Swiss National Science Foundation

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Comparing QCD+QED via full simulation versus the RM123 method: U-spin window contribution to a_μ^{HVP}

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On behalf of the RC* collaboration

8th g-2 Theory Initiative Workshop
10th September 2025

This work: arxiv:2506.19770 (accepted for publication in *JHEP*)

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Motivation

Comparing two methods for including the isospin breaking effects

RC* program: focus on non-perturbative QCD+ QED computations (of *masses of mesons and baryons, HVP, leptonic decays, etc.*) using C* boundary conditions as QED prescription

(see arXiv:1509.01636, arXiv:1908.11673)

This work (arXiv:2506.19770): We compute the HVP-like observable at finite lattice spacing and volume by exploiting two approaches

1. non-perturbative approach to QCD+QED: simulations with dynamical U(1) and $m_u \neq m_d$
2. isosymmetric QCD simulations + RM123 (arXiv:1303.4896) $\rightarrow$ perturbative expansion in δm_{ud} and α_{QED} at leading order, all sea-effects included

Comparing two methods for including the isospin breaking effects

Setup for the comparison: 1 QCD+QED and 1 isoQCD ensemble with $C^\star$ boundary conditions generated using openQ*D code

Main goal: compare the statistical precision at fixed number of configurations

Aspect	QCD+QED	isoQCD + RM123
Generation of Configurations [1]	(More expensive)	(Less expensive, isoQCD confs. can be reused)
Tuning[1]	(Complicated by additional parameters)	(Simpler)
Observable computation[2]	(All orders IB included)	(IB only at $O(\alpha)$, requires computation of many diagrams)
Statistical Noise	?	?

1 arXiv:2209.13183, arXiv:2212.09578 arXiv:2212.10894

2 arXiv:2212.11551, arXiv:2502.14845, arXiv:2502.03145

Parametrization of QCD+QED and isoQCD

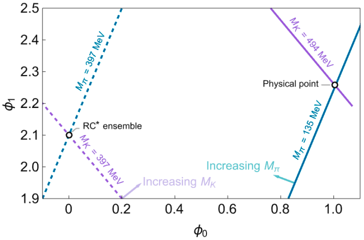
Lines of constant physics

Hadronic renormalization scheme for QCD+QED and isoQCD:

Observable	"Fix"	Physical value	Target RC* value
$\sqrt{8t_0}$	a	0.415*	0.415 fm
$\phi_0 = 8t_0(M^2_{K^\pm} - M^2_{\pi^\pm})$	$m_s - m_d$	0.992	0
$\phi_1 = 8t_0(M^2_{K^\pm} + M^2_{\pi^\pm} + M^2_{K^0})$	$m_u + m_d + m_s$	2.26	2.11
$\phi_2 = 8t_0(M^2_{K^0} - M^2_{K^\pm})/\alpha_R$	$m_u - m_d$	2.36	2.36, 0
$\phi_3 = \sqrt{8t_0}(M_{D_s^\pm} + M_{D^0} + M_{D^\pm})$	m_c	12.0	12.1
α_R	e^2	0.007297	α^{phys}, 0

* isoQCD computation by CLS: (arXiv:1608.08900)

- unphysical point $\rightarrow M_\pi \simeq 400$ MeV
- SU(3) symmetry in isoQCD
- U-spin symmetry $m_d = m_s$ in QCD+QED



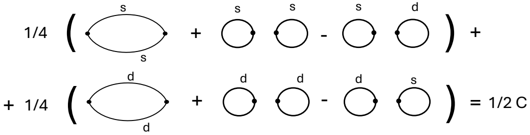
Observable: definition of $a_\mu^{\mathrm{U,w}}$

U-spin window observable $a_\mu^{\mathrm{U,w}}$

We define the HVP-related observable as:

$$a_\mu^{\mathrm{U,w}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}t\, G^{\mathrm{U}}(t)\tilde{K}(t; m_\mu) w_{\mathrm{I}}(t).$$

