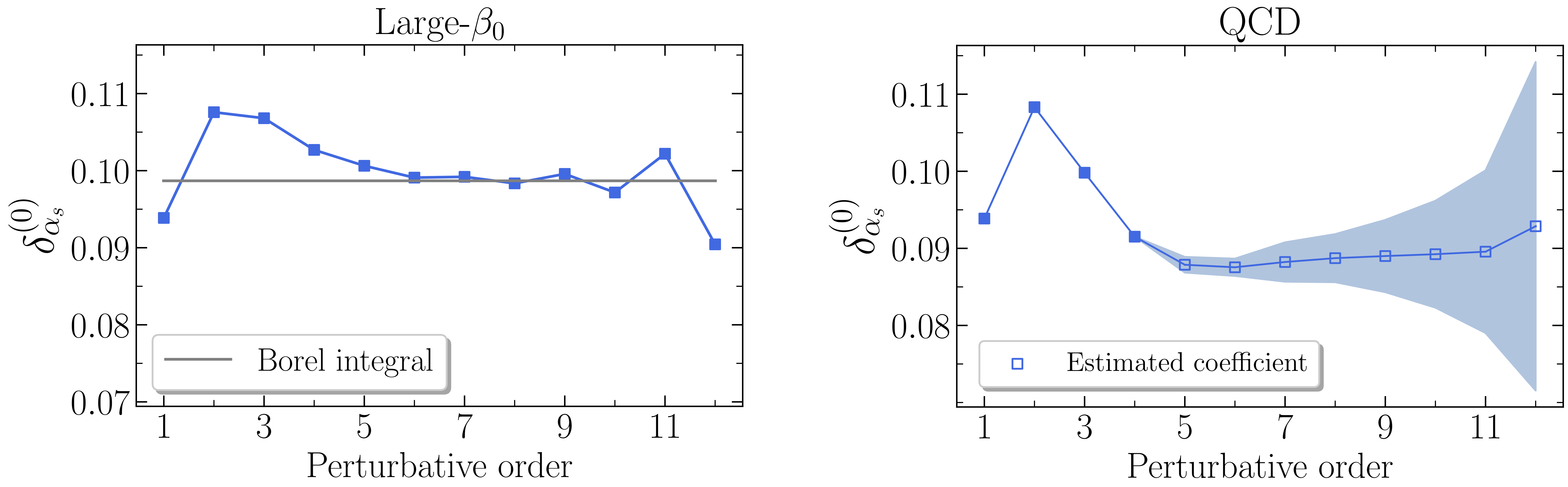

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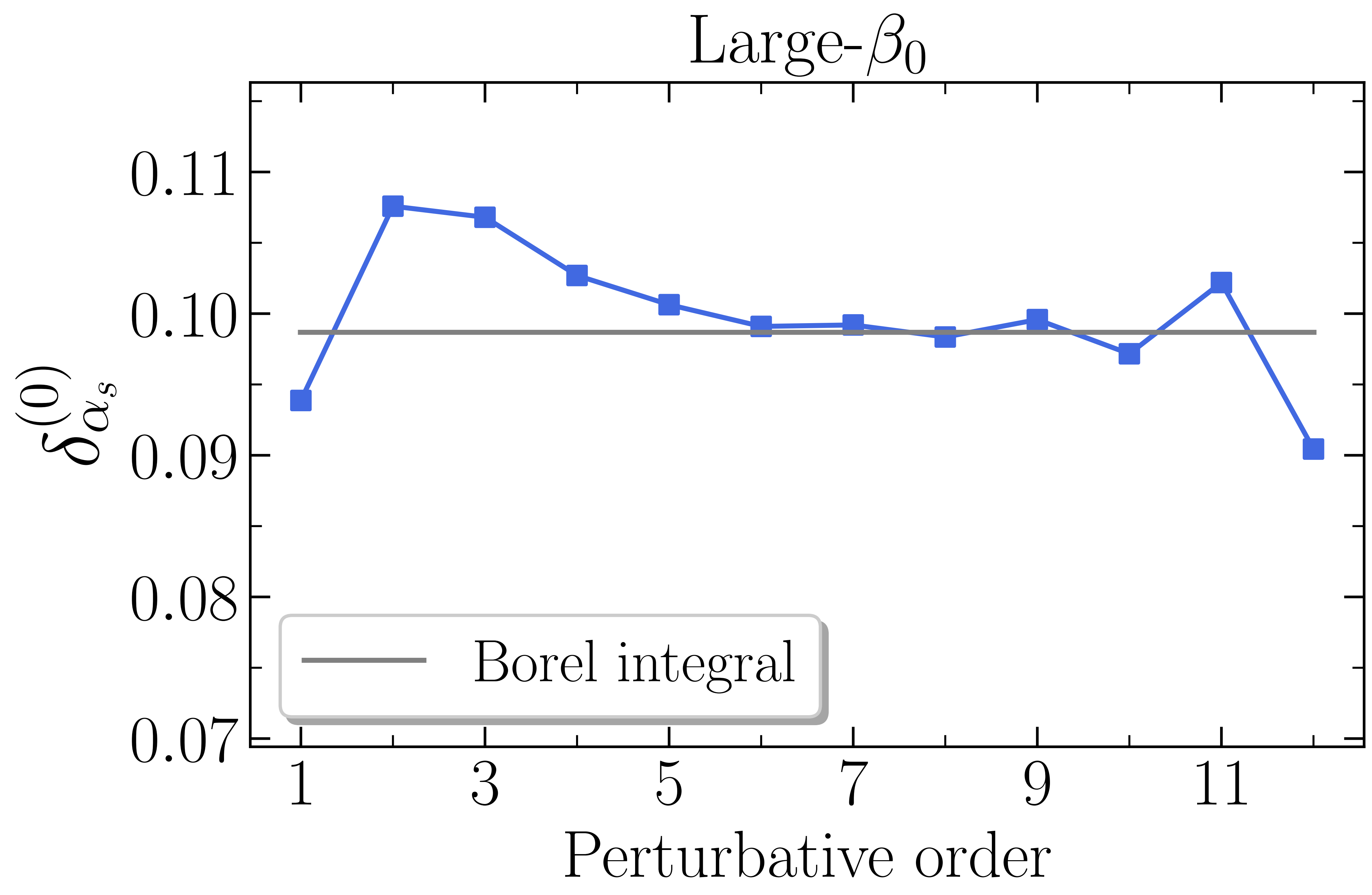

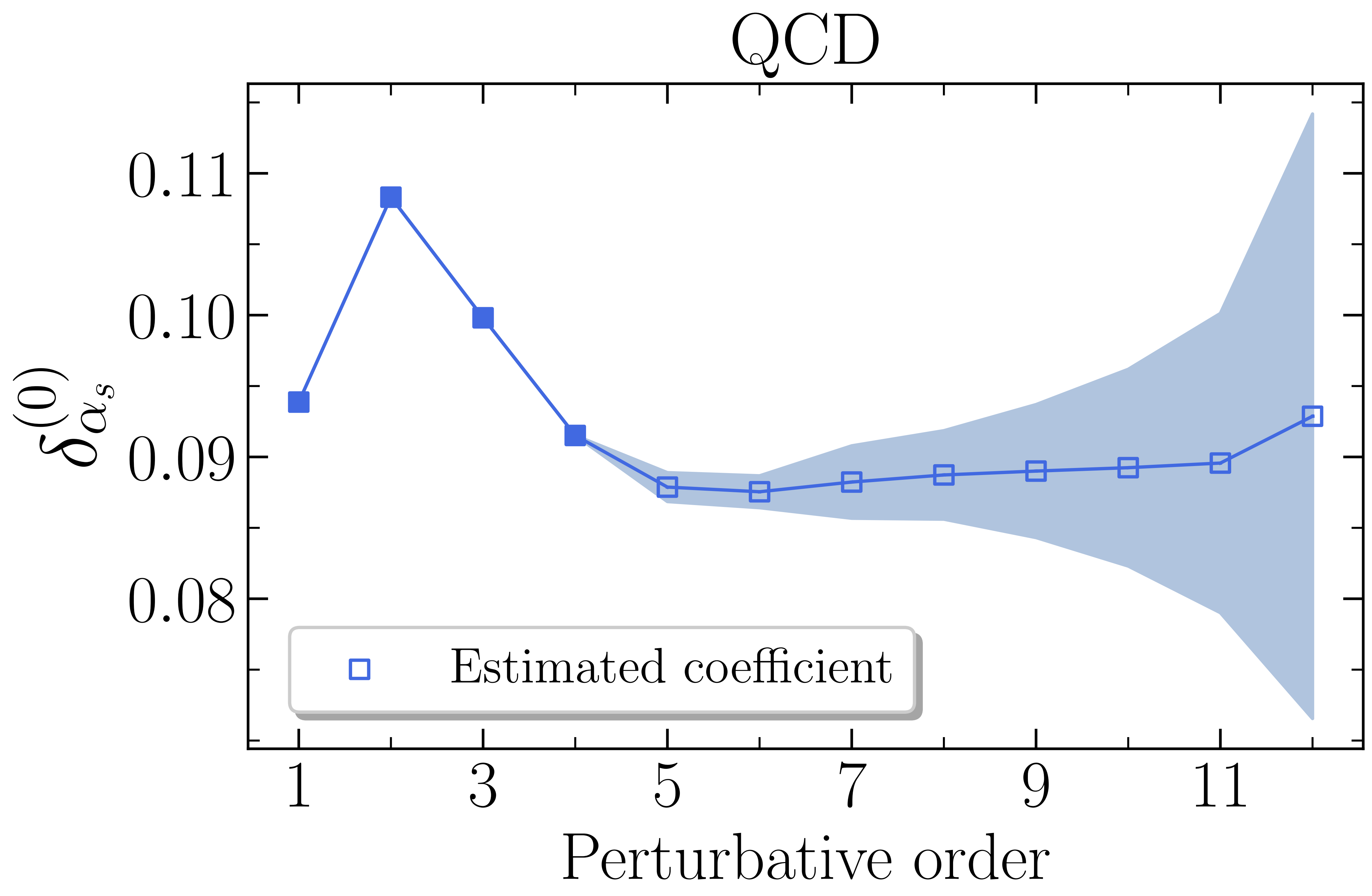

Another indication that the series is not a bad approximation provided we have at least 4 or 5 terms

pQCD: renormalon-free gluon-condensate scheme

- FOPT not the only choice for the renormalization scale.

