R-ratio and muon anomaly from solving inverse problem on the lattice

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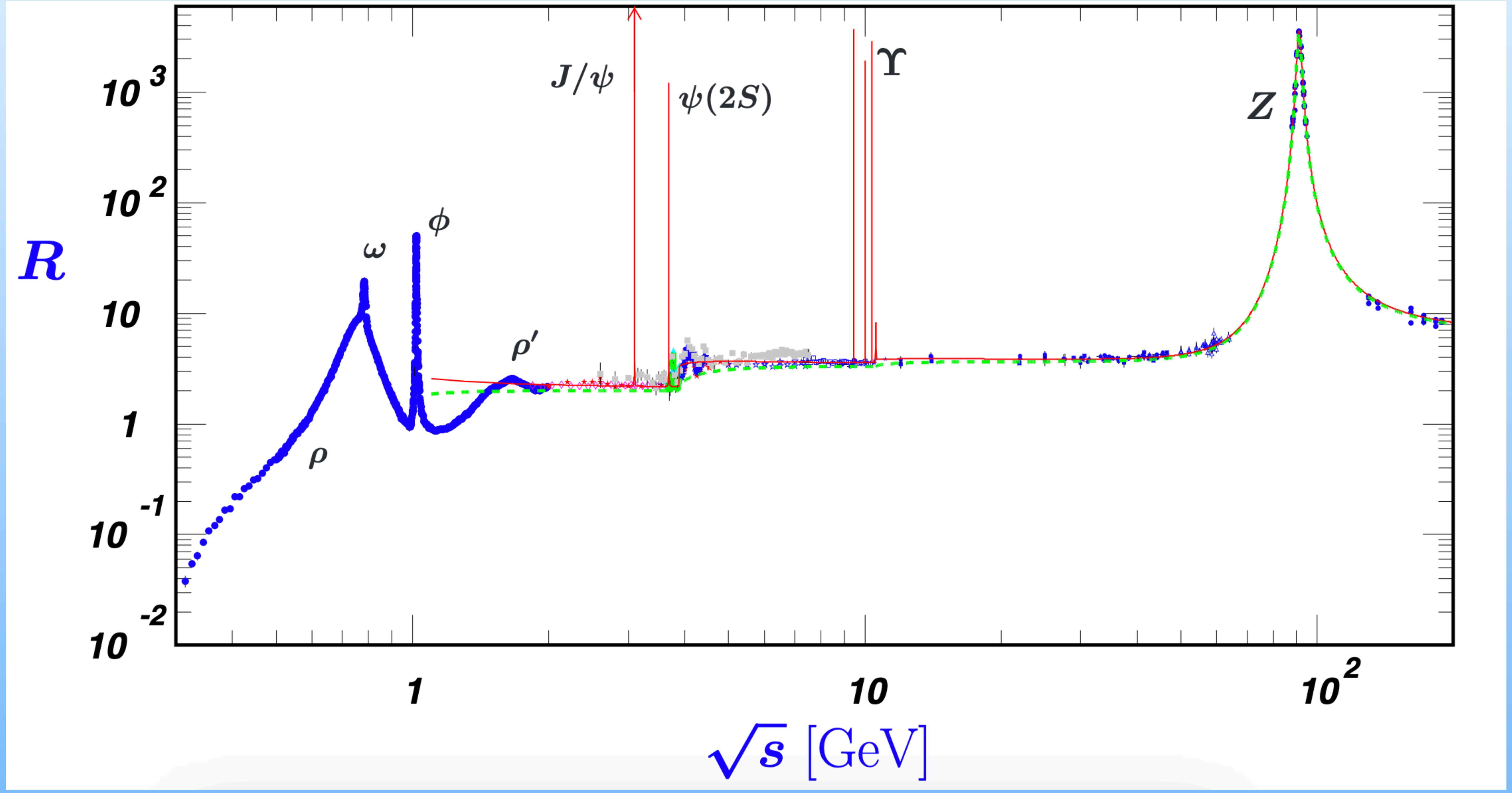
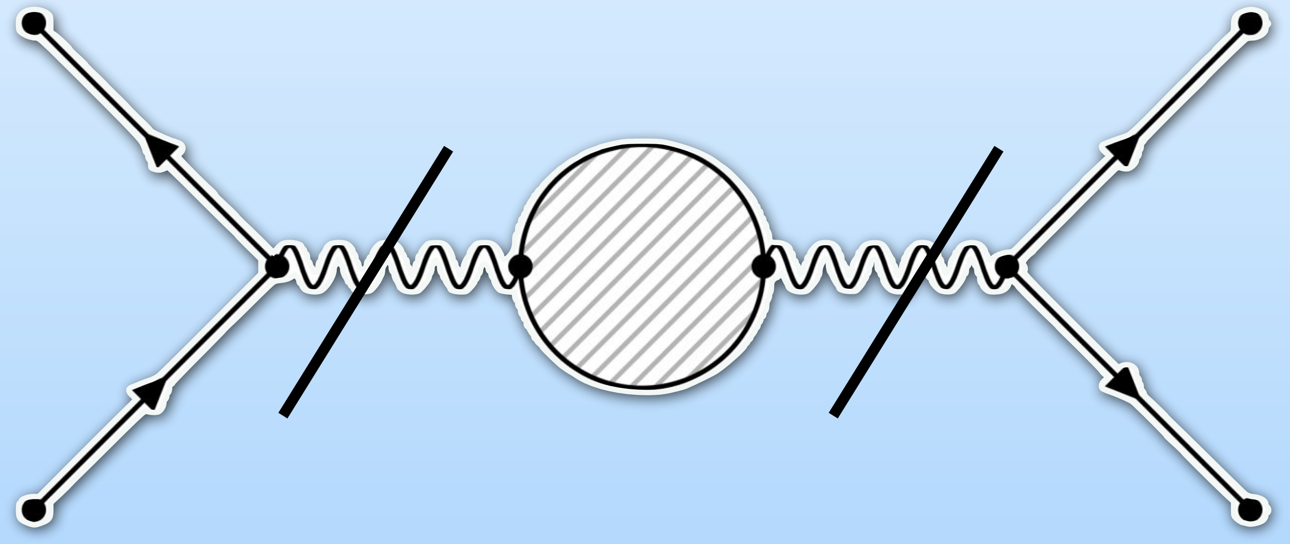
09/10/2025 The Eighth Plenary Workshop of the Muon g-2 Theory Initiative @ IJCLab

R-ratio from LQCD





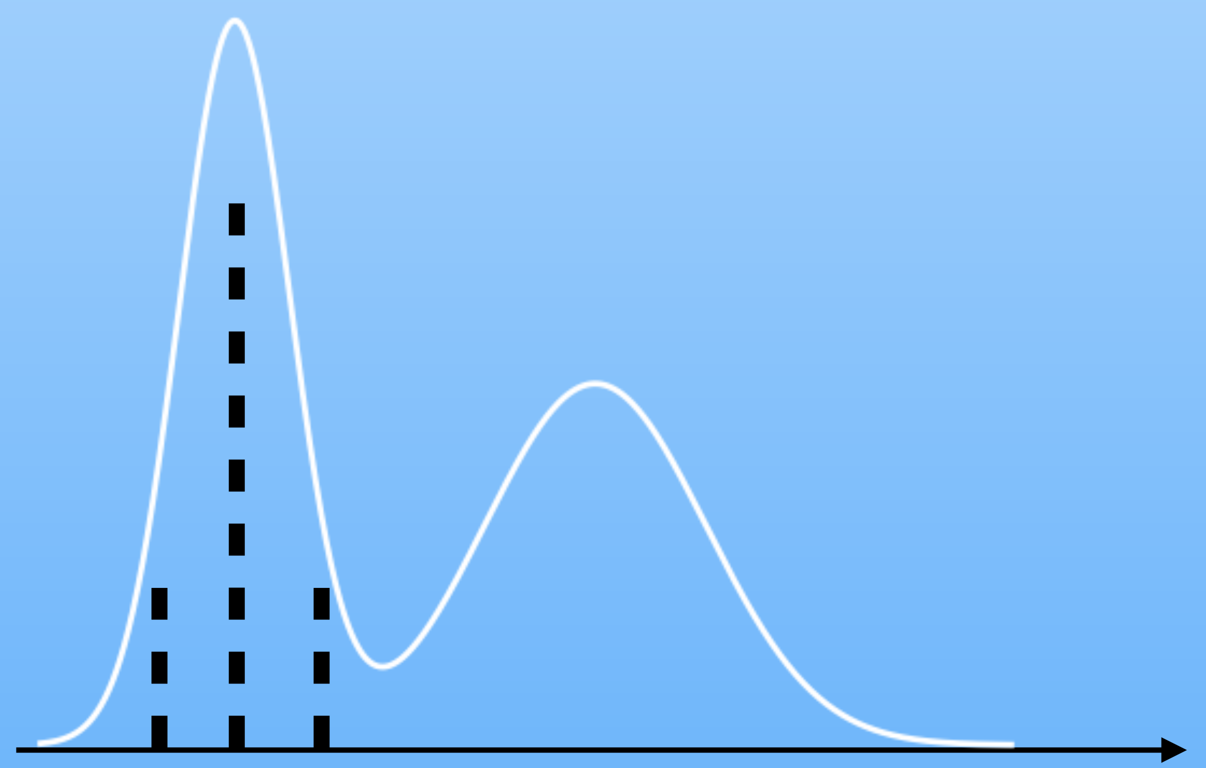
$$C_2(t) = \left\langle J_\mu^{\mathrm{em}}(t)\ J_\mu^{\mathrm{em}}(0) \right\rangle = \int d\omega \rho_L(\omega) e^{-\omega t}$$

$$R_L(\omega) = \frac{12\pi^2}{\omega^2} \rho_L(\omega)$$

R-ratio from LQCD

$$C_2(t) = \left\langle J_\mu^{\mathrm{em}}(t)\ J_\mu^{\mathrm{em}}(0) \right\rangle = \int d\omega \rho_L(\omega) e^{-\omega t}$$

$$R_L(\omega) = \frac{12\pi^2}{\omega^2} \rho_L(\omega)$$



$$\rho_L(\omega) = \sum_n A_n \delta\left(\omega, \omega_n\right)$$

✦ inverse problem, ill-posed, prior information needed for unique solution

Bayesian reconstruction (BR) algorithm

Y. Burnier and A. Rothkopf, PRL 111, 182003 (2013)

$$P[\rho\,|\,D, m] = \frac{P[D\,|\,\rho, I]}{P[D\,|\,m]} \int d\alpha P[\alpha\,|\,D, m]$$

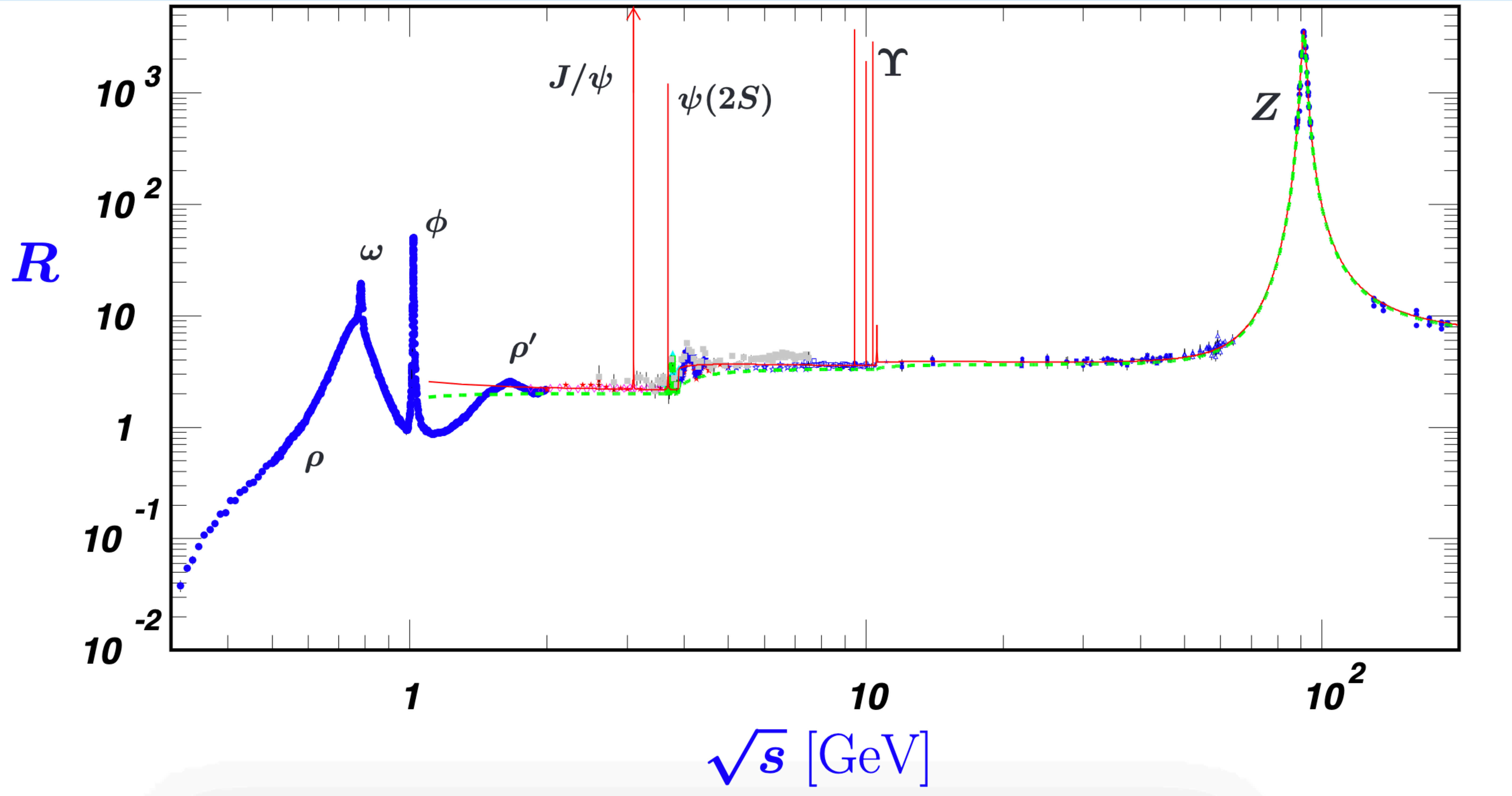
✦ finite-volume effects in the resonance region

$$\rho_L^S(\omega, \Delta) = \int d\omega' \mathcal{S}\left(\omega, \omega'\right) \rho_L\left(\omega'\right)$$

$$\rho\left(\omega\right) = \lim_{\Delta \to 0} \lim_{L \to \infty} \rho_L^S(\omega, \Delta)$$

