Lattice determination of the HO HVP contributions to the muon g-2

Arnau Beltran Martínez for the Mainz lattice group

Johannes Gutenberg-Universität Mainz

10^{th} of September 2025

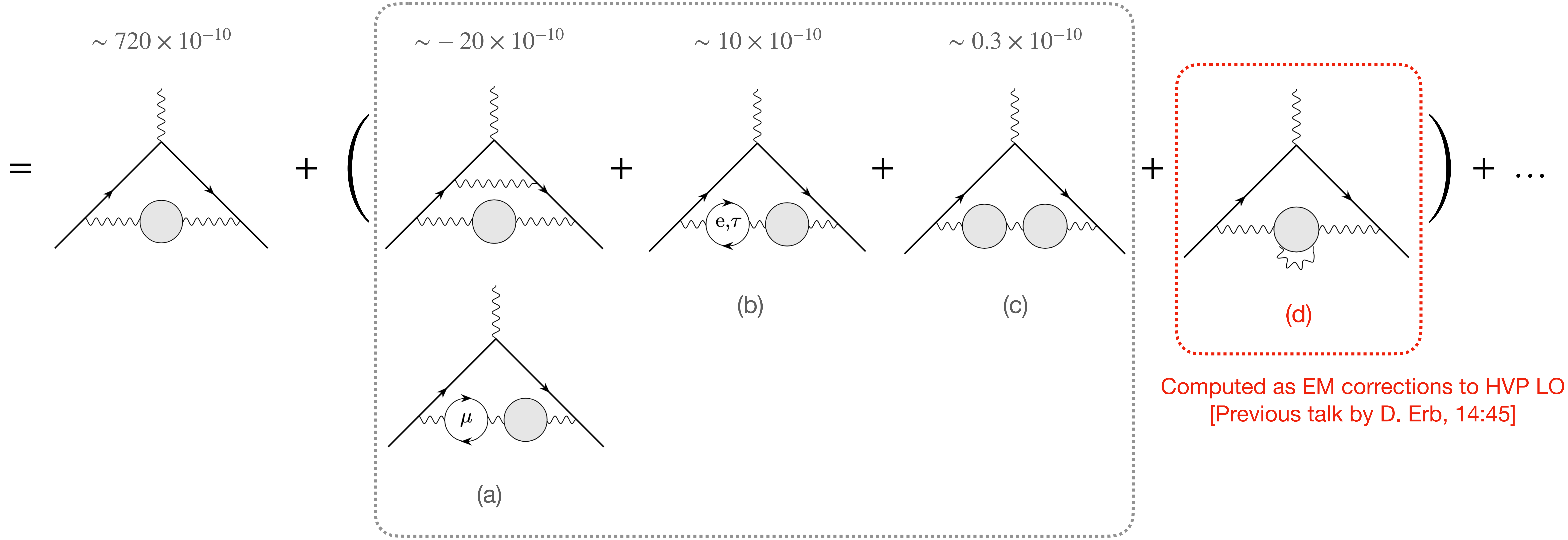
8th Plenary workshop of the Muon g-2 Theory Initiative 2025

- **Introduction**
- NLO TMR kernels
- Strategy
- Preliminary results
- Outlook and conclusions

HVP as a power expansion

The full HVP can be written as an expansion on $\alpha_{\rm em}$;

$$a_\mu^{\rm hvp} = a_\mu^{\rm hvp}[{\rm LO}] + a_\mu^{\rm hvp}[{\rm NLO}] + \ldots = \overbrace{a_\mu^{\rm hvp}[{\rm LO}]}^{\mathcal{O}(\alpha_{\rm em})^2} + \overbrace{\left(a_\mu^{\rm hvp}[{\rm NLOa}] + a_\mu^{\rm hvp}[{\rm NLOb}] + a_\mu^{\rm hvp}[{\rm NLOc}] + a_\mu^{\rm hvp}[{\rm NLOd}]\right)}^{\mathcal{O}(\alpha_{\rm em})^3} + \underbrace{\ldots}_{\mathcal{O}(\alpha_{\rm em})^{\geq 4}}$$



Computed as EM corrections to HVP LO
[Previous talk by D. Erb, 14:45]

Master formula

"Generalised" TMR master formula for HVP observables:

$$a_\mu^{\rm hvp}[(i)] = \left(\frac{\alpha_{\rm em}}{\pi}\right)^{2+\#_{N_i}} \int_0^\infty dt_1 \dots dt_m \, \tilde{f}^{(i)}(\hat{t}_1, \dots, \hat{t}_m) \, G(t_1) \times \dots \times G(t_m)$$

$i = \text{LO}, \text{NLOa}, \text{NLOb}, \text{NLOc} \dots$

m = # of QCD blobs

$$\tilde{f}^{(i)}(\hat{t}_1, \dots, \hat{t}_m) = \int_0^\infty \frac{d\hat{\omega}^2}{\hat{\omega}^2} f^{(i)}(\hat{\omega}^2) \prod_{j=1,\dots,m} \frac{4\pi^2}{m_\mu^2 \hat{\omega}^2} \left[\hat{\omega}^2 \hat{t}_j^2 - 4\sin^2 \frac{\hat{\omega}\hat{t}_j}{2}\right]$$

$$G(t) = -\frac{1}{3} \sum_{\vec{x}\in\Lambda} \sum_{k=0}^{3} \langle j_{\rm em}^k(\vec{x}, t) j_{\rm em}^k(0) \rangle$$

[S. Kuberski and A. Conigli, 11:25]

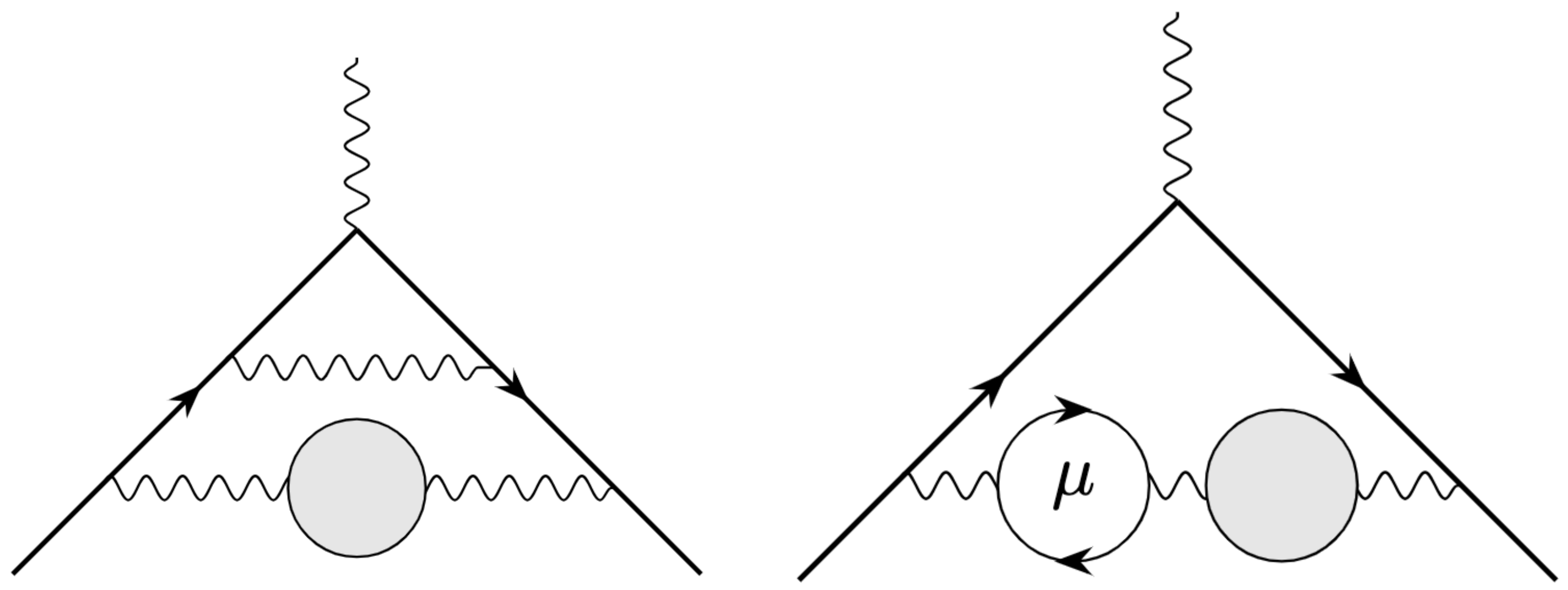
- Best case scenario: there's a close form solution for these integrals, e.g. LO and NLOc
- Sometimes an analytic solution cannot be found, e.g. NLOa and NLOb : "We need to get creative"

- Introduction
- **NLO TMR kernels**
- Strategy
- Preliminary results
- Outlook and conclusions

NLOa

$$\tilde{f}^{(\mathrm{NLOa})}(t) = \frac{8\pi^2}{m_\mu^2}\int_0^\infty \frac{d\hat{\omega}^2}{\hat{\omega}^2}\hat{f}^{(\mathrm{NLOa})}(\hat{\omega}^2)\left[\hat{\omega}^2\hat{t}^2 - 4\sin^2\frac{\hat{\omega}\hat{t}}{2}\right]$$

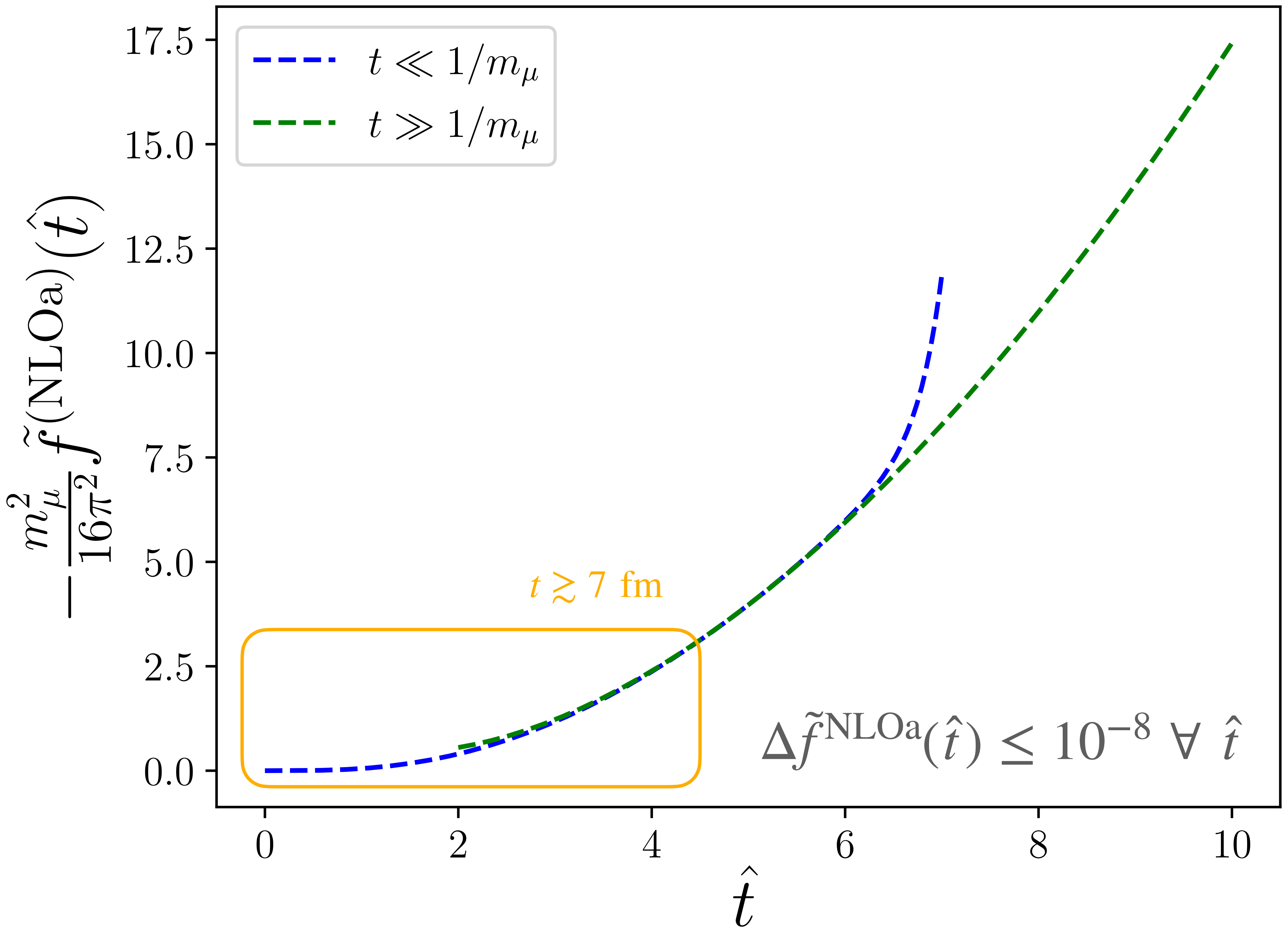
E.Balzani, S.Laporta and M.Passera. 2024 Physics Letters B



$t \ll 1/m_\mu$

$$\sim \sum_{\substack{n \geq 4 \\ n \in \text{even}}} \frac{\hat{t}^n}{n!}\left(a_n + b_n\pi^2 + c_n(\gamma_E + \ln\hat{t}) + d_n(\gamma_E + \ln\hat{t})^2\right)$$