- $G^{\mathrm{U}}(t)$ vector correlator with $V_\mu = \frac{1}{2}(\overline{\psi}_s\gamma_\mu\psi_s - \overline{\psi}_d\gamma_\mu\psi_d)$
- $w_{\mathrm{I}}(t)$ gives the intermediate window (arxiv:1801.07224)
- disconnected-free due to SU(3)/U-spin symmetry in our setup



Note: in isoQCD with SU(3) symmetry $\rightarrow a_\mu^{\mathrm{U,w}} = \frac{3}{4} a_\mu^{\mathrm{HVP,\, uds}}$

Lattice setup

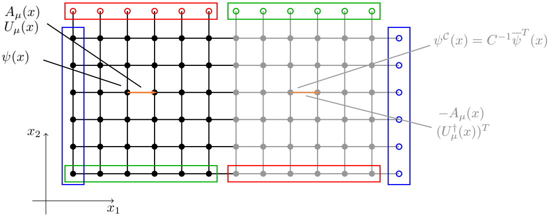
QED$_\mathrm{C}$ prescription

We use a local and gauge-invariant prescription for QED at finite volume:
C-periodicity of all fields in spatial directions

$$A_\mu(x + L_i\hat{i}) = -A_\mu(x), \quad U_\mu(x + L_i\hat{i}) = U^*_\mu(x)$$

$$\psi(x + L_i\hat{i}) = C^{-1}\bar{\psi}(x)^T, \quad \overline{\psi}(x + L_i\hat{i}) = -\psi^T(x)C$$

- The lattice is doubled in the $\hat{1}$ direction: $L_1 = 2L$, while $L_k = L$ for $k = 2, 3$.
- C^* BCs in other directions: $\psi(x + L_k\hat{k}) = \psi(x + \frac{L_1}{2}\hat{1})$ for $k = 2, 3$
- zero-mode removed by construction → charged-states propagation allowed



Lattice action

Gauge action:

- Lüscher-Weisz discretization of the action for SU(3)
- Wilson plaquette action in the compact formulation for $\mathrm{U}(1)$

Fermionic action:

$$S_{\mathrm{F}} = -\sum_f a^4 \sum_x \frac{1}{2}\chi_f^\top(x) KC(D_{\mathrm{w},f} + \delta D_{\mathrm{sw},f} + m_f)\chi_f(x).$$

- Notation for the fermion field

$$\chi = \begin{pmatrix} \psi \\ C^{-1}\bar{\psi}^\top \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

- $D_{\mathrm{w},f}$ Wilson-Dirac operator, couples to χ depending on q_f
- $\delta D_{\mathrm{sw},f} = \delta D_{\mathrm{sw}}^{\mathrm{SU}(3)} + D_{\mathrm{sw},f}^{\mathrm{U}(1)}$ SW improvement term

Discretization of the observable

We employ two discretizations of the vector correlator

$$a_\mu^{\mathrm{U,w}} = \left(\frac{\alpha}{\pi}\right)^2 a \sum_{t=0}^{T/2} G^{\mathrm{U},\ell}(t)\tilde{K}(t; m_\mu) w_{\mathrm{I}}(t), \qquad \ell = \mathrm{l,c} \tag{2}$$

where

$$G^{\mathrm{U,l}}(t) = (Z_{\mathrm{V}}^{\mathrm{m}})^2 G_{\mathrm{bare}}^{\mathrm{U,l}}(t), \quad G^{\mathrm{U,c}}(t) = Z_{\mathrm{V}}^{\mathrm{m}} G_{\mathrm{bare}}^{\mathrm{U,c}}(t). \tag{3}$$

- $\ell = l \rightarrow$ local-local discretization
- $\ell = c \rightarrow$ point-split local discretization