- Another popular choice is Contour Improved Perturbation Theory (CIPT), with a running scale that ressums the running of the strong coupling along the contour integration. But it is well known that FOPT and CIPT **do not agree.**

- CIPT was recently shown **to be incompatible** with the Operator Product Expansion.

Hoang & Regner ’20 (see also Golterman, Maltman and Peris ’23)

- But not everything is lost… The problem with CIPT is its sensitivity to IR renormalons, in particular the leading IR renormalon associated with the gluon condensate.

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- We can redefine the gluon condensate to remove its renormalon (renormalon-free gluon-condensate scheme, 'RF scheme') and then we are allowed to use a new version of CIPT and compare with FOPT.

Benitez-Rathgeb, DB, Hoang ‘22

$$\langle G^2\rangle^{(n)} = \langle G^2\rangle^{\rm RF} - R^4 N_g \sum_{n=0} r_n^{(4)} \alpha_s^{n+1}(R^2) + N_g c_0(R^2) \qquad \text{with} \qquad c_0(R^2) = R^4\left(\frac{2\pi}{\beta_1}\right) \mathrm{PV}\int_0^\infty du\, e^{-u/\bar{\alpha}_s(R^2)} \frac{1}{2-u}$$

This function is introduced such that the new GC is formally independent of scale R

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new RF gluon condensate

gluon condensate norm (renormalon residue)

$$\langle G^2\rangle^{(n)} = \langle G^2\rangle^{\mathrm{RF}} - R^4 N_g \sum_{n=0} r_n^{(4)} \alpha_s^{n+1}(R^2) + N_g c_0(R^2)$$

IR subtraction scale

with

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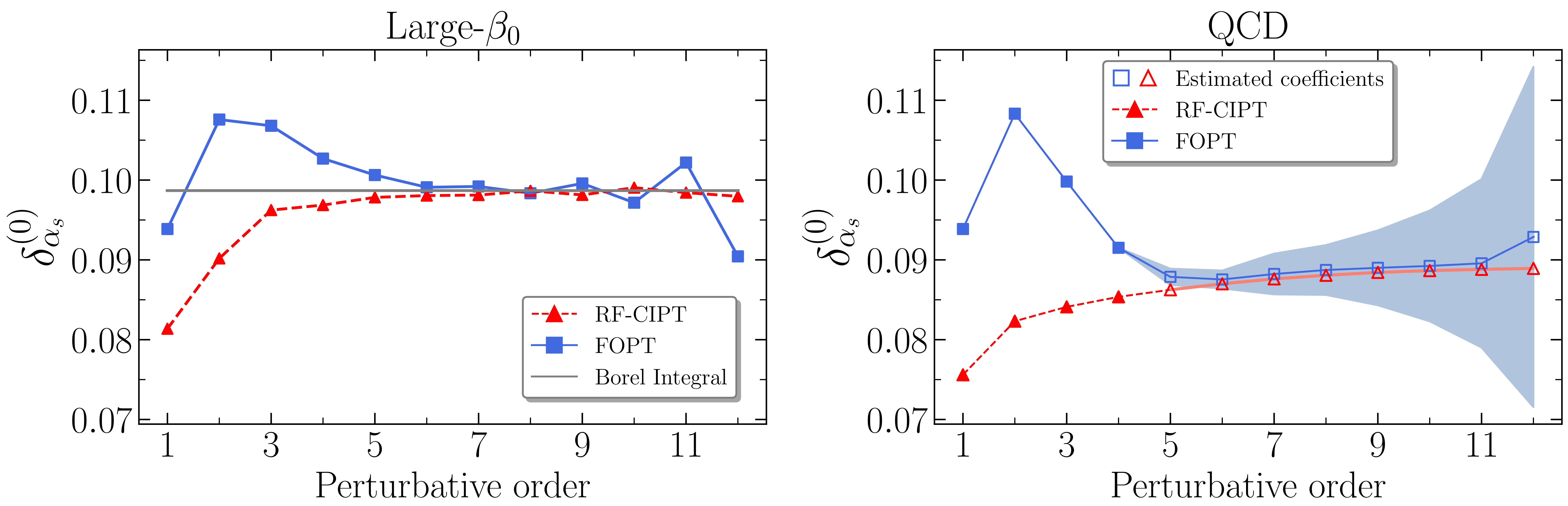
- Results for FOPT and RF-CIPT with $s = 4$ GeV^2.



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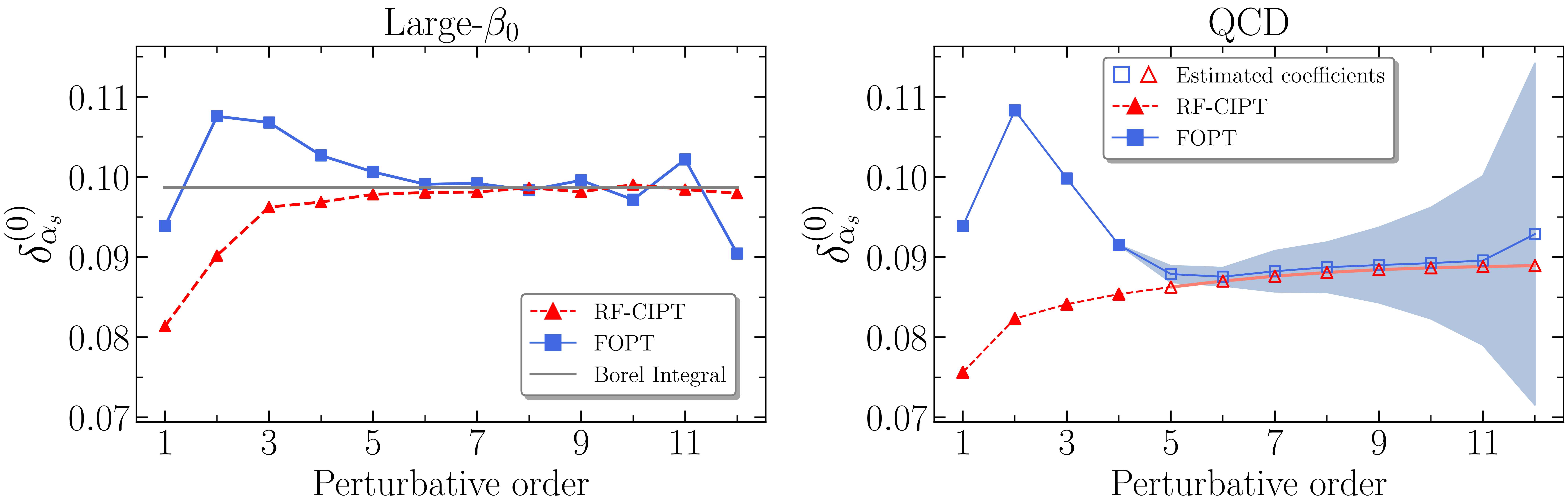

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3rd indication that the series is not a bad approximation provided we have at least 4 or 5 terms

Duality Violations

- pQCD is not all: beyond pQCD + OPE condensates there are potential quark-hadron duality violations (DVs)

- DVs cannot be obtained from first principles but can be parametrized based on reasonable assumptions about the QCD spectrum (asymptotic Regge trajectories and large N_c arguments)

$$\Pi(z) = \Pi_{\rm OPE}(z) + [\Pi(z) - \Pi_{\rm OPE}(z)] = \Pi_{\rm OPE}(z) + \Delta(z) \qquad \rho^{\rm DVs}_{\rm EM}(s) = \frac{1}{\pi}{\rm Im}\Delta(s)$$

- Asymptotically (large energies) one can show, given the assumptions above that (EM current, $I = 0$ and $I = 1$)

$$\rho^{\rm DVs}_{\rm EM}(s) = \underbrace{\frac{5}{9}e^{-\delta_1-\gamma_1 s}\sin(\alpha_1+\beta_1 s)}_{\text{I=1}} + \underbrace{\frac{1}{9}e^{-\delta_0-\gamma_0 s}\sin(\alpha_0+\beta_0 s)}_{\text{I=0}}$$

Catà, Golterman and Peris '05, '08
DB, Caprini, Golterman, Maltman and Peris '17
DB, Golterman, Maltman and Peris '18
also used in, e.g,
Benton, DB, Keshavarzi, Maltman and Peris '23, '24, '25