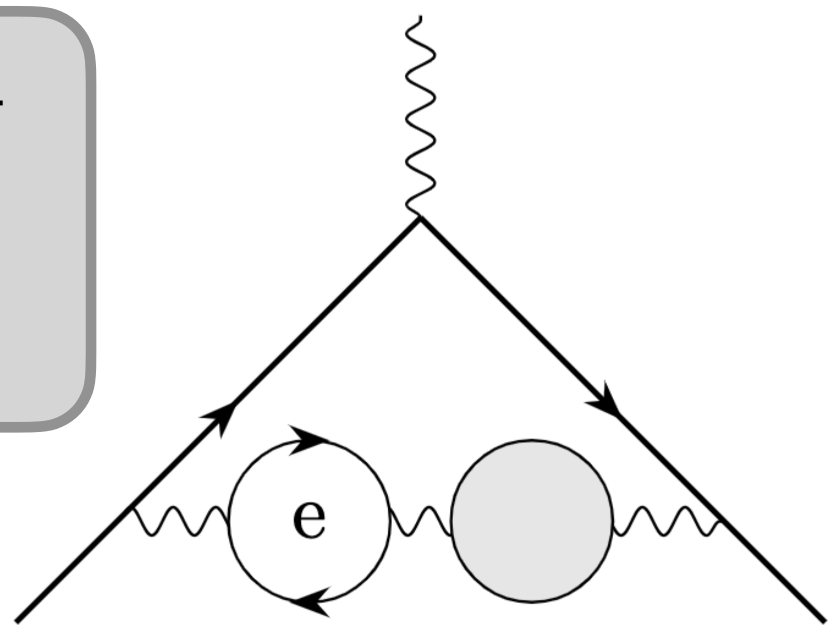
n	$\mathbf{a_n}$	$\mathbf{b_n}$	$\mathbf{c_n}$	$\mathbf{d_n}$
4	$\frac{317}{216}$	$-\frac{1}{3}$	$\frac{23}{18}$	0
6	$\frac{843829}{259200}$	$-\frac{371}{432}$	$\frac{877}{1080}$	$\frac{19}{36}$
8	$\frac{412181237}{5292000}$	$-\frac{233}{48}$	$-\frac{824603}{25200}$	$\frac{141}{20}$
28	$\frac{25114629392949805515668354977 3}{5545847763281826000000}$	$-\frac{678234361}{65}$	$-\frac{3787066553671821473}{20715766500}$	$\frac{1495034796}{65}$
30	$\frac{100792117463017684643555224178269168501}{5468055457076246304990720000 0}$	$-\frac{2551294690547}{60480}$	$-\frac{30599625762869165887553 3}{419236121304000}$	$\frac{64743309493}{720}$



NLOb

NLO TMR kernels

$$\tilde{f}^{(\mathrm{NLOb})}(t) = \frac{16\pi^2}{m_\mu^2}\int_0^\infty \frac{d\hat{\omega}^2}{\hat{\omega}^2}\hat{f}^{(\mathrm{LO})}(\hat{\omega}^2)F_e(\hat{\omega}^2, M)\left[\hat{\omega}^2\hat{t}^2 - 4\sin^2\frac{\hat{\omega}\hat{t}}{2}\right];\ F^l(\omega^2; m_l^2) = -\frac{8}{9} + \frac{\beta^2}{3} - \left(\frac{1}{2} - \frac{\beta^2}{6}\right)\beta\ln\frac{\beta-1}{\beta+2};\ \beta = \sqrt{1 + 4\frac{m_l^2}{\omega^2}}$$

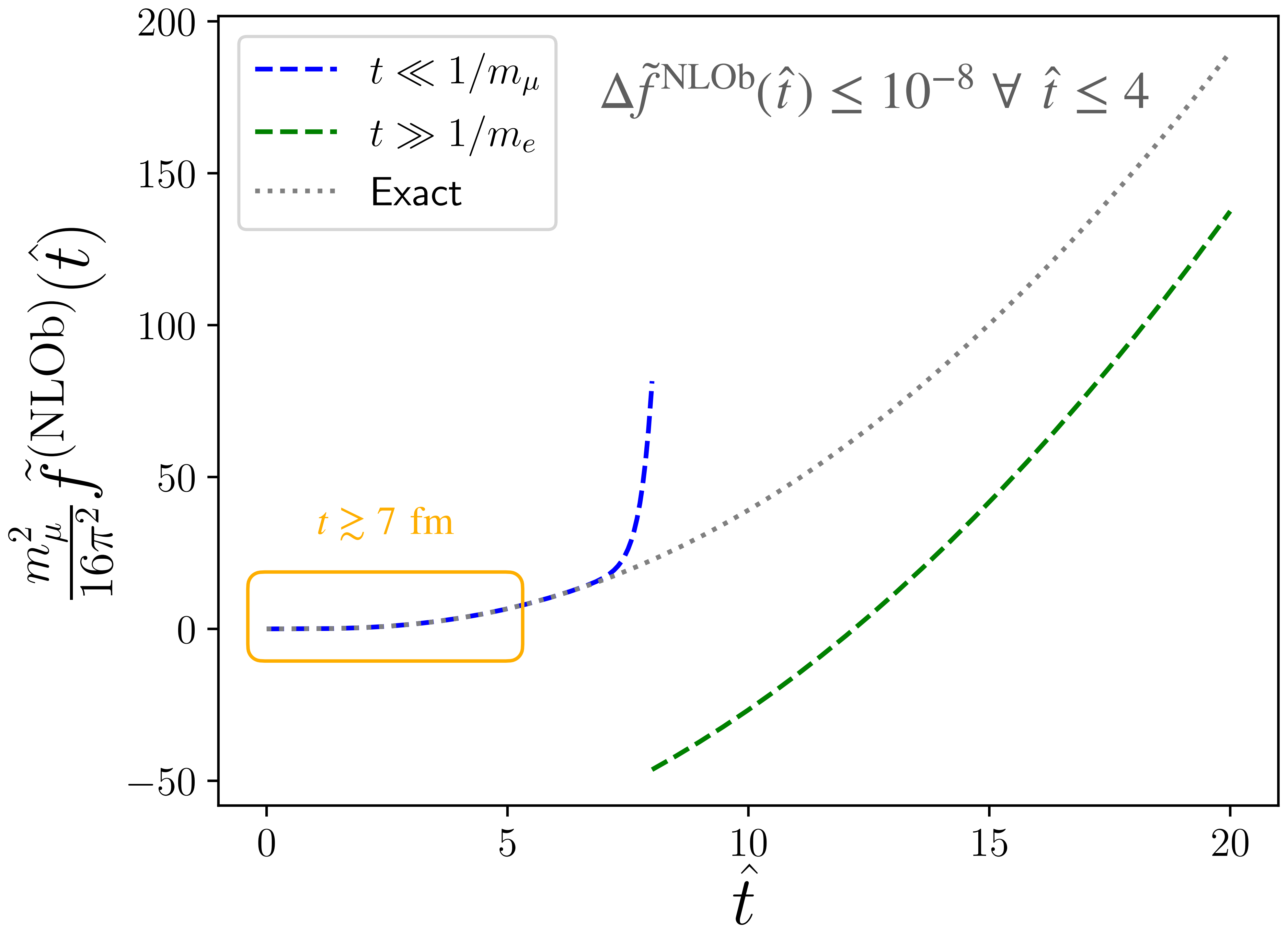


$t \ll 1/m_\mu$

$$\sim \sum_{\substack{n\geq 4,\ m \geq 0 \\ n \in \text{even}}} \frac{\hat{t}^n}{n!}M^m\left(a_{nm} + b_{nm}\pi^2 + c_{nm}\left(\gamma_E + \ln\hat{t}\right) + d_{nm}\left(\gamma_E + \ln\hat{t}\right)^2\right) \quad \text{where } M = m_e/m_\mu$$

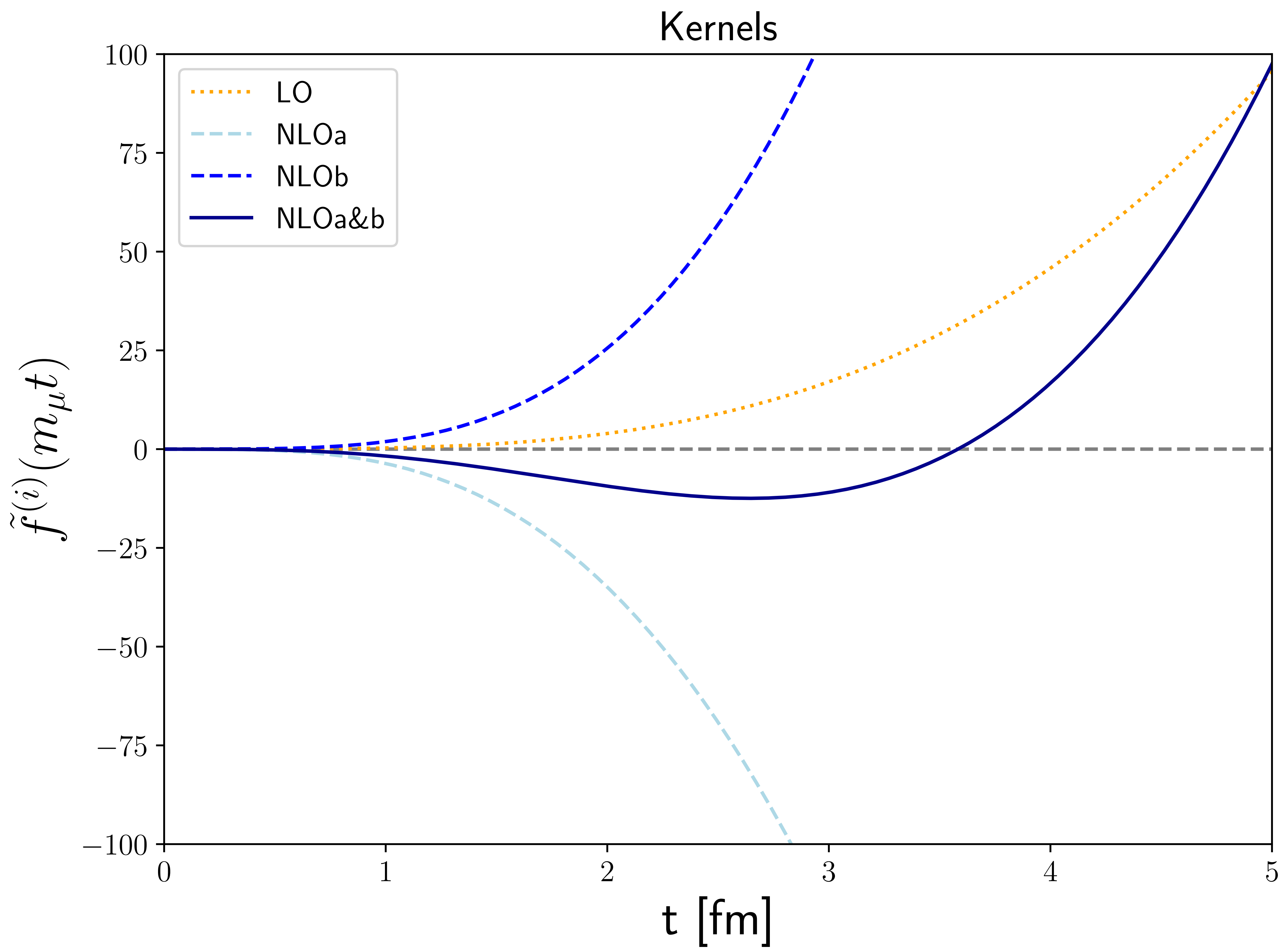
n	$m=0$	$m=2$	$m=4$	$m=6$
4	$-\frac{2}{9}\ln M - \frac{1}{18}$	1	$4 + 4\ln^2 M$	$-\frac{46}{27} + \frac{28\ln M}{9} - \frac{1}{3}8\ln^2 M$
6	$\frac{169\ln M}{90} - \frac{36931}{10800}$	$-\frac{2}{3}$	$-2 - 2\ln M$	$\frac{80}{27} + \frac{56\ln M}{9} + \frac{16\ln^2 M}{3}$
8	$\frac{1604\ln M}{105} - \frac{210047}{8820}$	$\frac{704}{105}$	$-\frac{10}{9} + \frac{2}{3}\ln M$	$-\frac{-38}{9} - \frac{8\ln M}{3}$

Some of the computed values for a_{mn}



NLOa&b

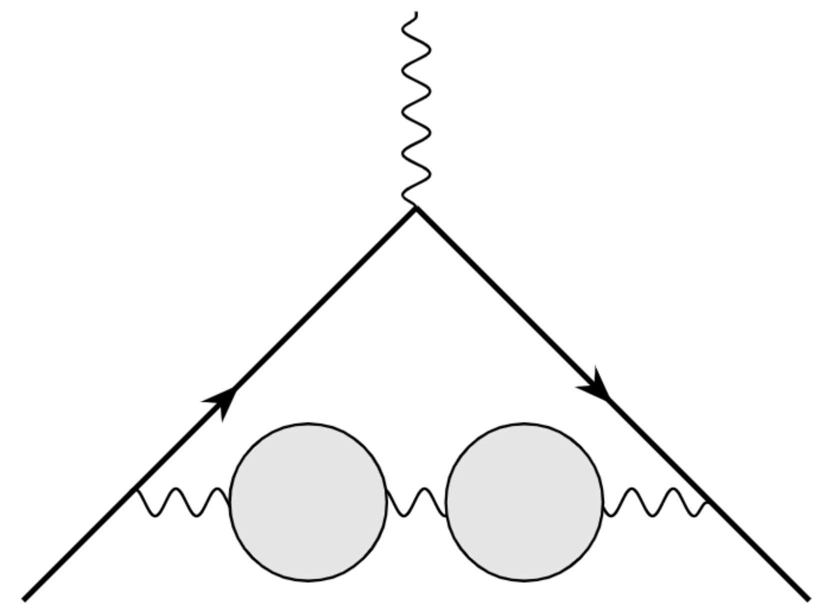
$$\tilde{f}^{(\mathrm{NLOa\&b})}(t) = \tilde{f}^{(\mathrm{NLOa})}(t) + \tilde{f}^{(\mathrm{NLOb})}(t)$$