- mass-dependent renormalization factor through

$$\lim_{t\to\infty} \frac{Z_{\mathrm{V}}^{\mathrm{m}} G_{\mathrm{bare}}^{\mathrm{U,l}}(t)}{G_{\mathrm{bare}}^{\mathrm{U,c}}(t)} = 1. \tag{4}$$

Computation of the observable: 1. QCD+QED

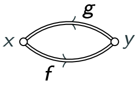
1. Sampling of configurations from

$$p(U,z) \propto \prod_f \mathrm{Pf}(CKD_f[U,z])\, \mathrm{e}^{-S_{\mathrm{g,SU(3)}}[U]-S_{\mathrm{g,U(1)}}[z]}$$

   - U(1) link: $z_\mu(x) = e^{ieaA_\mu(x)}$
   - $\mathrm{Pf}(CKD_f[U,z]) = (\det(CKD_f[U,z])^{1/2}$

2. Compute two-point correlation functions on QCD+QED configurations:

$$\mathcal{O}(x,y) = \mathrm{Tr}\{D_f^{-1}(y|x)\Gamma_A D_g^{-1}(x|y)\Gamma_B\} =$$

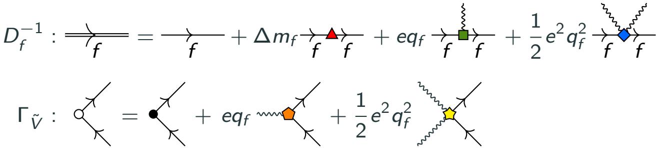


   - $\Gamma_{A,B}$ pseudoscalar, vector local or point-split vertex

1. Expansion of the probability density $\mathrm{Pf}(CKD_f[U,z])e^{-S_{\mathrm{g,SU(3)}}[U]}e^{-S_{\mathrm{g,U(1)}}[z]}$

$$\underbrace{\mathrm{Pf}(CKD_f^{(0)}[U])e^{-S_{\mathrm{g,SU(3)}}[U]}}_{\text{sampled } p(U)}\;\underbrace{e^{-S_\gamma[A]}\Big[1+O(\Delta D_f[U,A])\Big]}_{\text{ob. comp.}}$$

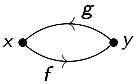
2. Expansion of the observable through



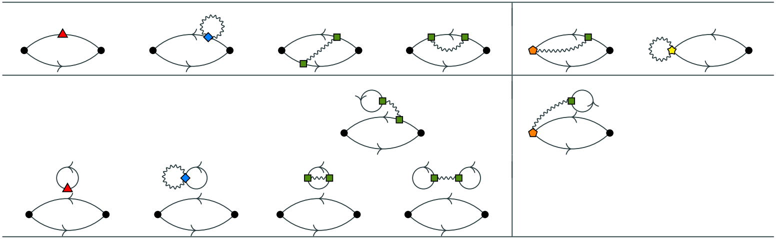
Computation of the observable: 2. isoQCD+RM123

3. Computation of the LO two-point correlator functions

$$\mathcal{O}^{(0)}(x, y) = \mathrm{Tr}\{D_f^{(0)\,-1}(y|x)\Gamma_A D_g^{(0)\,-1}(x|y)\Gamma_B\} =$$



4. Computation of the IB corrections



Results

Ensembles

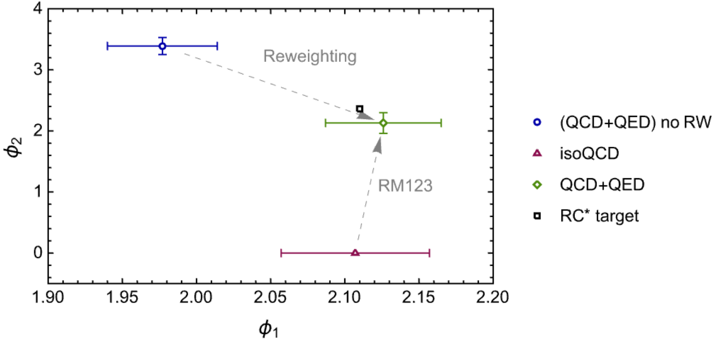
Setup:

- 2 gauge ensembles with $\alpha = 0$ and $\alpha \simeq \alpha^{\text{phys}}$
- same lattice volume and $\beta \rightarrow$ different a
- $O(a)$-improved Wilson fermions with improvement coefficients $c_{\text{sw}}^{SU(3)} = 2.18859$ (arxiv:1805.01661) and $c_{\text{sw}}^{U(1)} = 1$
- different hopping parameters κ_q for quarks
- statistics: 2000 configurations

ensemble	lattice	β	α	κ_u	$\kappa_d = \kappa_s$	κ_c
A400a00	64×32^3	3.24	0	0.13440733	0.13440733	0.12784
A380a07	64×32^3	3.24	0.007299	0.13457969	0.13443525	0.12806355

Realistic scenario: reweighting vs RM123

- mass-reweighting (RW) needed for the QCD+QED ensemble
- RW computed in previous work (arxiv:2209.13183)$\rightarrow$ no effect on the uncertainties



Note: $\phi_1 = 8t_0(M^2_{K^\pm} + M^2_{\pi^\pm} + M^2_{K^0})$ and $\phi_2 = 8t_0(M^2_{K^0} - M^2_{K^\pm})/\alpha$

Two comparison strategies

1. Fixed bare parameters:
   - isoQCD+RM123: we use the exact quark mass shifts from $\kappa^{-1}_{QCD+QED} - \kappa^{-1}_{isoQCD}$

$$a\Delta m_{\mathrm{u}} = -0.00476435, \quad a\Delta m_{\mathrm{d,s}} = -0.00077259, \quad a\Delta m_{\mathrm{c}} = -0.00682735 \tag{5}$$

   - we compare the QCD+QED results for all obs: $t_0, \phi_i, a_\mu^{\mathrm{U,w}}$

$$X^{n.p.} \quad \text{vs} \quad X(0) + e^2 \partial_{e^2} X(0) + \sum_{f=\mathrm{u,d,s,c}} \Delta m_f \partial_{m_f} X(0) \tag{6}$$

2. Fixed line of constant physics (LCP):
   - isoQCD+RM123: we tune the quark mass shifts through 4 eqs.

$$\phi_i^{(0)} + e^2 \partial_{e^2} \phi_i + \sum_f \Delta m_f \, \partial_{m_f} \phi_i - \Delta_L \phi_i = \phi_i^\star . \tag{7}$$

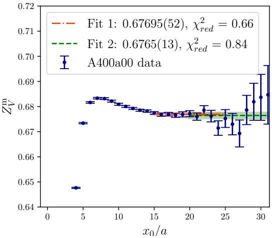
     The uncertainties on Δm_f propagates to $a_\mu^{\mathrm{U,w}}$
   - n.p. QCD+QED we propagate the tuning error to the final observable through

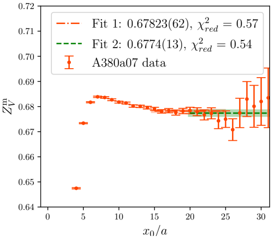
$$(\mathrm{d}a_\mu^{\mathrm{U,w}})^2 = \sum_f \left(\partial_{m_f} a_\mu^{\mathrm{U,w}}\right)^2 (\mathrm{d}m_f)^2, \quad \mathrm{d}m_f(\vec{\phi}) = \sum_i (J^{-1})_{fi} \, \mathrm{d}\phi_i , \tag{8}$$

