Update from Mainz on a_μ^{hvp}

Simon Kuberski for the Mainz lattice group

Eighth Plenary Workshop
of the Muon $g-2$ Theory Initiative,
IJCLab Orsay,
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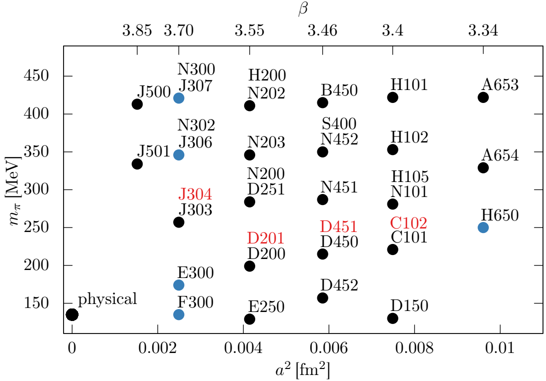


- $\mathrm{O}(a)$ improved Wilson-clover fermions.
- Six values of $a \in [0.039, 0.099]$ fm.
- Open boundary conditions in the temporal direction.
- $a\mathrm{Tr}[M_\mathrm{q}] = 2am_\mathrm{l} + am_\mathrm{s} = \mathrm{const.}$ and $m_\mathrm{s} \approx m_\mathrm{s}^\mathrm{phys}$ to stabilize the strange-quark interpolation.

- New ensemble / improved statistics with respect to [Djukanovic et al., 2411.07969].
- Generating a third ensemble with $m_\pi \approx m_\pi^\mathrm{phys}$
  - 256×128^3 lattice F300 at 0.05 fm
  - target: increase precision and further constrain $(am_\pi)^2$ effects.

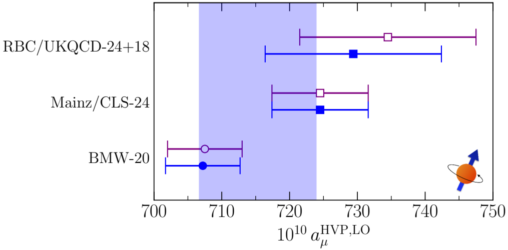
- Work in isospin decomposition of the electromagnetic current

$$j_\mu^{\mathrm{em}} = \tfrac{2}{3}\bar{u}\gamma_\mu u - \tfrac{1}{3}\bar{d}\gamma_\mu d - \tfrac{1}{3}\bar{s}\gamma_\mu s + \tfrac{2}{3}\bar{c}\gamma_\mu c + \ldots = j_\mu^{I=1} + j_\mu^{I=0} + \tfrac{2}{3}\bar{c}\gamma_\mu c + \ldots\,,$$

- $\mathrm{O}(a)$ improved correlation functions with
  - local-local (LL) and local-conserved (LC) vector currents
  - two different lines of constant physics for the $\mathrm{O}(a)$ improvement (set 1/ set 2).

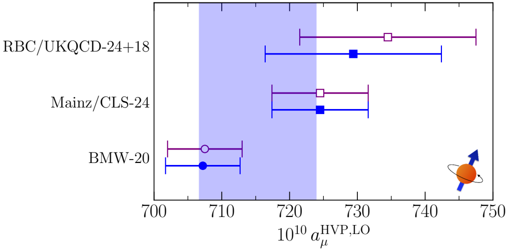
- Scale setting:
  - Our current scheme is based on f_π to set the scale.
  - f_π-rescaling reduces chiral dependence of the isovector contribution.
    → Less and less important because of precise data at m_π^{phys}.
  - Use $\sqrt{t_0}$ as preferred intermediate theory scale otherwise.
  - Evaluation of $(a_\mu^{\mathrm{hvp}})^{\mathrm{LD}}$ also done in the WP25 scheme, based on w_0.

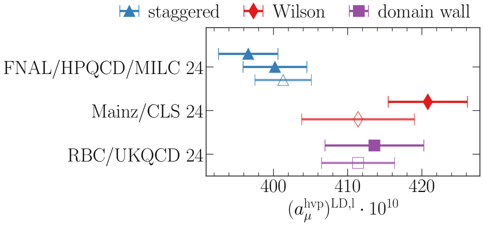
The Mainz/CLS result for $a_\mu^{\rm hvp}$



- Just below 1% precision for $a_\mu^{\rm hvp}$ [Djukanovic et al., 2411.07969].

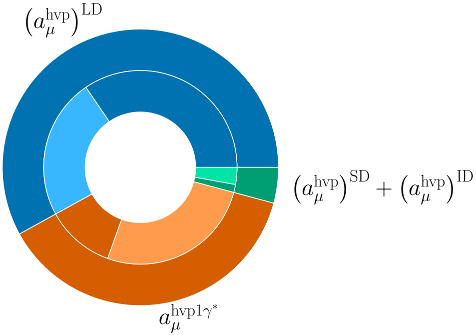
The Mainz/CLS result for a_μ^{hvp}





- Just below 1% precision for a_μ^{hvp} [Djukanovic et al., 2411.07969].
- We observe a rather strong scheme dependence, Mainz-f_π vs. WP25, for $(a_\mu^{\text{hvp}})^{\text{LD}}$.
- Tensions in the long-distance regime need to be understood.

  → Conservative averaging procedure in the White Paper.



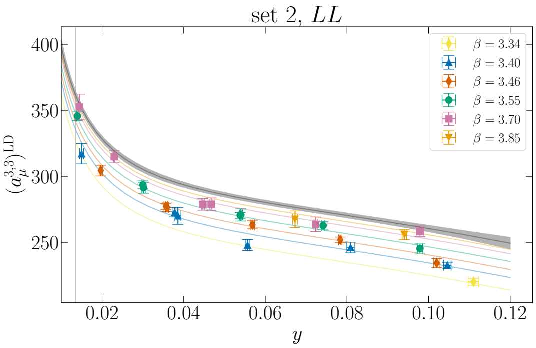
dark: statistics
light: systematics

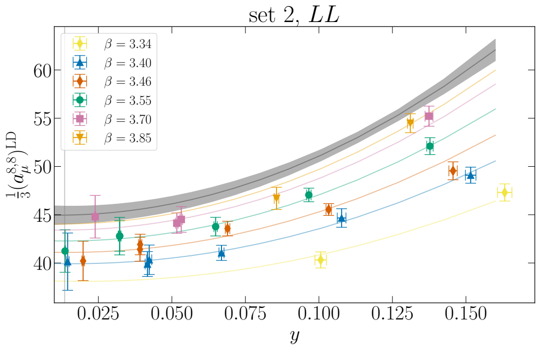
Error budget dominated by:

- LD: statistical uncertainty.
- LD: continuum extrapolation.
- LD: finite-volume correction.
- Isospin-breaking effects.