NLOc

NLO TMR kernels

$$\tilde{f}^{(\mathrm{NLOc})}(t,\tau) = \frac{8\pi^2}{m_\mu^2}\int_0^\infty \frac{d\hat{\omega}^2}{\hat{\omega}^2}\hat{f}^{(\mathrm{LO})}(\hat{\omega}^2)\left[\hat{\omega}^2\hat{t}^2 - 4\sin^2\frac{\hat{\omega}\hat{t}}{2}\right]\left[\hat{\omega}^2\hat{\tau}^2 - 4\sin^2\frac{\hat{\omega}\hat{\tau}}{2}\right]$$



$$\frac{m_\mu^4}{32\pi^4}\tilde{f}^{(\mathrm{NLOc})}\left(\hat{t},\hat{\tau}\right) = \frac{\hat{\tau}^2\hat{t}^2}{4} + \frac{\hat{t}^2}{\hat{\tau}^2} + \frac{\hat{\tau}^2}{\hat{t}^2} - \frac{1}{2}\left(\hat{t}^2+\hat{\tau}^2\right) + \frac{1}{6} - 2(1+\gamma_E) + 2\hat{t}^2(\ln\hat{\tau}+\gamma_E) + 2\hat{\tau}^2(\ln\hat{t}+\gamma_E) + 2\left(\hat{t}^2-1\right)\ln\hat{t} + 2\left(\hat{\tau}^2-1\right)\ln\hat{\tau}$$

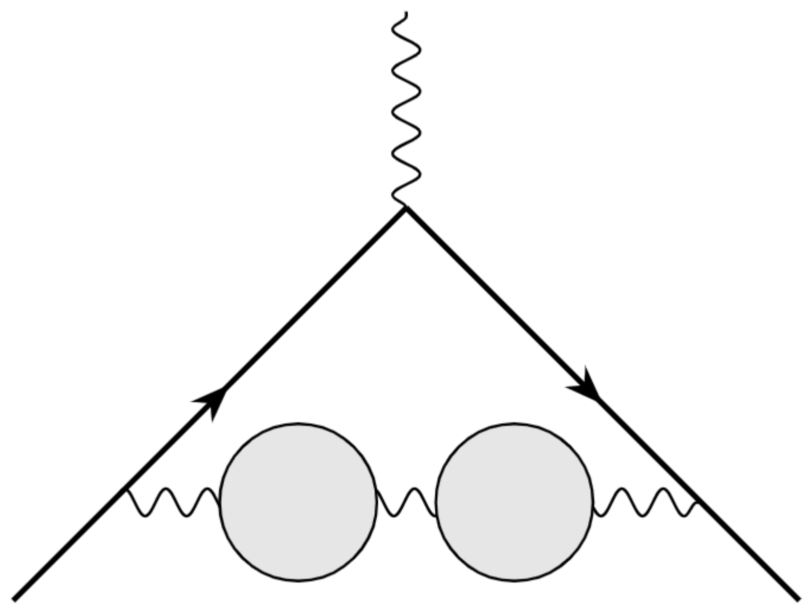
$$+\left[1-(\hat{t}+\hat{\tau})^2\right]\ln(\hat{t}+\hat{\tau}) + \left[1-(\hat{t}-\hat{\tau})^2\right]\ln|\hat{t}-\hat{\tau}| + \left(\frac{\hat{t}^2}{6}-2\right)K_0(2\hat{t}) + \left(\frac{\hat{\tau}^2}{6}-2\right)K_0(2\hat{\tau}) + \left(1-\frac{1}{12}(\hat{t}+\hat{\tau})^2\right)K_0(2(\hat{t}+\hat{\tau})) + \left(1-\frac{1}{12}(\hat{t}-\hat{\tau})^2\right)K_0(2|\hat{t}-\hat{\tau}|)$$

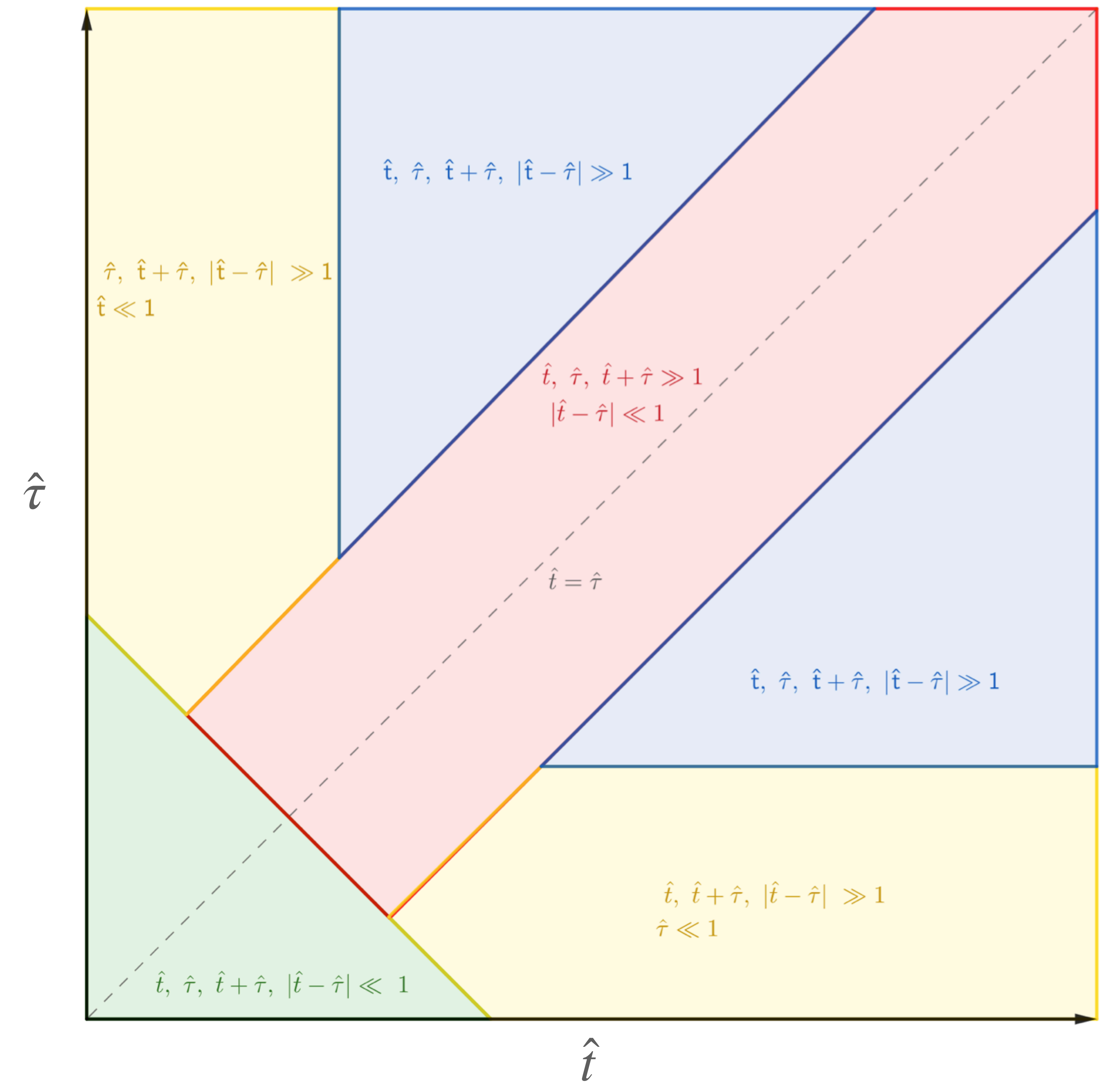
$$-\left(\frac{2\hat{t}^2}{\hat{\tau}}+\frac{\hat{\tau}}{12}\right)K_1(2\hat{\tau}) - \left(\frac{2\hat{\tau}^2}{\hat{t}}+\frac{\hat{t}}{12}\right)K_1(2\hat{t}) + \frac{1}{24}|\hat{t}-\hat{\tau}|K_1(2|\hat{t}-\hat{\tau}|) + \frac{1}{24}(\hat{t}+\hat{\tau})K_1(2(\hat{t}+\hat{\tau})) + \left(\frac{\hat{t}^2}{12}+\frac{\hat{\tau}^2}{4}-\frac{15}{16}\right)G^{2,1}_{1,3}\left(\hat{t}^2\,\middle|\,\begin{matrix}\frac{3}{2}\\0,1,\frac{1}{2}\end{matrix}\right)$$

$$+\left(\frac{\hat{\tau}^2}{12}+\frac{\hat{t}^2}{4}-\frac{15}{16}\right)G^{2,1}_{1,3}\left(\hat{\tau}^2\,\middle|\,\begin{matrix}\frac{3}{2}\\0,1,\frac{1}{2}\end{matrix}\right) + \left(\frac{15}{32}-\frac{1}{24}(\hat{t}+\hat{\tau})^2\right)G^{2,1}_{1,3}\left((\hat{t}+\hat{\tau})^2\,\middle|\,\begin{matrix}\frac{3}{2}\\0,1,\frac{1}{2}\end{matrix}\right) + \left(\frac{15}{32}-\frac{1}{24}(\hat{t}-\hat{\tau})^2\right)G^{2,1}_{1,3}\left((\hat{t}-\hat{\tau})^2\,\middle|\,\begin{matrix}\frac{3}{2}\\0,1,\frac{1}{2}\end{matrix}\right)$$

NLO TMR kernels

$$\tilde{f}^{(\mathrm{NLOc})}(t,\tau) = \frac{8\pi^2}{m_\mu^2}\int_0^\infty \frac{d\hat{\omega}^2}{\hat{\omega}^2}\hat{f}^{(\mathrm{LO})}(\hat{\omega}^2)\left[\hat{\omega}^2\hat{t}^2 - 4\sin^2\frac{\hat{\omega}\hat{t}}{2}\right]\left[\hat{\omega}^2\hat{\tau}^2 - 4\sin^2\frac{\hat{\omega}\hat{\tau}}{2}\right]$$





$$\Delta\tilde{f}^{\mathrm{NLOc}}(\hat{t},\hat{\tau}) \leq 10^{-8}\ \forall\ \hat{t},\ \hat{\tau}$$

- Introduction
- NLO TMR kernels
- **Strategy**
- Preliminary results
- Outlook and conclusions

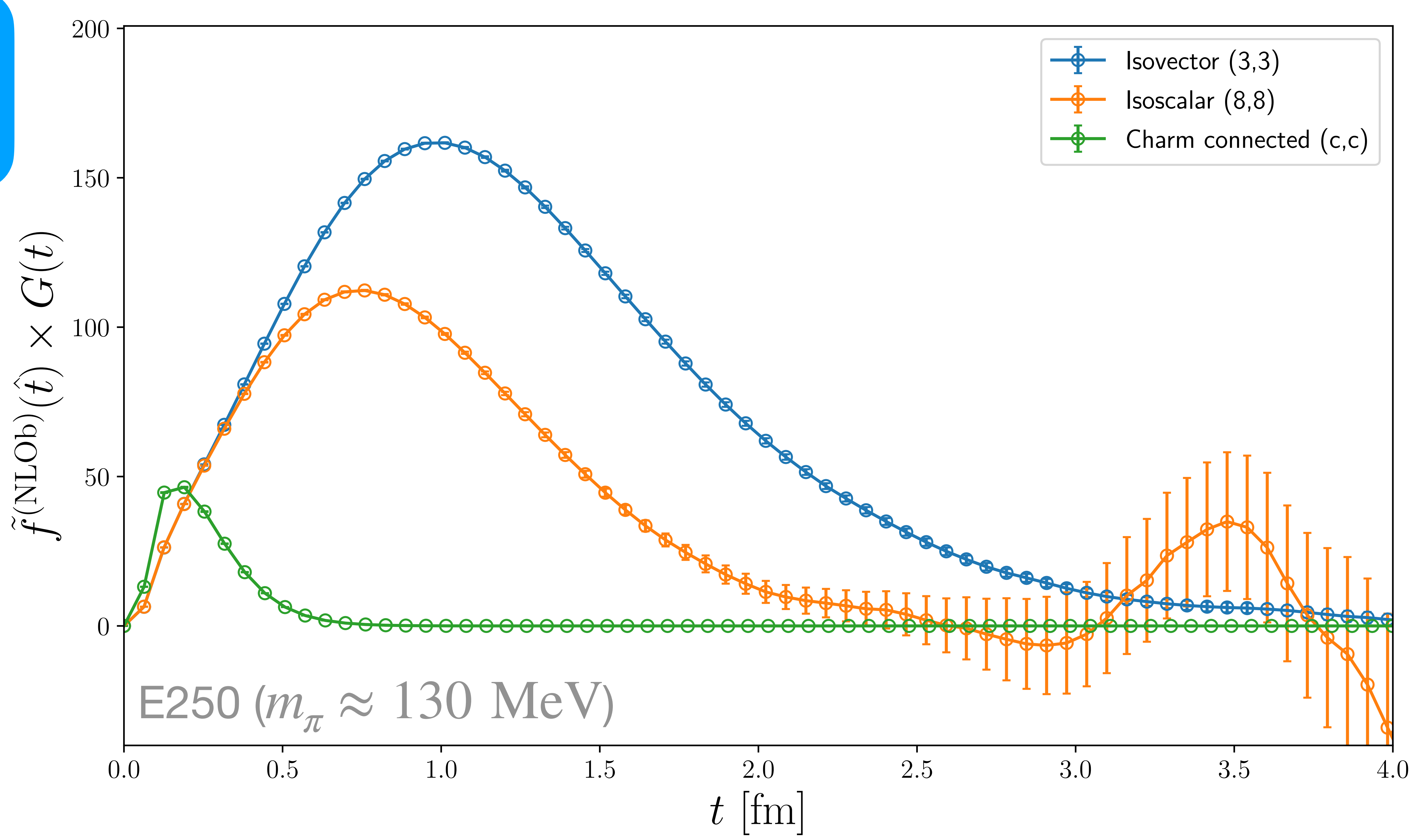
Isospin splitting — Strategy

Gell-Mann matrices allow for an isospin basis decomposition:

$$G(t) = G^{3,3}(t) + 1/3G^{8,8}(t) + 4/9G^{c,c}(t) + \ldots$$

Observable unavoidably gains intrinsic dependencies:

$$G(t) = G(t; a, m_\pi, m_K, V) \to a_\mu = a_\mu(a, m_\pi, m_K, V)$$



Isoscalar (8,8) and charm (C,C) :