   - we compare only $a_\mu^{\mathrm{U,w}}$

Non-perturbative computation of $a_\mu^{\mathrm{U,w}}$

- Determination of Z_V^m

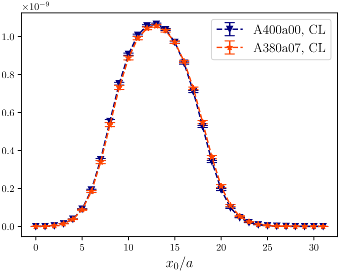




$$Z_{\mathrm{V}}^{\mathrm{m}}(t) = \frac{G_{\mathrm{bare}}^{\mathrm{U,c}}(t)}{G_{\mathrm{bare}}^{\mathrm{U,l}}(t)}$$

- Results:

	$a_\mu^{\mathrm{U,w}} \times 10^{11}$	
	$\ell = \mathrm{l}$	c
isoQCD	1083(6)	1086(5)
non-pert. QCD+QED	1082(7)	1085(7)



RM123: corrections to the scale

We expand the condition $t_0^2\langle E(t_0)\rangle = 0.3$ around $t_0^{(0)}$ and get

$$\delta t_0 = \frac{-t_0 \Delta\varepsilon \cdot \nabla\langle E(t_0)\rangle}{2\langle E(t_0)\rangle + t_0\langle d/dt E(t)|_{t=t_0}\rangle}. \tag{9}$$

- $\Delta\varepsilon = (e^2, \Delta m_\mathrm{u}, \Delta m_\mathrm{d}, \Delta m_\mathrm{s}, \Delta m_\mathrm{c})$
- only sea-quark effects contribute to $\nabla\langle E(t_0)\rangle$

Results in lattice units:

$\hat{t}_0^{(0)}$	$\partial_{am_\mathrm{u}}\hat{t}_0$	$\partial_{am_\mathrm{c}}\hat{t}_0$	$\partial_{e^2}\hat{t}_0$
7.400(69)	-76(24)	-26.5(8.1)	-6.1(1.9)

Correction to the scale with fixed bare parameters (input exact Δm_f)

$$\delta\hat{t}_0 = 0.10(3) \to \frac{\delta a}{a^{(0)}} = -\frac{\delta t_0}{2t_0^{(0)}} = -0.0069(19)$$

RM123: corrections to ϕ_i

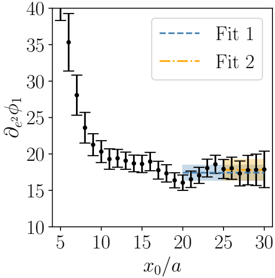
$$\delta\phi_i = e^2\partial_{e^2}\phi_i + \sum_f \Delta m_f\,\partial_{m_f}\phi_i - \Delta_L\phi_i$$

- $\Delta_L\phi_i$: universal FV corrections to ϕ_i from the ones on the masses

$$\Delta M(L) = \frac{e^2}{4\pi}\left(\frac{\zeta(1)}{2L} + \frac{\zeta(2)}{\pi M L^2}\right)$$

$\Delta_L\phi_1$	$\Delta_L\phi_2$	$\Delta_L\phi_3$
-0.00651(4)	0.446(3)	-0.00323(2)

- $\partial_\varepsilon\phi_i$: combination of meson masses and scale derivatives + fit procedure



Correction to ϕ_i with fixed bare parameters (input exact Δm_f):

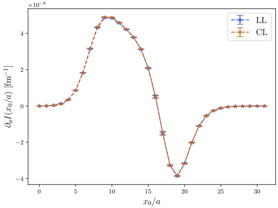
$$\delta\phi_0 = 0.0 \quad \delta\phi_1 = 0.084(82) \quad \delta\phi_2 = 2.52(14) \quad \delta\phi_3 = 0.053(47)$$

