Update on HLbL in soft kinematics

Jan-Niklas Toelstede
University of Zurich and Paul Scherrer Institute

in collaboration with Emilis Kaziukėnas, Massimiliano Procura, Peter Stoffer

12.09.2025

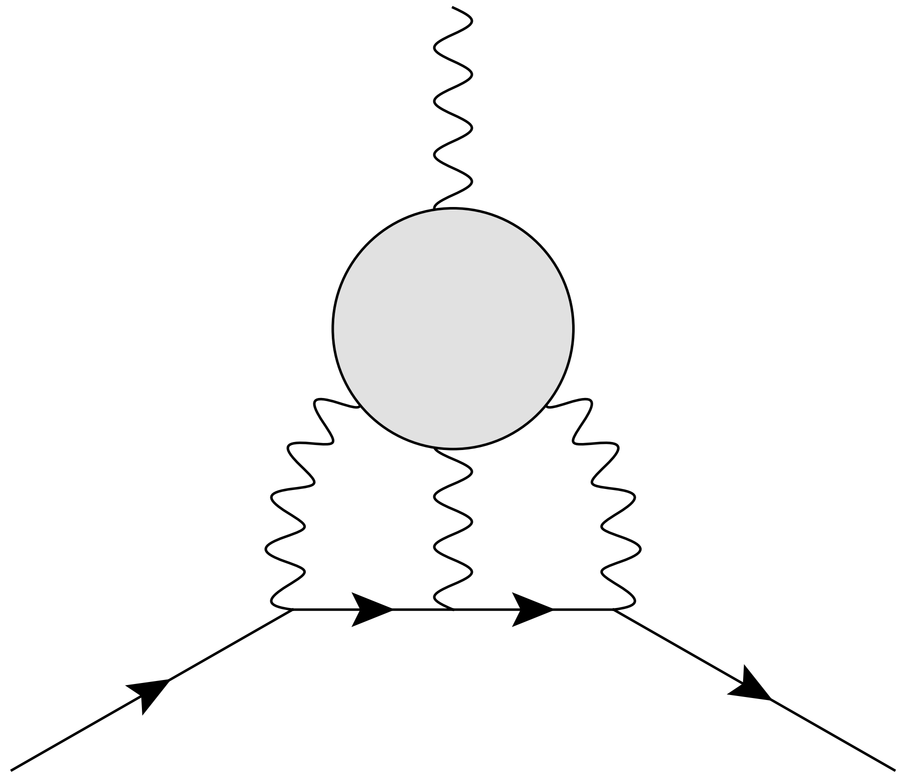
Content

1. Short review on HLbL in soft kinematics
2. Current progress in $\gamma^*\gamma \to \pi\pi$ and $\pi\pi \to \pi\pi\gamma$
3. Subtraction constants and asymptotic constraints

Content

1. Short review on HLbL in soft kinematics
2. Current progress in $\gamma^*\gamma \to \pi\pi$ and $\pi\pi \to \pi\pi\gamma$
3. Subtraction constants and asymptotic constraints

Dispersive Number for HLbL



$$a_\mu^{\text{HLbL,disp}} = 103.3(8.8) \cdot 10^{-11}$$ [Aliberti et. al., 2025]

$$a_\mu^{\text{HLbL,Lat-av}} = 122.5(9.0) \cdot 10^{-11}$$

$$a_\mu^{\text{HLbL,WP20}} = 92(19) \cdot 10^{-11}$$ [Aoyama et. al., 2020]

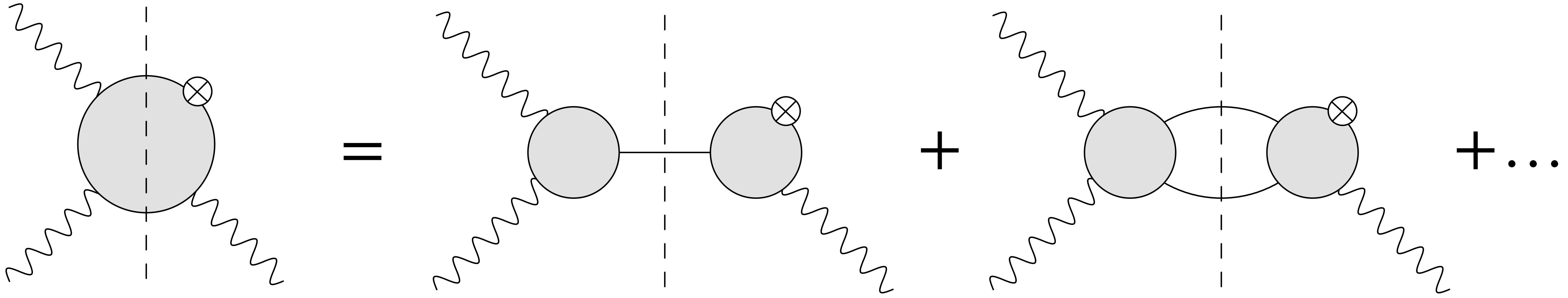
➡ We already met the precision goal of $< 10\,\%$