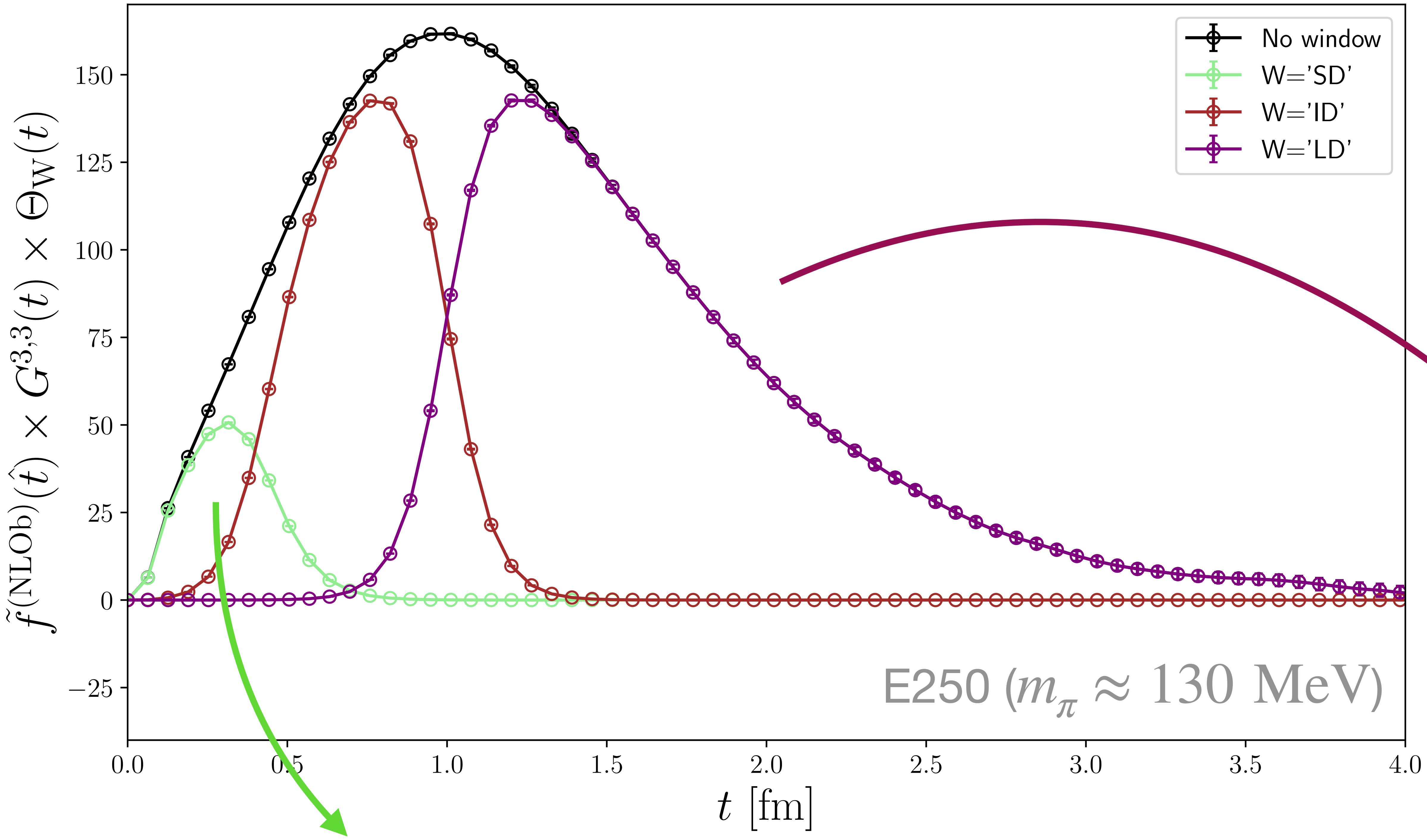
$$a_\mu(a, m_\pi, m_K, V) \xrightarrow{\text{HP}} a_\mu(a, m_\pi, m_K, \infty) \xrightarrow{\text{extr.}} a_\mu(0, m_\pi^{\text{ph}}, m_K^{\text{ph}}, \infty)$$

Isovector (3,3) :

$$a_\mu(a, m_\pi, m_K, V) \xrightarrow{\text{HP\&MLL}} a_\mu(a, m_\pi, m_K, V_{\text{ref}}) \xrightarrow{\text{extr.}} a_\mu(0, m_\pi^{\text{ph}}, m_K^{\text{ph}}, V_{\text{ref}}) \xrightarrow{\chi\text{PT corr.}} a_\mu(0, m_\pi^{\text{ph}}, m_K^{\text{ph}}, \infty)$$

The hadronic vacuum polarisation contribution to the muon g-2 at long distances, *JHEP* 04 (2025) 098

Window splitting Strategy



$$\tilde{f}(t) = \tilde{f}(t)\left(\Theta_{SD}(t) + \Theta_{ID}(t) + \Theta_{LD}(t)\right)$$

LD piece: Signal-to-noise problem and big dependence on **scale setting**. Smoother continuum limit.

Bounding method and **spectroscopy analysis** to mitigate signal-to-noise problem.

The hadronic vacuum polarisation contribution to the muon g-2 at long distances, *JHEP* 04 (2025) 098

SD piece: Multiplicative **tree level improvement** improve continuum limit. Around the region $t \to 0$, one expects terms $\sim a^2 \ln a^2$ to emerge. These are hard to constrain in the continuum extrapolation.

We proceed with a **subtracted kernel:**

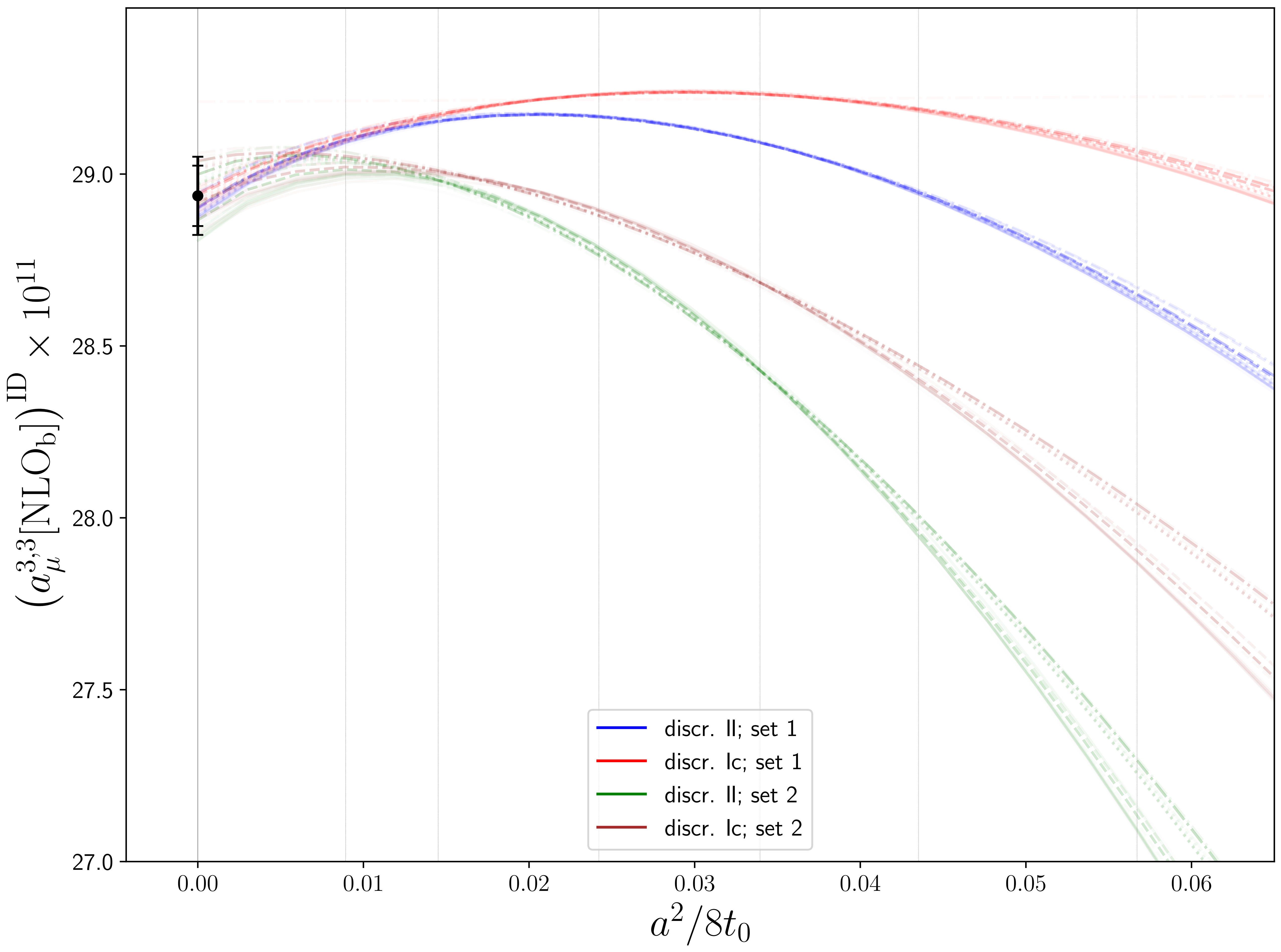
$$\tilde{f}^{(i)}(\hat{t})\Theta_{SD}(t) \longrightarrow \tilde{f}^{(i)}_{sub}(\hat{t}) = \tilde{f}^{(i)}(\hat{t})\Theta_{SD}(t) - \Theta_{SD}(0)C_4^{(i)}(\hat{t})\left(\frac{16\pi}{Q^2}\right)^2 \sin^4\frac{Qt}{4} \quad \text{where} \quad \tilde{f}^{(i)}(\hat{t}) \sim C_4^{(i)}(\hat{t})\hat{t}^4 + \mathcal{O}\left(\hat{t}^6\right)$$

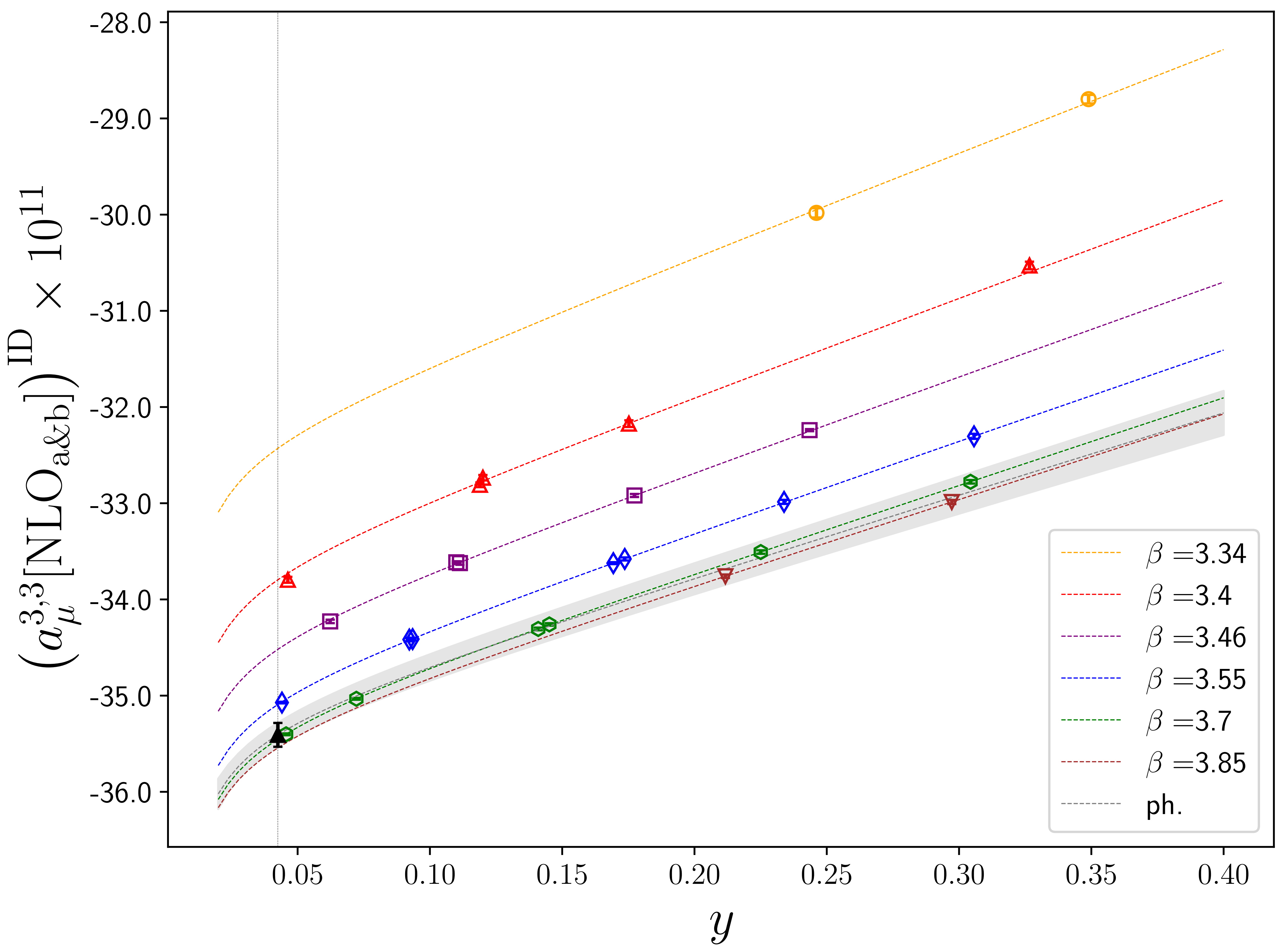
The hadronic vacuum polarisation contribution to the muon g-2: the short-distance contribution from lattice QCD, *JHEP* 03 (2024) 172