M. T. Hansen et al., Phys. Rev. D 96, 094513 (2017)

R-ratio from LQCD



✦ resonance region:

$$R_L^S(\omega, \Delta) = \int d\omega' \mathcal{S}(\omega, \omega') R_L(\omega')$$

$$R^S(\omega, \Delta) = \lim_{L\to\infty} R_L^S(\omega, \Delta) \leftrightarrow R_E^S(\omega, \Delta)$$

✦ high-energy region: flat, hadronic tensor

J. Liang et al., PRD101, 114503 (2020)

Numerical setup

Label	L/T	Mpi (MeV)	a (fm)	L (fm)
48I	48/96	139	0.11406	5.47
64I	64/128	139	0.08365	5.35
24D	24/64	139	0.1940	4.656
32D	32/64	139	0.1940	6.208
48D	48/96	139	0.1940	9.312

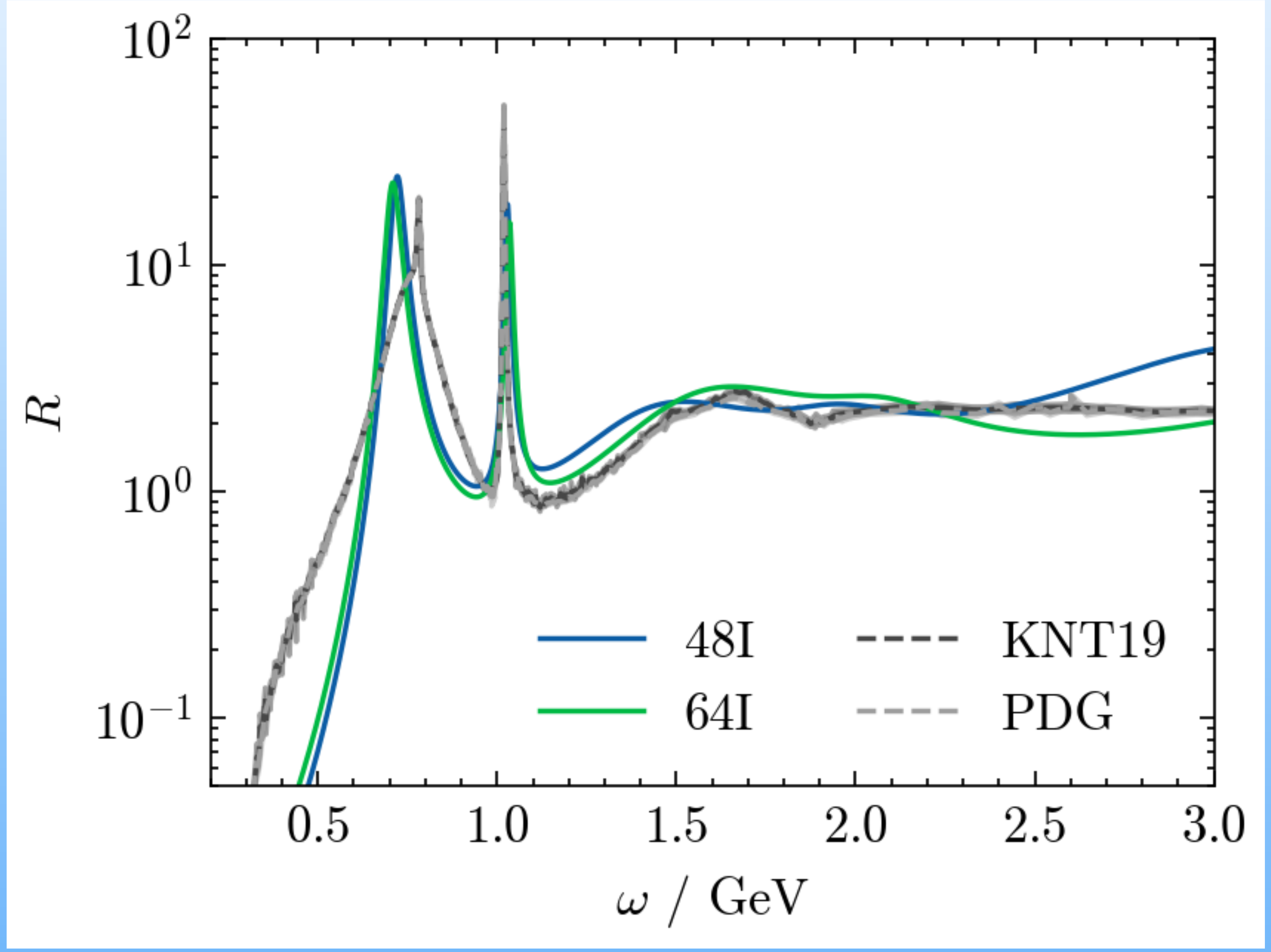
R. Arthur et al., PRD87, 094514 (2013)
T. Blum et al., PRD93, 074505 (2016)
P. Boyle et al., PRD 93, 054502 (2016)

- Overlap fermions on RBC/UKQCD domain wall gauge ensembles at the physical point with different lattice spacings and volumes

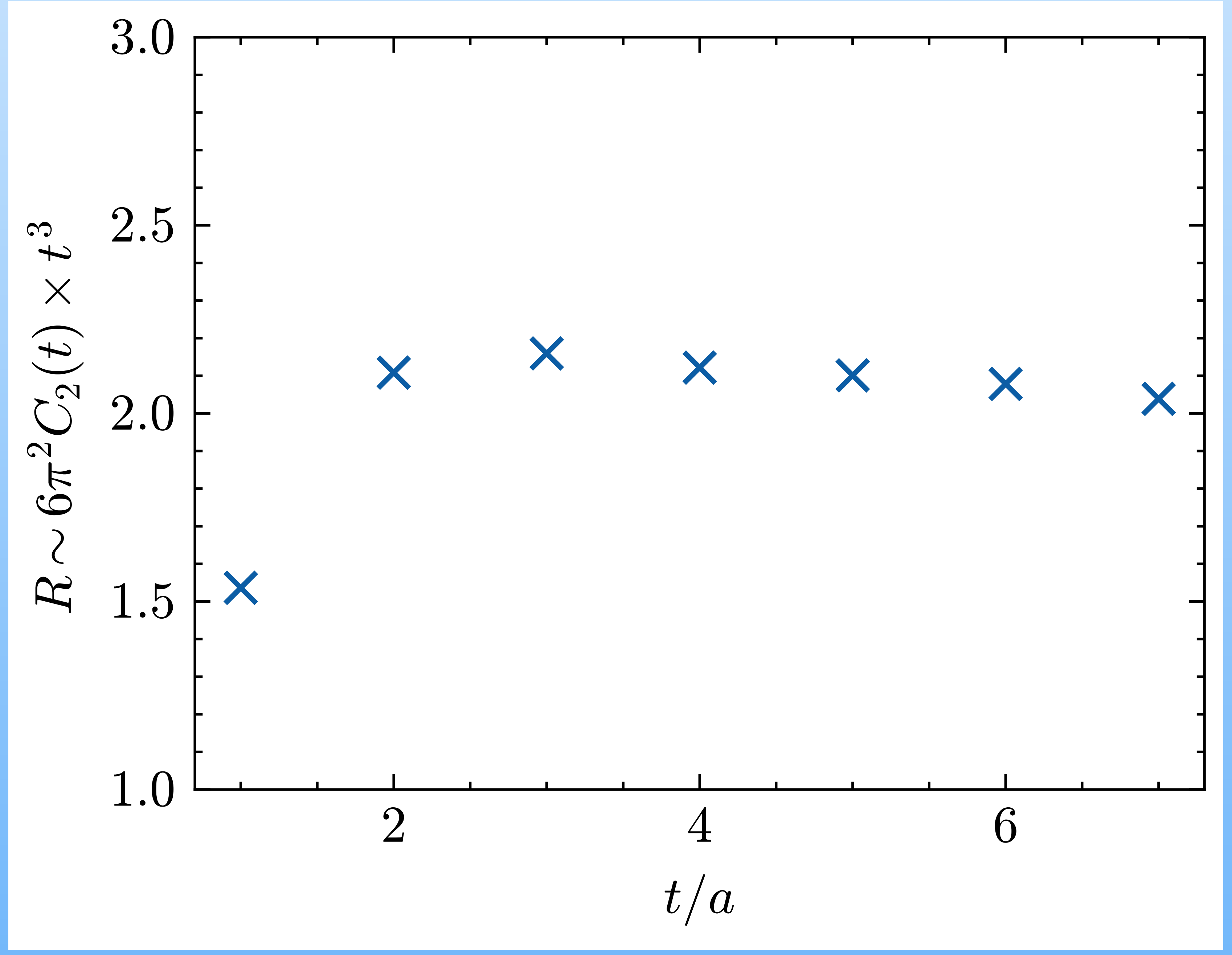
Gen Wang et al., Phys. Rev. D 107, 034513 (2023)

- High-precision current-current correlation functions for both u/d and s
- Careful systematic uncertainty study