➡ Tension between data-driven and lattice about $(1-2)\sigma$

HLbL in soft kinematics

[Lüdtke, Procura, Stoffer 2023]

$$\Pi^{\mu\nu\lambda\sigma}(q_i) = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma}\,\Pi_i(s,t,u)\,, \quad q_i^2 \neq 0$$



unsubtracted DR

$$\check{\Pi}_i(s) \sim \frac{1}{\pi}\int ds' \frac{\mathrm{Im}\check{\Pi}_i(s')}{s'-s}$$

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \sqrt{1-\tau^2}\, Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1,Q_2,\tau)\bar{\Pi}_i(Q_1,Q_2,\tau)$$

$$q_1^2 = -Q_1^2\,, \qquad q_2^2 = -Q_2^2\,,$$
$$q_3^2 = -Q_3^2 = -Q_1^2 - Q_2^2 - 2Q_1Q_2\tau\,, \qquad \boxed{q_4^2 = 0}$$

- ➡ All contributions (S,P,V,A,T) described without spurious kinematic singularities
- ➡ Reshuffling of intermediate states relevant to matching on SDCs
  [Colangelo et. al., 2020][Leutgeb et. al. 2019+20+22][Bijnens et. al. 2019+20+21+22+23]

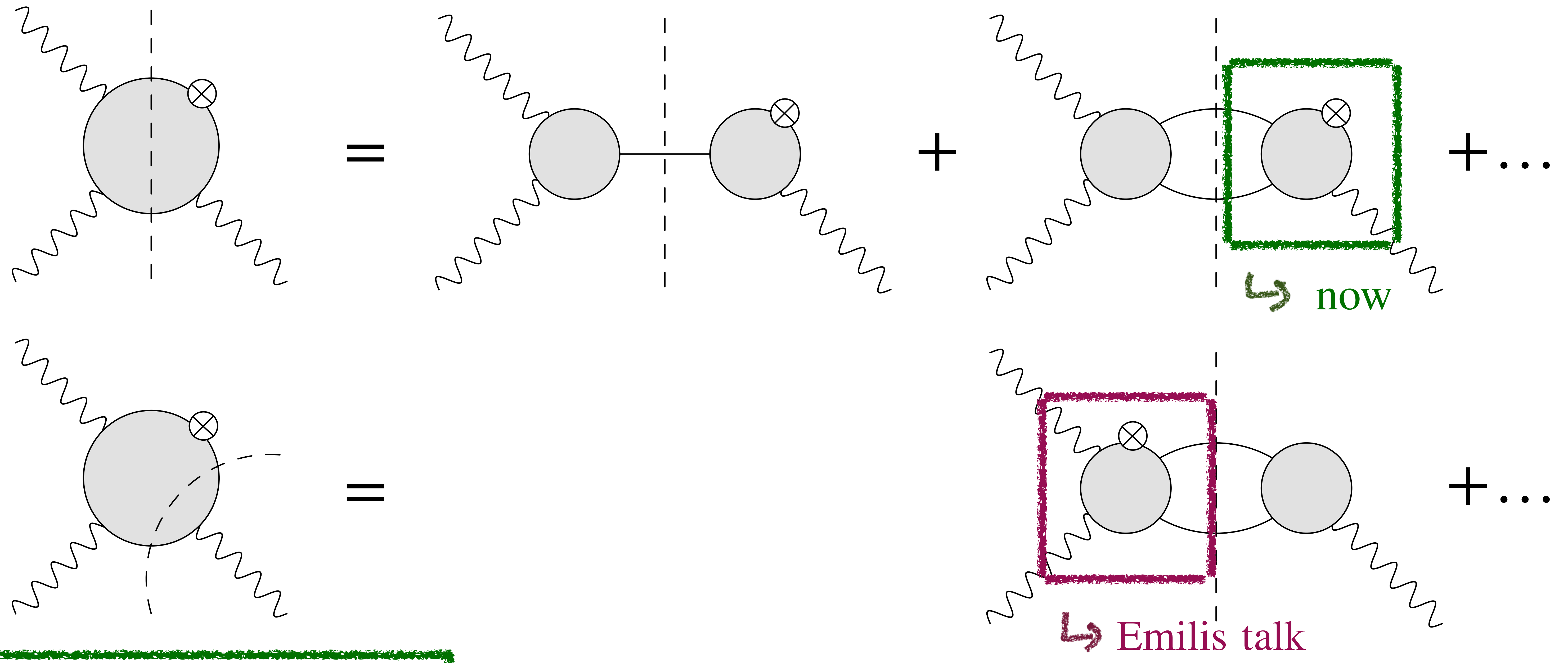
HLbL in soft kinematics

[Lüdtke, Procura, Stoffer 2023]