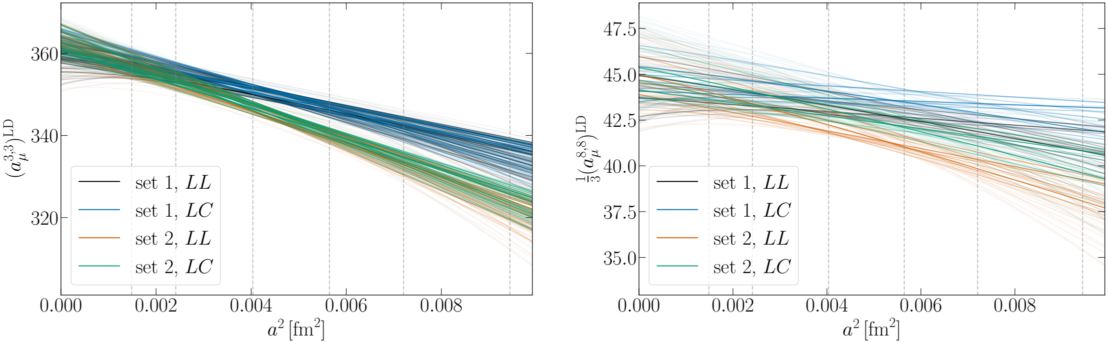
→ Work on all of them to improve significantly.

LD: Statistical uncertainty

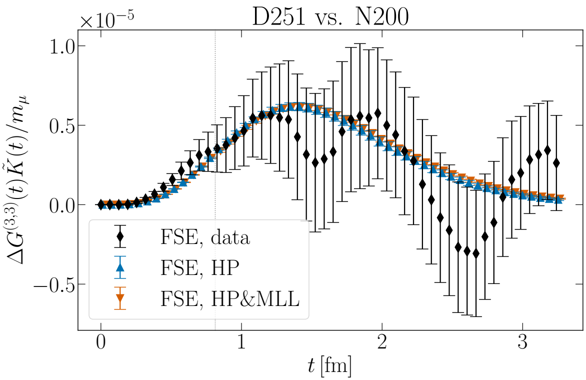


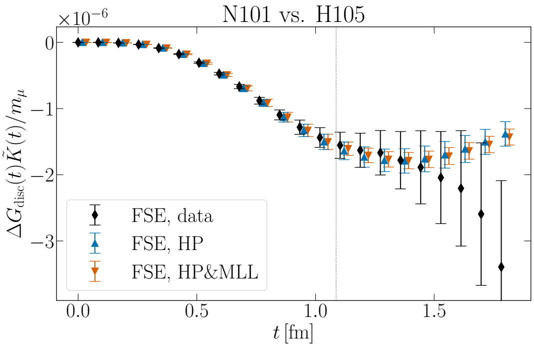


- Isovector contribution on new F300 ensemble still “imprecise”.
- Better precision at fine a requires longer MC chains.
- Quark-disconnected diagram in gauge noise on many ensembles.

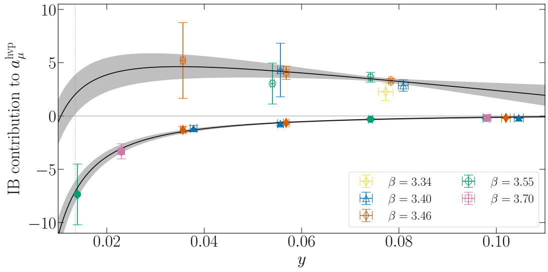


- Pure a^2 fits dominate, but we cannot exclude some curvature.
- Better precision at fine a requires longer MC chains.
- Cutoff effects strongly depend on the choice of the scale setting quantity, af_π, $\sqrt{t_0}/a$, w_0/a, ...





- Correct to $(m_\pi L)^{\mathrm{ref}} = (m_{\pi^0})_{\mathrm{phys}} \cdot 6.272\,\mathrm{fm} \approx 4.290$ at finite a.
- Estimate $(a_\mu^{\mathrm{hvp}})^{\mathrm{LD}}(L=\infty) - (a_\mu^{\mathrm{hvp}})^{\mathrm{LD}}(L_{\mathrm{ref}}) = 16.7(1.5)$ based on HP and MLL: vary the parameters, compare the models, estimate missing contributions.
- Smaller uncertainty requires data at large L.



Performed a combined fit of

- fully connected data (open symbols, positive),
- disconnected QED correction (closed symbols, negative) [Parrino et al., 2501.03192],

estimating the missing pieces.

We are missing:

- More precise data at small pion mass.
- The quark-disconnected strong isospin-breaking correction.
- Diagrams beyond the electroquenched approximation.
- Isospin-breaking in scale setting.

→ dominated by systematic uncertainties.

See also:
[Volodymyr Biloshytskyi, Wed 2:30]
[Dominik Erb, Wed 2:45]

Outlook

- Several sources of uncertainty have to be addressed to significantly improve our results.
- Another factor of two improvement for a_μ^{hvp} will take some time.
- In the meantime we exploit our data set for related observables:
  - The NLO HVP contribution [Arnau Beltran, Wed 2:55]
  - The hadronic contribution to the running of α and the electroweak mixing angle [Alessandro Conigli, now]

The hadronic contribution to the running of α and the electroweak mixing angle

Alessandro Conigli

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Why we care

Relevant quantities for precision tests of Standard Model

- Electromagnetic coupling: [PDG 2024]

$$\alpha(q^2 = 0) = 1/137.035\,999\,178(8)$$
$$\alpha(M_Z^2) = 1/127.930(8)$$

- Hadronic contribution as main source of uncertainty
- Observed tensions with phenomenological estimates

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Standard approach

- Experimental input
- Dispersion theory

Lattice determination

- First-principles calculation
- Exact flavour separation

Time Momentum Representation

- Electroweak couplings as a function of the momentum transfer q^2

$$\alpha(-q^2) = \alpha/(1 - \Delta\alpha(-q^2)), \qquad \sin^2\theta_W(-q^2) = \sin^2\theta_W(1 + \Delta\sin^2\theta_W(-q^2))$$