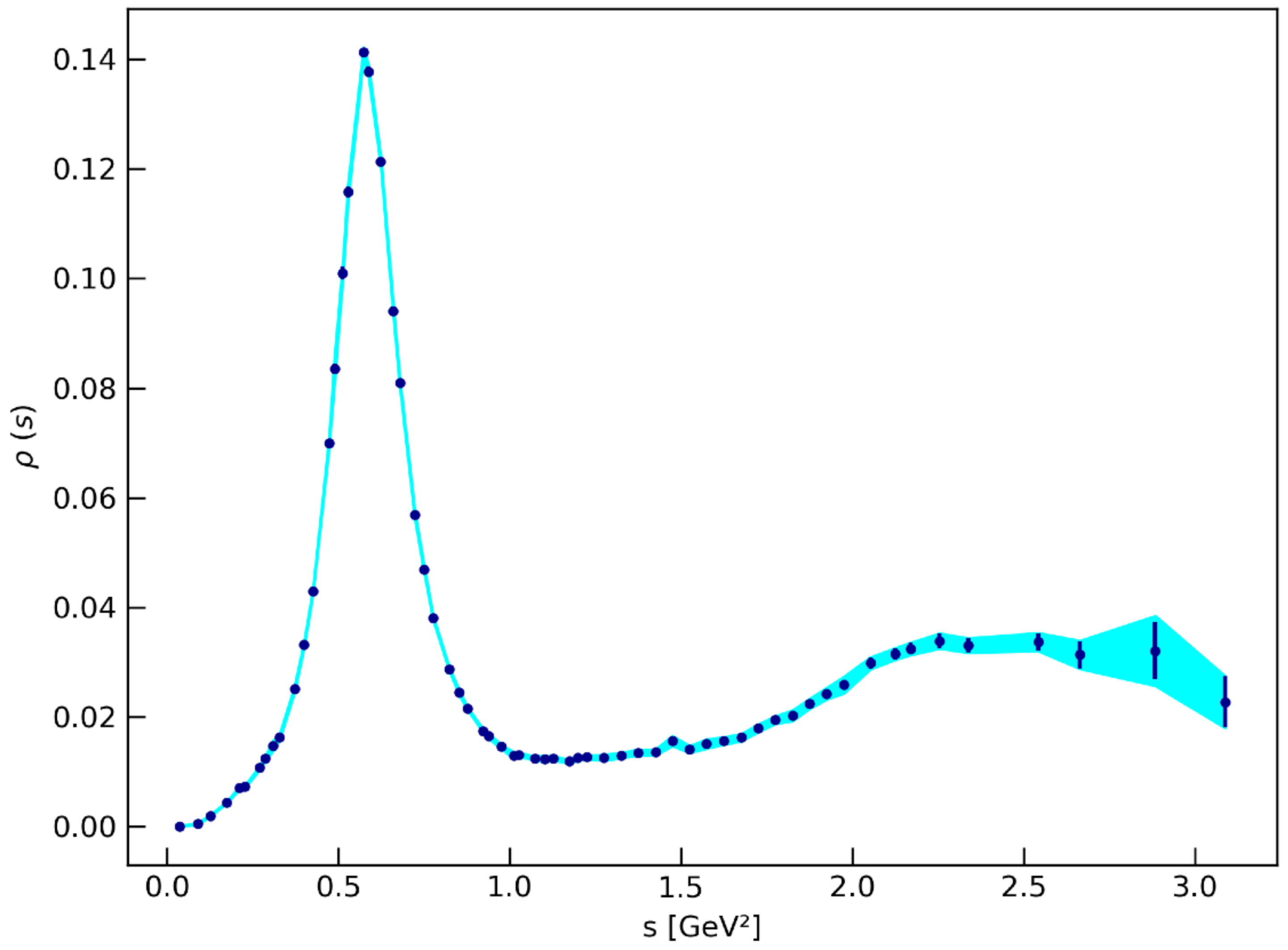
- There are four parameters *per channel*, because DVs are associated with resonances.

- Additional assumption: $\gamma_0 = \gamma_1$ and $\beta_0 = \beta_1$. All 6 parameters fixed with external information.

- $I = 1$ parameters can be fixed from fits to tau decay data while $I = 0$ can be fixed from fits to exclusive R(s) data.

Duality Violations: parameters

Isospin 1



I=1 parameters

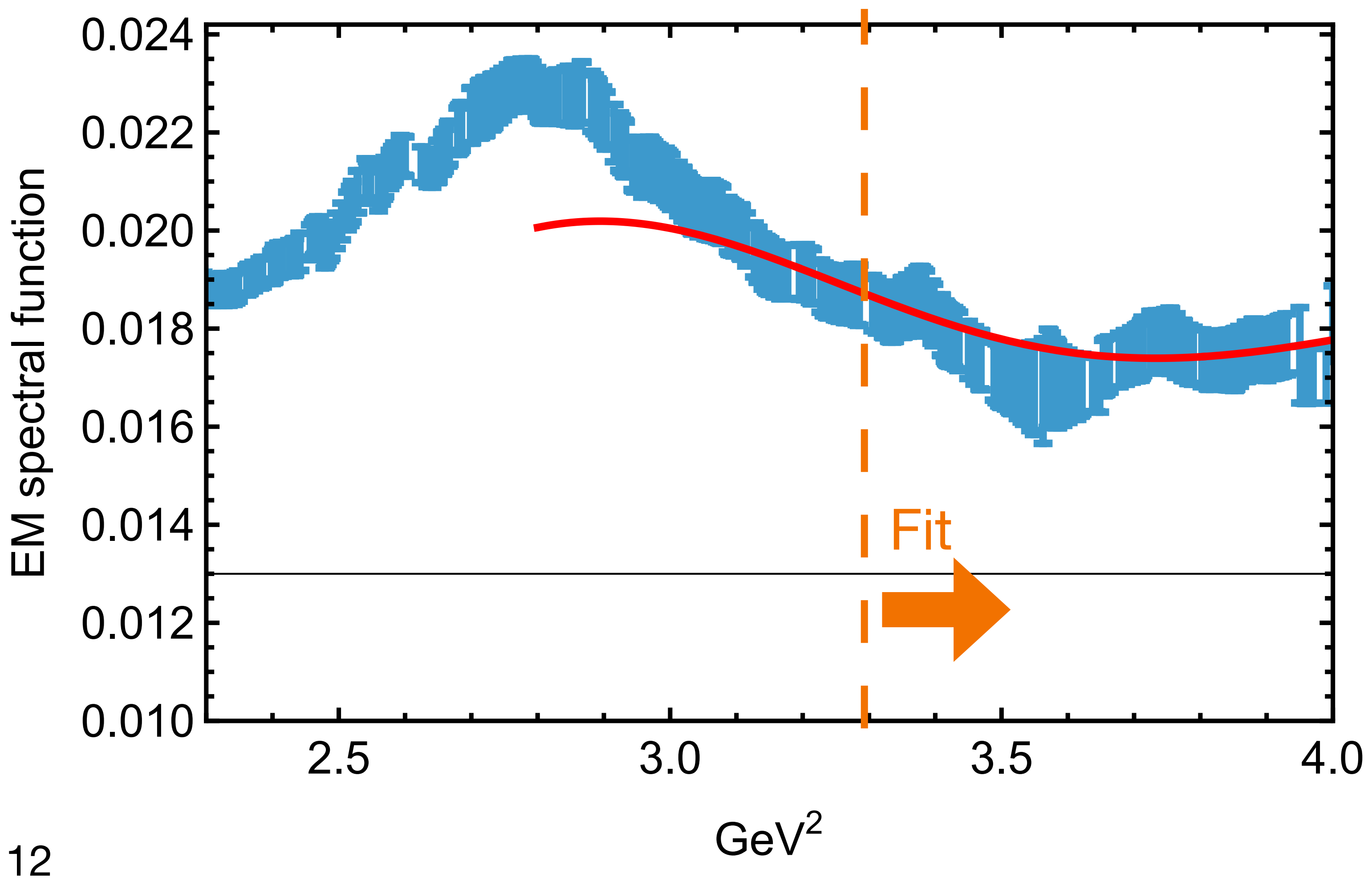
$$\delta_1 = 3.01(39)$$
$$\gamma_1 = 0.87(24)\ \text{GeV}^{-2}$$
$$\alpha_1 = -1.34(73)$$
$$\beta_1 = 3.78(38)\ \text{GeV}^{-2}$$

- Fits to sum rules from our most recent inclusive vector-isovector spectral function

DB, A. Eiben, M. Golterman, K. Maltman, L. Mansur, and S. Peris, 2502.08147, Phys. Rev D (2025)

(see poster by Lucas Mansur)

Isospin 0



I=0 parameters

$$\delta_0 = 0.96(22)$$
$$\alpha_0 = 0.80(27)$$

- Updated fit to the *exclusive* R_{uds}(s) data from the KNT compilation

DB, Golterman, Keshavarzi, Maltman, Nomura, Peris and Teubner, ‘18

Full $R_{uds}(s)$

$$R_{uds}(s) = 12\pi^2 \rho_{\rm EM}(s) = N_c \sum_{q=u,d,s} Q_q^2 \left(1 + \delta_{\alpha_s}^{(0)} + \delta_{\rm DVs} + \delta_{m_q^2} + \delta_{\rm EM}\right)$$

$\delta_{\alpha_s}^{(0)}$:
$$\delta_{\alpha_s}^{(0)}(s) = a_s(s) + 1.6398\, a_s^2(s) - 10.284 a_s^3(s) - 106.88 a_s^4(s) - (499.6 \pm 140) a_s^5(s) + \cdots$$

$\delta_{\rm DVs}$:
$$\delta_{\rm DVs} = 6\pi^2 \left(\frac{5}{9} e^{-\delta_1 - \gamma_1 s} \sin(\alpha_1 + \beta_1 s) + \frac{1}{9} e^{-\delta_0 - \gamma_0 s} \sin(\alpha_0 + \beta_0 s) \right)$$

$\delta_{m_q^2}$:
$$\delta_{m_q^2} = \frac{m_s^2(s)}{s} \left(1 + 2a_s + \frac{227}{12} a_s^2 + \cdots \right)$$

