Lattice calculation of the hadronic light-by-light contribution to the muon $g-2$

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8th Plenary workshop of the Muon g-2 Theory Initiative - Orsay

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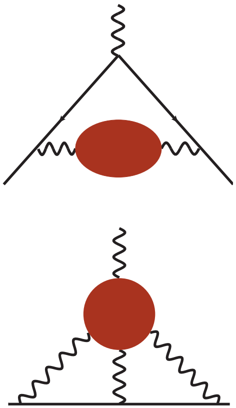
anr agence nationale de la recherche

CPT Centre de Physique Théorique

- Magnetic moment of charged leptons :

$$\vec{\mu} = g_\ell \left(\frac{Qe}{2m_\ell}\right) \vec{S} \qquad \Rightarrow \qquad a_\ell = \frac{g_\ell - 2}{2} = \frac{\alpha}{2\pi} + O(\alpha^2)$$

Contribution	$a_\mu \times 10^{11}$
- **QED** (10^{th} order)	$116\ 584\ 718.931 \pm 0.104$
- **Electroweak**	153.6 ± 1.0
- **Strong interaction**	
HVP (LO)	$7\ 041 \pm 61$
HVP (NLO + NNLO)	-85.9 ± 0.7
HLbL	112.6 ± 9.6
Standard Model	$116\ 592\ 033 \pm 62$
Experiment	$116\ 592\ 072 \pm 15$



- Challenging

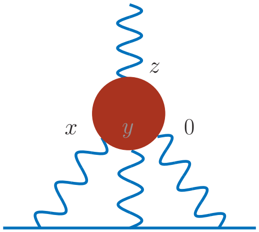
→ hadronic light-by-light tensor $\Pi_{\mu\nu\lambda\sigma}(p_1, p_2, p_3) = \int_{x,y,z} \Pi_{\mu\nu\lambda\sigma}(x, y, z) e^{-i(q_1 x + q_2 y + q_3 z)}$

→ depends on four momenta, multi-scale system

- Precision goal : $< 10\%$ (with controlled uncertainties)

→ requires first principle approach : data-driven dispersive framework / lattice QCD

Based on the coordinate-space approach developed by the Mainz group



$$a_\mu^{\mathrm{HLbL}} = \frac{m_\mu e^6}{3} \int \mathrm{d}^4y \int \mathrm{d}^4x \, \mathcal{L}_{[\rho,\sigma],\mu\nu\lambda}(x,y) \, i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$$

- QED part of the diagram : $\mathcal{L}_{[\rho,\sigma],\mu\nu\lambda}(x,y)$ ("QED kernel")

→ computed in the continuum and infinite volume
→ $\mathcal{L}_{[\rho,\sigma],\mu\nu\lambda}(x,y)$ computed semi-analytical : Mainz group [JHEP 04 (2023) 040]
→ efficient implementation is needed

→ we use a slightly different kernel compared to Mainz' 22

- Hadronic correlation function : $i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int \mathrm{d}^4z \, z_\rho \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \rangle$

Calculation performed at the physical pion mass (staggered quarks)

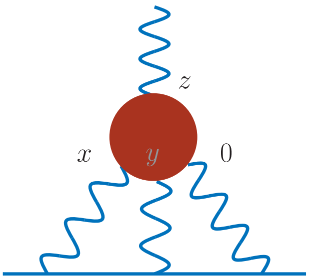
→ Statistical precision at long distances
→ Finite volume effects needs to considered
→ Continuum extrapolation

Comment : contractions are performed on the fly (we don't save lattice-size objects on disk ...)

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \int \mathrm{d}^4 y \int \mathrm{d}^4 x \, \mathcal{L}_{[\rho,\sigma],\mu\nu\lambda}(x,y) \, i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$$

with

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int \mathrm{d}^4 z \, z_\rho \, \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \rangle$$