Leading hadronic contribution

$$\Delta\alpha_{\text{had}}(-q^2) = 4\pi\alpha\bar{\Pi}^{\gamma\gamma}(-q^2), \qquad (\Delta\sin^2\theta_W)_{\text{had}}(-q^2) = -4\pi\alpha/\sin^2\theta_W\bar{\Pi}^{Z\gamma}(-q^2)$$

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Time Momentum Representation (TMR) [Bernecker, Meyer 2011; Francis *et al.* 2013]

$$\Pi(-q^2) = \int_0^\infty \mathrm{d}t G(t) K(t, q^2) \qquad G(t) = -\tfrac{1}{3}\int \mathrm{d}\vec{x} \sum_{k=1}^{3} \langle j_k^{\gamma(Z)}(x) j_k^{\gamma}(0)\rangle$$

- In the $SU(3)$-flavour basis $j_k^a = \bar{q}\gamma_k(\lambda_a/2)q$, $a = 3, 8, 0$

$$G_{\mu\nu}^{\gamma\gamma}(x) = G_{\mu\nu}^{33}(x) + \frac{1}{3}G_{\mu\nu}^{88}(x) + \frac{4}{9}G_{\mu\nu}^{cc}(x)$$

$$G_{\mu\nu}^{Z\gamma}(x) = \left(\frac{1}{2} - \sin^2\theta_W\right)G_{\mu\nu}^{\gamma\gamma}(x) - \frac{1}{6\sqrt{3}}G_{\mu\nu}^{08}(x) - \frac{1}{18}G_{\mu\nu}^{cc}(x)$$

Computational Strategy

- We propose the following decomposition for the subtracted HVP

$$\begin{aligned}\bar{\Pi}(Q^2) &= [\Pi(Q^2) - \Pi(Q^2/4)] + [\Pi(Q^2/4) - \Pi(Q^2/16)] + [\Pi(Q^2/16) - \Pi(0)] \\ &= \qquad \widehat{\Pi}(Q^2) \qquad + \qquad \widehat{\Pi}(Q^2/4) \qquad + \qquad \bar{\Pi}(Q^2/16)\end{aligned}$$

- Clear separation of the different euclidean regions gives access to $0.25\ \mathrm{GeV}^2 \leq Q^2 \leq 12.0\ \mathrm{GeV}^2$

HV: $4.0\ \mathrm{GeV}^2 \leq Q^2 \leq 12.0\ \mathrm{GeV}^2$

MV: $1.0\ \mathrm{GeV}^2 \leq Q^2/4 \leq 3.0\ \mathrm{GeV}^2$

LV: $0.25\ \mathrm{GeV}^2 \leq Q^2/16 \leq 0.75\ \mathrm{GeV}^2$

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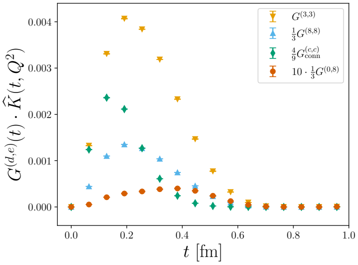

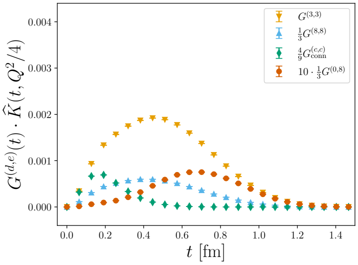

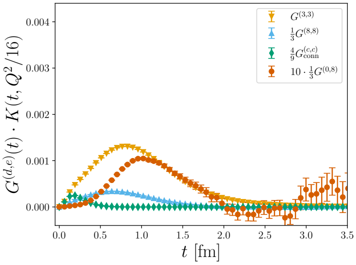

Computational Strategy

- Two discretisations of the vector current, the local (L) and point-split conserved (C)

Set 1: Improvement coefficients from large-volume [1811.08209]

Set 2: Improvement coefficient from SF setup [1805.07401, 2010.09539]

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- Isovector contribution [S. Kuberski *et al.* 2024]

$$\bar{\Pi}^{(3,3)}(Q^2) = \bar{\Pi}^{(3,3)}_{\text{sub}}(Q^2) + b^{(3,3)}(Q^2, Q_m^2)$$

where

$$\widehat{K}(x_0, Q^2, Q_m^2)_{\text{sub}} = \frac{16}{Q^2}\sin^4\left(\frac{Qx_0}{4}\right) - \frac{Q^2}{Q_m^4}\sin^4\left(\frac{Q_m x_0}{2}\right), \qquad b^{(3,3)}(Q^2, Q_m^2) = \frac{Q^2}{4Q_m^2}\left(\Pi^{(3,3)}(4Q_m^2) - \Pi^{(3,3)}(Q_m^2)\right)$$

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$$\bar{\Pi}^{(3,3)}(Q^2) = \bar{\Pi}^{(3,3)}_{\text{sub}}(Q^2) + b^{(3,3)}(Q^2, Q_m^2)$$

- Isoscalar contribution

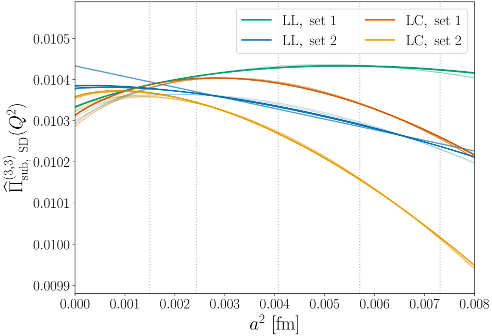
$$\text{HV}: \widehat{\Pi}^{(8,8)} = \widehat{\Pi}^{(3,3)} + \widehat{\Delta}_{\text{ls}}(Q^2), \qquad \text{MV, LV}: \bar{\Pi}^{(8,8)} = \int_0^\infty \mathrm{d}x_0 G^{(8,8)} K(x_0, Q^2)$$

where

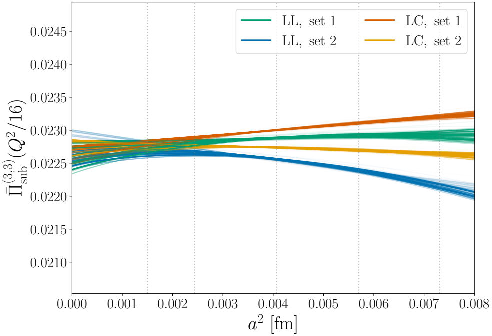
$$\Delta_{\text{ls}}(Q^2) = G^{(8,8)} - G^{(3,3)} \propto \alpha_s(m_s^2 - m_l^2), \qquad \widehat{K}(x_0, Q^2) = \frac{16}{Q^2}\sin^4\left(\frac{Qx_0}{4}\right), \qquad K(x_0, Q^2) = x_0^2 - \frac{4}{Q^2}\sin^2\left(\frac{Qx_0}{2}\right)$$