$\delta_{\rm EM}$:
$$\delta_{\rm EM} = \frac{\alpha_{\rm EM}}{4\pi}$$

Full $R_{uds}(s)$

$\sqrt{s}$ (GeV)	R_{uds}	$\delta^{(0)}_{\alpha_s}$	$\delta_{\rm DVs}$	$\delta_{m_q^2}$	$\delta_{\rm EM}$
2.0	2.12(14)	0.0879(21)	−0.030(69)	0.002960(56)	0.00058
2.5	2.181(19)	0.0822(15)	0.0060(93)	0.001621(28)	0.00058
3.0	2.1576(46)	0.0776(12)	−0.0004(19)	0.001009(17)	0.00058
3.5	2.1502(22)	0.0739(11)	−0.00003(13)	0.000683(12)	0.00058

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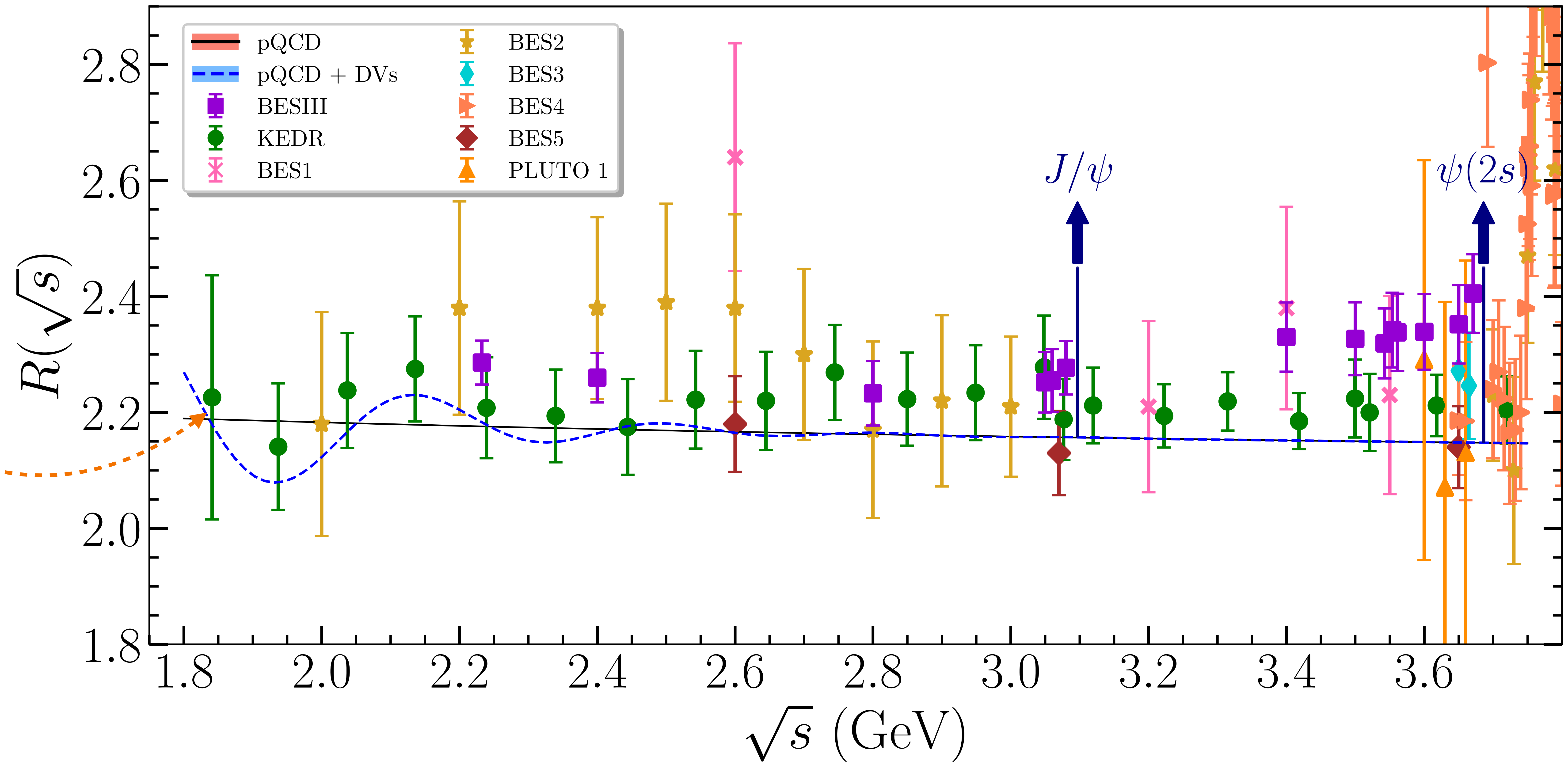
The description with DVs (not a fit!) follows the oscillations observed in the KEDR data for lower energies



Diogo Boito

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But with very larger errors…

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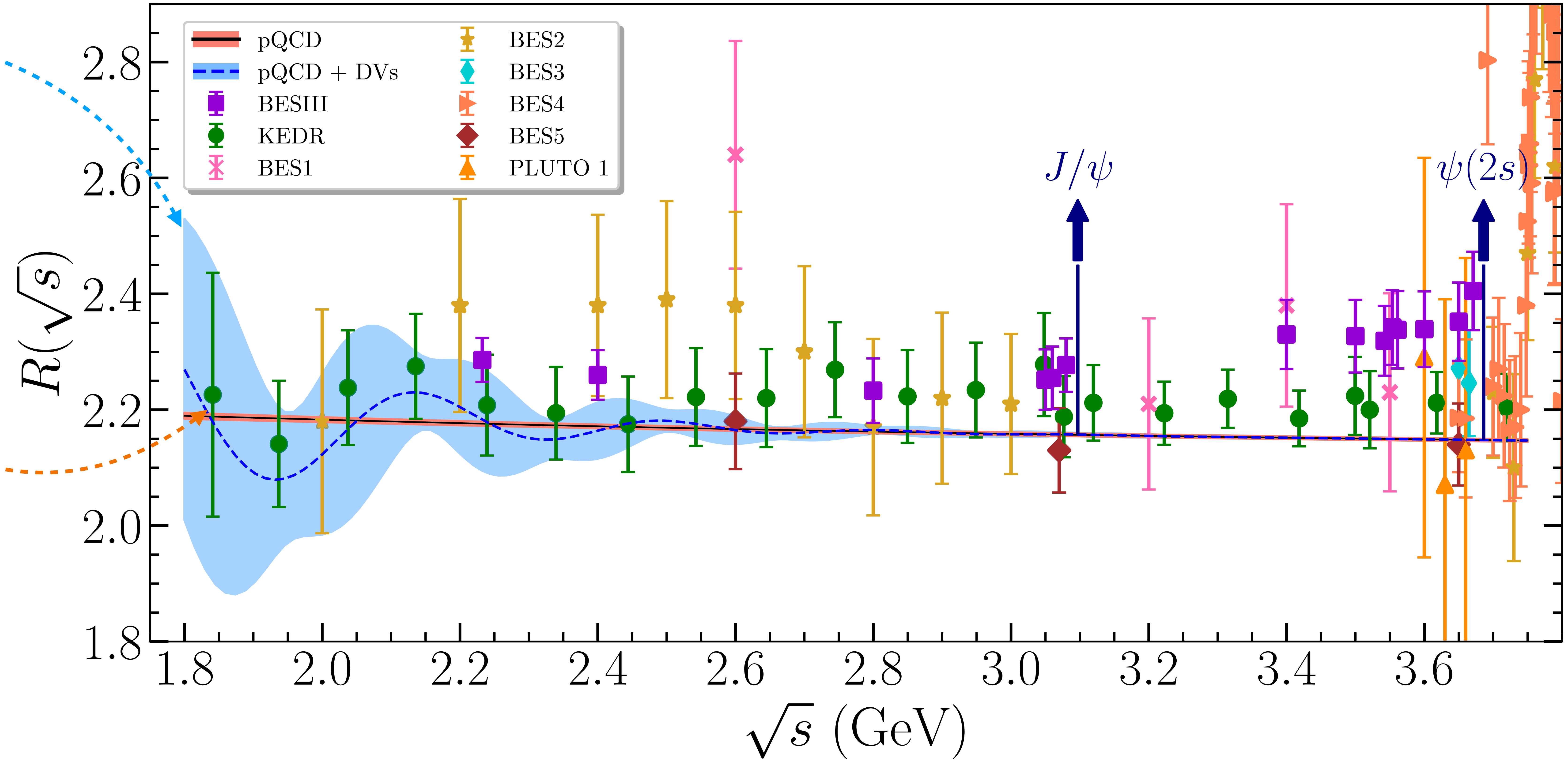