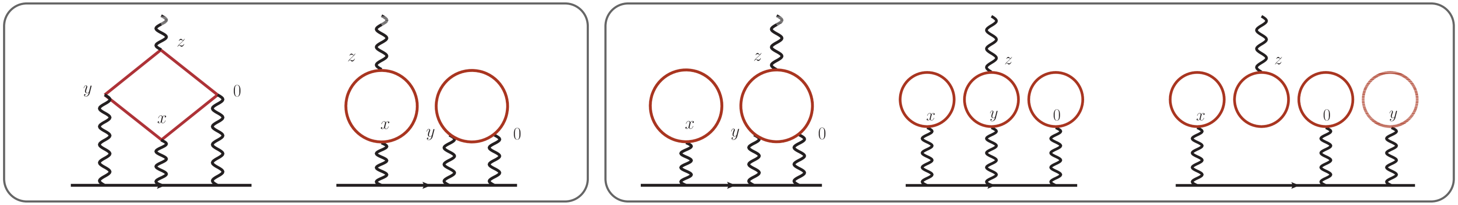


- sums over x and z are done explicitly over the lattice
- use rotational symmetry to write $\mathrm{d}^4 y = 4\pi^2 |y|^3 \mathrm{d}|y|$
- sample the remaining 1D integral over $|y|$ with $O(10)$ points

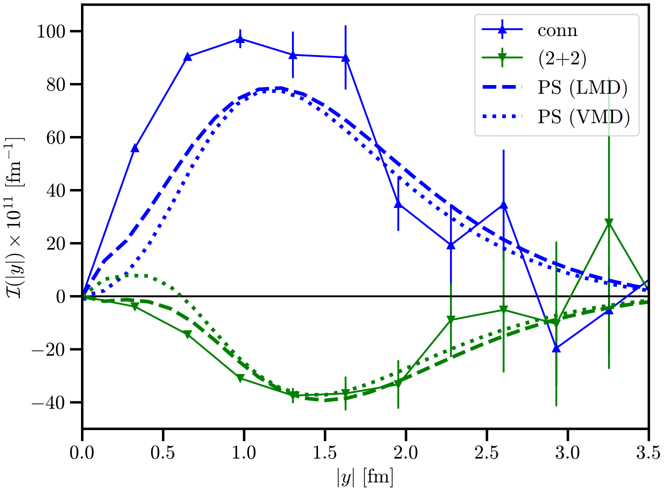
$$a_\mu(|y|) = \int_0^{|y|} \mathcal{I}(|y'|) \, \mathrm{d}|y'|$$

- Wick contractions : $i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z\, z_\rho \,\langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0)\rangle$

"connected" "$(2+2)$" sub-leading diagrams



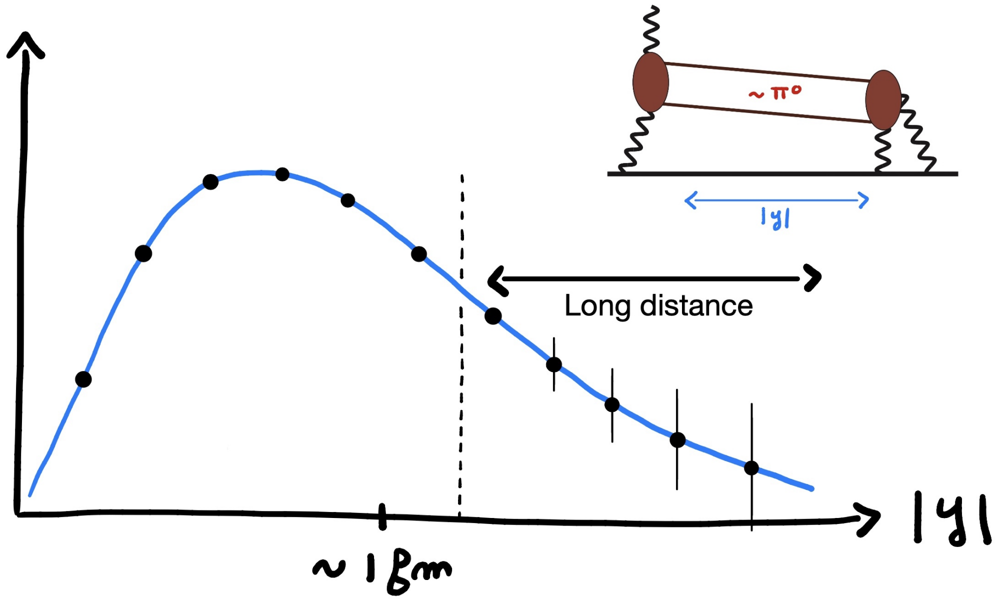
$\approx +220 \times 10^{-11}$ $\approx -100 \times 10^{-11}$ $\mathcal{O}(1 \times 10^{-11})$ (computed in this work)