The isovector channel: cutoff dependence

HV: $Q^2 = 9.0\ \mathrm{GeV}^2$



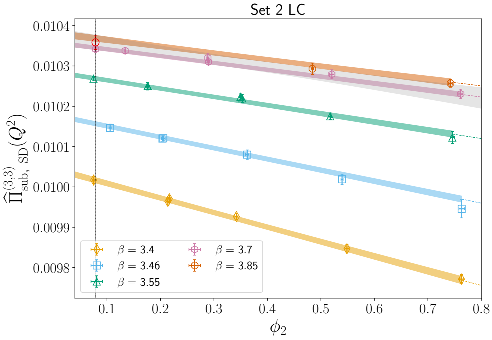
LV: $Q^2/16 = 0.56\ \mathrm{GeV}^2$



- Dependence of $\bar{\Pi}^{(3,3)}$ on a^2 at physical quark masses for HV and LV virtuality regions
- Each line represents a fit in the model average: four sets of data differ by $O(a^2)$
- $\sqrt{t_0}$ from a combination of f_π and f_K

The isovector channel: chiral dependence

HV: $Q^2 = 9.0\ \text{GeV}^2$



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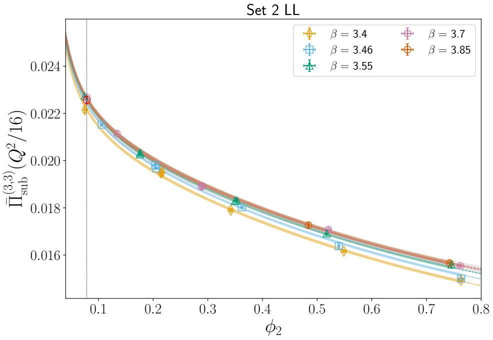
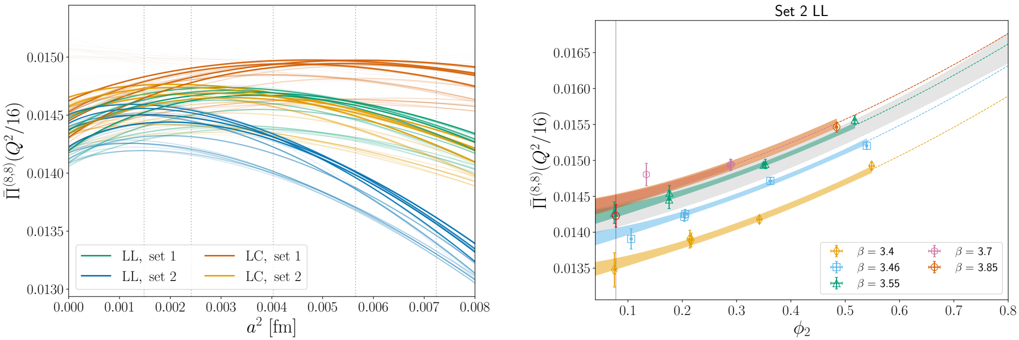


- Dependence of $\bar{\Pi}^{(3,3)}$ on $\phi_2 = 8t_0 m_\pi^2$ for HV and LV virtuality regions
- Well constrained chiral dependence across the range of pion masses
- Fits with chirally divergent terms dominating the model average in LV region

The isoscalar channel

- Chiral-continuum dependence of $\bar{\Pi}^{(8,8)}$ in the LV region at $Q^2/16 = 0.56\ \mathrm{GeV}^2$



- Divergent behaviour of light-connected and disconnected pieces is cancelled
- Bounding method in the LV region to mitigate noise at large Euclidean distances

Isospin breaking effects

- **Lattice calculation:**
  - $\rightarrow$ Down to $m_\pi \sim 200$ MeV
  - $\rightarrow$ Only connected diagrams
- **Phenomenological model:**
  - a Charged pion loops with VMD form factors [2501.03192]
  - b Pseudoscalar meson exchanges (π^0, η, η', η_c) [2209.02149]
  - c Strong IB, based on $\omega - \rho$ mixing [2505.24344]

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Final IB-effects estimate at $Q^2 = 9\ \mathrm{GeV}^2$

$$(\bar{\Pi}^{(\gamma,\gamma)})^{\gamma^\star} = 7(14) \times 10^{-5}$$

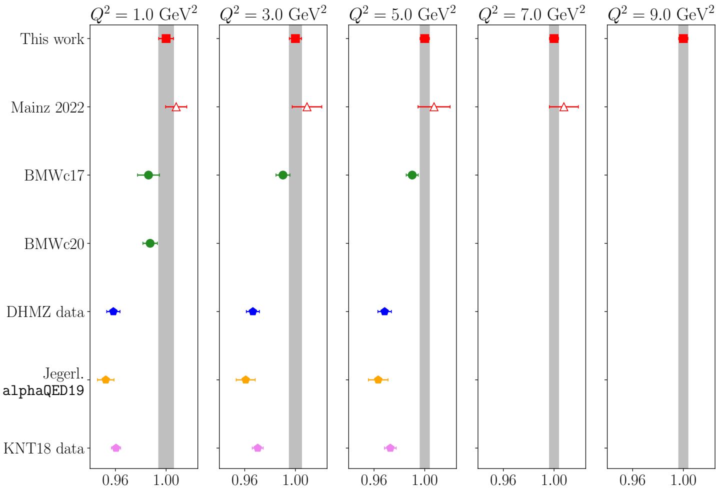
Quoted uncertainty corresponds to

1. Size of the dominant contribution (strong IB)
2. Size of lattice purely-connected results

Comparison with other determinations at space-like momenta

Ratio of $\Delta\alpha_{\text{had}}$ divided by our results for different Q^2

- Agreement with Mainz 2022
  [M. Cè et al., 2021]
- 1-2 σ disagreement between this work and [Borsanyi et al., 2018; Borsanyi et al., 2021]
- 3-6 σ tension between this work and [Keshavarzi, Nomura, and Teubner 2020; Davier et al. 2020; Jegerlehner 2020]