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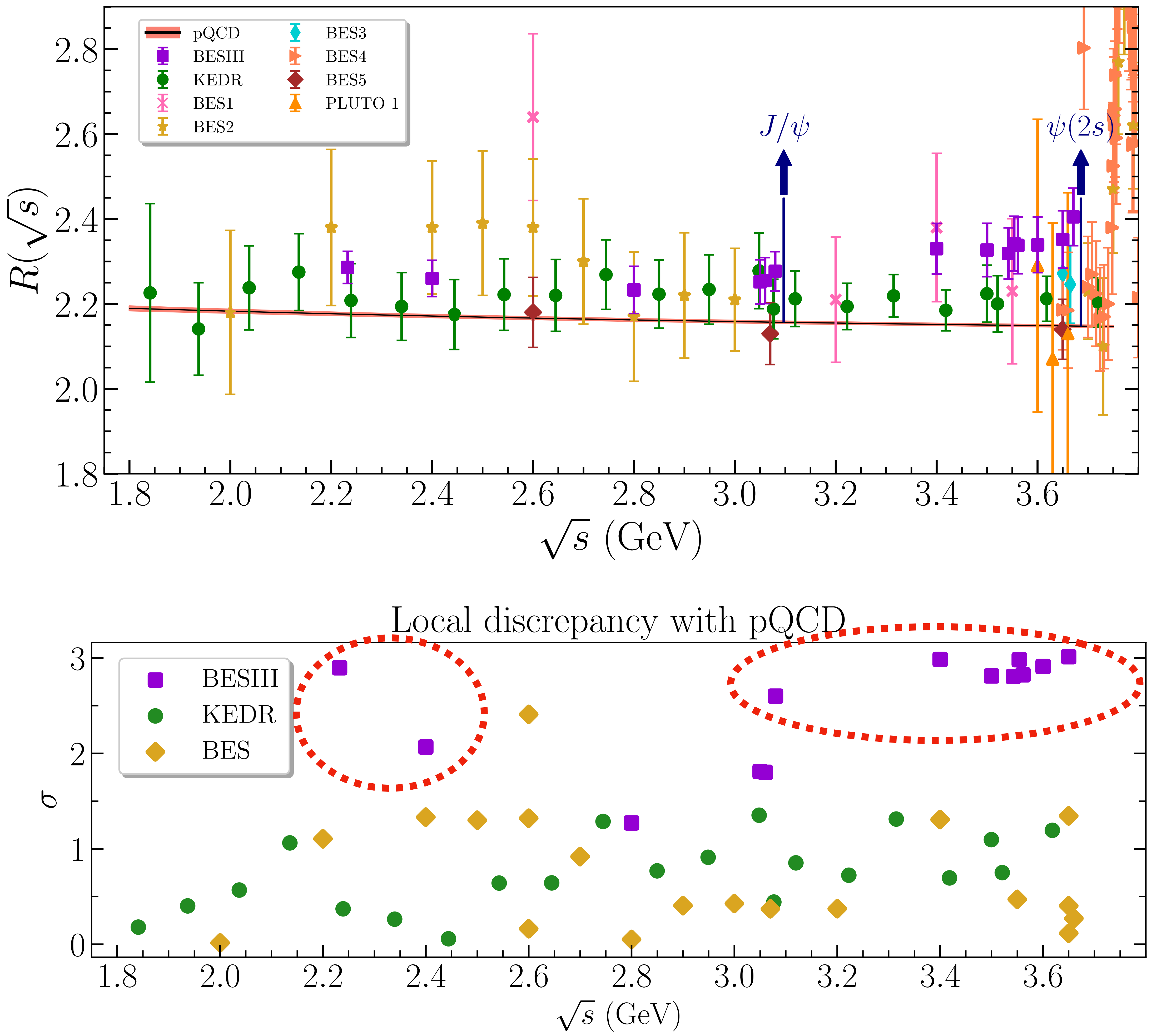
Local discrepancies

Discrepancies with respect to pQCD



(all results include m_s and EM corrections)

Local discrepancies

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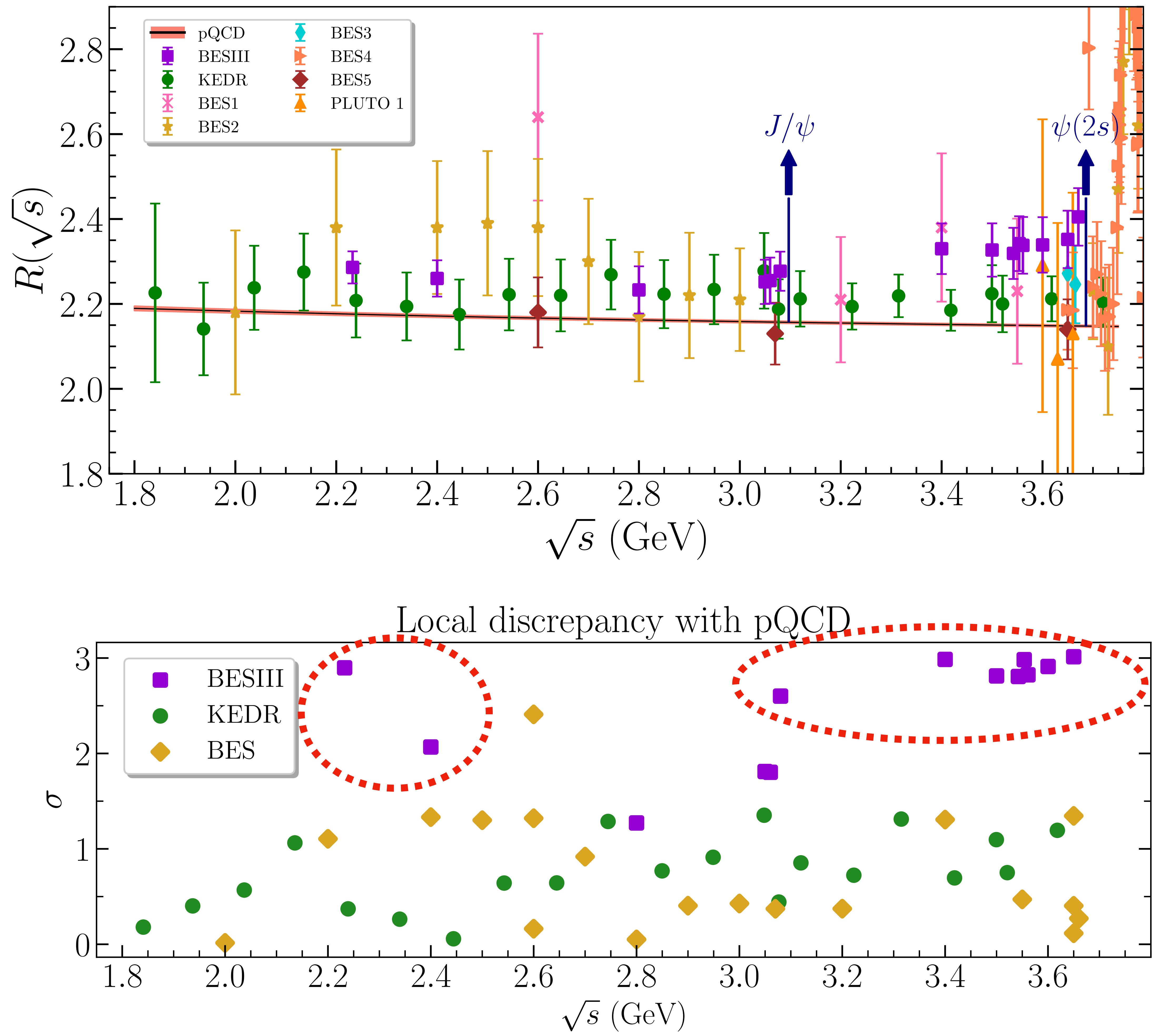
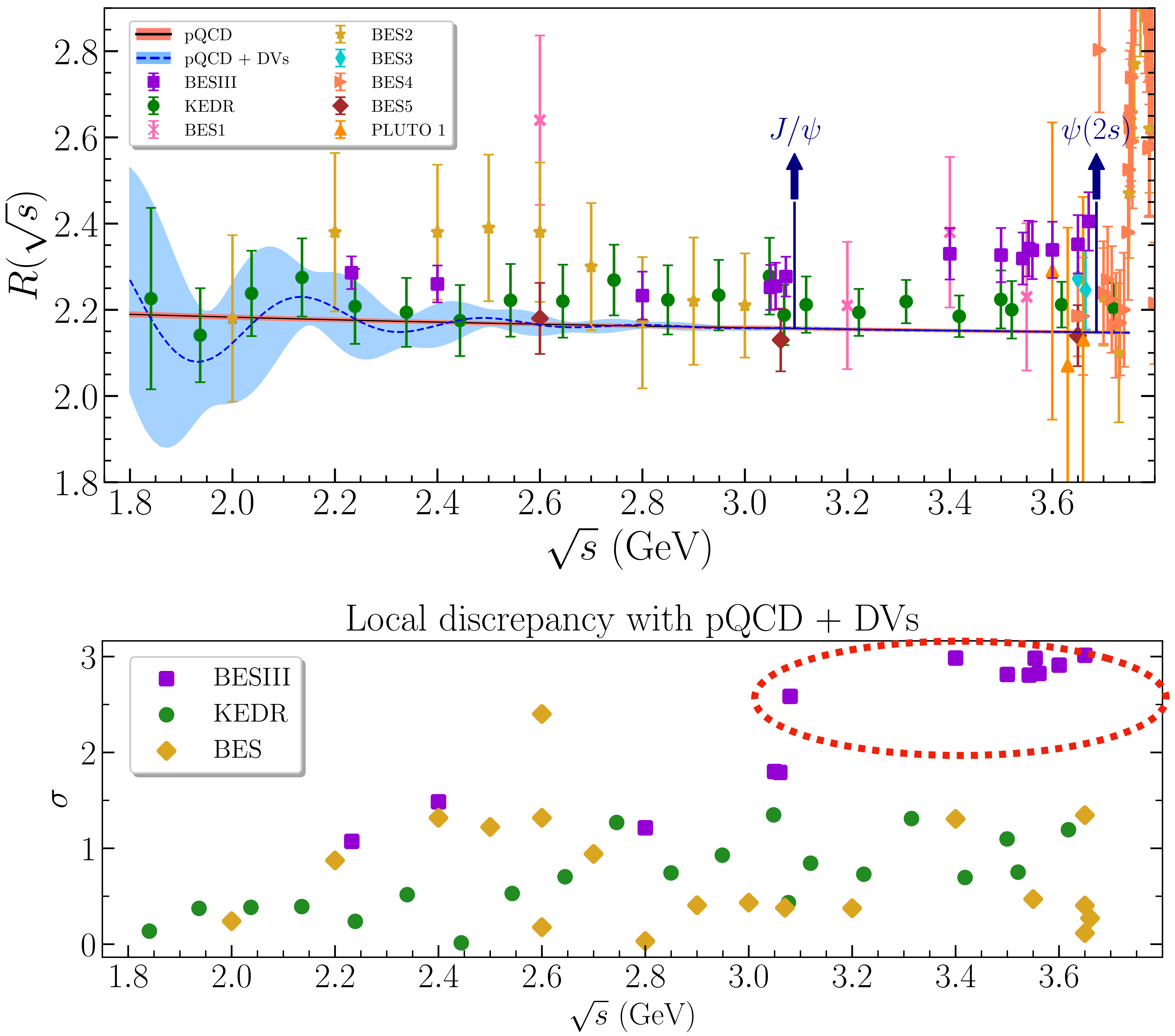