- pion-exchange = main contribution

$$a_\mu^{\rm hlbl,conn} \longleftarrow +\frac{34}{9} \times a_\mu^{\pi^0-{\rm pole}}$$
$$a_\mu^{\rm hlbl,2+2} \longleftarrow -\frac{25}{9} \times a_\mu^{\pi^0-{\rm pole}}$$

- large cancellation expected [J. Bijnens et. al '16]

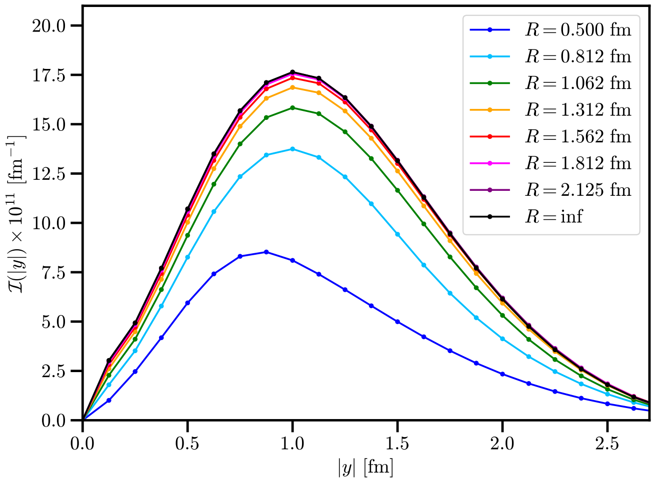


- Long distances $(|y| > 1 \text{ fm})$:
  - → statistical noise increases exponentially with $|y|$ (situation even worse at $m_\pi = m_\pi^{\text{phys}}$)
  - → finite-volume effects
  - → dominated by the pion-exchange contribution

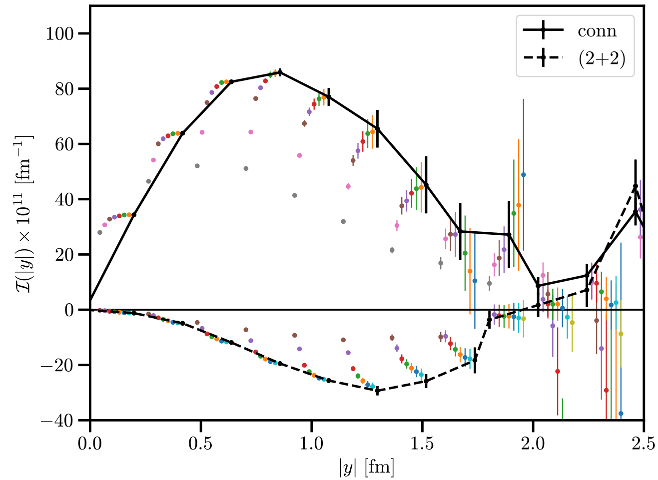
$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} 2\pi^2 \int \mathrm{d}|y|\ |y|^3 \int \mathrm{d}^4x\, \mathcal{L}_{[\rho,\sigma],\mu\nu\lambda}(x,y)\, i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$$

- Cut the $x-$integral if $|x| > R$ and $|x-y| > R$:

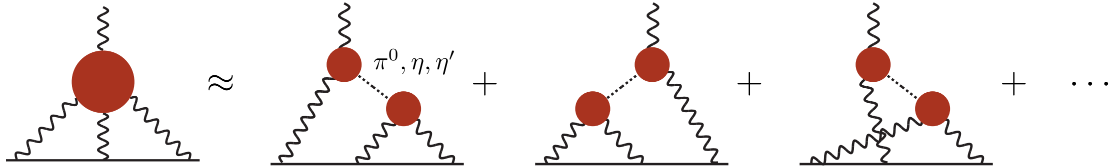
pion-pole (VMD)