Hadronic running at the Z-pole

- Convert lattice results for $\Delta\alpha^{(5)}_{\rm had}(-Q_0^2)$ to an estimate of $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$
- Euclidean Splitting Technique: the Adler function approach [Jegerlehner, hep-ph/9901386, arXiv:0807.4206]

$$\Delta\alpha^{(5)}_{\rm had}(M_z^2) = \Delta\alpha^{(5)}_{\rm had}(-Q_0^2) + \left[\Delta\alpha^{(5)}_{\rm had}(-M_z^2) - \Delta\alpha^{(5)}_{\rm had}(-Q_0^2)\right] + \left[\Delta\alpha^{(5)}_{\rm had}(M_z^2) - \Delta\alpha^{(5)}_{\rm had}(-M_z^2)\right]$$

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- $\Delta\alpha_{\mathrm{had}}^{(5)}(-Q_0^2)$: input from Lattice QCD

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- $\Delta\alpha^{(5)}_{\rm had}(-Q_0^2)$: input from Lattice QCD
- From the Adler function $D(Q^2)$, known in massive pQCD at three-loops

$$\left[\Delta\alpha^{(5)}_{\rm had}(-M_z^2) - \Delta\alpha^{(5)}_{\rm had}(-Q_0^2)\right]_{\rm pQCD/Adler} = \frac{\alpha}{3\pi}\int_{Q_0^2}^{M_Z^2} \frac{{\rm d}Q^2}{Q^2} D(Q^2)$$

Determination of $D(Q^2)$

- **AdlerPy** [R. F. Hernández, 2311.04849; J. Erler et al., 2308.05740]
- Jegerlehner's software package **pQCDAdler**

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- Perturbation theory: $\left[\Delta\alpha_{\mathrm{had}}^{(5)}(M_z^2) - \Delta\alpha_{\mathrm{had}}^{(5)}(-M_z^2)\right] = 0.000\,045(2)$ [Jegerlehner, CERN Yellow Report, 2020]

Evaluation of $\Delta\alpha^{(5)}_{\text{had}}(M_Z^2)$

- **Lattice input:**
  $\Delta\alpha^{(5)}_{\text{had}}(-Q_0^2)$ for $3\ \text{GeV}^2 \leq Q_0^2 \leq 12\ \text{GeV}^2$
- **Perturbative running:**

  Evaluation of the Adler function with three different methods

  1. AdlerPy
  2. pQCD-bfmom
  3. pQCD-updated

  Input values from FLAG24 for 1 and 3

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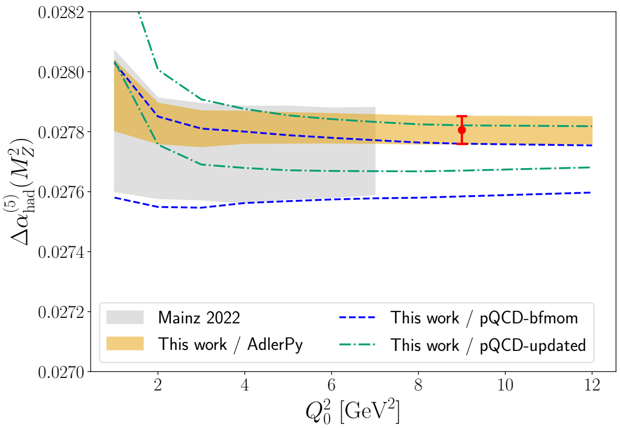
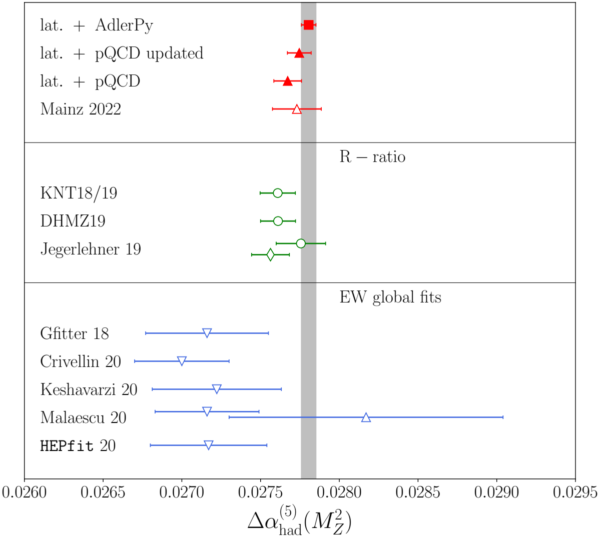
  Input values from FLAG24 for 1 and 3
- **Threshold energy:** $Q_0^2 = 9\ \mathrm{GeV}^2$



Comparison with phenomenology and electroweak fits

- Evidence of a small tension with R-ratio determinations and electroweak global fits
- Running from $-Q_0^2$ to $-M_Z^2$ is correlated
  - $\rightarrow$ Tension at space-like Q^2 is largely washed out when running up to the Z-pole

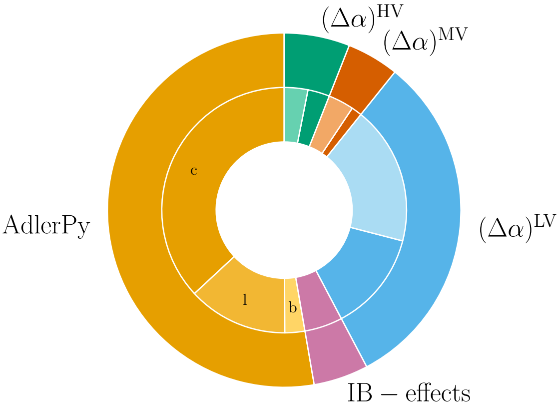


Main contribution to the variance of $\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2)$

- **Threshold energy:** $Q_0^2 = 9\ \mathrm{GeV}^2$
- Good balance between lattice and pQCD

Error budget dominated by:

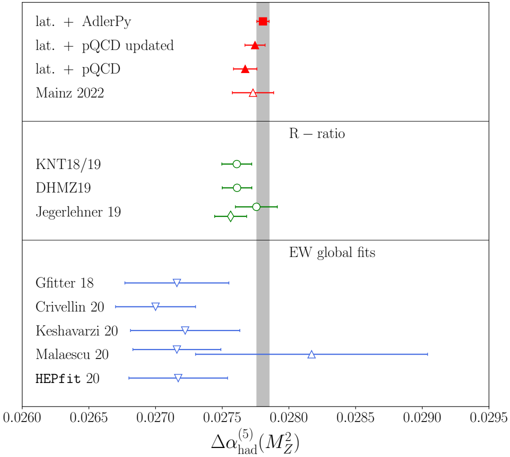
- **AdlerPy**: charm contribution to $D(Q^2)$
- **Lattice**: $(\Delta\alpha)^{\mathrm{LV}}$ contribution
- **IB-effects:** by far subleading



Conclusions

Summary

- Computation of $\Delta\alpha_{\mathrm{had}}$ and $(\Delta \sin^2\theta_W)_{\mathrm{had}}$ in the space-like interval $0 \leq Q_0^2 \leq 12\ \mathrm{GeV}^2$
  - → Factor 3 increased precision w.r.t Mainz 2022
    [M. Cè *et al.* 2022]
- Significant tension between lattice estimate for $\Delta\alpha_{\mathrm{had}}(-Q^2)$ and the data-driven approach
- Small tension with electroweak global fits at the Z-pole



Thank You!

Related works of the Mainz group:

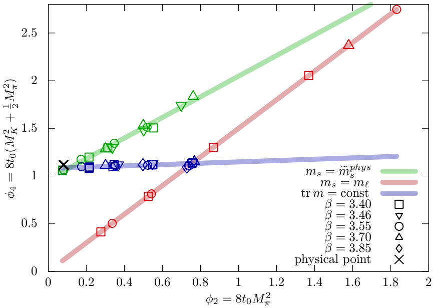
- IB with CSS [D. Erb, Wed 14:45]
- NLO HVP [A. Beltran, Wed 16:55]

Lattice setup - CLS ensembles

[Lüscher and Schaefer, JHEP 1107 036 - JHEP 1502 043 - 1712.04884 - 2003.13359]

- Lüscher-Weisz tree-level improved gauge action
- $N_f = 2 + 1$ non-perturbatively $\mathcal{O}(a)$-improved Wilson fermions
- Open boundary conditions in time for fine values of the lattice spacings
  - $\rightarrow$ Reliable error estimates
- Chiral trajectory $\Phi_4 \propto \mathrm{Tr}(M_q) = \Phi_4^{\text{phys}}$
  - $\rightarrow$ 4 ensembles on $m_s \approx m_s^{\text{phys}}$ to account for small mistuning

[Plot by J. Simeth]



Lattice spacings :
$a = 0.087,\ 0.077,\ 0.065,\ 0.050,\ 0.039$ fm

Pion masses :
$130\ \text{MeV} \leq m_\pi \leq 420\ \text{MeV}$

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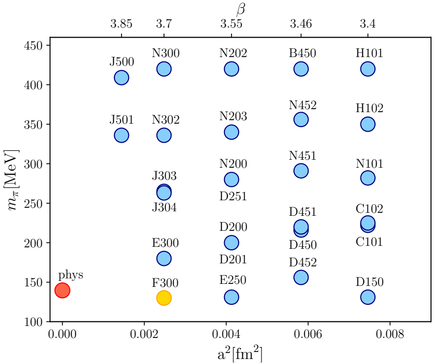

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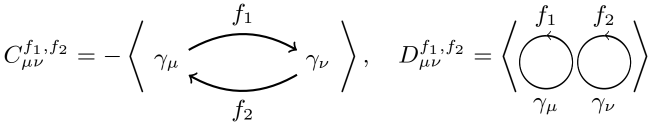
Lattice correlators

In the $SU(3)$ flavour basis and the isospin-symmetric limit:

- $I = 1$ contribution: $G^{33}_{\mu\nu}(x) = \frac{1}{2} C^{\ell\ell}_{\mu\nu}(x)$
- $I = 0$ contribution: $G^{88}_{\mu\nu}(x) = \frac{1}{6}\left[C^{\ell\ell}_{\mu\nu}(x) + 2C^{ss}_{\mu\nu}(x) + D^{\ell-s,\ell-s}_{\mu\nu}(x)\right]$
- Z-γ mixing: $G^{08}_{\mu\nu}(x) = \left[C^{\ell\ell}_{\mu\nu}(x) - C^{ss}_{\mu\nu}(x) + D^{2\ell+s,\ell-s}_{\mu\nu}(x)\right]$

where the connected and disconnected Wick's contractions read



The relevant correlators are therefore given by

$$G^{\gamma\gamma}_{\mu\nu}(x) = G^{33}_{\mu\nu}(x) + \frac{1}{3} G^{88}_{\mu\nu}(x) + \frac{4}{9} G^{cc}_{\mu\nu}(x)$$

$$G^{Z\gamma}_{\mu\nu}(x) = \left(\frac{1}{2} - \sin^2\theta_W\right) G^{\gamma\gamma}_{\mu\nu}(x) - \frac{1}{6\sqrt{3}} G^{08}_{\mu\nu}(x) - \frac{1}{18} G^{cc}_{\mu\nu}(x)$$

Computational strategy

- Two discretisations of the vector current, the local (L) and point-split (C)

Set 1: Improvement coefficients from large-volume [1811.08209]

Set 2: Improvement coefficient from SF setup [1805.07401, 2010.09539]

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- Isovector contribution [S. Kuberski *et al.* 2024]

$$\bar{\Pi}^{(3,3)}(Q^2) = \bar{\Pi}^{(3,3)}_{\text{sub}}(Q^2) + b^{(3,3)}(Q^2, Q^2_m)$$

where

$$\widehat{K}(x_0, Q^2, Q^2_m)_{\text{sub}} = \frac{16}{Q^2}\sin^4\left(\frac{Qx_0}{4}\right) - \frac{Q^2}{Q^4_m}\sin^4\left(\frac{Q_m x_0}{2}\right), \qquad b^{(3,3)}(Q^2, Q^2_m) = \frac{Q^2}{4Q^2_m}\left(\Pi^{(3,3)}(4Q^2_m) - \Pi^{(3,3)}(Q^2_m)\right)$$