- Introduction
- NLO TMR kernels
- Strategy
- **Preliminary results**
- Outlook and conclusions

Extrapolations

Preliminary results

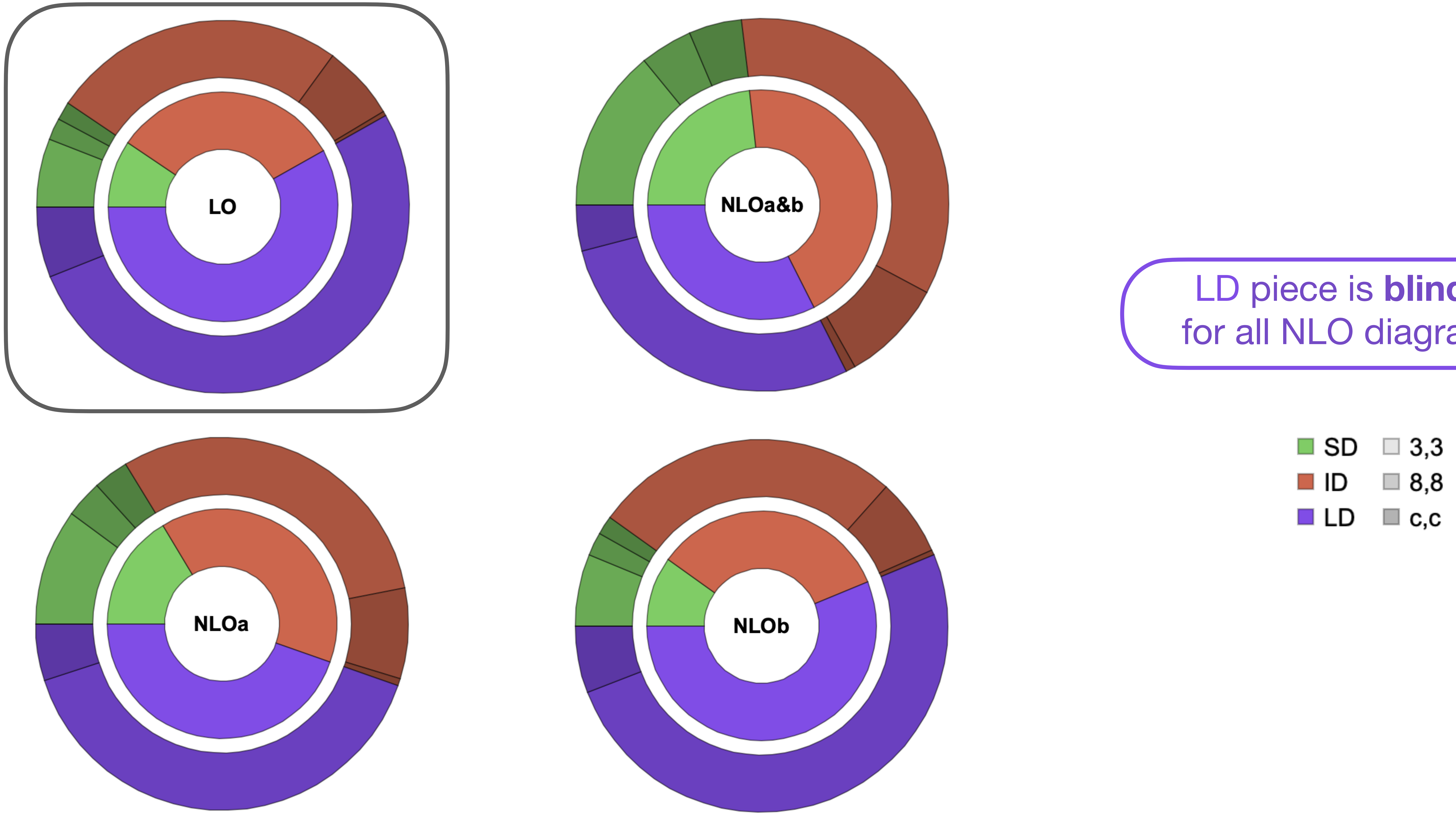




Typical fit for the iso-vector (ll set 2, with f_π rescaling); $\chi^2/\chi^2_{\mathrm{exp}} \approx 1.1$

Relative contributions to central value

Preliminary results



LD piece is **blinded** for all NLO diagrams!

Conclusion and outlook

- **Methodology** for a pure lattice NLO HVP computation through the TMR method.
- Observable seems to be more '**short-distanced**' than the LO HVP, but has some challenges of its own.
- **Sub-percentage precision** is to be expected after unblinding. Possible to compete and compare with data-driven results.
- Statistics dominated result $\mathrm{err}_{\mathrm{stat}} \gtrsim \mathrm{err}_{\mathrm{syst}}$

- Still in need of a 'full power' computation of the sub-leading **NLOc piece**.
- Ongoing study of the benefits an f_π **rescaling** can have when setting the scale.
- **Isobreaking effects** must be included.
- Other small contributions: bottom quark, charm quenching, top loop …

Backup slides

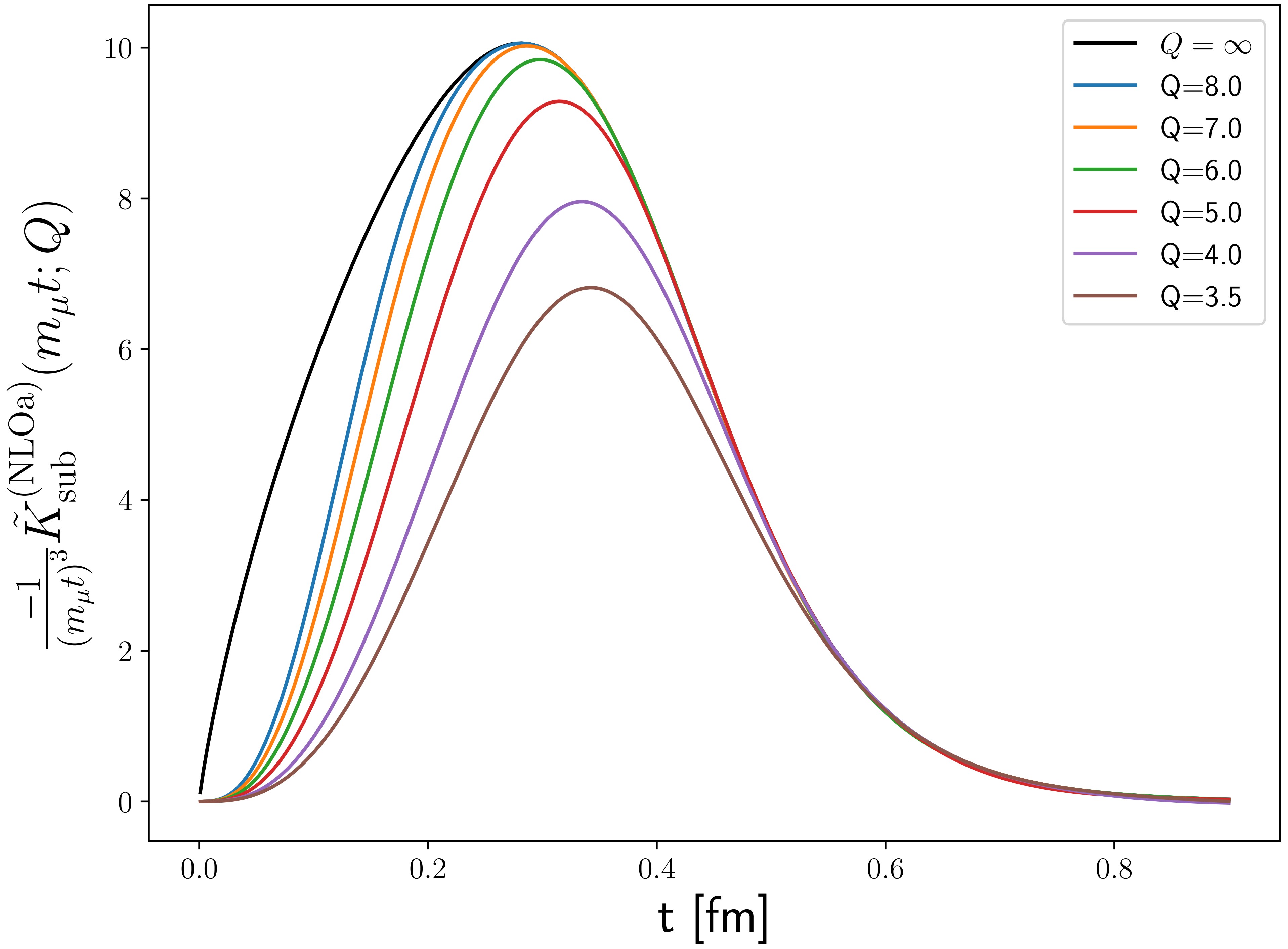
Subtracted Kernel on the NLOa

- For ‘LO’ and ‘NLOb’ :

$$C_4^{(\mathrm{LO})} = 1/9$$

$$C_4^{(\mathrm{NLOb})} = \frac{-1}{27}(1 + 4\ln M) + \frac{2M^2}{3} - \frac{4M^3\pi^2}{9} + \ldots$$

$$\left(a_\mu^a[(i)]\right)^{\mathrm{SD}} = \left(a_\mu^a[(i)]\right)^{\mathrm{SD}}_{\mathrm{sub}}(Q^2) + \Theta_{\mathrm{SD}}(0)\left(\frac{\alpha}{\pi}\right)^{2+\#_{N_i}}\left(\frac{4\pi m_\mu}{Q}\right)^2 C_4^{(i)}\left(\Pi^a(Q^2) - \Pi^a(Q^2/4)\right)$$



- For ‘NLOa’ : $C_4^{(\mathrm{NLOa})} = \frac{317}{324} - \frac{2\pi^2}{9} + \frac{23}{27}(\gamma_E + \ln\hat{t})$

$$\left(a_\mu^a[(i)]\right)^{\mathrm{SD}} = \left(a_\mu^a[(i)]\right)^{\mathrm{SD}}_{\mathrm{sub}}(Q^2) + \Theta_{\mathrm{SD}}(0)\left(\frac{\alpha}{\pi}\right)^3\frac{16\pi^2 m_\mu^2}{Q^2}\left[\frac{317}{324} - \frac{2\pi^2}{9} + \frac{23}{27}\left(\gamma_E + \ln\frac{m_\mu}{\Lambda}\right)\right]\left[\Pi^{(3,3)}(Q^2) - \Pi^{(3,3)}(Q^2/4)\right]$$

$$+\Theta_{\mathrm{SD}}(0)\frac{23}{27}\left(\frac{16\pi m_\mu}{Q^2}\right)^2\left(\frac{\alpha}{\pi}\right)^3\int_0^\infty dt\ \ln\Lambda t\ G^{(3,3)}(t)\sin^4\frac{Qt}{4}$$

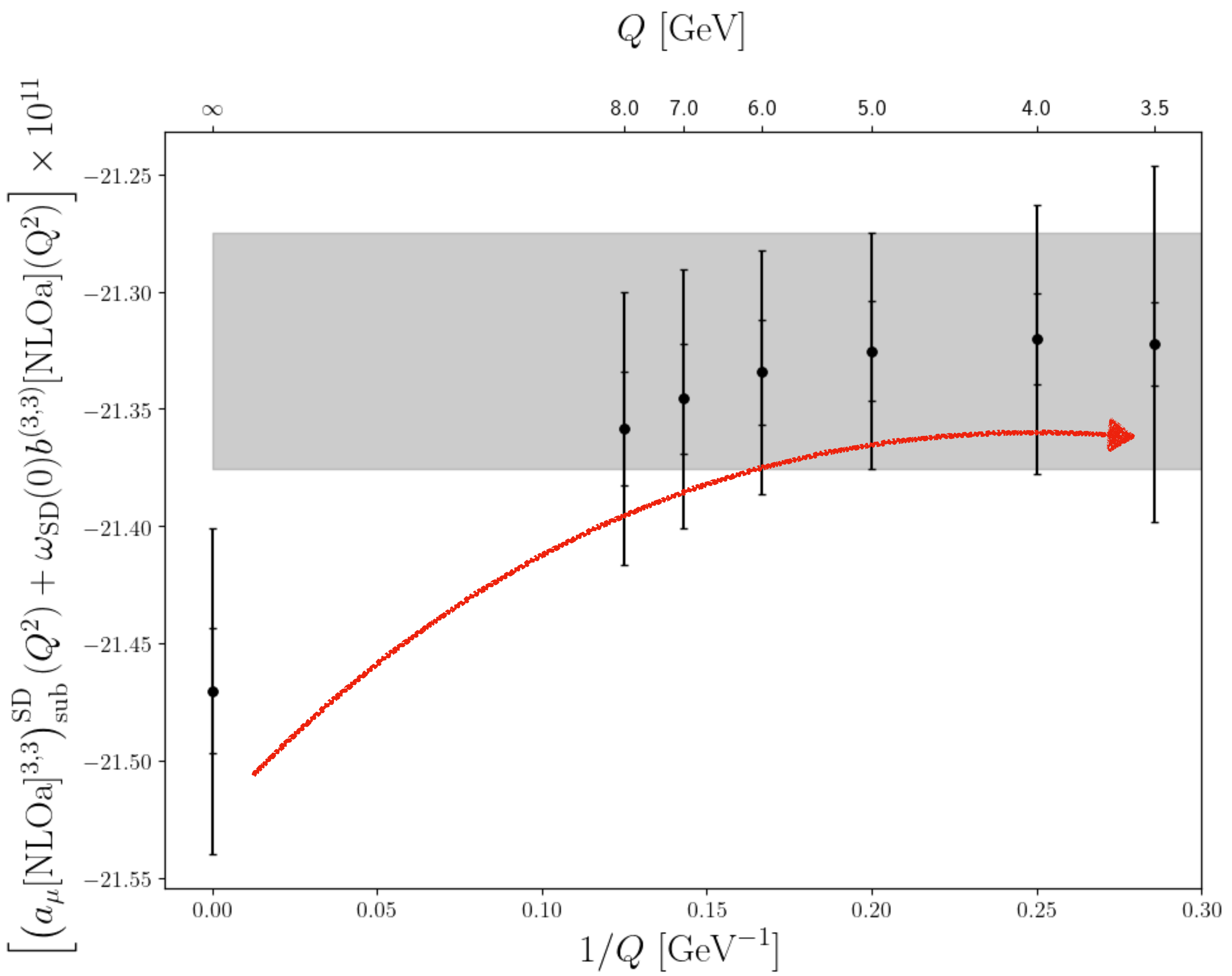
Can be computed to relative “low precision” through pQCD