$$\Pi^{\mu\nu\lambda\sigma}(q_i) = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma}\, \Pi_i(s,t,u)\,, \quad q_i^2 \neq 0$$

unsubtracted DR

$$\check{\Pi}_i(s) \sim \frac{1}{\pi}\int ds' \frac{\mathrm{Im}\check{\Pi}_i(s')}{s'-s}$$



$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \sqrt{1-\tau^2}\, Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1,Q_2,\tau)\bar{\Pi}_i(Q_1,Q_2,\tau)$$

$$q_1^2 = -Q_1^2\,, \qquad q_2^2 = -Q_2^2\,,$$
$$q_3^2 = -Q_3^2 = -Q_1^2 - Q_2^2 - 2Q_1Q_2\tau\,, \qquad q_4^2 = 0$$

➡ All contributions (S,P,V,A,T) described without spurious kinematic singularities

➡ Reshuffling of intermediate states relevant to matching on SDCs
[Colangelo et. al., 2020][Leutgeb et. al. 2019+20+22][Bijnens et. al. 2019+20+21+22+23]

Content

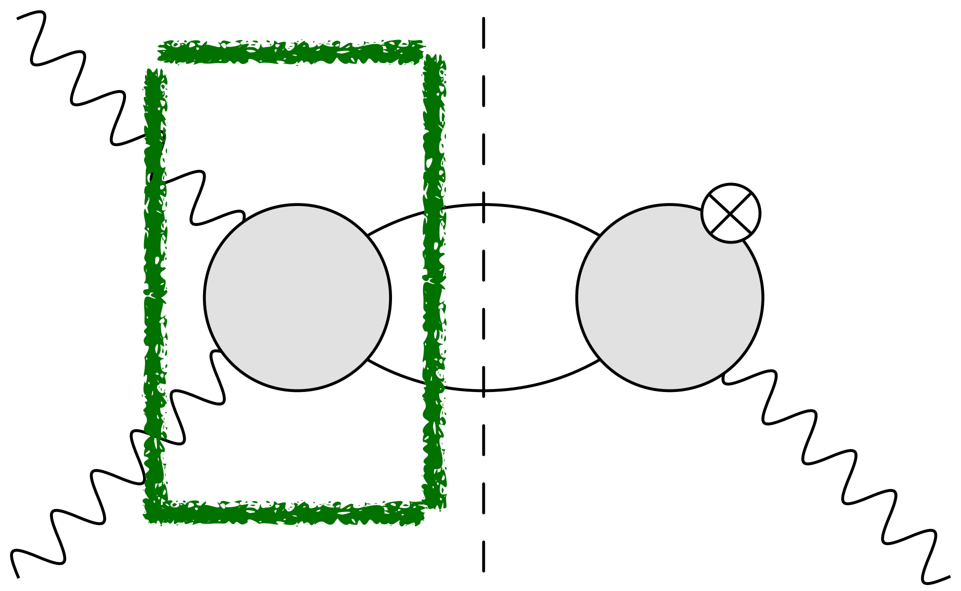
1. Short review on HLbL in soft kinematics

2. Current progress in $\gamma^*\gamma \to \pi\pi$ and $\pi\pi \to \pi\pi\gamma$

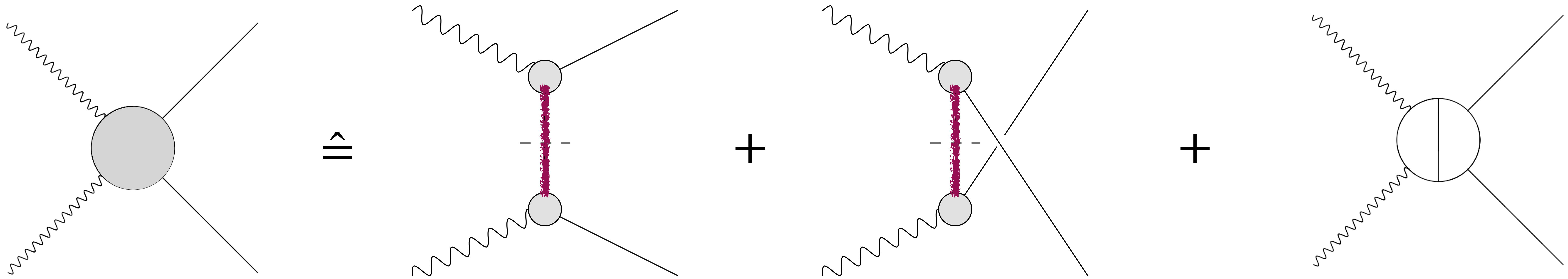
3. Subtraction constants and asymptotic constraints