lattice data



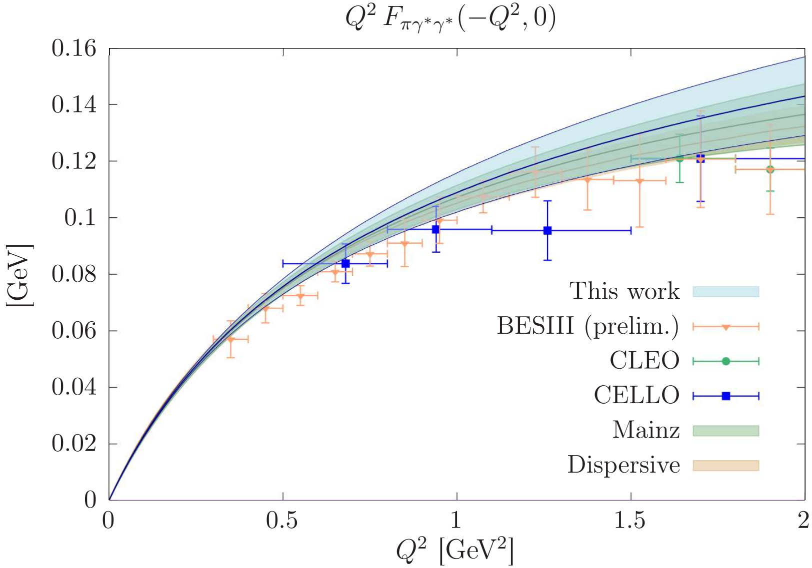
- long-distance contributes only to the noise
  $\rightarrow$ value of the cut R depends on $|y|$: value chosen on ensembles with high statistical precision



[Jegerlehner & Nyffeler '09]

$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau)\; \mathcal{F}_{P\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\; \mathcal{F}_{P\gamma^*\gamma^*}(-Q_2^2, 0) + w_2(Q_1,Q_2,\tau)\; \mathcal{F}_{P\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\; \mathcal{F}_{P\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

Integrand concentrated at spacelike momenta below 2 GeV



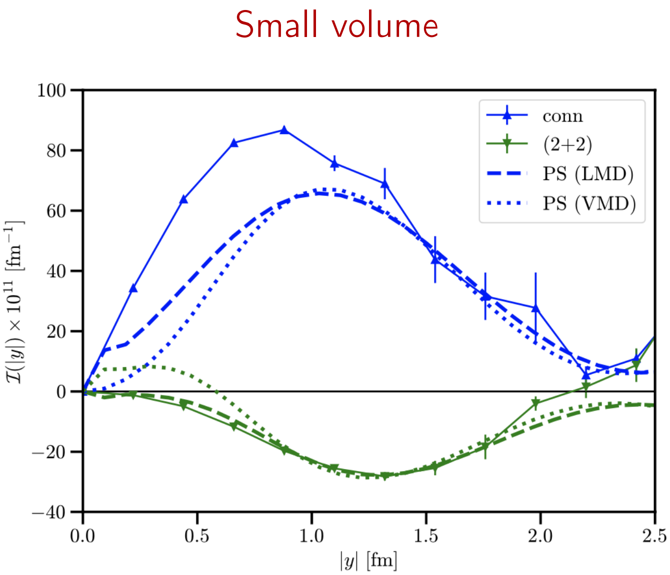
- **Transition form factors**
  [Phys.Rev.D 111 (2025) 5, 054511]

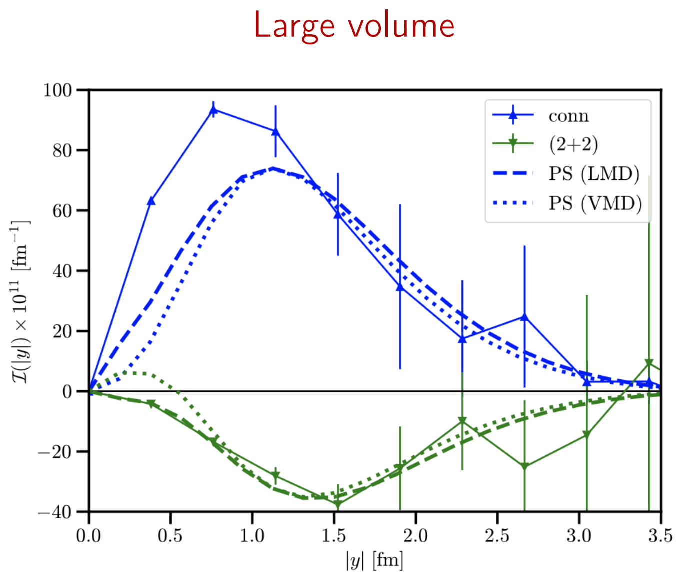
$$\epsilon_{\mu\nu\alpha\beta}\, q_1^\alpha\, q_2^\beta\; \mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2) = i \int \mathrm{d}^4x\, e^{iq_1 x}\, \langle 0 | T\{J_\mu(x) J_\nu(0)\} | \pi^0(\vec{p}) \rangle$$