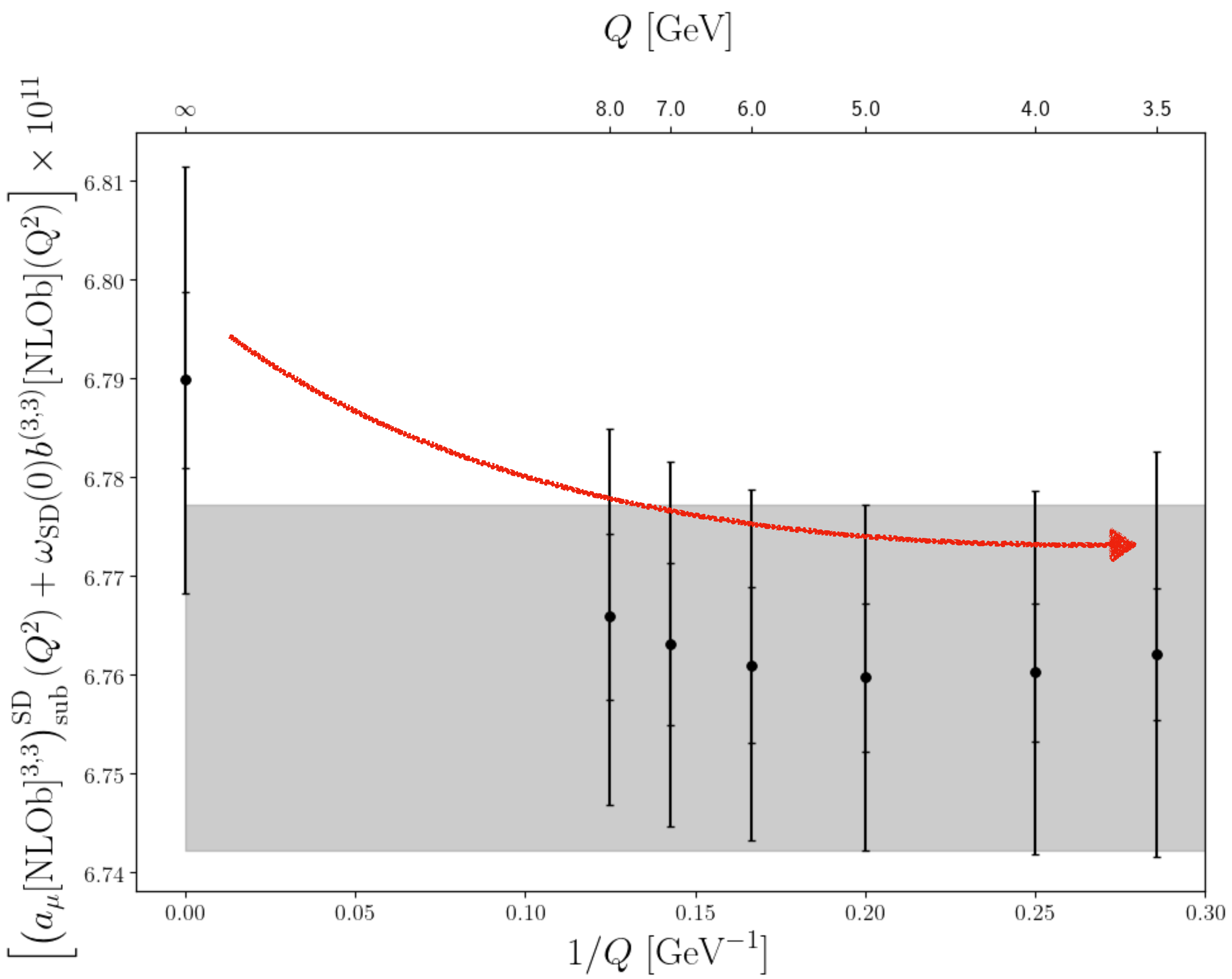
n	a_n	b_n	c_n	d_n
4	$\frac{317}{216}$	$-\frac{1}{3}$	$\frac{23}{18}$	0
6	$\frac{843829}{259200}$	$-\frac{371}{432}$	$\frac{877}{1080}$	$\frac{19}{36}$
8	$\frac{412181237}{5292000}$	$-\frac{233}{48}$	$-\frac{824603}{25200}$	$\frac{141}{20}$

Massless correlators of vector, scalar and tensor currents in position space at orders α_s^3 and α_s^4 : explicit analytical results, *Nucl.Phys.B* 844 (2011) 266-288

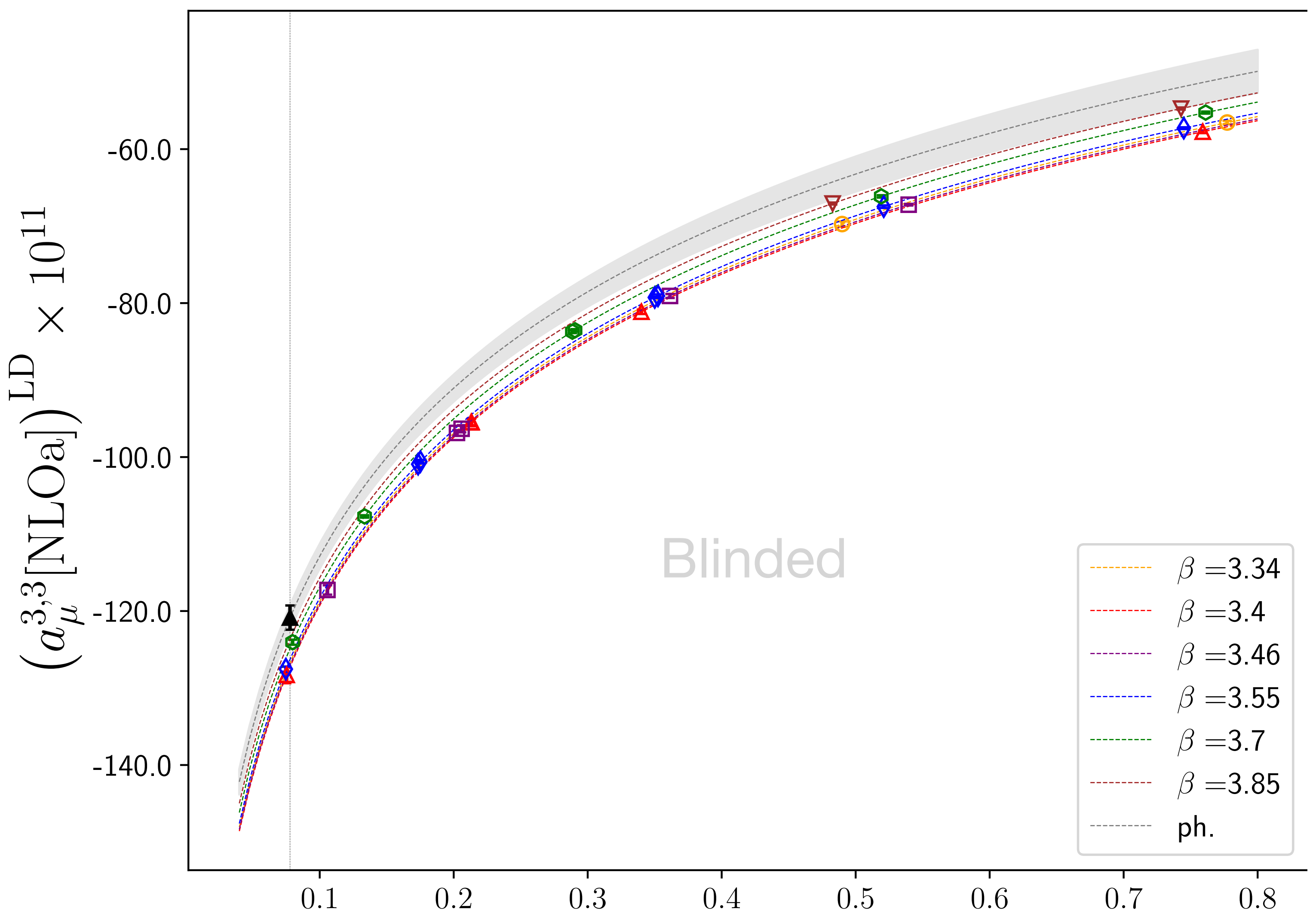
Subtracted Kernel on the NLOa

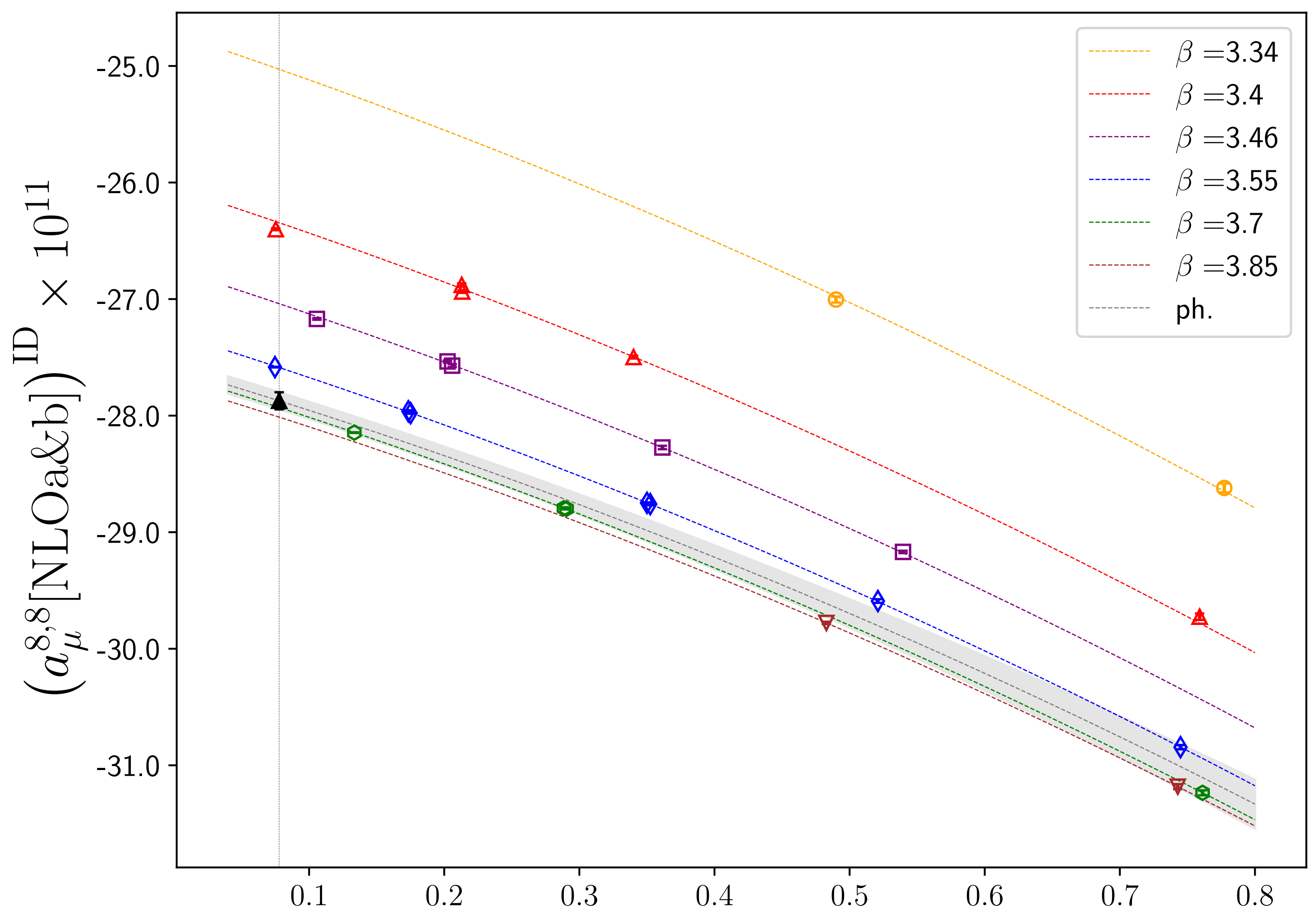
$a^2 \ln a^2$ terms slowly disappear as $1/Q$ increases