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- Isoscalar contribution

$$\mathrm{HV}: \widehat{\Pi}^{(8,8)} = \widehat{\Pi}^{(3,3)} + \widehat{\Delta}_{\mathrm{ls}}(Q^2), \qquad \mathrm{MV},\ \mathrm{LV}: \ \bar{\Pi}^{(8,8)} = \int_0^\infty \mathrm{d}x_0 G^{(8,8)} K(x_0, Q^2)$$

where

$$\Delta_{\mathrm{ls}}(Q^2) = G^{(8,8)} - G^{(3,3)} \propto \alpha_s (m_s^2 - m_l^2), \qquad \widehat{K}(x_0, Q^2) = \frac{16}{Q^2}\sin^4\left(\frac{Qx_0}{4}\right), \qquad K(x_0, Q^2) = x_0^2 - \frac{4}{Q^2}\sin^2\left(\frac{Qx_0}{2}\right)$$

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- Charm connected contribution

$$\bar{\Pi}^{(c,c)} = \bar{\Pi}^{(c,c)}_{\mathrm{sub}}(Q^2) + b^{(c,c)}(Q^2)$$

where

$$b^{(c,c)}(Q^2) = 2b^{(3,3)}(Q^2, Q_m^2) + \Delta_{lc} b(Q^2, Q_m^2), \qquad \Delta_{lc} b(Q^2, Q_m^2) = \frac{Q^2}{4Q_m^2}\left(\Pi(4Q_m^2) - \Pi(Q_m^2)\right)$$

FVC and noise reduction strategies

- Finite-volume corrections via spacelike [Hansen and Patella, 1904.10010, 2004.03935] and timelike [Meyer, 1105.1892][Lellouch and Lüscher, hep-lat/0003023] pion form factor
  - → Correct to $(m_\pi L)_{\mathrm{ref}} = 4.29$ for $a \neq 0$ [Borsanyi et al., 2002.12347] [Djukanovic et al., 2411.07696]
  - → Correct to $L = \infty$ in the continuum at the physical point

FVC and noise reduction strategies

- Finite-volume corrections via spacelike [Hansen and Patella, 1904.10010, 2004.03935] and timelike [Meyer, 1105.1892][Lellouch and Lüscher, hep-lat/0003023] pion form factor
- Low-Mode Averaging (LMA)
  - → Improved estimator for the light-connected correlation function
  - → LMA is used for all ensembles with $m_\pi < 280$ MeV

FVC and noise reduction strategies

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- Bounding Method in isovector and isoscalar channels

FVC and noise reduction strategies

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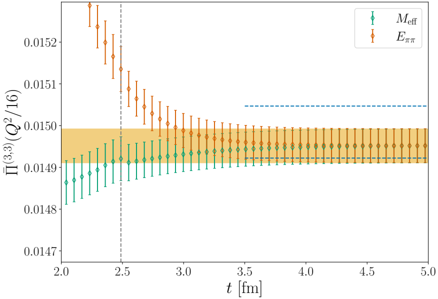
- Low-Mode Averaging (LMA)
- Bounding Method in isovector and isoscalar channels
- Spectral reconstruction of the $\pi\pi$ contribution in isovector channel [Djukanovic et al., 2411.07696][Nolan Miller @ Lattice24]

- $N_{\pi\pi} = 4$ states used to reconstruct the correlator
- Solves signal-to-noise problem beyond $t = 2.5$ fm
- Increase the precision by a factor of 2 on physical point ensemble E250: 0.2% for $\bar{\Pi}^{(3,3)}$



Isovector channel: tree-level improvement

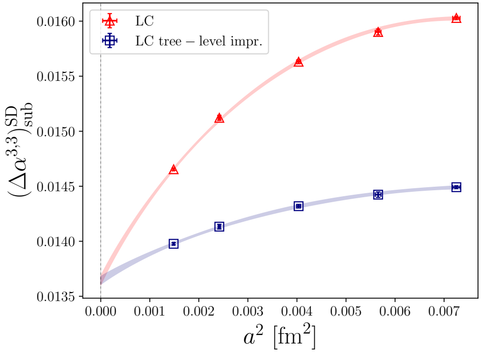
- Reduction of cutoff effects in the short Euclidean distance
  [ETM 2022; M. Cè *et al.* 2021; S. Kuberski *et al.* 2024]

- Continuum extrapolation of $(\Delta\alpha^{3,3})^{\rm SD}_{\rm sub}$ at the $SU(3)$-symmetric point $M_\pi = M_K \approx 415$ MeV

- Tree-level improvement based on massless perturbation theory

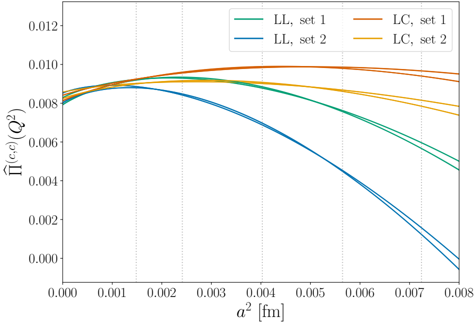
$$\mathcal{O}(a) \to \mathcal{O}(a)\frac{\mathcal{O}^{\rm tl}(0)}{\mathcal{O}^{\rm tl}(a)}$$

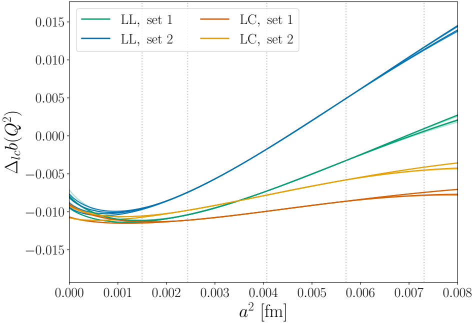
- Cutoff effects at $a = 0.087$ fm reduced from 19% to 7%



Charm connected contribution - HV region

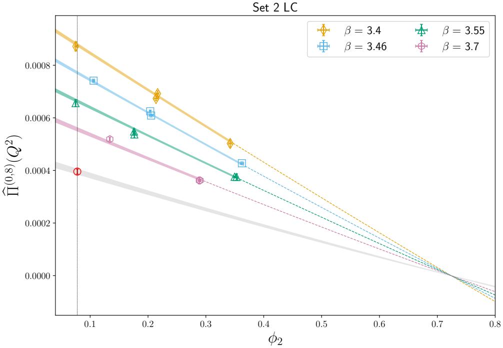
- $\widehat{\Pi}^{(c,c)} = (\widehat{\Pi}^{(c,c)})_{\text{sub}} + b^{(c,c)}$ chiral-continuum extrapolation at $Q^2 = 9\ \text{GeV}^2$
- $b^{(c,c)}(Q^2) = 2b^{(3,3)}(Q^2) + \Delta_{\text{lc}} b$
- Very good agreement despite significantly different cutoff effects