→ computed on the same set of ensembles

→ all π^0, η and η' TFFs have been computed

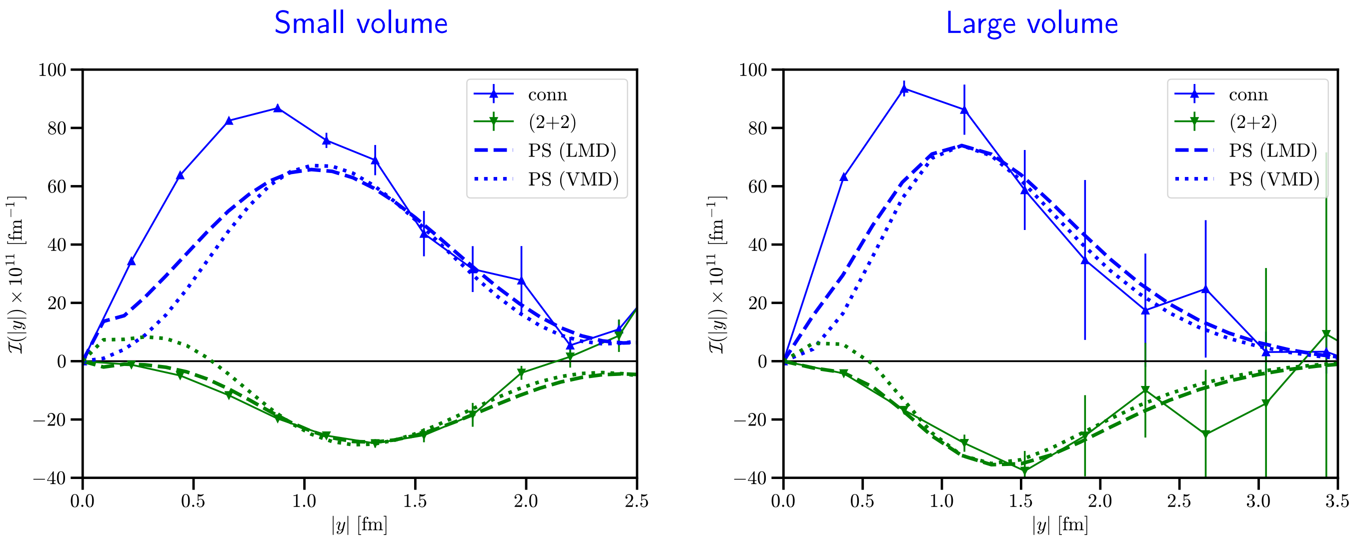
→ model independent parameterization





- Pion exchange computed in finite volume and position-space (assuming VMD or LMD parameterisation)
  → good description of our data at large $|y|$
- Strategy : replace lattice data by pion exchange for $|y| > y_{\mathrm{cut}}$
  → keep y_{cut} large such that this correction is small compared to statistical uncertainty.

- Based on a computation of the pion-pole contribution in finite volume
  → we assume finite-size effects are dominated by PS-pole (i.e. neglect FSE from pion loop, ...)