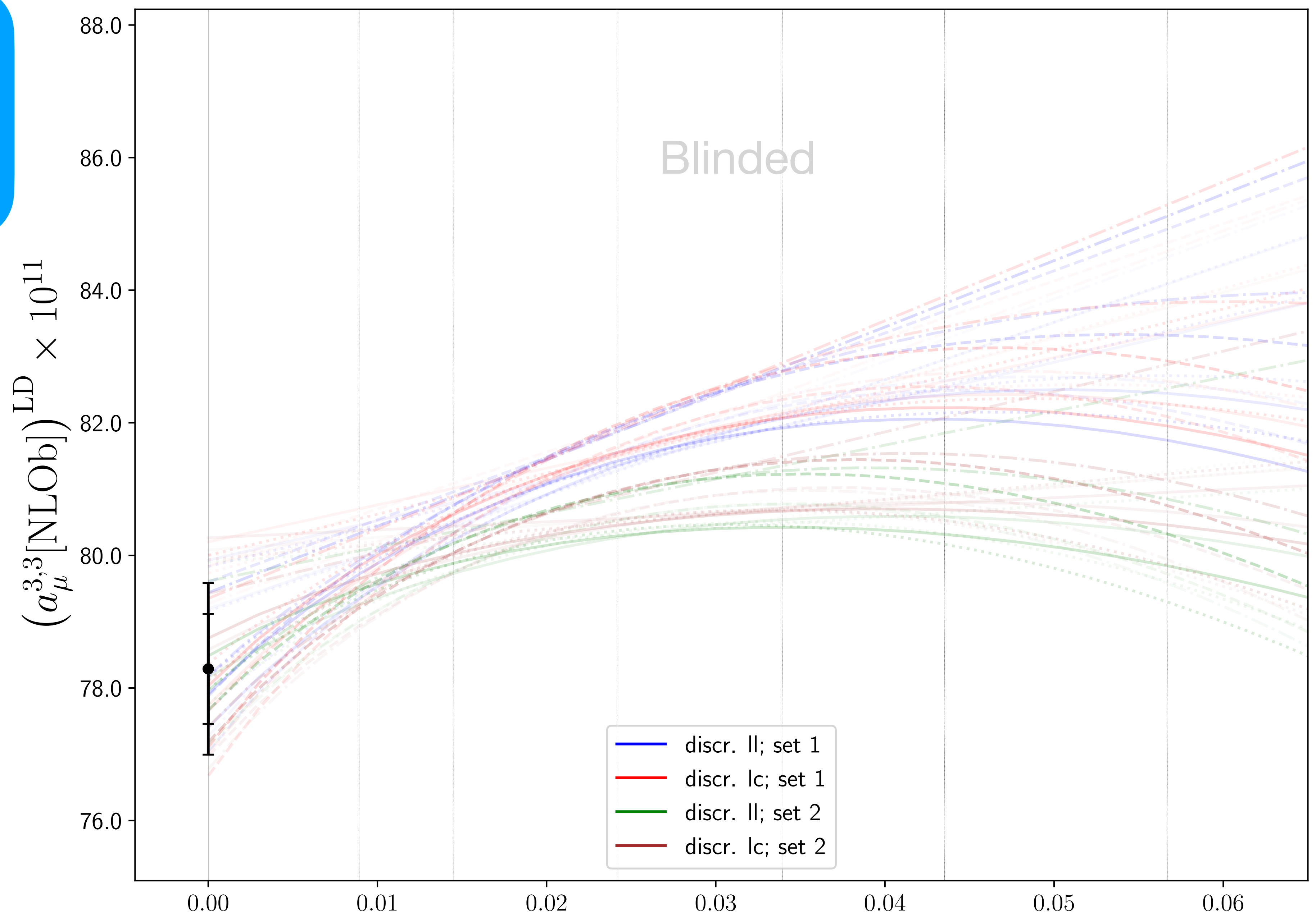


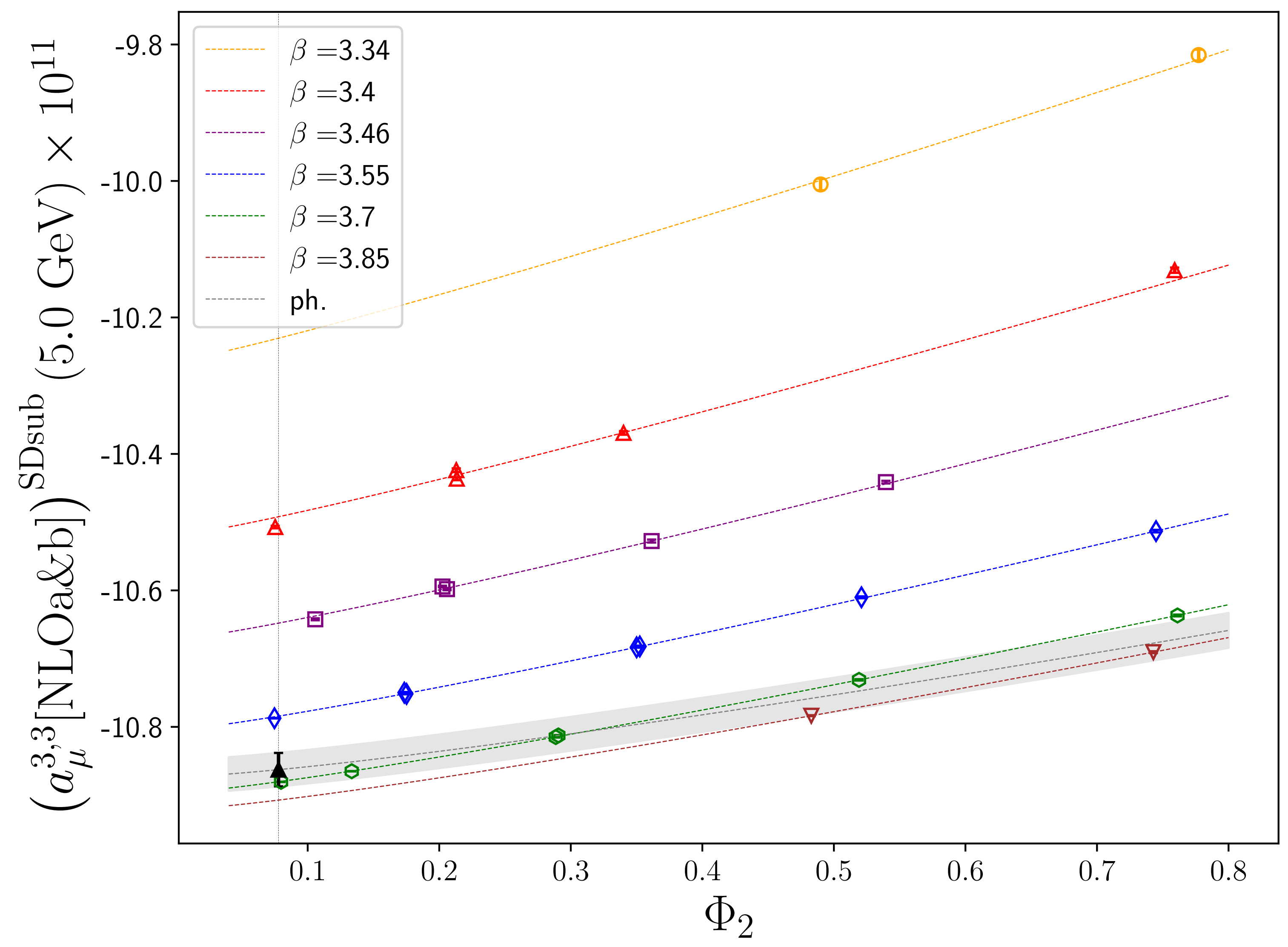


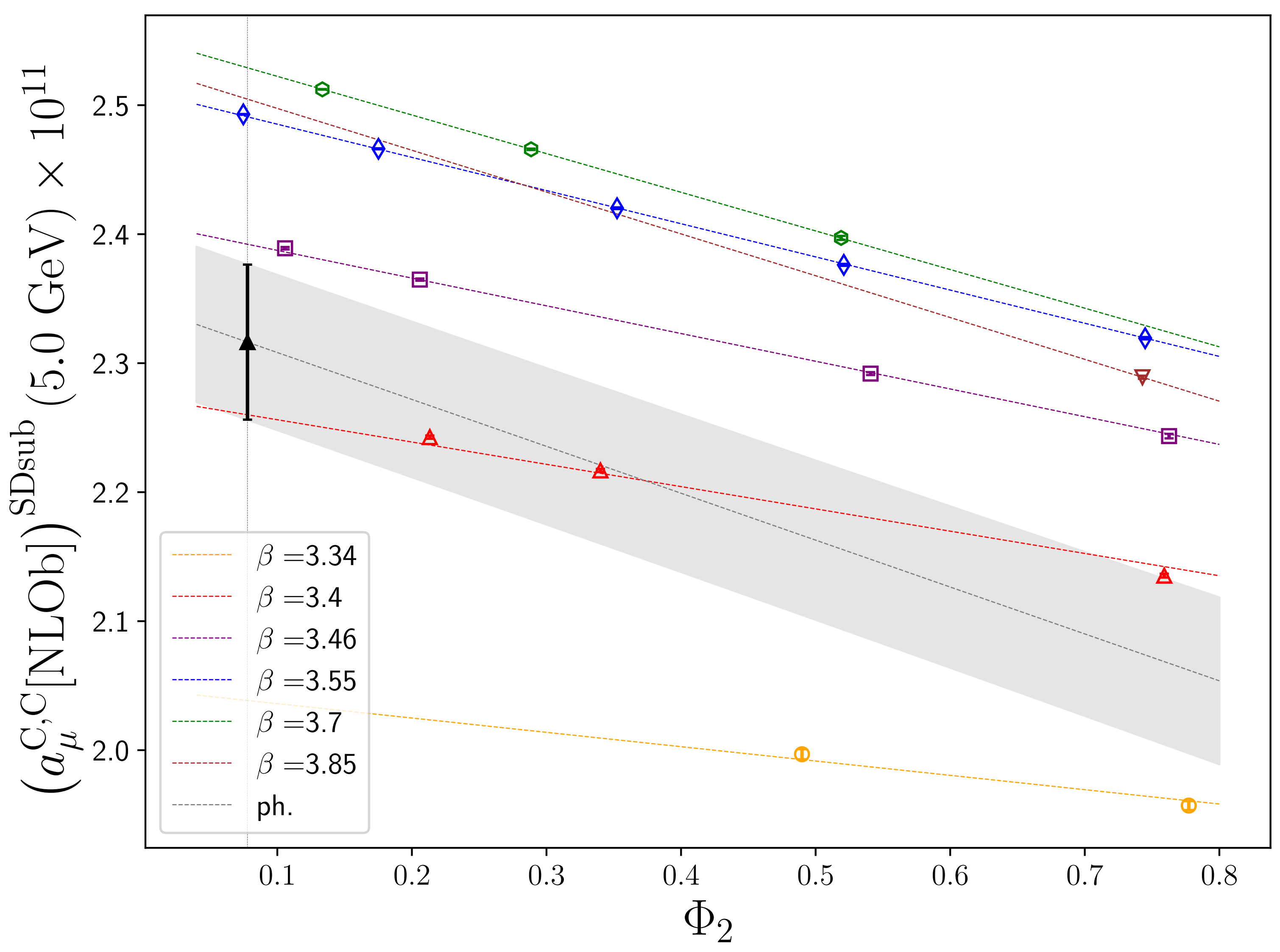
More extrapolations Backup (Results)