Results without smearing

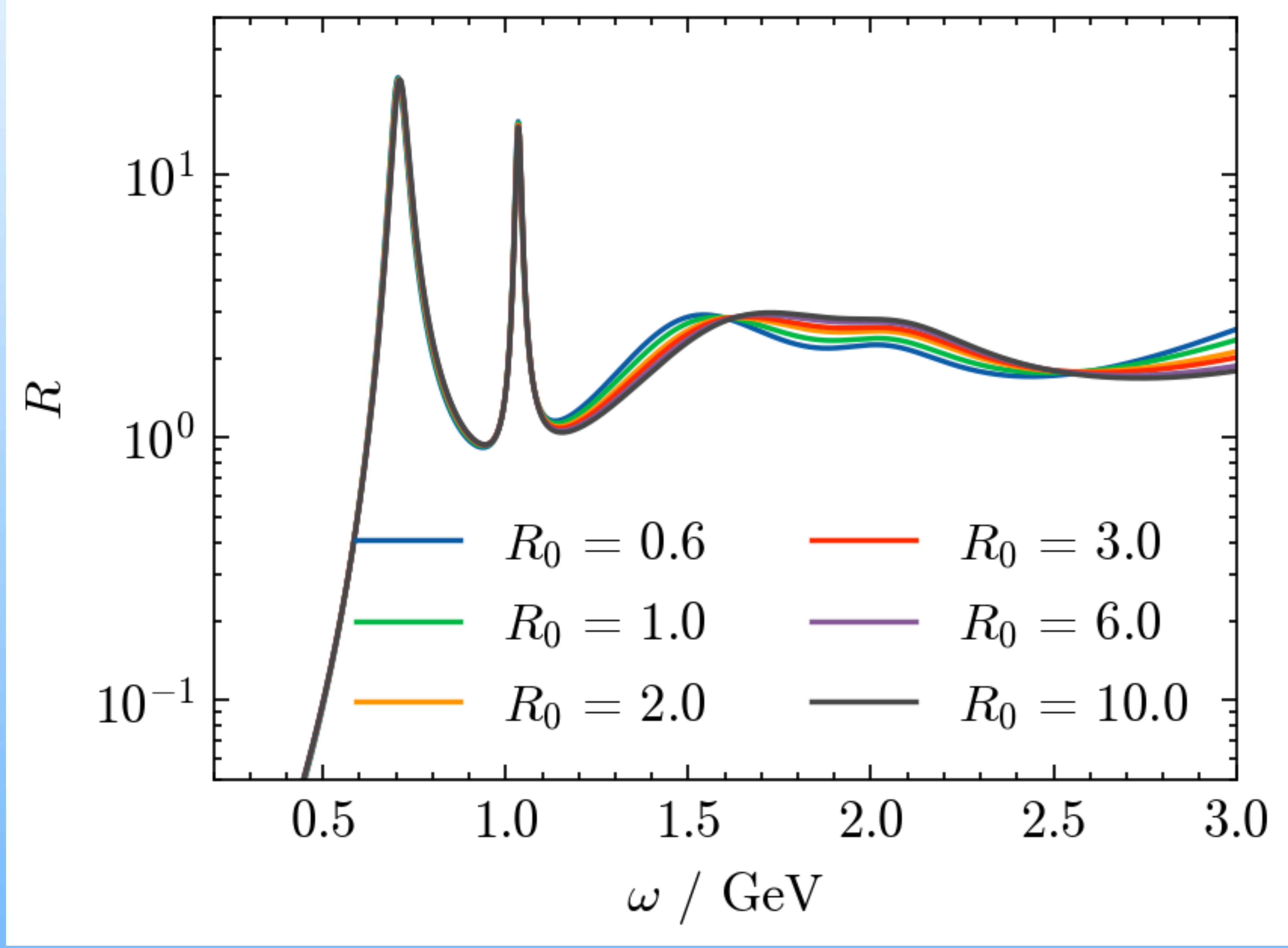


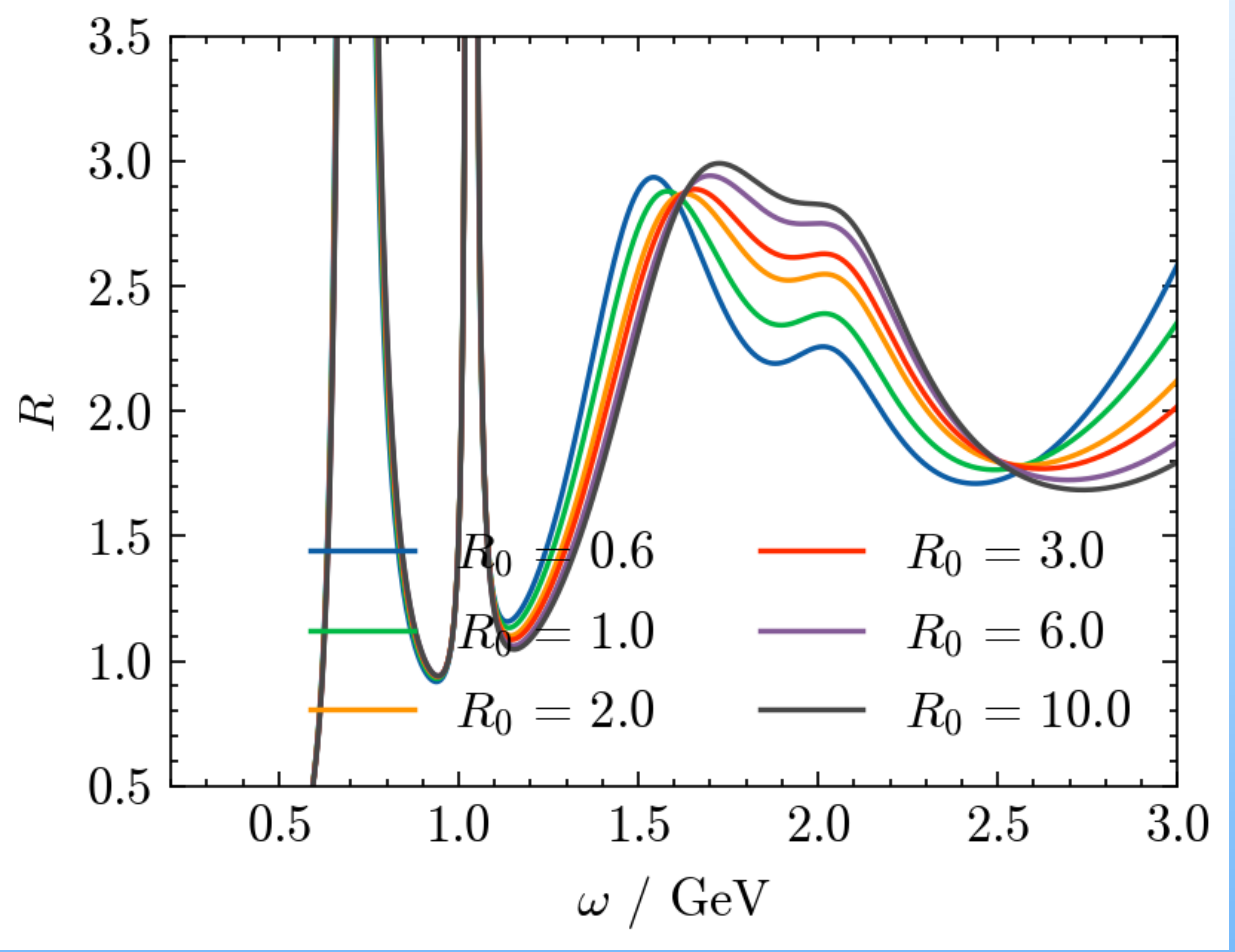
$$C_2(t) = \int_0^\infty e^{-\omega t} \frac{R}{12\pi^2} \omega^2 d\omega \sim \frac{R}{6\pi^2 t^3}$$



Finite-volume effects around ρ peak, additional ϕ peek from applying separate BR on light and strange correlators