Discrepancies with respect to pQCD + DVs



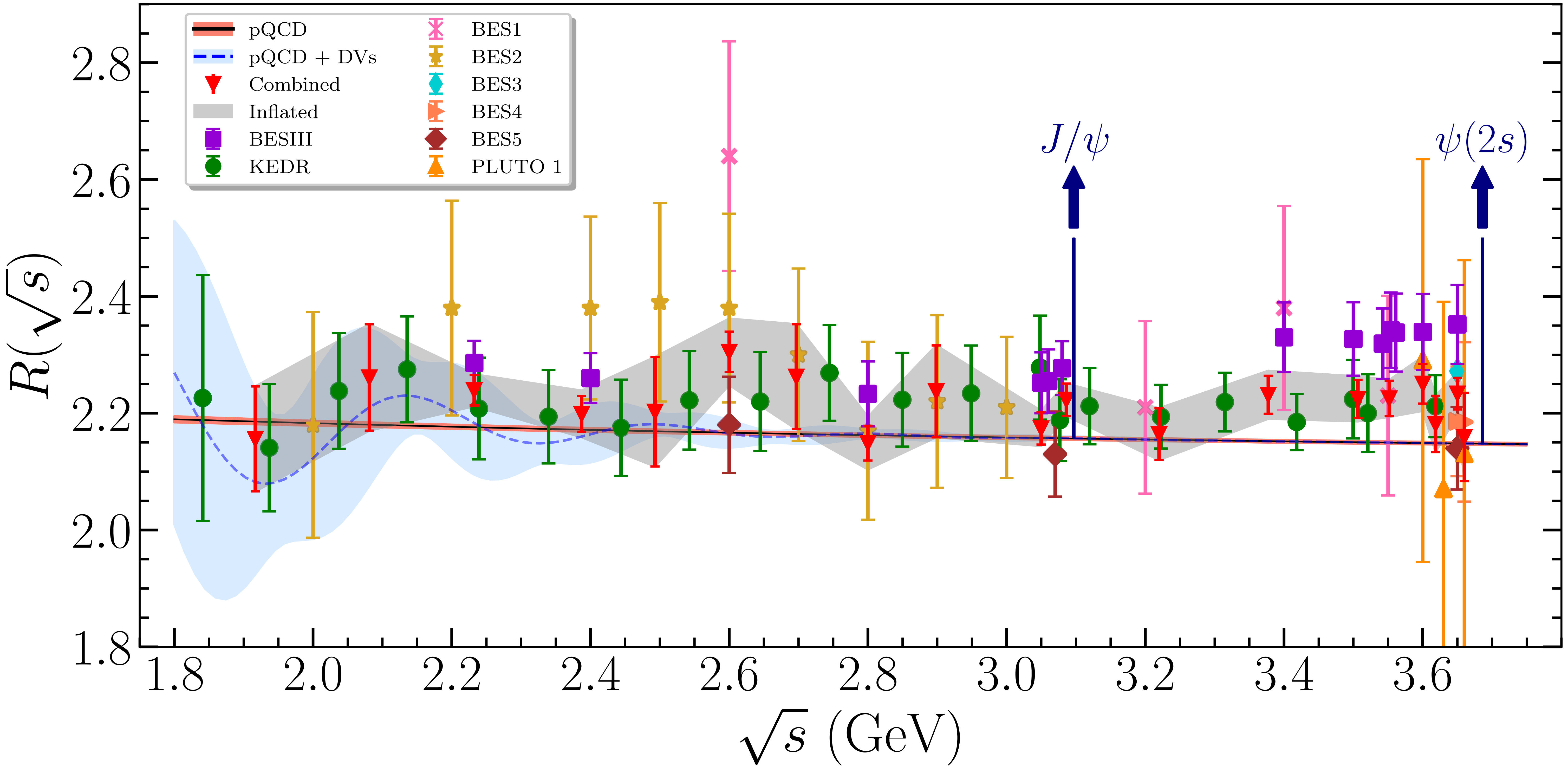
(all results include m_s and EM corrections)

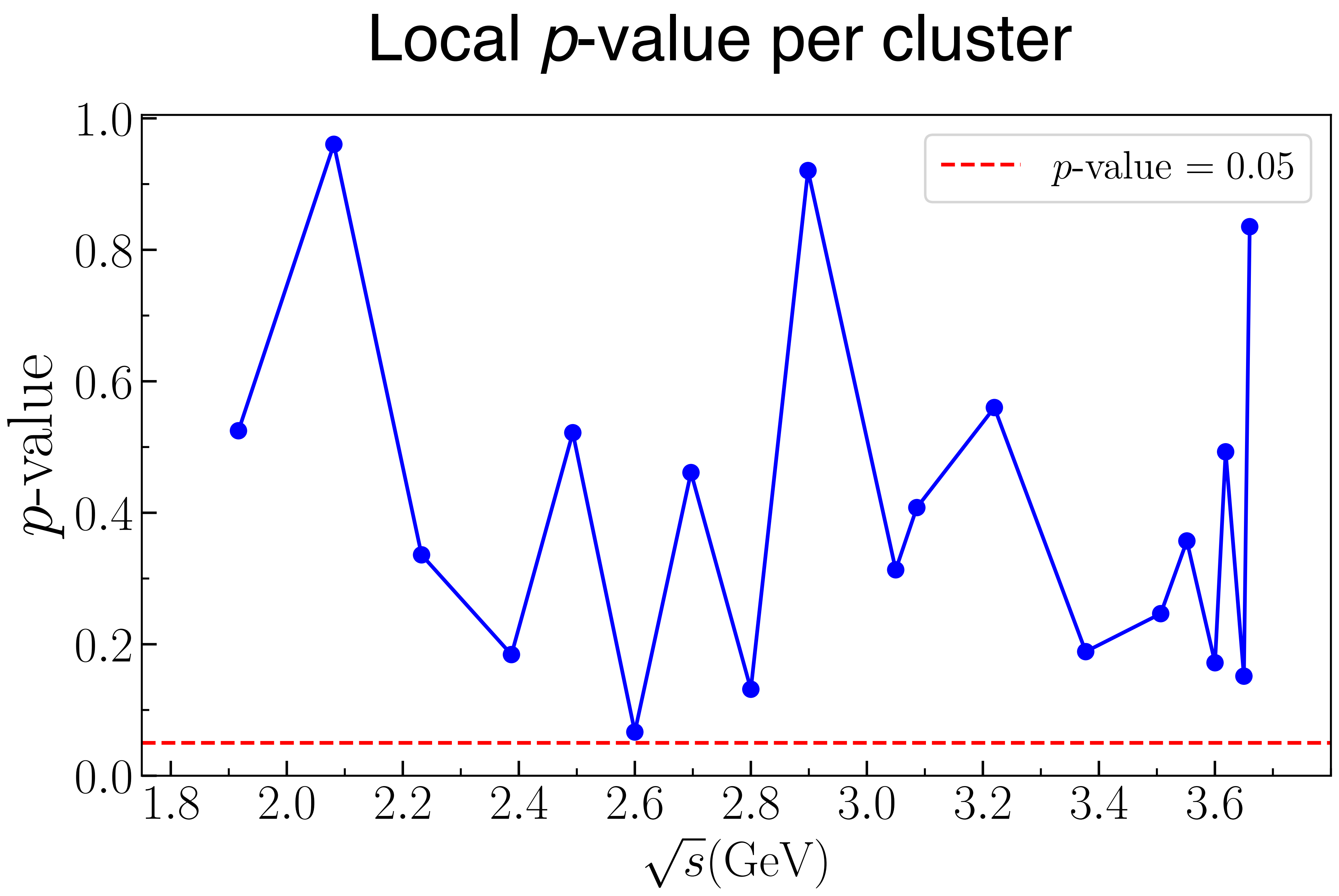
Combination of the experimental data

- To investigate local discrepancies in the experimental data and obtain integrated results, we have combined the available data for R_{uds}.
- We used the KNT algorithm (but more modest analysis, restricted to the the interval 1.8 GeV to 3.66 GeV)