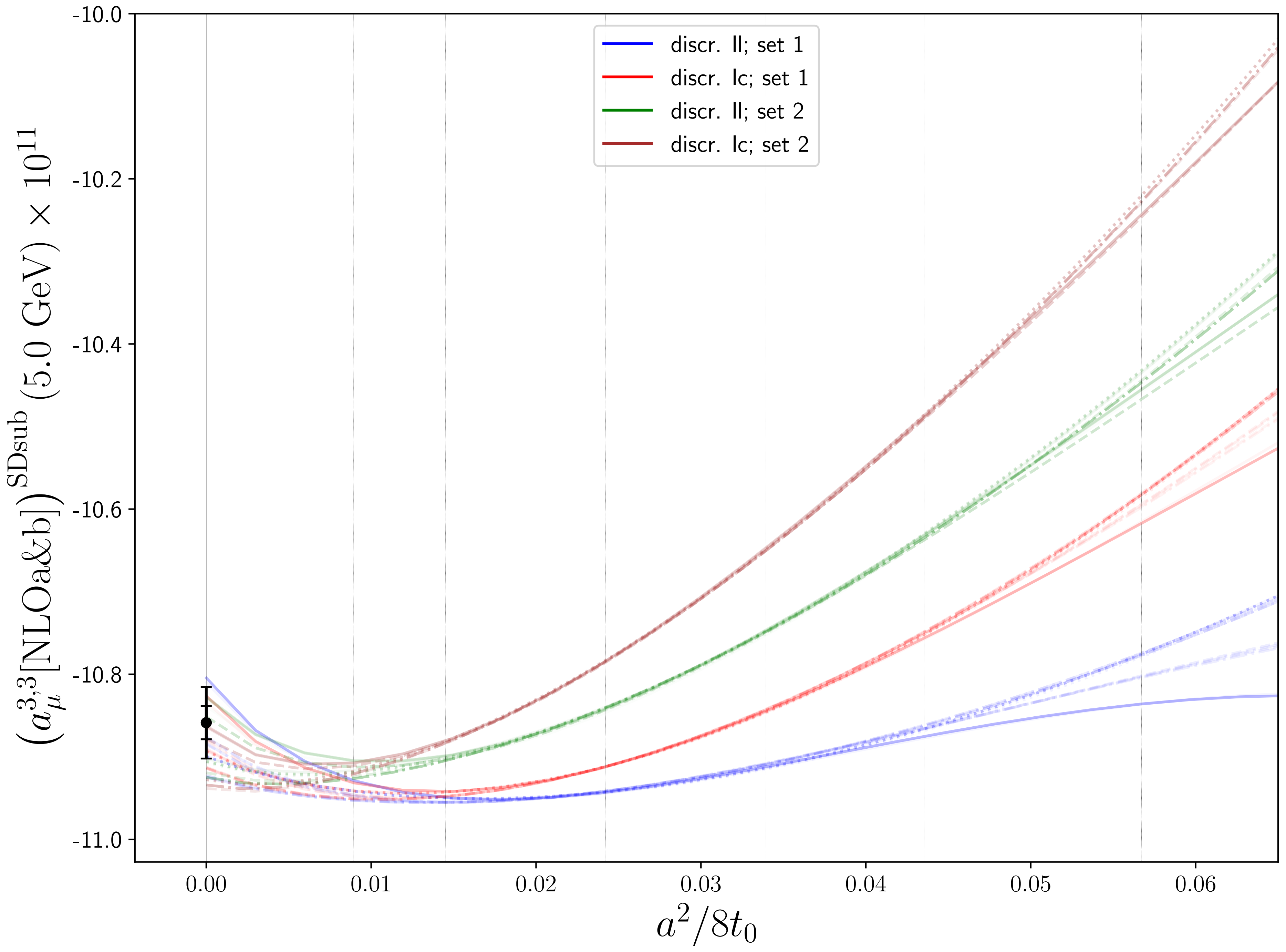




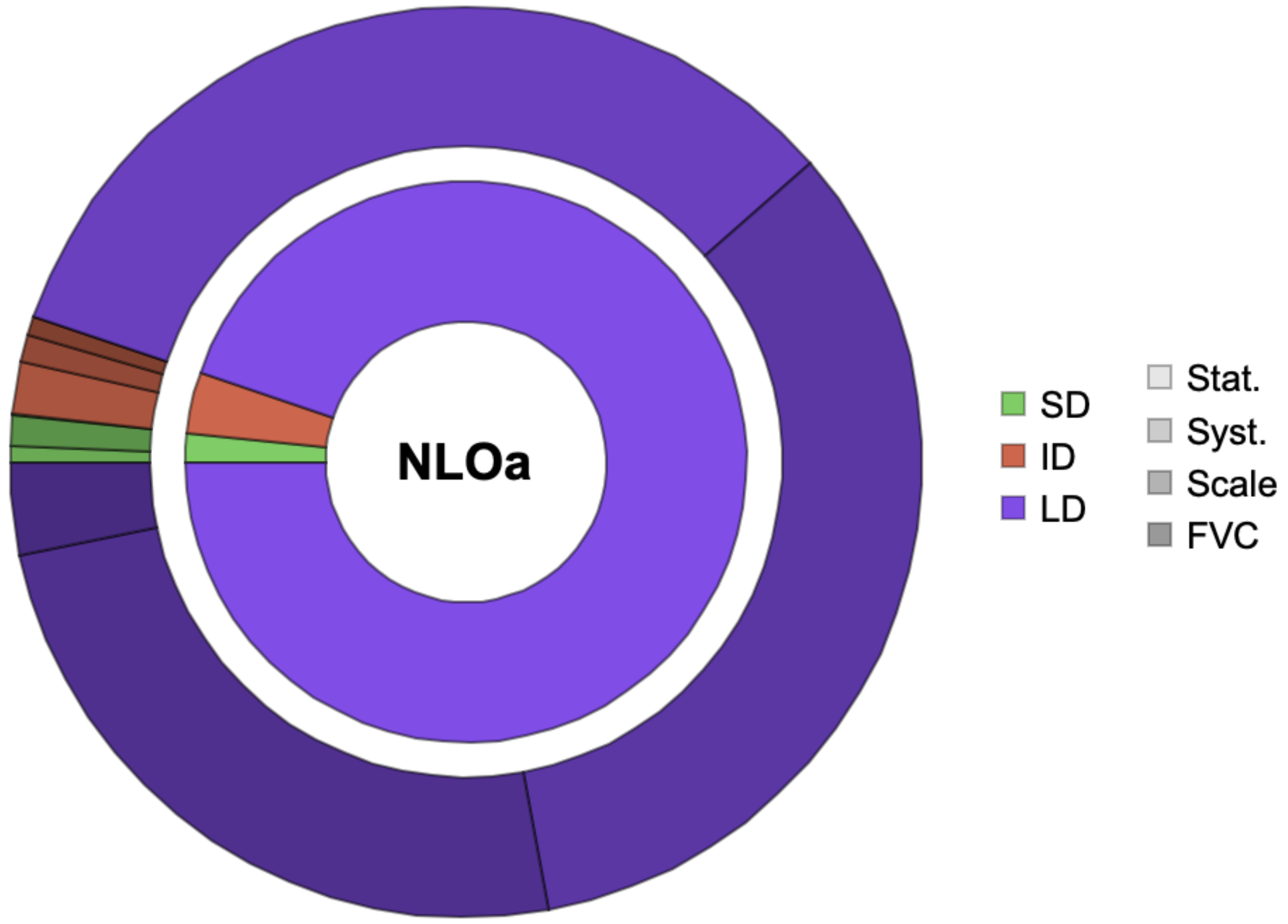


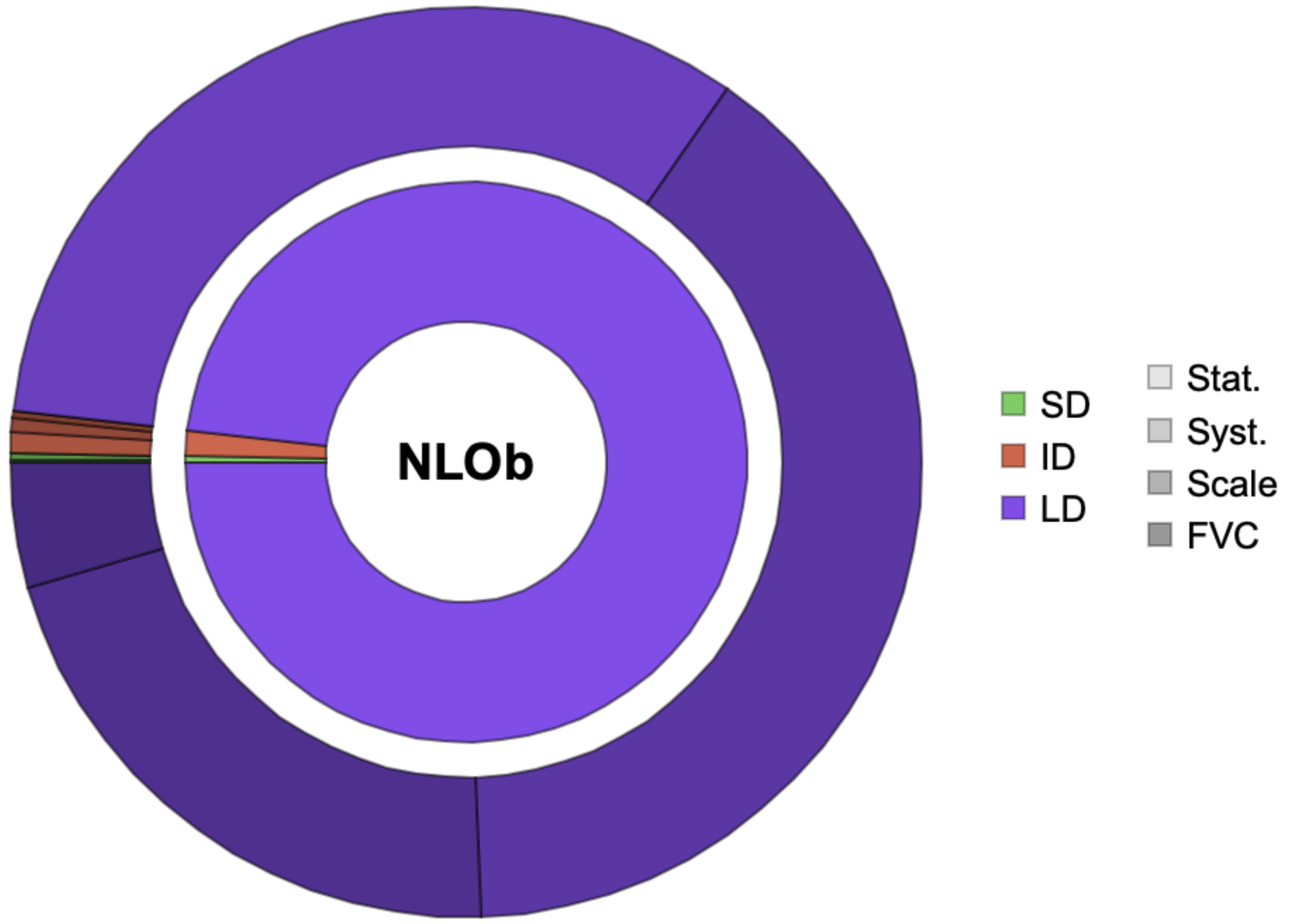


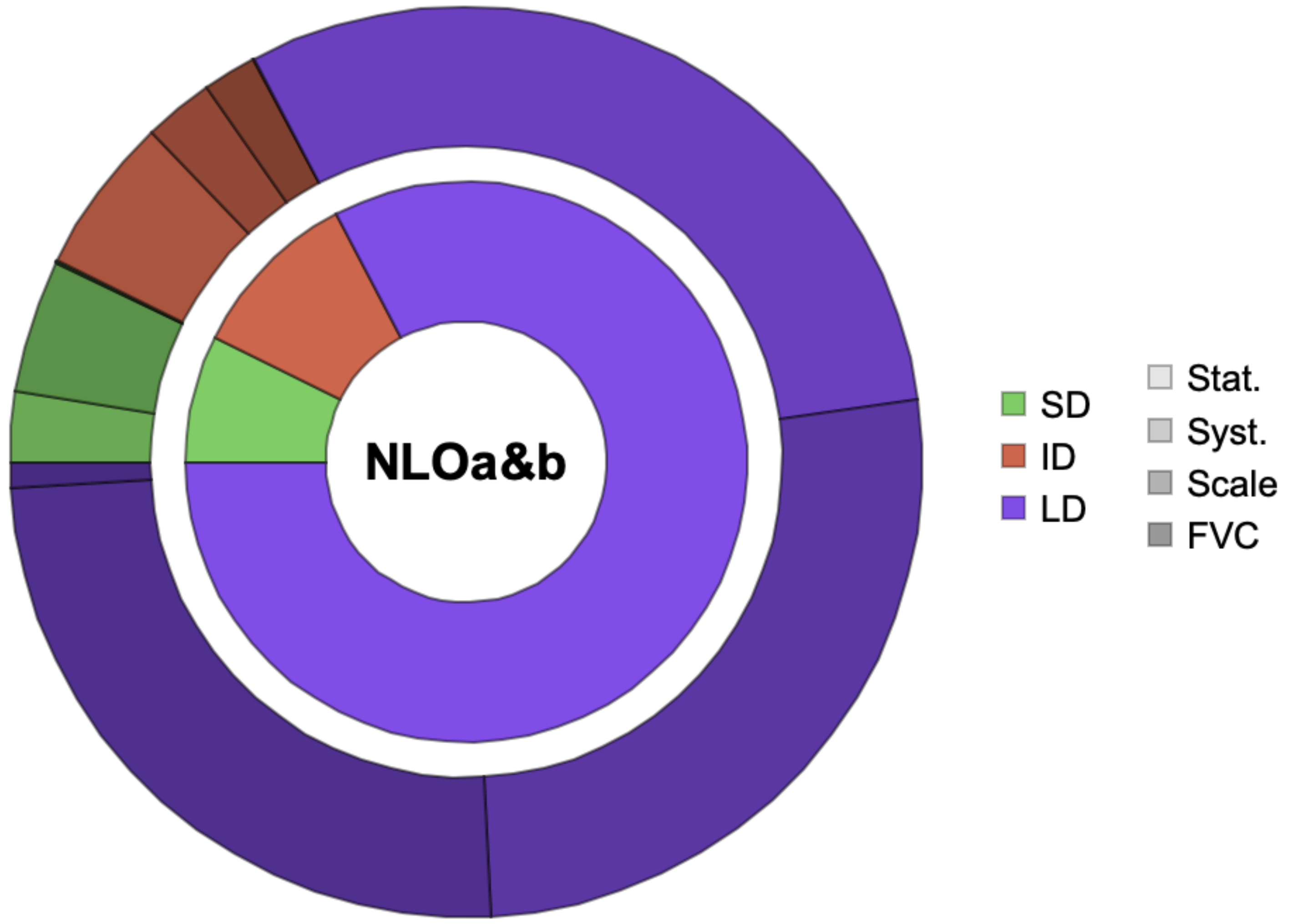




Relative contributions to the error







NLOc strategy

- Sub-leading contribution so less precision can be accepted.
- No window splitting.
- HP&MLL methods used to correct directly to ∞ volume (no V_{ref} in between step)
- Must take care of the crossed terms coming form the $G(t) \times G(\tau)$ product.

Using $\tilde{f}^{(4c)}(\hat{t}, \hat{\tau}) = \tilde{f}^{(4c)}(\hat{\tau}, \hat{t})$:

$$G(t)G(\tau) = G^{33}(t)G^{33}(\tau) + \frac{1}{9}G^{88}(t)G^{88}(\tau) + \frac{16}{81}G^{cc}(t)G^{cc}(\tau) + \frac{2}{3}G^{33}(t)G^{88}(\tau) + \frac{8}{9}G^{33}(t)G^{cc}(\tau) + \frac{8}{27}G^{88}(t)G^{cc}(\tau)$$

$$\Delta\left[G^{33}G^{33}\right] = \Delta G^{33}\Delta G^{33} + 2G^{33}\Delta G^{33}$$
$$\Delta\left[G^{88}G^{88}\right] = \Delta G^{88}\Delta G^{88} + 2G^{88}\Delta G^{88}$$
$$\Delta\left[G^{33}G^{88}\right] = \Delta G^{33}\Delta G^{88} + G^{33}\Delta G^{88} + \Delta G^{33}G^{88}$$
$$\Delta\left[G^{33}G^{cc}\right] = \Delta G^{33}G^{cc}$$

Other results

Previous lattice result (Fermilab Lattice, HPQCD, and MILC Collaborations)

Higher-order hadronic-vacuum-polarization contribution to the muon g-2 from lattice QCD

$$a_\mu^{\rm hvp}[\mathrm{NLO}]_{\rm lat} = -93(12) \times 10^{-11}$$

Final 2025 White Paper average

The anomalous magnetic moment of the muon in the Standard Model (2025)

$$a_\mu^{\rm hvp}[\mathrm{NLO}]_{\rm data-driven} = \begin{cases} -98.3(4) \times 10^{-11} & \mathrm{KNT19} \\ -100.8(6) \times 10^{-11} & \mathrm{KNT19/CMD-3} \end{cases} \implies \mathrm{a}_\mu^{\rm hvp}[\mathrm{NLO}]_{\rm WP} = -99.6(1.3) \times 10^{11}$$