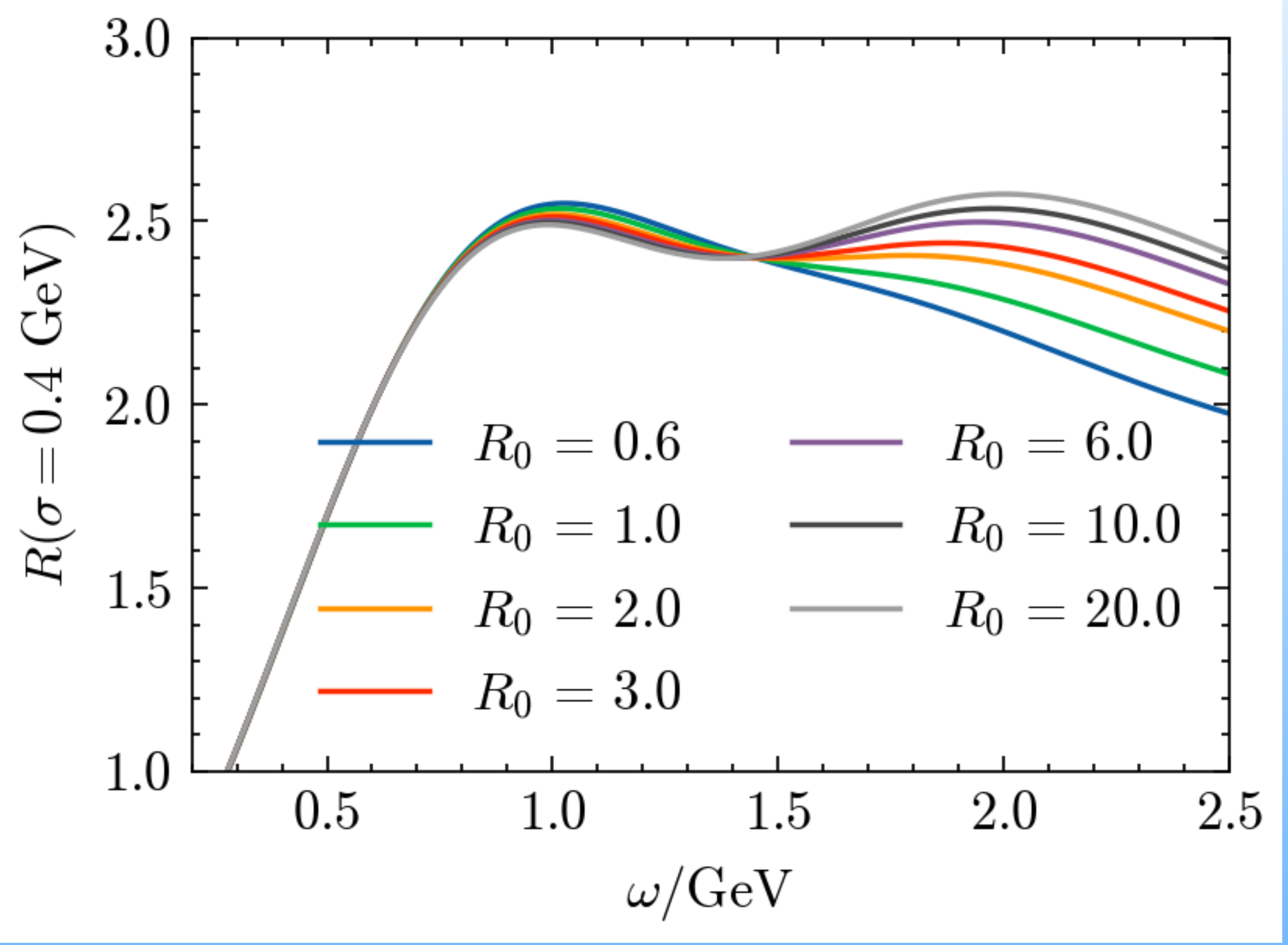
Prior dependence

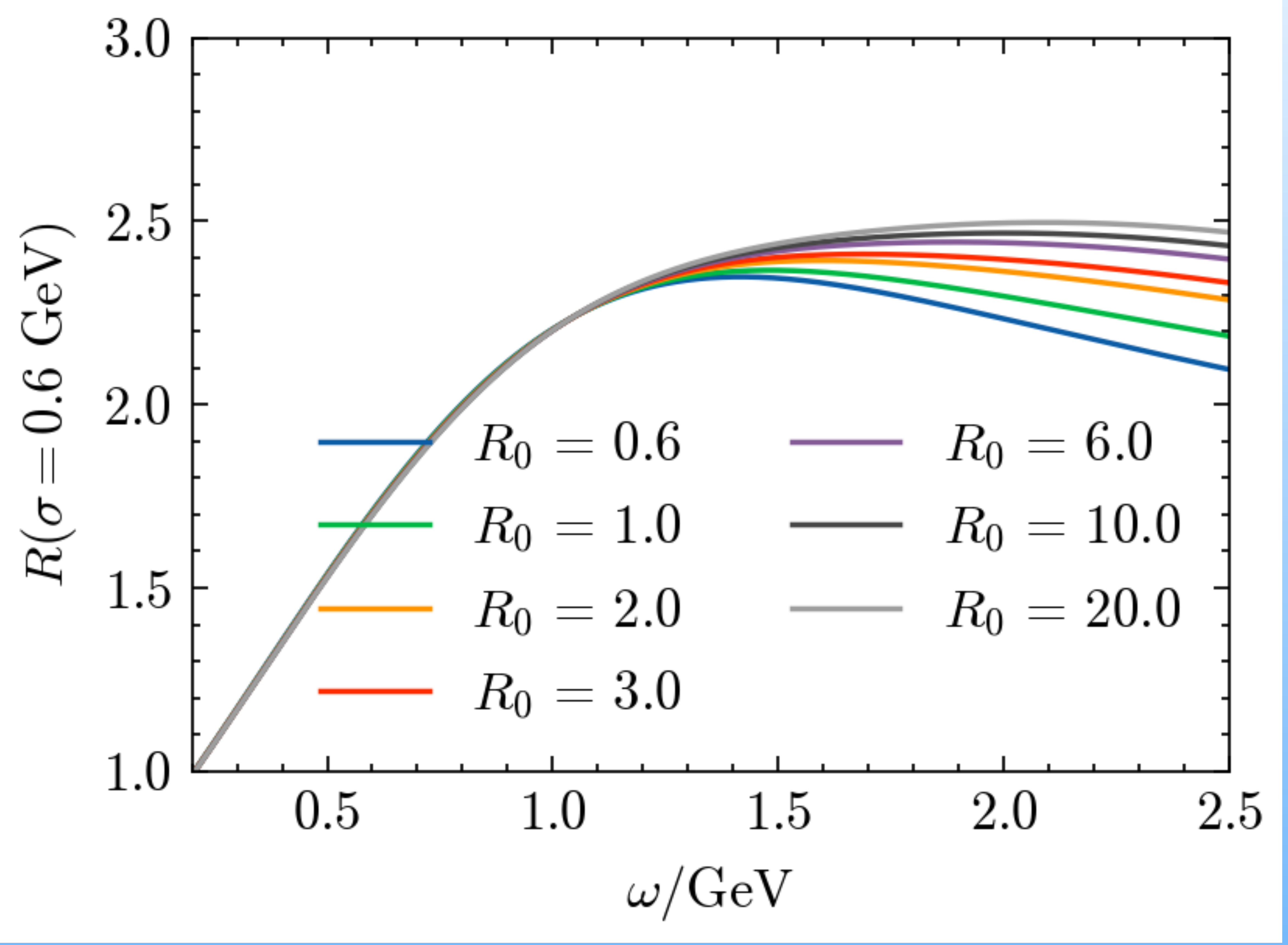




Constant priors in R, 64I

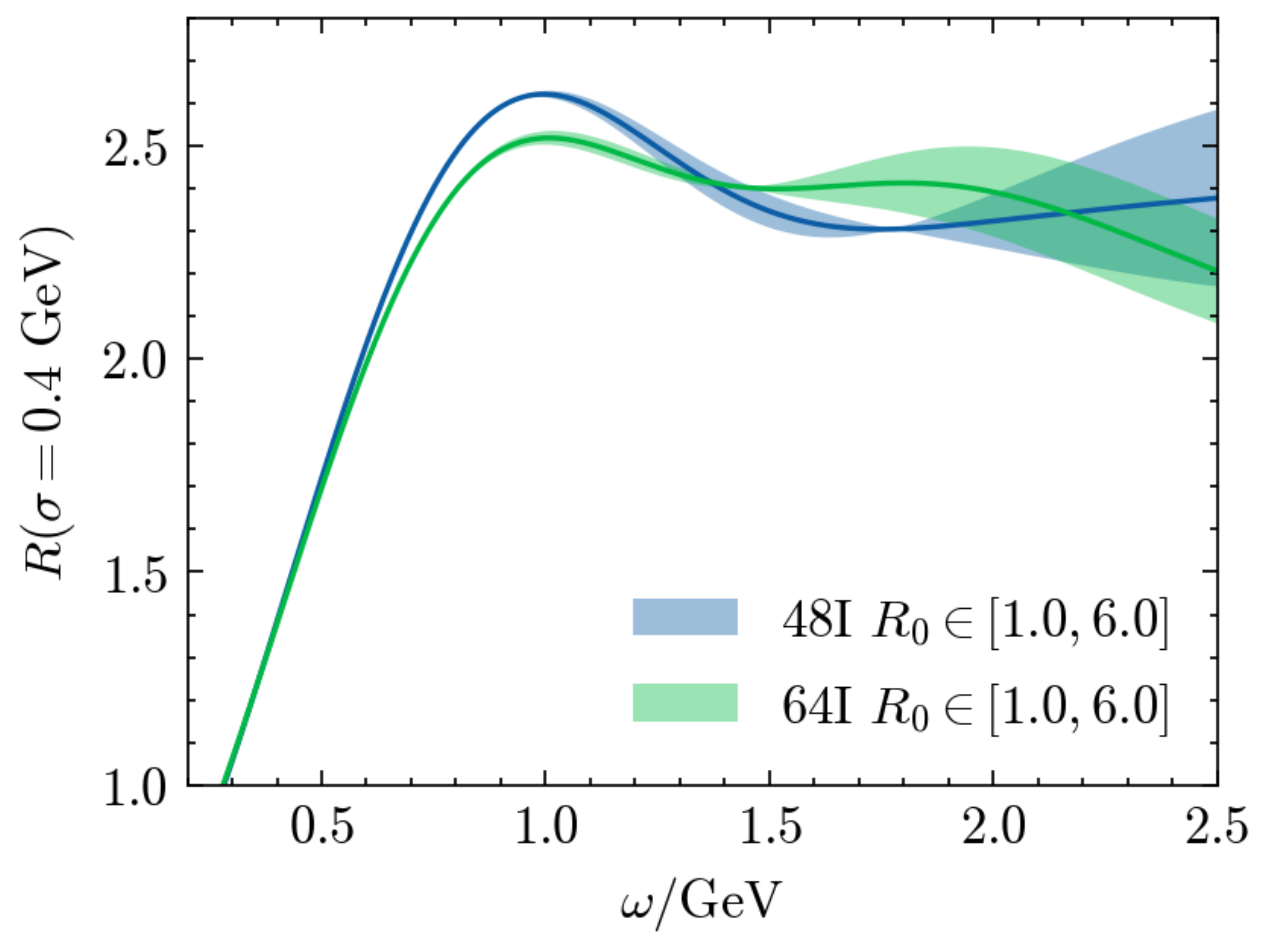
Smearing

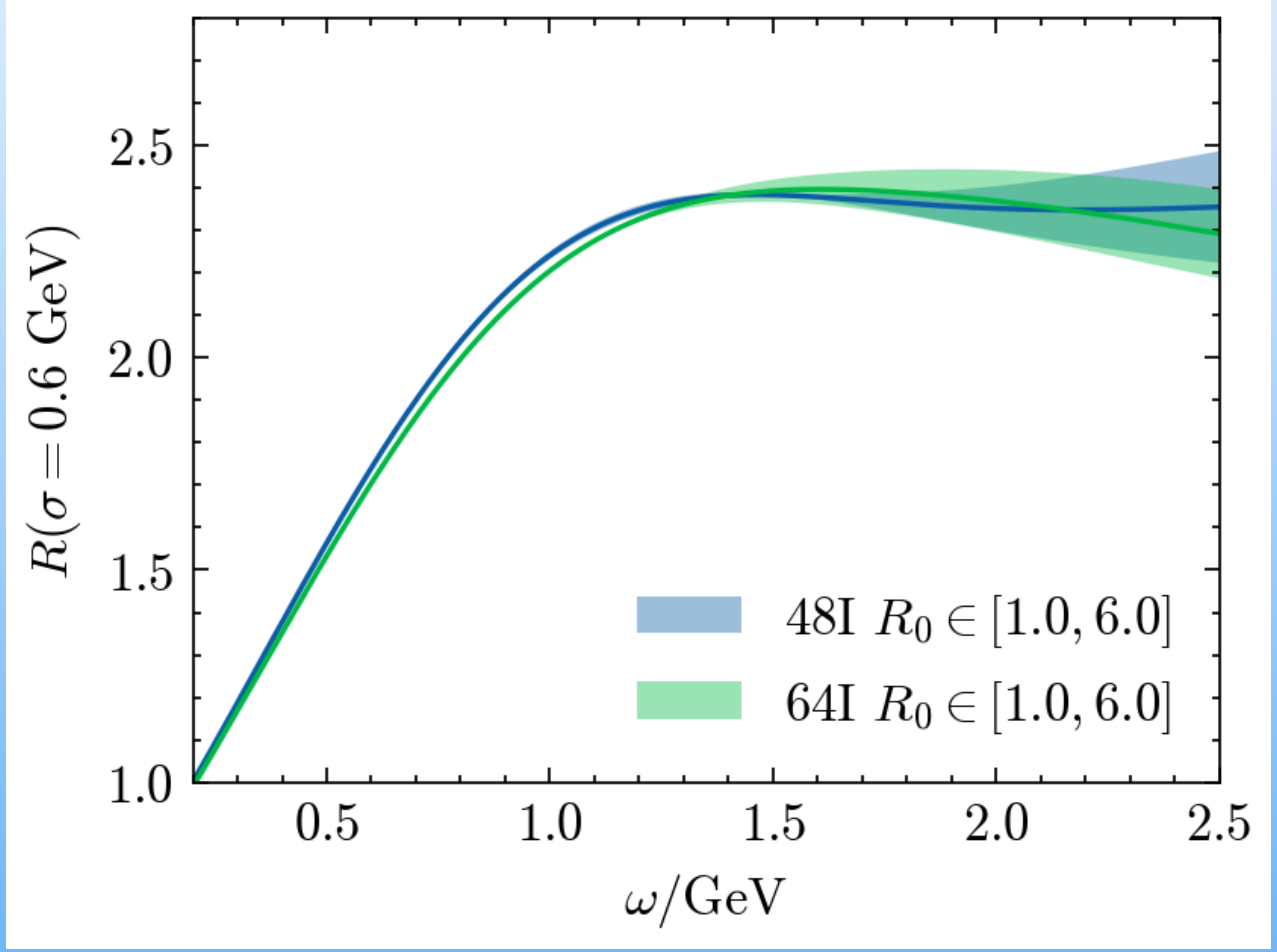




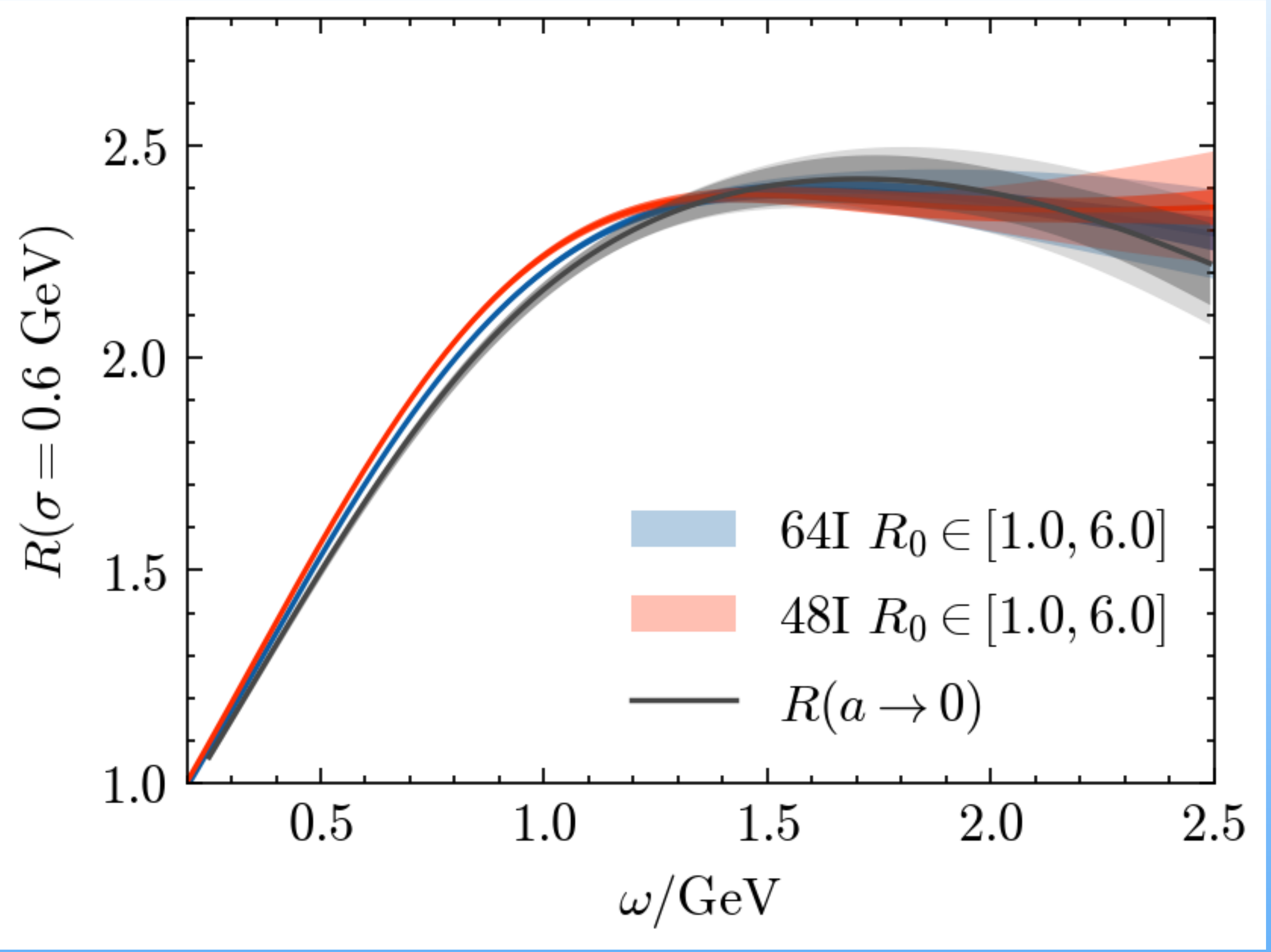
$$R_L^S(\omega, \Delta) = \int d\omega' \mathcal{S}_\Delta(\omega, \omega') R_L(\omega') \qquad \mathcal{S}_\Delta(\omega, \omega') \sim \exp\left(-\frac{(\omega - \omega')^2}{2\Delta^2} \right)$$