Keshavarzi, Nomura and Teubner '18

$$\chi^2/\text{dof} = 41.7/37 = 1.13 \qquad p = 27.5\%$$



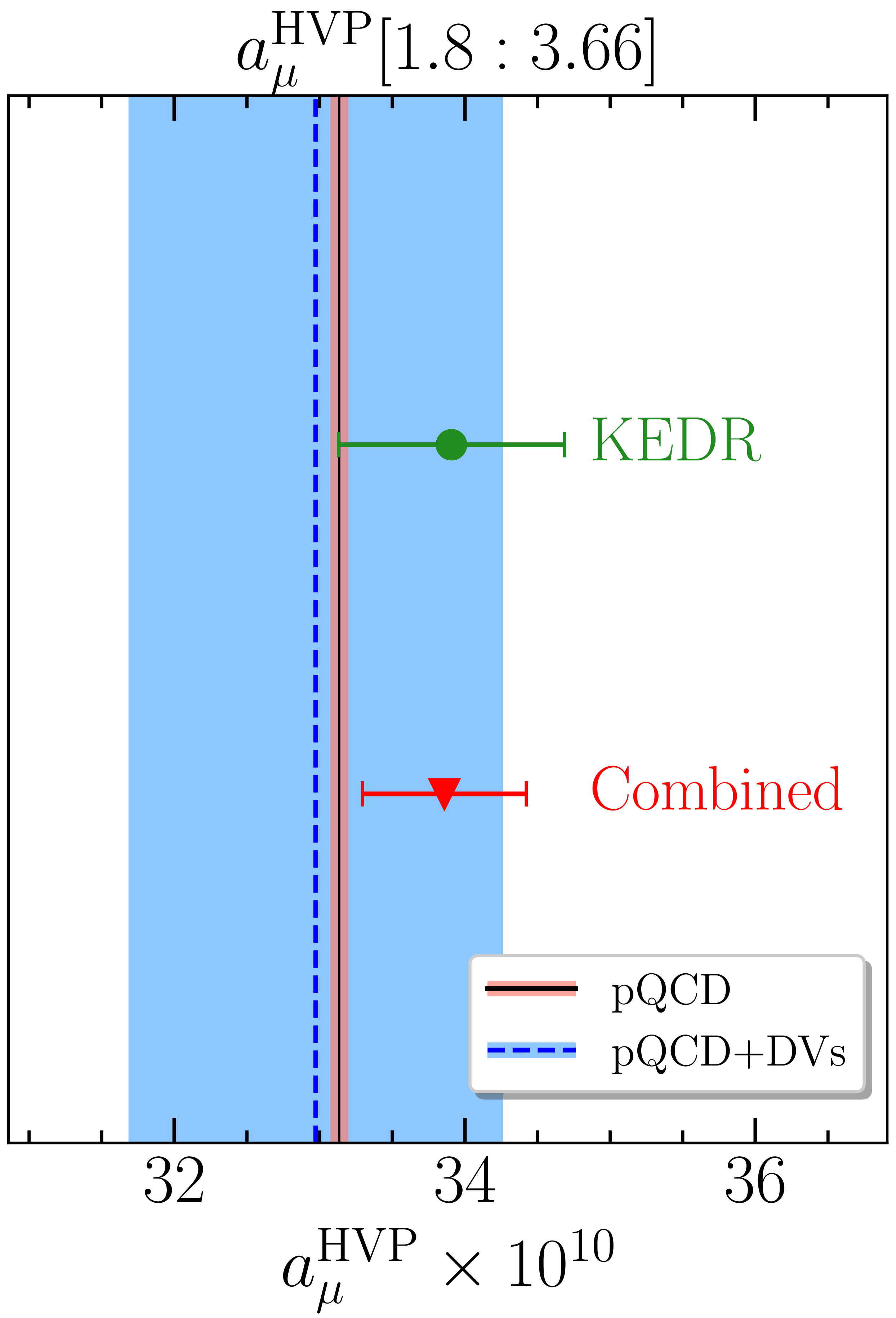


(BES-III data above 3.4 GeV are highly positively correlated)

Contributions to a_μ^{HVP}

$a_\mu^{\mathrm{HVP}}[s_1 : s_2] \times 10^{10}$

$\sqrt{s}$ [GeV]	pQCD	pQCD+ DVs	BESIII	KEDR	BES	Comb.
[1.8 : 3.66]	33.135(51)	33.0(1.3)	—	33.91(78)	—	33.86(56)
[2.23 : 3.66]	17.875(23)	17.85(18)	19.21(39)	18.34(34)	19.21(60)	18.42(23)



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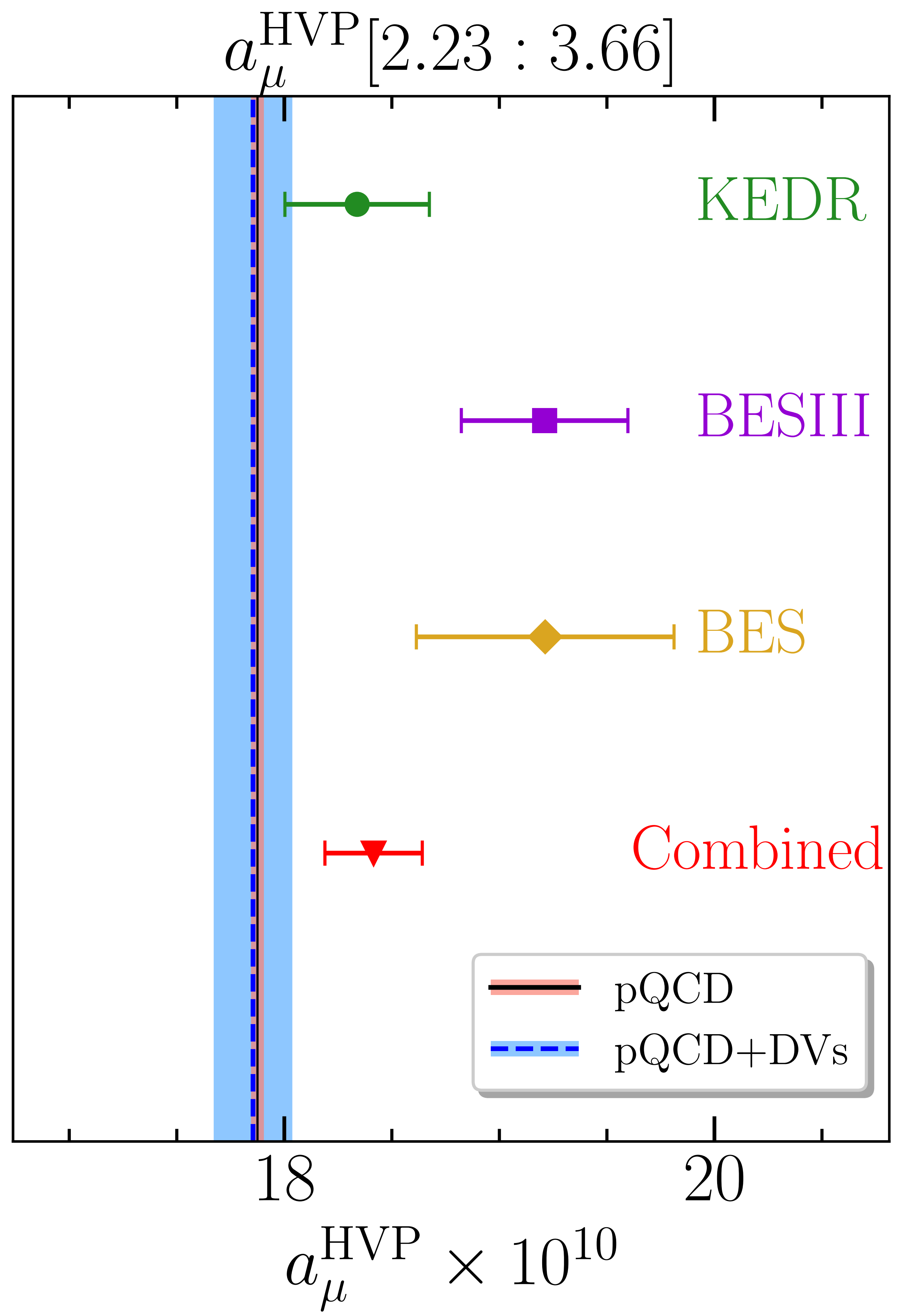

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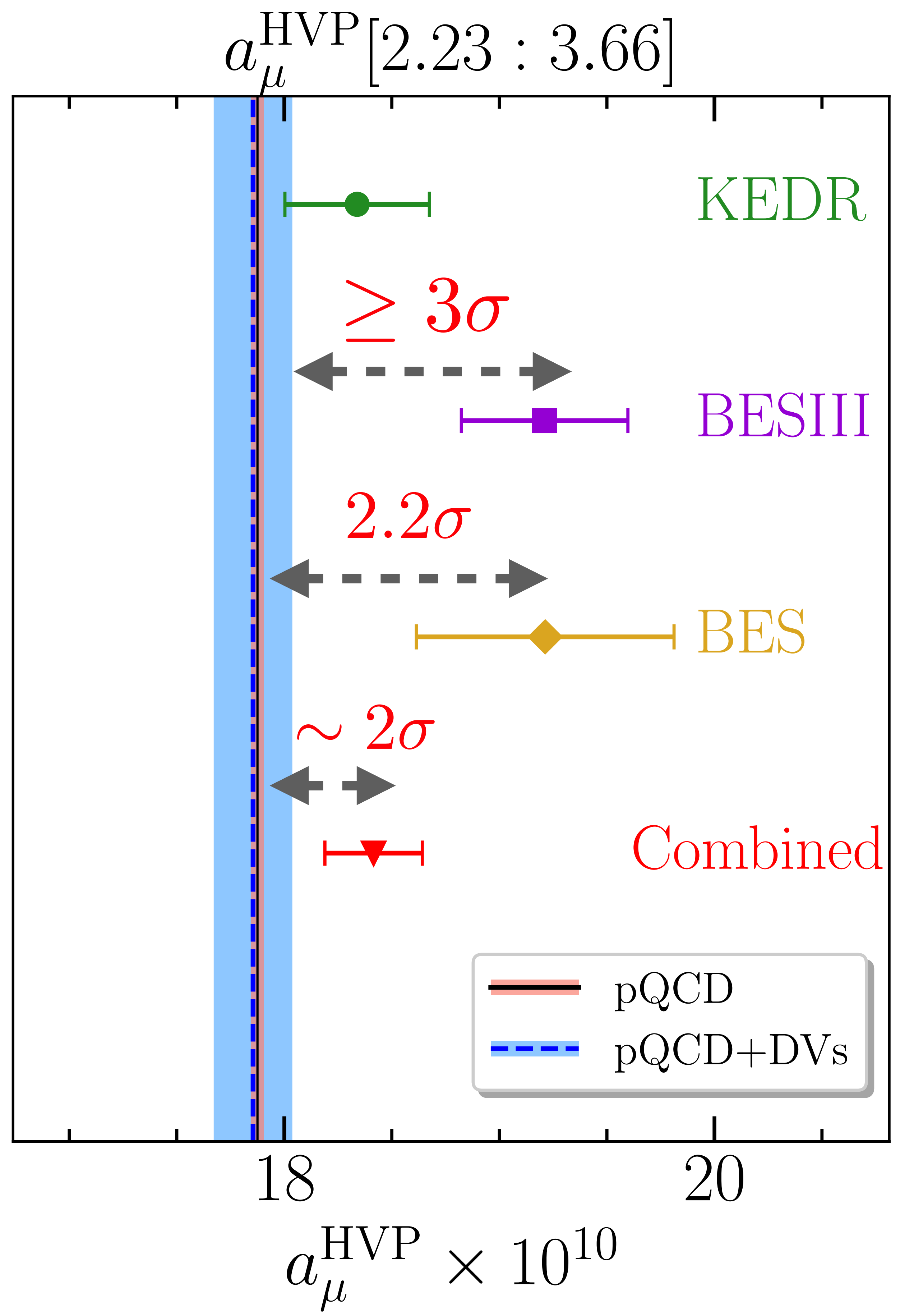