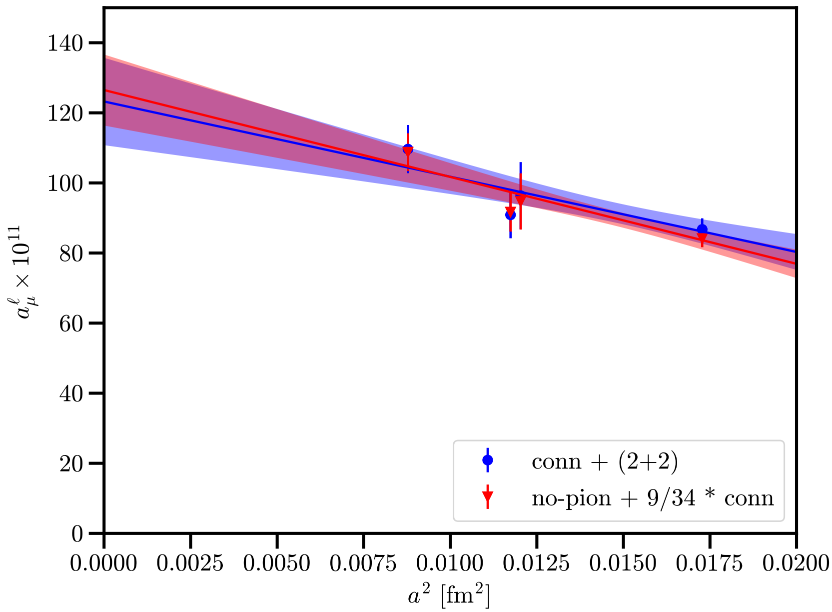
→ Good description of long-distance lattice data $|y| > 1.5$ fm (little model dependence)

→ Connected contribution computed on large/small volumes compatible after FSE correction :

$$V_1 \; : \; a_\mu^{\rm conn} = 118.8(2.9)_{\rm stat} \times 10^{-11} \quad \Rightarrow \quad a_\mu^{\rm conn,cor} = 151.9(2.9)_{\rm stat} \times 10^{-11}$$

$$V_2 \; : \; a_\mu^{\rm conn} = 141.9(11.9)_{\rm stat} \times 10^{-11} \quad \Rightarrow \quad a_\mu^{\rm conn,cor} = 143.6(11.9)_{\rm stat} \times 10^{-11}$$

- **Only large-volume ensembles are used in our final analysis**

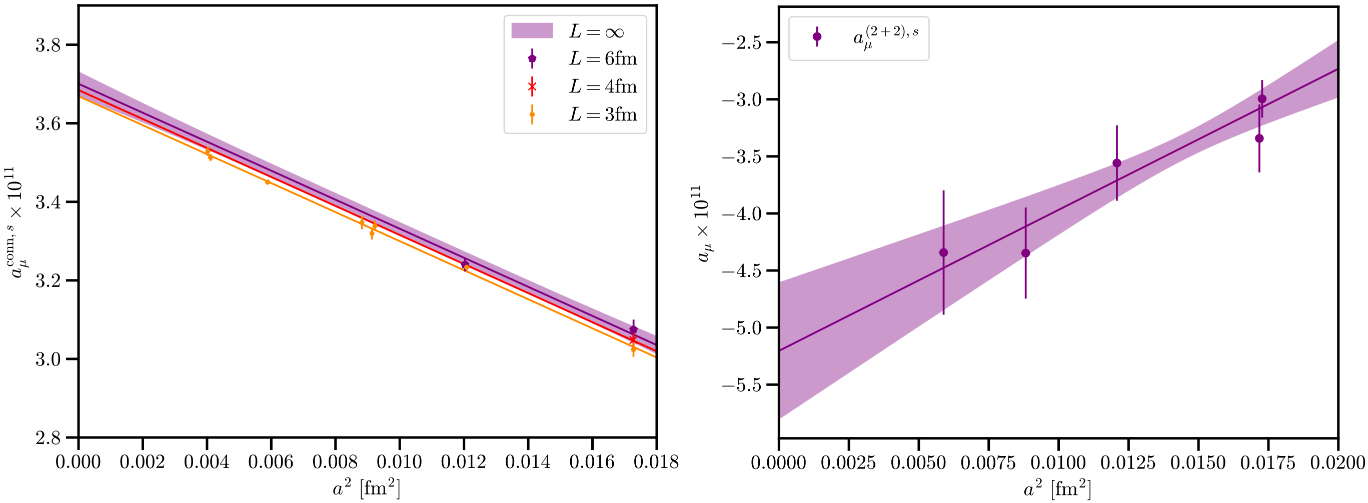


- 3 lattice spacings
- all simulations are performed at the physical pion-mass

$$a_\mu^{\text{conn},l} = 220.1(13.0)_{\text{stat}}(3.8)_{\text{syst}} \times 10^{-11}$$
$$a_\mu^{(2+2),l} = -101.1(12.4)_{\text{stat}}(3.2)_{\text{syst}} \times 10^{-11}$$
$$a_\mu^{l} = 122.6(11.5)_{\text{stat}}(1.8)_{\text{syst}} \times 10^{-11}$$