Uncertainty due to prior

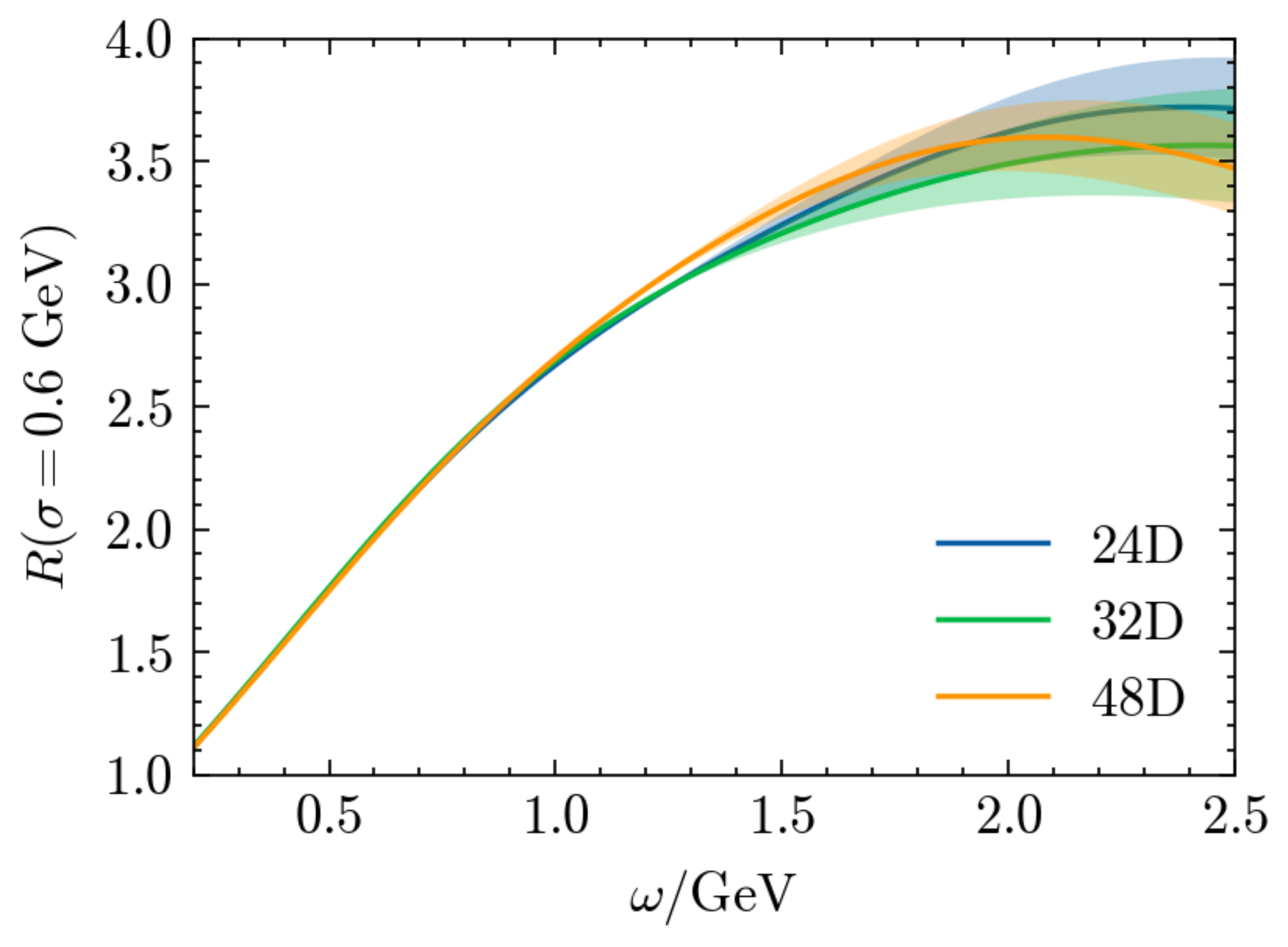




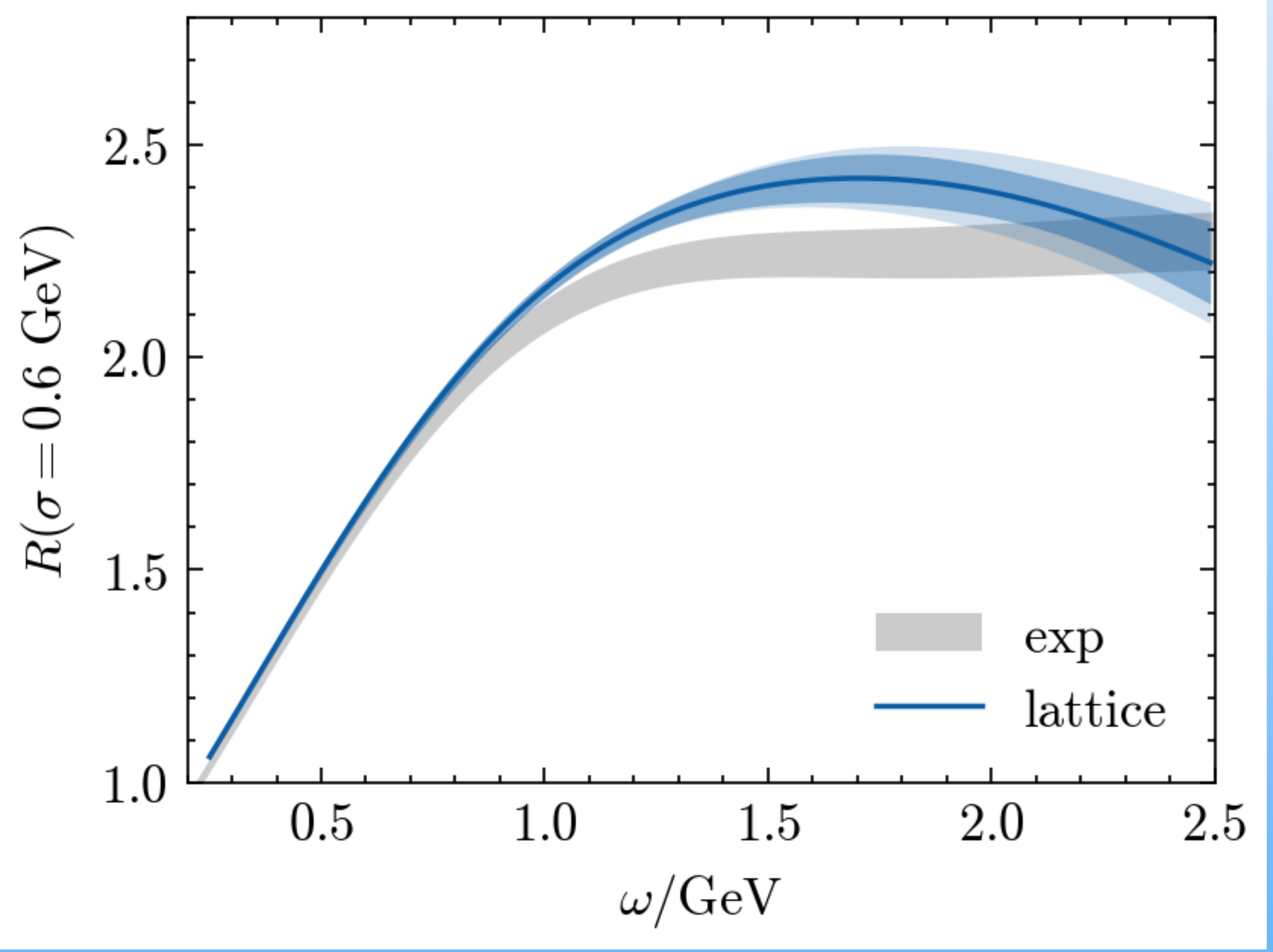
Continuum extrapolation

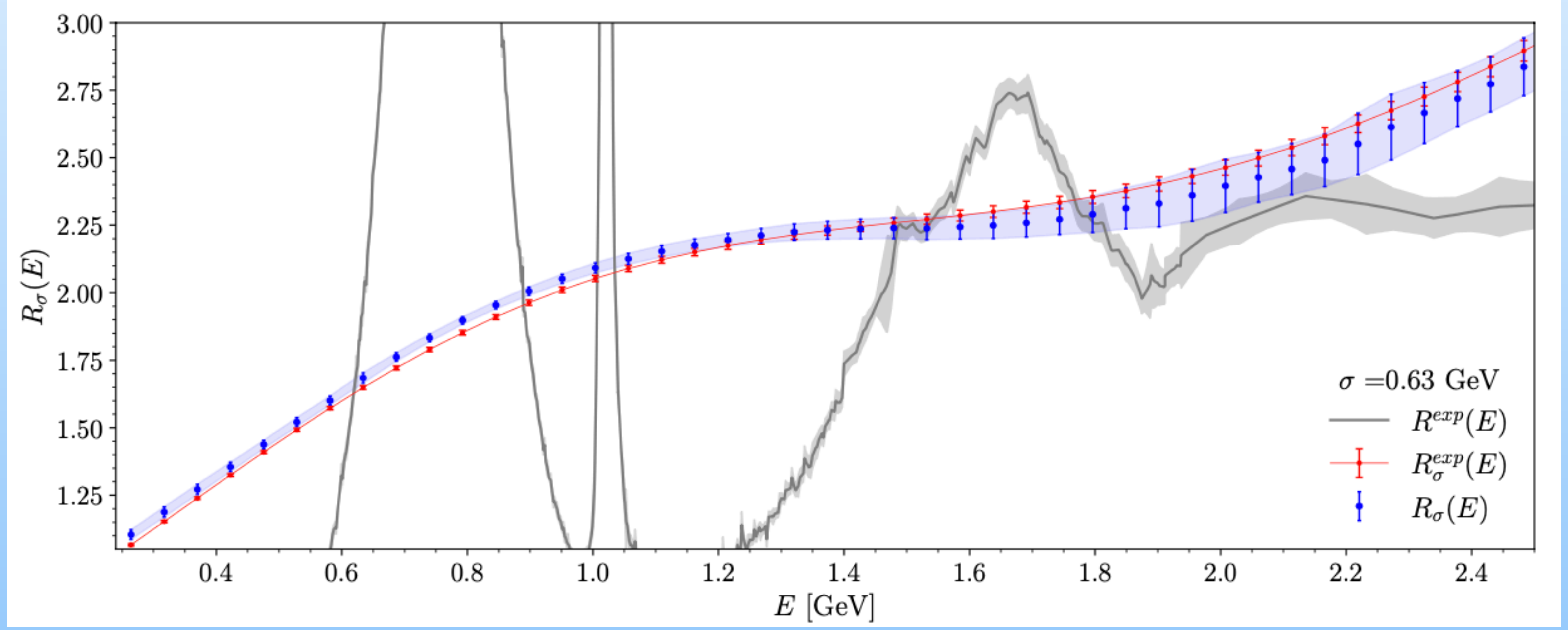


Volume dependence



R ratio with all systematic uncertainties

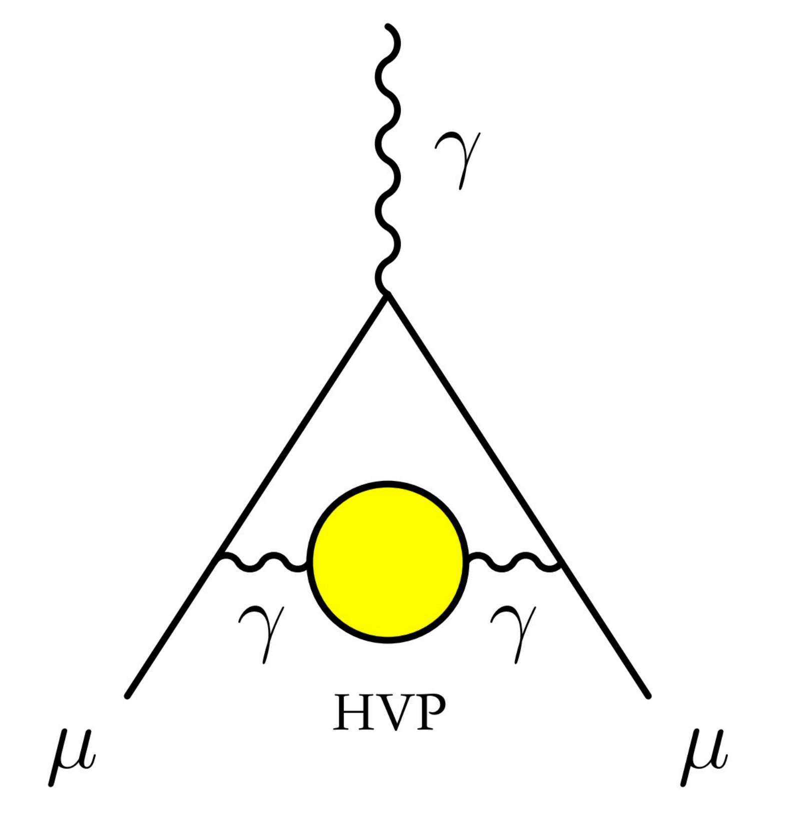




M. T. Hansen et al., Phys. Rev. D 99, 094508 (2019)

C. Alexandrou et al., Phys. Rev. Lett. 130, 241901 (2023)

HVP from solving inverse problem



$$a_\mu^{\mathrm{HVP}}[\mathrm{LO}] = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{s_{\mathrm{thr}}}^{\infty} ds \frac{K(s)}{s^2} R(s)$$

$$C_2(t) = \int \mathrm{d}\omega R(\omega) \frac{\omega^2}{12\pi^2} e^{-\omega t}$$

$$T(\omega) = \frac{2K(\omega)R(\omega)}{\omega^3}$$

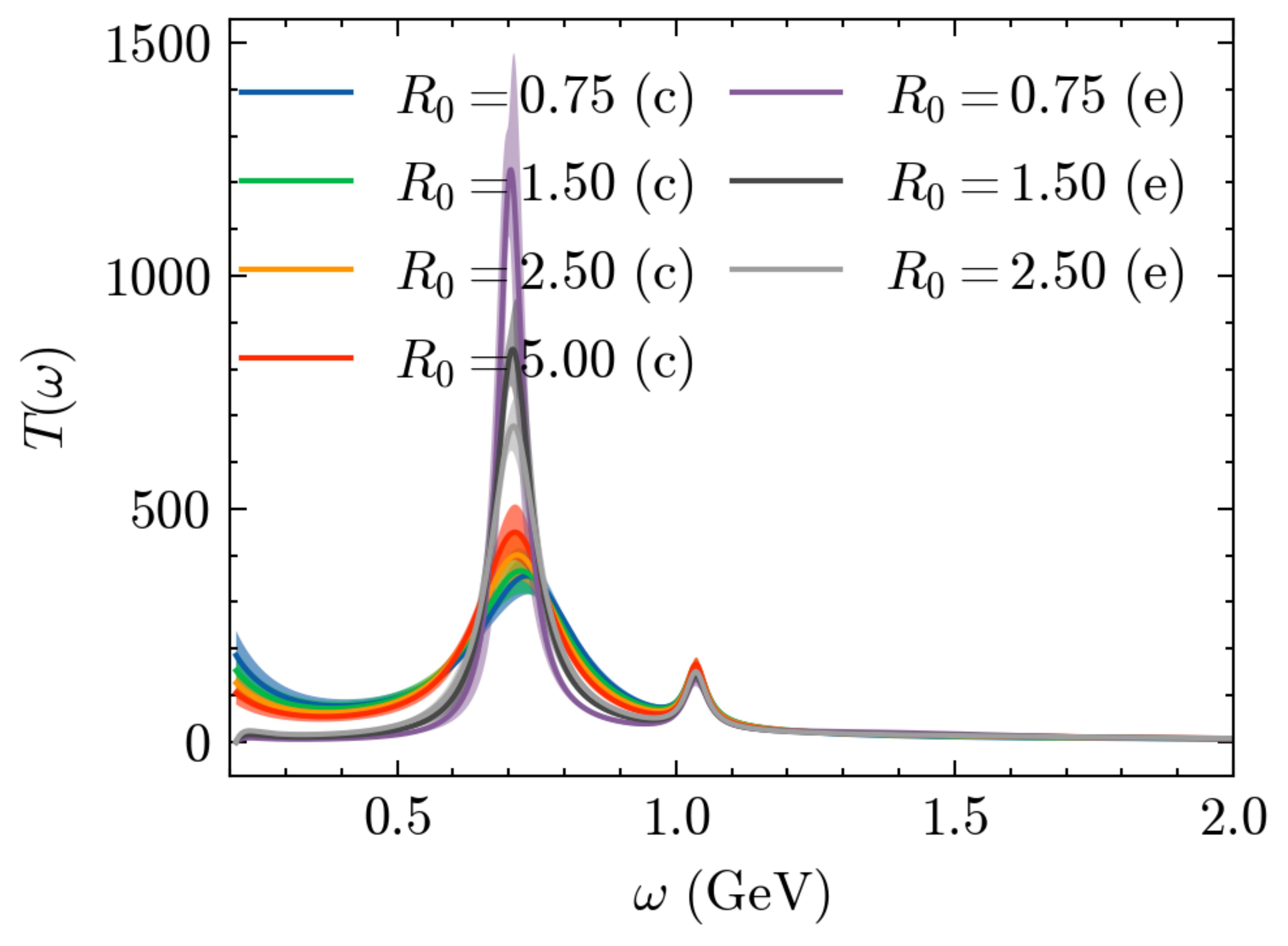
$$N(\omega, t) = \frac{\omega^5 e^{-\omega t}}{24\pi^2 K(\omega)}$$