Note: U-symmetry $\rightarrow \phi_0 = 0$

RM123: corrections to $a_\mu^{\mathrm{U,w}}$

$$a_\mu^{\mathrm{U,w}} = \left(\frac{\alpha}{\pi}\right)^2 \sum_{\hat{t}=0}^{\hat{T}/2} \underbrace{\hat{\tilde{K}}\left(\hat{t}; am_\mu\right) w_{\mathrm{I}}\left(a\hat{t}\right)}_{\delta_a a_\mu^{\mathrm{U,w}}} \underbrace{Z_{\mathrm{V}}^{\mathrm{m}}}_{\delta_{Z_V} a_\mu^{\mathrm{U,w}}} \underbrace{\hat{G}_{\mathrm{bare}}^{\mathrm{U},c}\left(\hat{t}\right)}_{\delta_G a_\mu^{\mathrm{U,w}}}$$

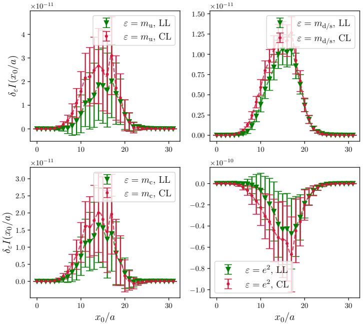
- Integrand for $\partial_a a_\mu^{\mathrm{U,w}}$:



- Derivatives of $Z_{\mathrm{V}}^{\mathrm{m}}$:

$\partial_{am_{\mathrm{u}}} Z_{\mathrm{V}}^{\mathrm{m}}$	$\partial_{am_{\mathrm{d,s}}} Z_{\mathrm{V}}^{\mathrm{m}}$	$\partial_{am_{\mathrm{c}}} Z_{\mathrm{V}}^{\mathrm{m}}$	$\partial_{e^2} Z_{V}^{\mathrm{m}}$
-1.30(26)	-1.75(27)	-0.39(10)	-0.11(2)

- Integrand for $\delta_G a_\mu^{\mathrm{U,w}}$:



Corrections to $a_\mu^{\mathrm{U,w}}$ at fixed bare parameters (input exact Δm_f):

units of 10^{-11}	$\ell = \mathrm{l}$	$\ell = \mathrm{c}$
$\delta_G a_\mu^{\mathrm{U,w}}$	11(16)	12(17)
$\delta_Z a_\mu^{\mathrm{U,w}}$	3.8(3.4)	1.9(1.7)
$\delta_a a_\mu^{\mathrm{U,w}}$	-8.4(2.3)	-8.4(2.4)
$\delta a_\mu^{\mathrm{U,w}}$	7(18)	6(18)

Results at fixed bare parameters

Results:

	$\hat{t}_0$	ϕ_1	ϕ_2	ϕ_3	$a_\mu^{\mathrm{U,w}} \times 10^{11}$ $\ell = \mathrm{l}$	c
isoQCD+RM123$\vert_{\mathrm{eq}}$	7.400(69)	2.257(34)	2.20(14)	12.100(44)	1078(5)	1080(5)
isoQCD+RM123	7.502(81)	2.198(92)	2.53(14)	12.151(66)	1090(18)	1092(18)
n.p. QCD+QED	7.523(94)	2.128(34)	2.37(12)	12.103(47)	1082(7)	1085(7)

Observations:

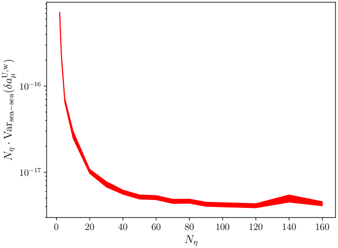
- valence-valence effects precisely computed $\rightarrow$ no increased uncertainties for the EQ results
- $\hat{t}_0$ precision not affected by sea-quark effects, mild effect on ϕ_3
- exact cancellation for the sea-quark effects on ϕ_2 $\rightarrow$ uncertainty remains the same
- ϕ_1 and $a_\mu^{\mathrm{U,w}}$ almost three times noisier in the full RM123 setup

Results at fixed bare parameters

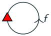
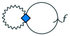
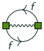
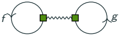
- Decomposition of the IB correction to $a_\mu^{\mathrm{U,w}}$

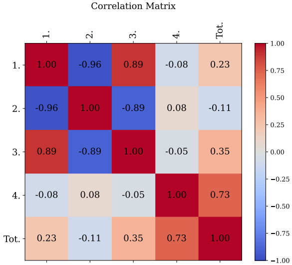
Class	$\ell = \mathrm{l}$	c
vv	-6.6(4)	-6.6(4)
sv	-0.24(1)	-0.24(1)
ss	14(18)	13(18)
Tot. $\delta a_\mu^{\mathrm{U,w}}$	7(18)	6(18)

Uncertainty dominated by sea-sea effects!