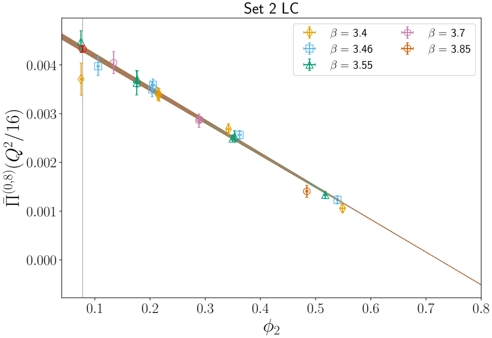




$\bar{\Pi}^{08}$ contribution

- $(\Delta \sin^2 \theta_W)^{0,8}$ chiral-continuum extrapolation at $Q^2 = 9$ GeV
- Results for MV (left) and LV (right) virtuality regions
- $SU(3)_f$ breaking → parametrically suppressed at short distance





Chiral-continuum extrapolations

Functional forms for isovector and charm connected contributions

- Our general ansatz for the chiral dependence reads

$$\mathcal{O}(\phi_2) = \mathcal{O}(\phi_2^{\rm phys}) + \gamma_1(\phi_2 - \phi_2^{\rm phys}) + \gamma_2(f_{\chi,1}(\phi_2) - f_{\chi,1}(\phi_2^{\rm phys})) + \gamma_3(f_{\chi,2}(\phi_2) - f_{\chi,2}(\phi_2^{\rm phys}))$$

  where

$$f_{\chi,1} \in \{1/\phi_2, \phi_2 \log(\phi_2),\ \phi_2^2\}, \qquad f_{\chi,2} \in \{1/\phi_2,\ \phi_2^2\}$$

- To account for a small mistuning from $m_s^{\rm phys}$

$$\mathcal{O}(\phi_4) = \mathcal{O}(\phi_4^{\rm phys}) + \gamma_0\big(\phi_4 - \phi_4^{\rm phys}\big).$$

- Cutoff effects are described by the general form

$$\mathcal{O}(a) = \beta_2 \frac{a^2}{8t_0} + \beta_3\left(\frac{a^2}{8t_0}\right)^{3/2} + \beta_4\left(\frac{a^2}{8t_0}\right)^2 + \delta_2\frac{a^2}{8t_0}\big(\phi_2 - \phi_2^{\rm phys}\big)$$
$$+ \delta_3\left(\frac{a^2}{8t_0}\right)^{3/2}\big(\phi_2 - \phi_2^{\rm phys}\big) + \epsilon_2\frac{a^2}{8t_0}\big(\phi_4 - \phi_4^{\rm phys}\big)$$

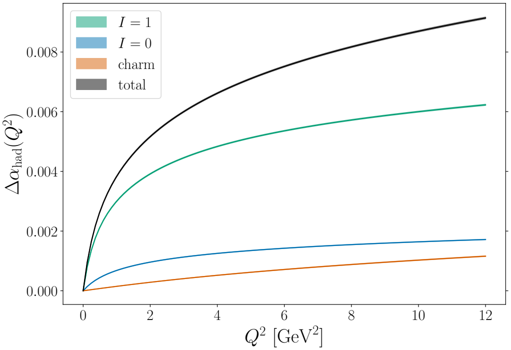
Chiral-continuum extrapolations

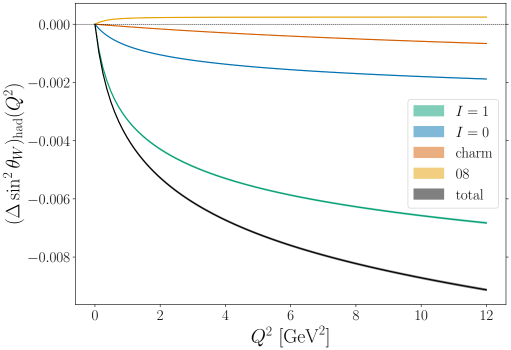
Functional forms for Δ_{ls} and $\bar{\Pi}^{08}$, $SU(3)$-flavour breaking quantities.

- Expected to depend at leading order on $m_l - m_s$
- Defining $\Phi_\delta = \Phi_4 - \frac{3}{2}\Phi_2$, our general ansatz reads

$$\mathcal{O}(\Phi_\delta, \phi_2, a) = \Phi_\delta \left(\gamma_1 + \gamma_2 \Phi_\delta + \beta_2 \frac{a^2}{8t_0} + \beta_3 \left(\frac{a^2}{8t_0} \right)^{3/2} + \gamma_0 \Phi_4 \right)$$

Running of $\Delta\alpha_{\text{had}}$ and $(\sin^2\theta_W)_{\text{had}}$





- Total HVP contribution to $\Delta\alpha_{\text{had}}$ and $(\Delta\sin^2\theta_W)_{\text{had}}$
- Results in the range $0 \leq Q^2 \leq 12\ \text{GeV}^2$
- Rational approximation of the running through a multi-points Padé Ansatz

Evaluation of $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$

- **Lattice input:**
  $\Delta\alpha_{\text{had}}^{(5)}(-Q_0^2)$ for $3\ \text{GeV}^2 \leq Q_0^2 \leq 12\ \text{GeV}^2$
- **Perturbative running:**
  Evaluation of the Adler function with three different methods
  1. AdlerPy
  2. pQCD-bfmom
  3. pQCD-updated

  Input values from FLAG24 for 1 and 3

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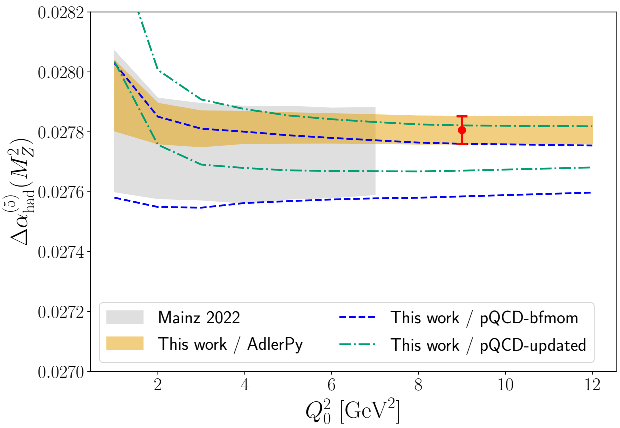

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