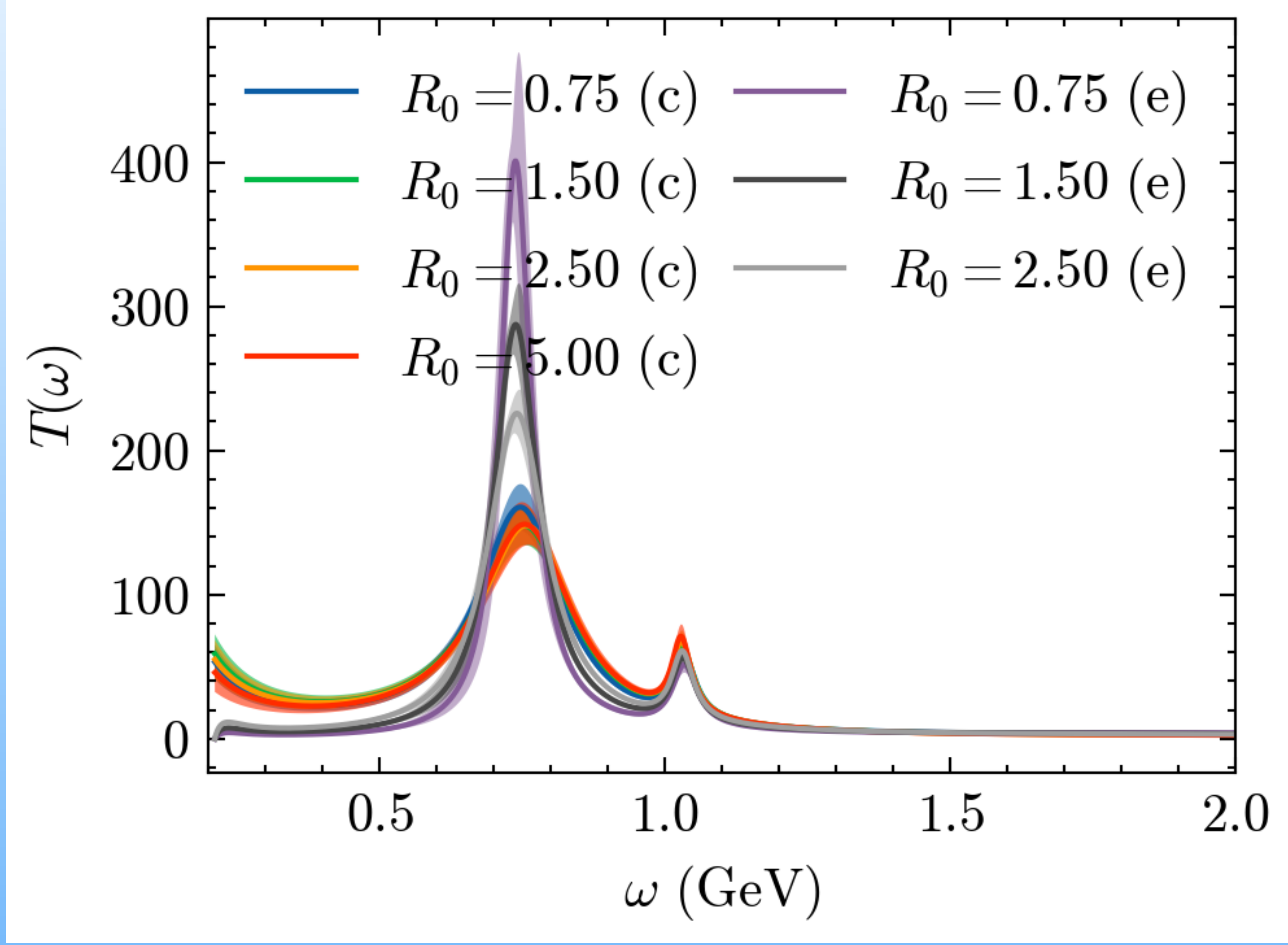
$$C_2(t) = \int \mathrm{d}\omega T(\omega) N(\omega, t)$$

$$a_\mu^{\mathrm{HVP}}[\mathrm{LO}] = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{2m_\mu}^{\infty} \mathrm{d}\omega T(\omega)$$

smearing free!

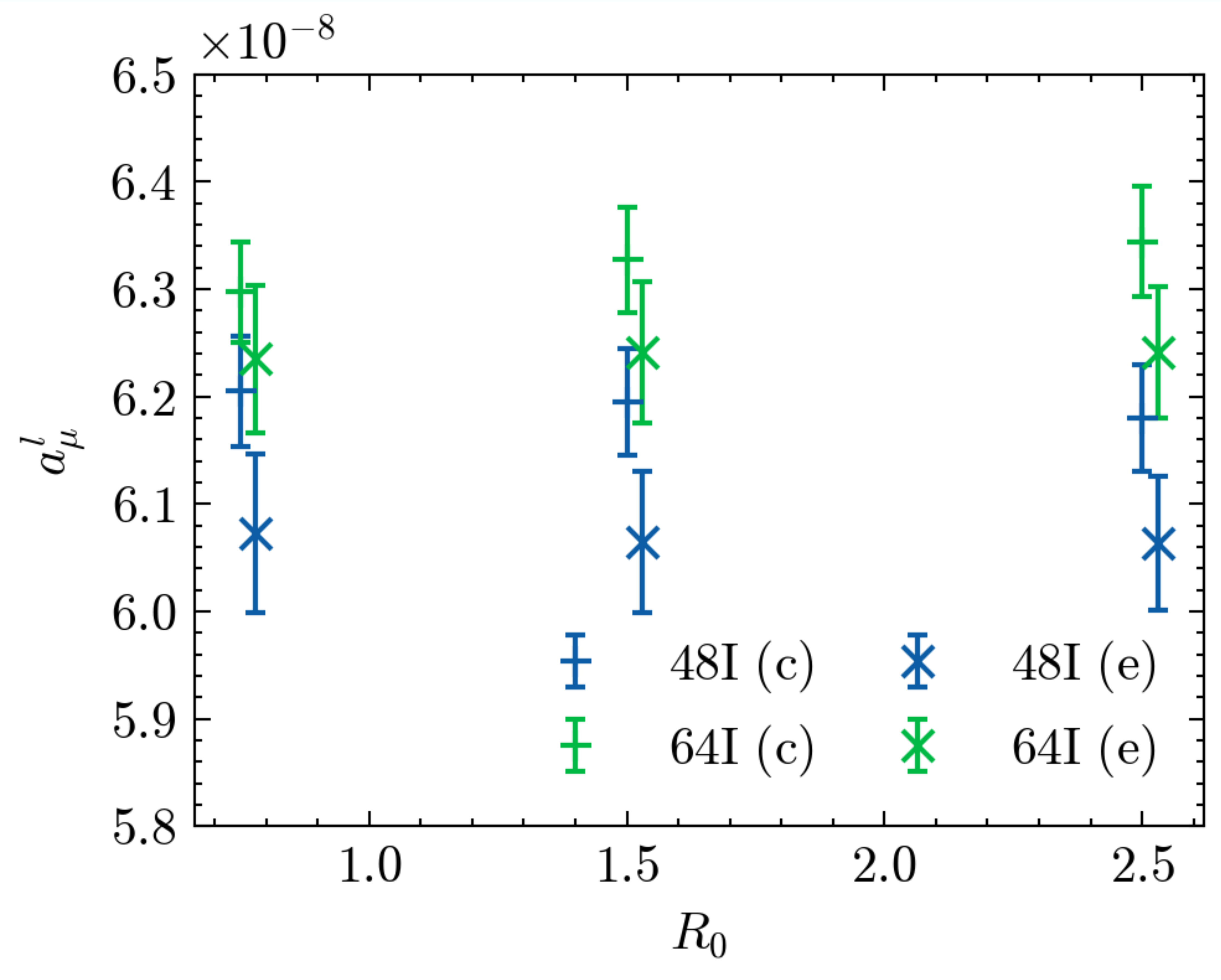
Visualization of $T(\omega)$

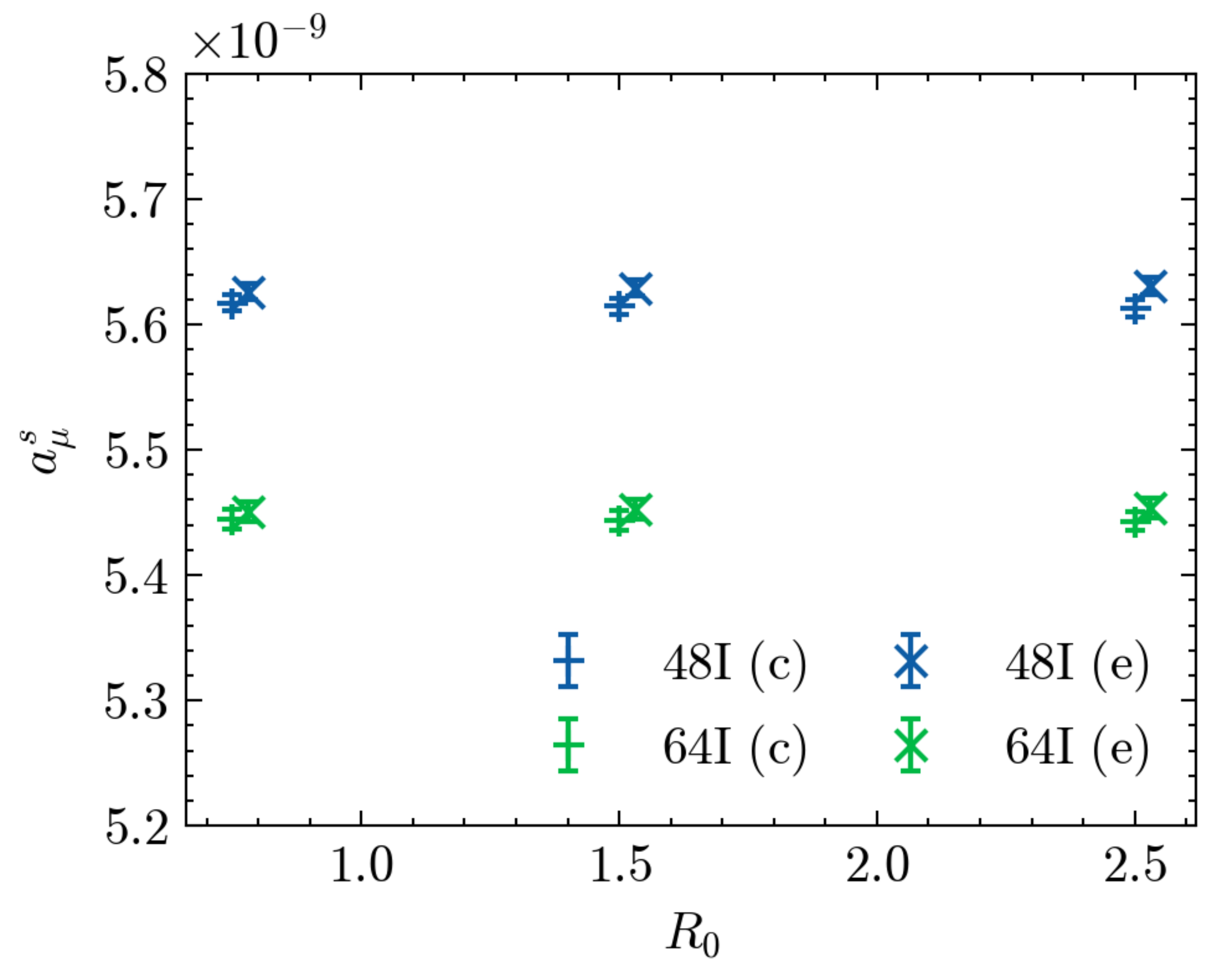




constant prior in R_0 in $\frac{2K}{\omega^3}R_0$ and a modified form $\left[1 + e^{-c\left(\omega - 2m_\mu\right)}\right]\frac{2K}{\omega^3}R_0$, 64I and 48I