- Discretization effects dominated by pion-pole

- More lattice spacings are used for strange / charm quark contributions



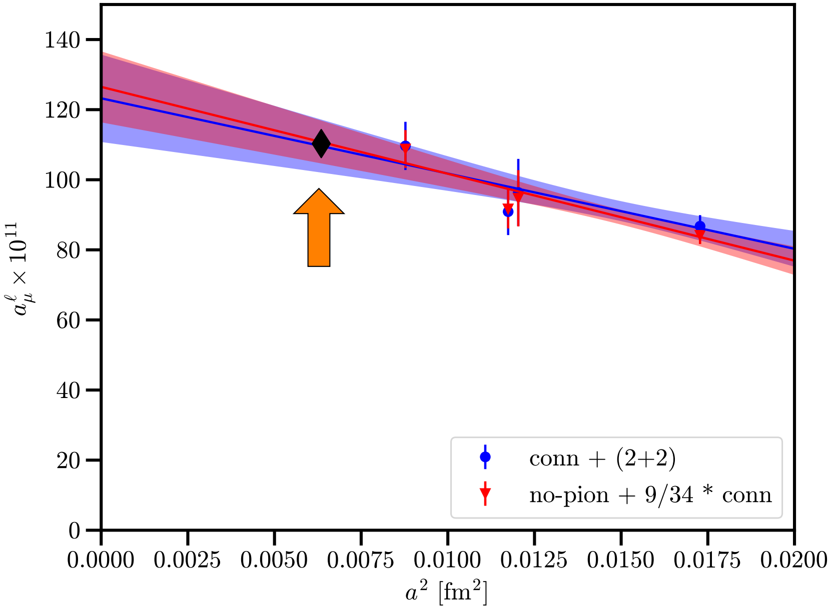
- Much higher statistical precision
  - $\rightarrow$ finite-volume effects are small but not negligible
  - $\rightarrow m_K \neq m_K^{\mathrm{phys}}$: slight mistuning needs to be considered in the analysis
  - $\rightarrow$ scale-uncertainty not negligible (especially for the charm quark contribution)

- Sum over all contributions

Contribution	$a_\mu \times 10^{11}$
Light Connected	$220.1(13.0)_{\rm stat}(3.8)_{\rm syst}$
Light 2+2	$-101.1(12.4)_{\rm stat}(3.2)_{\rm syst}$
Light Total	$122.6(11.5)_{\rm stat}(1.8)_{\rm syst}$
Strange Connected	$3.694(17)_{\rm stat}(20)_{\rm syst}$
Strange 2+2	$-5.4(0.8)_{\rm stat}(0.2)_{\rm syst}$
Strange Total	$-1.7(0.8)_{\rm stat}(0.3)_{\rm syst}$
Charm Connected	$3.92(1)_{\rm stat}(26)_{\rm syst}$
Charm 2+2	$-0.185(51)_{\rm stat}(04)_{\rm syst}$
Charm Total	$3.73(5)_{\rm stat}(26)_{\rm syst}$
subleading Disc.	$0.82(18)_{\rm stat}(17)_{\rm syst}$
Total	$125.5(11.5)_{\rm stat}(1.9)_{\rm syst}$

[Phys.Rev.D 98 (2018) 7, 074501]

Add a fourth lattice spacing $a = 0.078$ fm : $L = 80^3 \times 128$ ($L = 6.3$ fm)



- Use the same strategy to correct for finite volume effects (π^0 TFF is known)
- Focus on the connected and leading-disconnected light quark contribution
- Accumulated statistic : precision should match the one at coarser lattice spacings

- Complete calculation now published [Phys.Rev.D 111 (2025) 11, 114509]
  - → 3 lattice spacings at the physical pion mass
  - → dedicated calculation of the pion transition form factor : finite-volume effects, long-distance contribution
  - → precision <10%
  - → ongoing : add a fourth lattice spacing to improve the continuum limit extrapolation

$$a_\mu^{\mathrm{HLbL}} \times 10^{11}$$