Conclusions

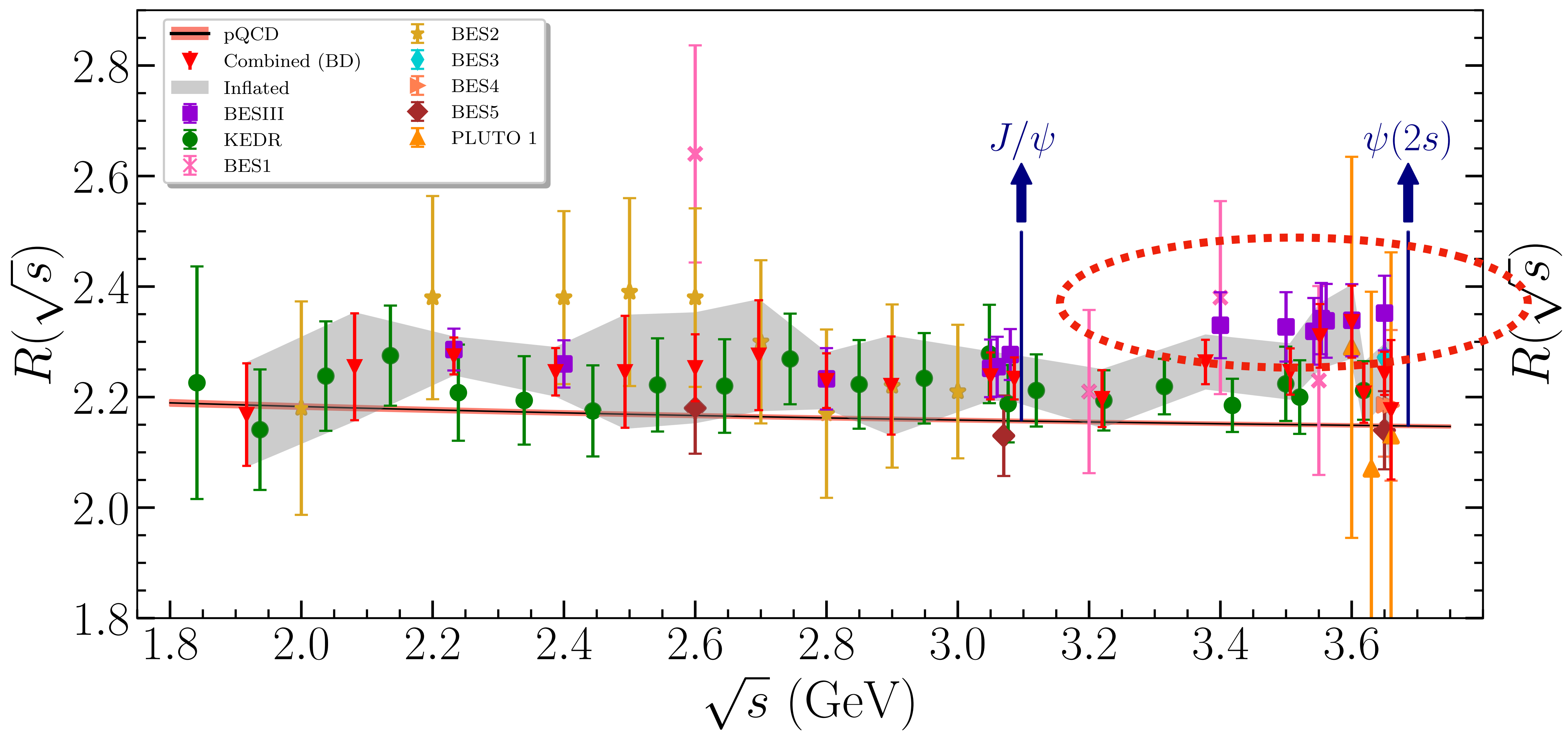
- pQCD **in good shape**: several indications that there is nothing problematic with the series
- Very little room for changes in pQCD. Uncertainty under control.
- DVs significant below 2.5 GeV and improve agreement between theory and data (but large errors)
- Local discrepancies reach 3σ for the BES-III data set (mainly for $\sqrt{s} \geq 3.4\mathrm{GeV}$, but data are correlated)
- Data combination suggests that data sets are locally compatible
- We cannot provide a mechanism to account for the discrepancies between QCD and some of BES-III data points
- More BES-III data will be welcome see talk by Weiping Wang

Extra

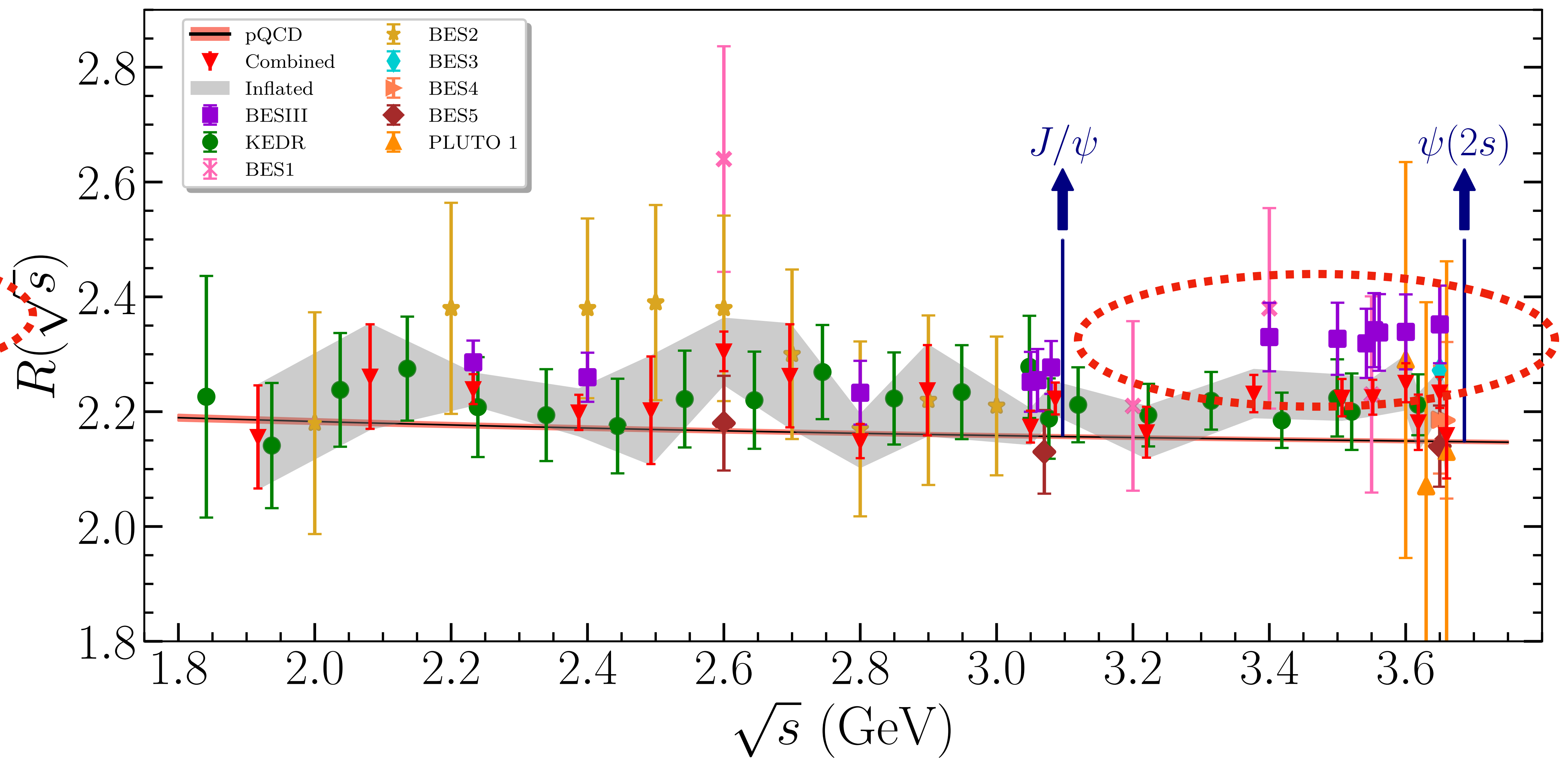
Combining the data: correlations

- Data combination considering only the correlations within a given cluster (*block-diagonal (BD)* combination)

Block-diagonal



Standard fit



- Block-diagonal data combination closer to BES-III
- Long-distance correlations not included in the fit (only included in error propagatinos), but statistical analysis possible using Bruno & Sommer '22