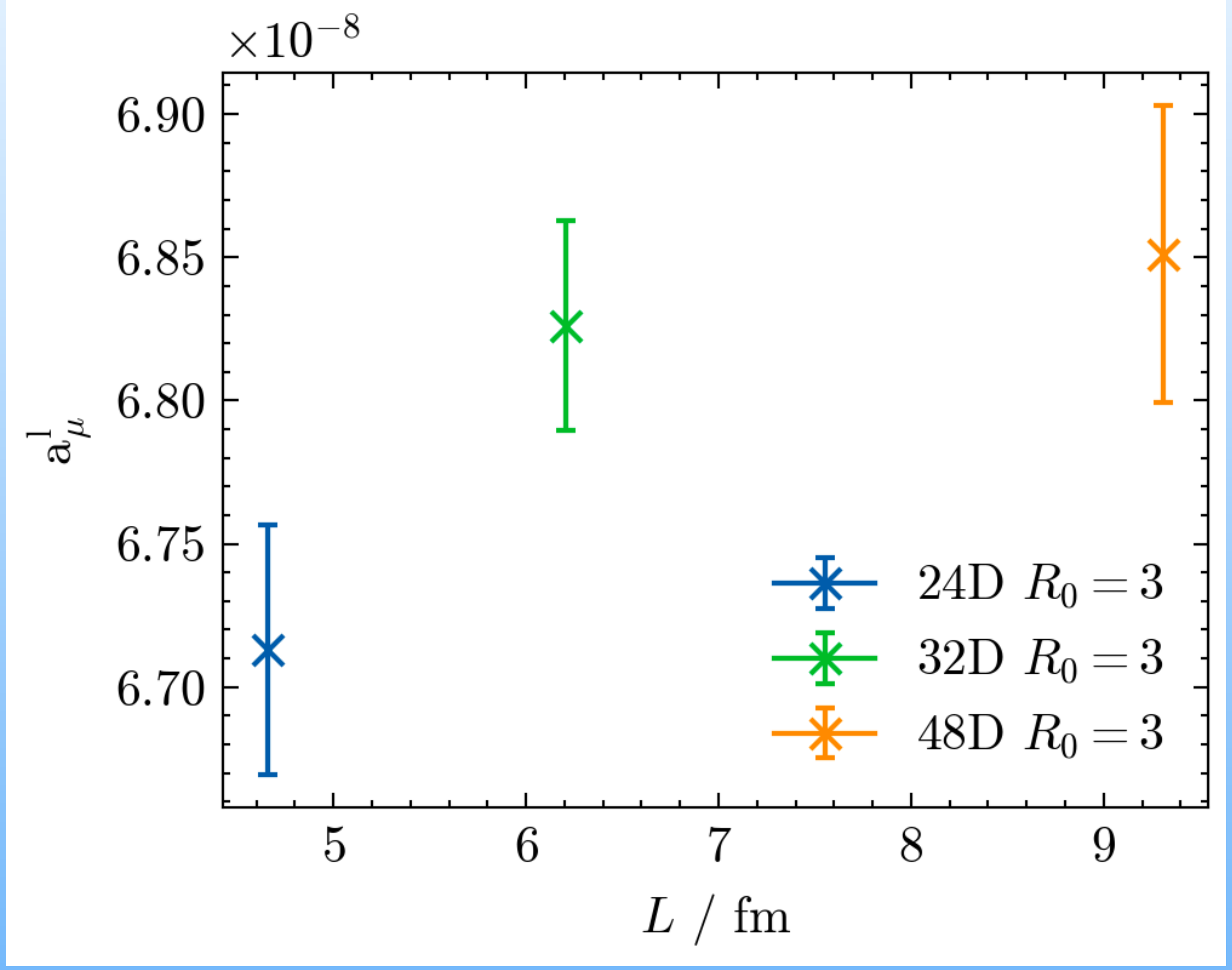
Prior dependence

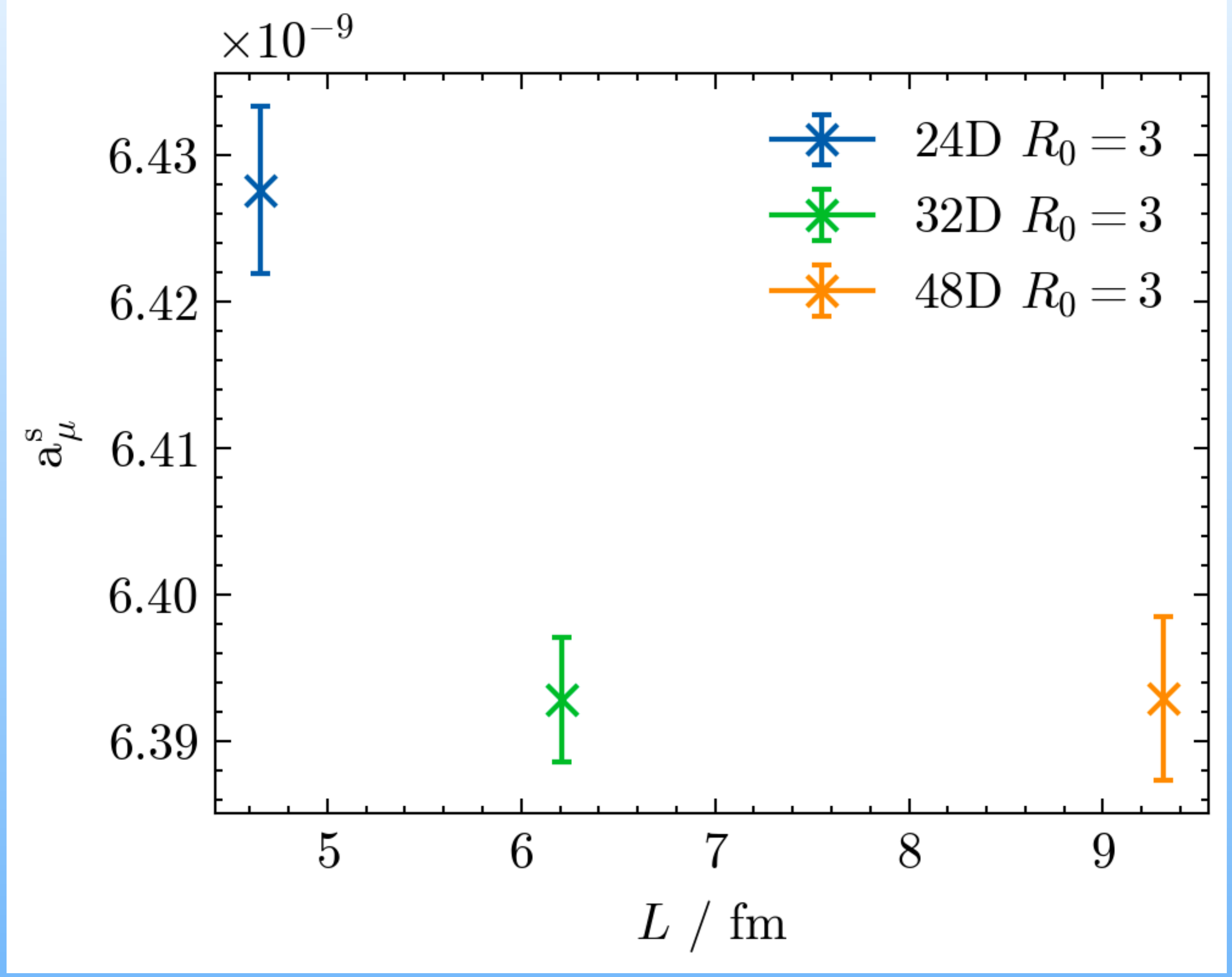




~1.5% prior uncertainty with light quarks, finite-volume? ~0.4% prior uncertainty with strange quarks

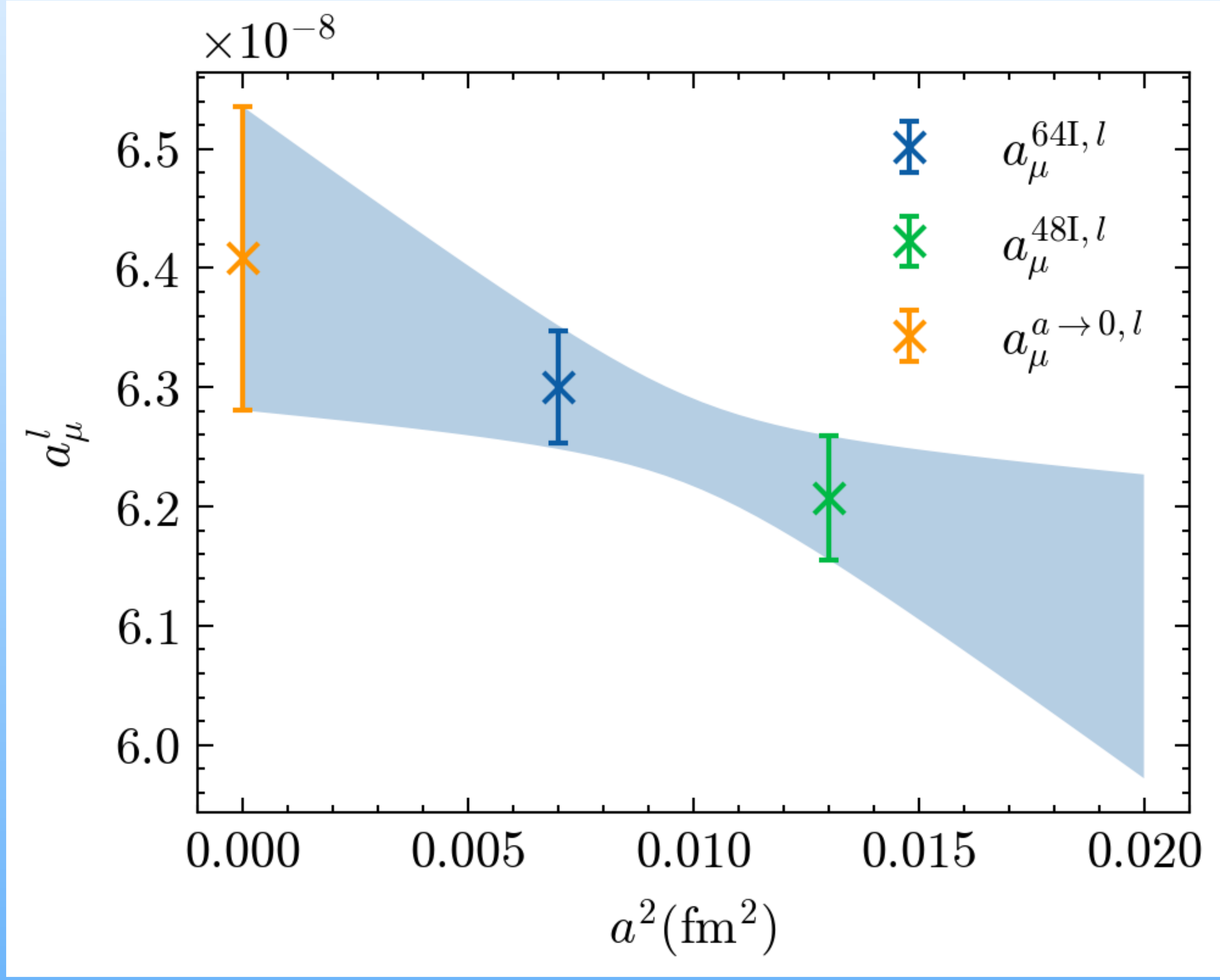
Volume dependence





~1.6% finite volume correction with light quarks and ~0.5% with strange quarks on ~4.8 fm lattices

Continuum extrapolation



$$a_\mu^{48\mathrm{I},\,l} = 620.7(5.2) \times 10^{-10}$$

$$a_\mu^{64\mathrm{I},\,l} = 630.0(5.2) \times 10^{-10}$$

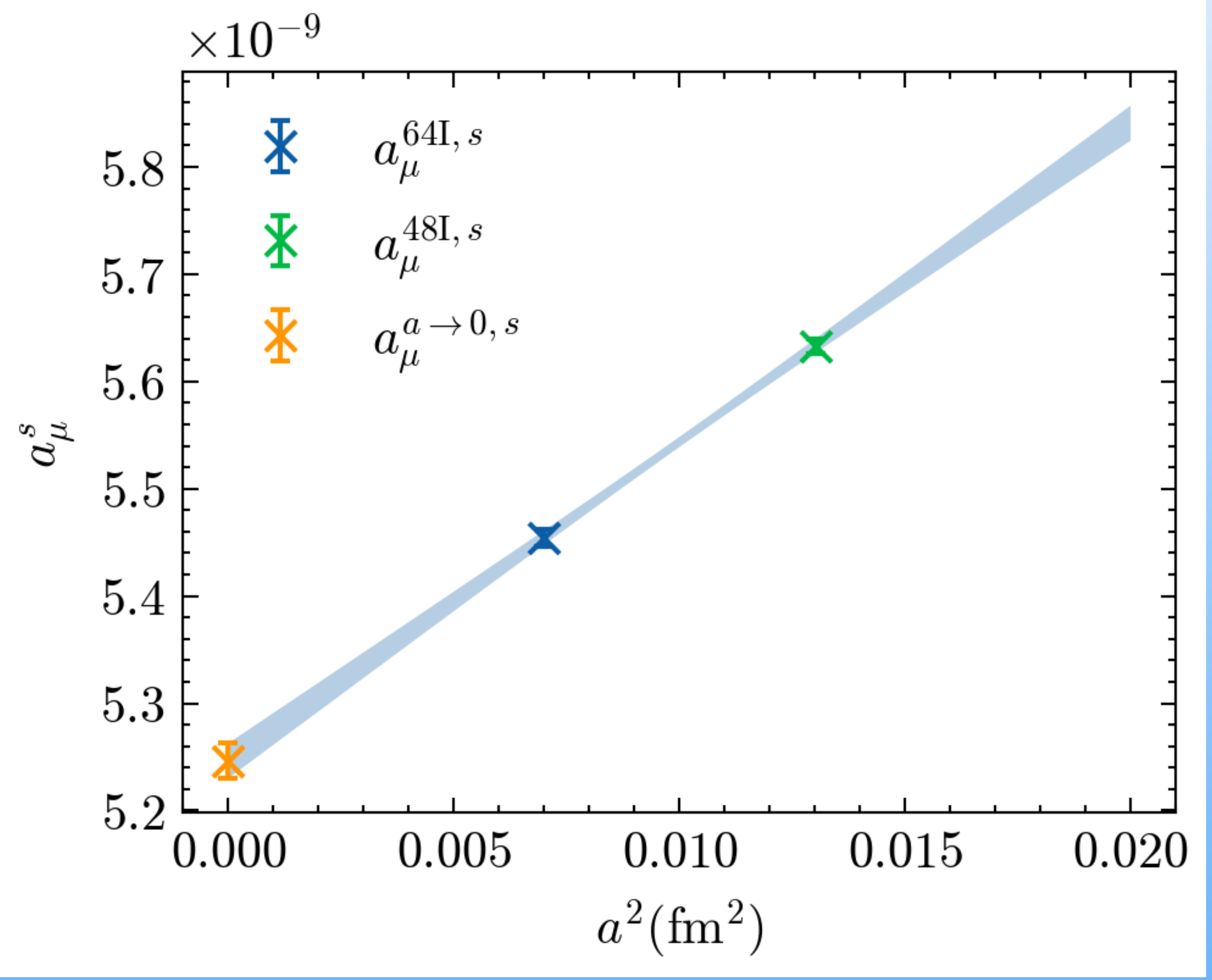
$$a_\mu^{a\to 0,\,l} = 641(13)(10)(10) \times 10^{-10}$$

same date with bounding method:

$$a_\mu^{48\mathrm{I},\,l} = 615.9(12.2) \times 10^{-10}$$

$$a_\mu^{64\mathrm{I},\,l} = 641.4(16.4) \times 10^{-10}$$

Continuum extrapolation



$$a_\mu^{48\mathrm{I},\,s} = 56.331(66) \times 10^{-10}$$

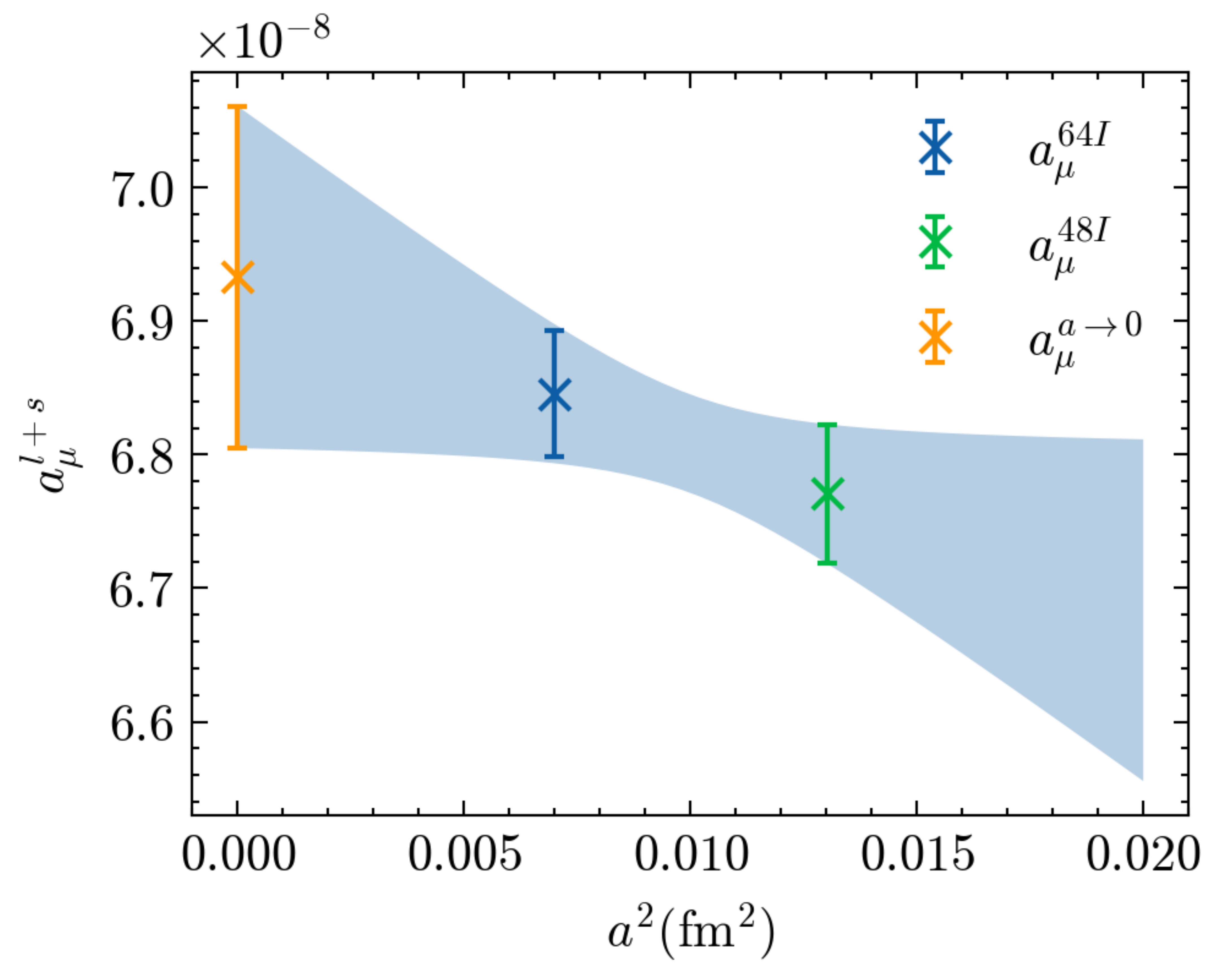
$$a_\mu^{64\mathrm{I},\,s} = 54.544(66) \times 10^{-10}$$

$$a_\mu^{a\to 0,\,s} = 52.46(16)(21)(26) \times 10^{-10}$$

same date with bounding method:

$$a_\mu^{48\mathrm{I},\,s} = 57.7(1) \times 10^{-10}$$

$$a_\mu^{64\mathrm{I},\,s} = 55.3(1) \times 10^{-10}$$



$$a_\mu^{48\mathrm{I},\ l+s} = 677(5) \times 10^{-10}$$

$$a_\mu^{64\mathrm{I},\ l+s} = 685(5) \times 10^{-10}$$

$$a_\mu^{a\to 0,\ l+s} = 693(13)(10)(11) \times 10^{-10}$$

Window method:

$$a_\mu^{W=[0.4-1.0]\ \mathrm{fm}} = 207(2) \times 10^{-10} + 26.8(1) \times 10^{-10}$$

Wang G et al. PRD, 2023, 107: 034513

BMW21:

$$a_\mu^{\mathrm{con},\ l+s} = 687.1(2.1)_{\mathrm{tot}} \times 10^{-10}$$

Sz. Borsanyi et al. Nature 593.7857 (Apr. 2021)

Summary & Take-home

- ✦ We employ the BR method to tackle inverse problems on the lattice with sophisticated error analysis
- ✦ The R-ratio is reconstructed. With proper smearing, the lattice results match the (smeared) experimental data well
- ✦ A promising alternative way for calculating a_μ^{HVP} is provided, including discussion on systematic uncertainties
- ✦ New way to study the finite-volume correction and real-time dynamics

Thank you for your attention

Bayesian Reconstruction

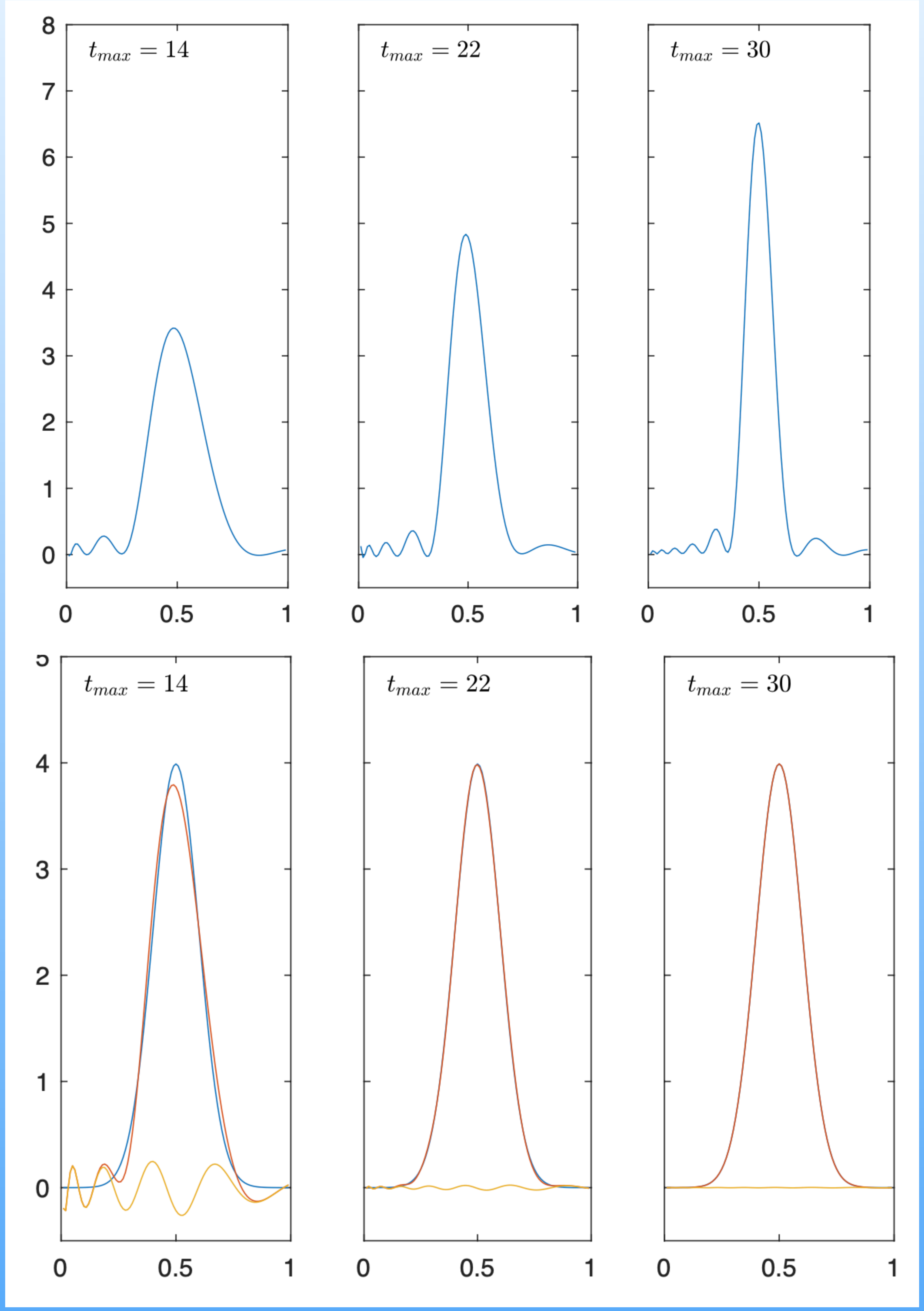
$$P[\rho \,|\, D, \alpha, m] \propto e^{Q(\rho)}$$

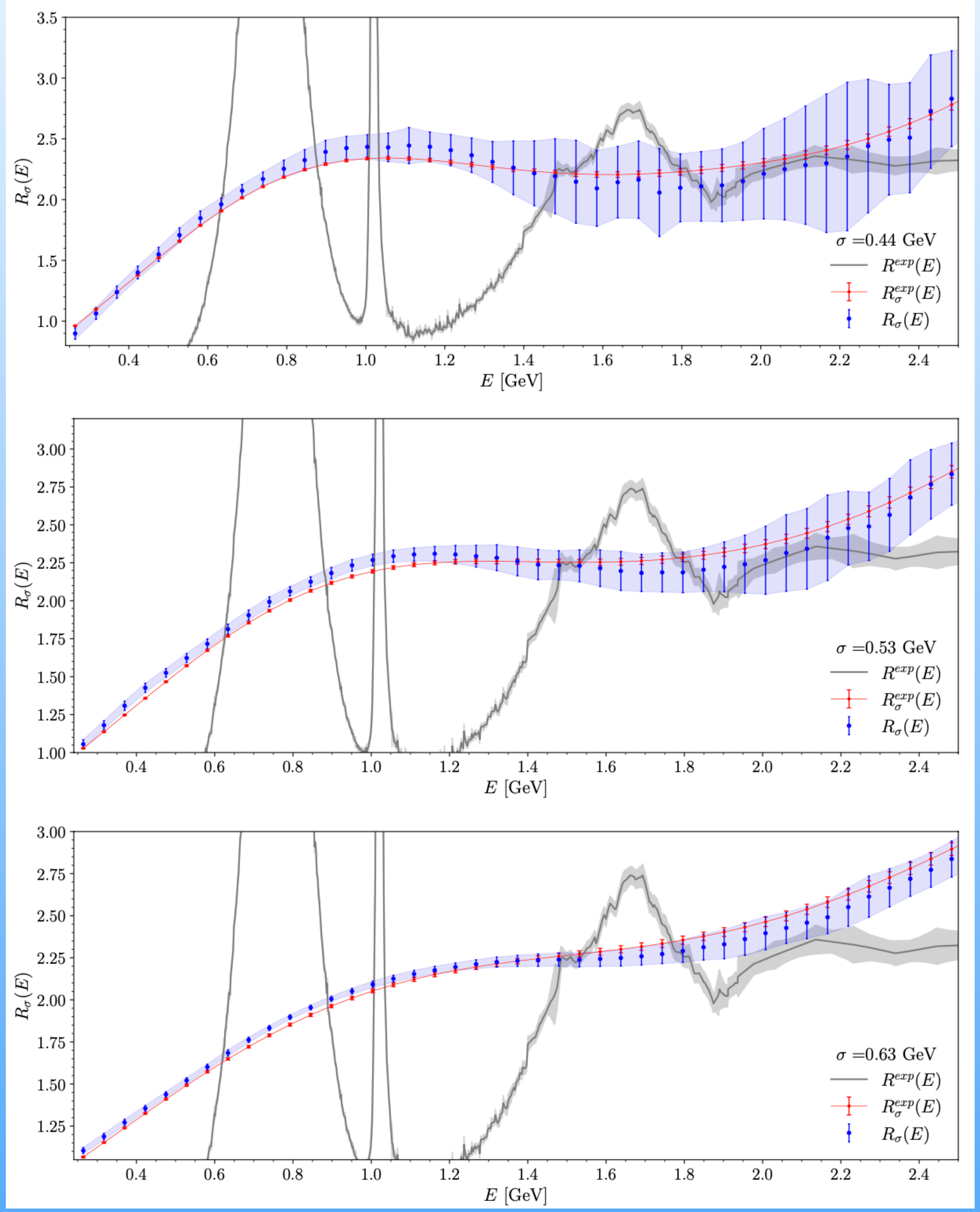
$$Q = \alpha S - L - \gamma(L - N_\tau)^2$$

$$S = \sum_{\omega} \left[1 - \frac{\rho(\omega)}{m(\omega)} + \log\left(\frac{\rho(\omega)}{m(\omega)}\right)\right]\Delta\omega$$

$$P[\rho \,|\, D, m] = \frac{P[D \,|\, \rho, I]}{P[D \,|\, m]} \int d\alpha P[\alpha \,|\, D, m]$$

- ✦ Hyper parameter α is integrated over
- ✦ Maximum search is in the entire parameter space($O(10^3)$)
- ✦ High precision architecture (e.g.,512-bit floating point number).





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