- Detailed sea-sea effects to $\delta a_\mu^{\mathrm{U,w}} \times 10^{11}$

	Sea-diagram	$\ell = \mathrm{l}$	c
1.	$\sum_f \Delta m_f$ 	14(30)	13(30)
2.	$e^2 \sum_f q_f^2$ 	-19(43)	-18(43)
3.	$e^2 \sum_f q_f^2$ 	7(19)	6(19)
4.	$e^2 \sum_{fg} q_f q_g$ 	13(14)	12(14)
Tot.	$\delta a_\mu^{\mathrm{U,w}}(ss)$	14(18)	13(18)



Results at fixed lines of constant physics (LCP)

Further step: Change the strategy to determine the quark mass shifts and compare

	From fixed bare parameters	From fixing LCP
$a\Delta m_{\mathrm{u}}$	-0.00476435	-0.00477(17)
$a\Delta m_{\mathrm{d,s}}$	-0.00077259	-0.00082(17)
$a\Delta m_{\mathrm{c}}$	-0.00682735	-0.0083(28)

- from fixed bare parameters: $\kappa^{-1}_{QCD+QED} - \kappa^{-1}_{isoQCD}$
- from fixing LCP: solution of $\phi_i^{\text{isoQCD+ RM123}} = \phi_i^\star$

Final results for $a_\mu^{\mathrm{U,w}}$:

	From fixing bare parameters		From fixed LCP	
	ℓ=l	ℓ= c	ℓ=l	ℓ=c
isoQCD+RM123$\|_{\mathrm{eq}}$	1078(5)	1080(5)	1084(5)	1087(5)
isoQCD+RM123	1090(18)	1092(18)	1093(20)	1094(21)
n.p. QCD+QED	1082(7)	1085(7)	1082(8)	1085(7)

- no significant changes observed between the two strategies

Conclusion & Outlook

Summary: We compared two methods for computing $a_\mu^{\mathrm{U.w}}$ in QCD+QED:

1. non-perturbative setup with QED at all orders
2. isoQCD+RM123: IB effects including sea-quark contributions at order $O(\alpha)$ (A. Cotellucci et al., see poster at the workshop)

with

- same unphysical pion mass, single volume and lattice spacing
- focus on the statistical uncertainty

Our conclusion:

$$a_\mu^{\mathrm{U,w}} \times 10^{11} = \begin{cases} 1094(21) & \mathrm{RM123} \\ 1085(7) & \mathrm{non\text{-}perturbative} \end{cases} .$$

To be clarified:

- Dependence on the pion mass?
- Dependence on the QED prescription?
- Observable dependence?

Conclusion & Outlook

Summary: We compared two methods for computing $a_\mu^{\mathrm{U.w}}$ in QCD+QED:

1. non-perturbative setup with QED at all orders
2. isoQCD+RM123: IB effects including sea-quark contributions at order $O(\alpha)$
   (A. Cotellucci et al., see poster at the workshop)

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- focus on the statistical uncertainty

Our conclusion:

$$a_\mu^{\mathrm{U,w}} \times 10^{11} = \begin{cases} 1094(21) & \mathrm{RM123} \\ 1085(7) & \text{non-perturbative} \end{cases} .$$

To be clarified:

- Dependence on the pion mass?
- Dependence on the QED prescription?
- Observable dependence?

Thank you for listening!