Subprocess $\gamma^*\gamma \to \pi\pi$

$$W^{\mu\nu} = \sum_{i=1}^{5} T_i^{\mu\nu} A_i(s,t,u)$$



➡ Pole and rescattering terms contribute to dispersion relations



➡ Roy-Steiner equation for partial waves and Omnès solution

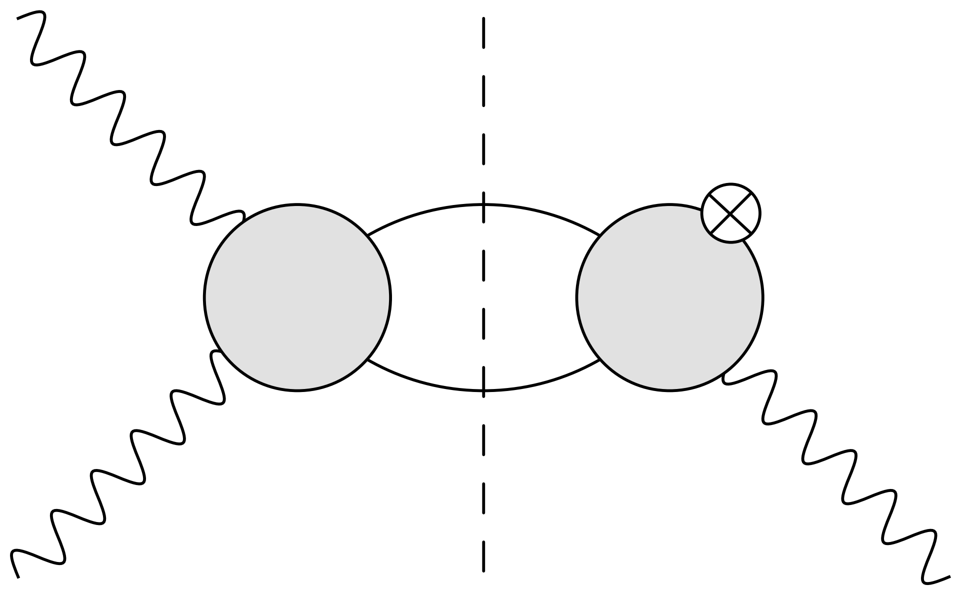
[Hoferichter, Stoffer 2023]

$$h_J(s) = \underline{\Delta_J(s)} + \frac{1}{\pi}\sum_{J'\geq J}\int_{4m_\pi^2}^{\infty} ds' K_{JJ'}(s,s') \star \mathrm{Im}\; h_{J'}(s')$$

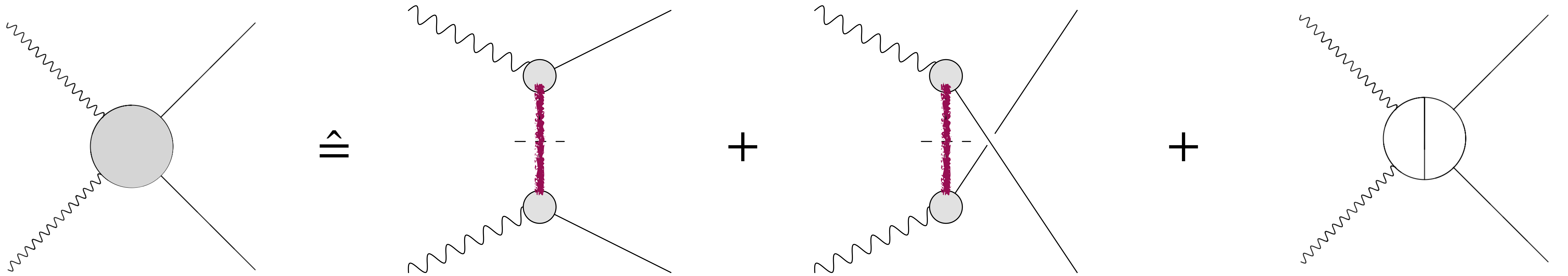
$$h_J(s) = \Delta_J(s) + \Omega_J(s)\,\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\Delta_J(s')\sin\delta_J(s')}{|\Omega_J(s')|\,s'(s'-s)}\,, \qquad \Omega_J(s) = \exp\left(\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\delta_J(s')}{s'(s'-s)}\right)$$

Subprocess $\gamma^*\gamma \to \pi\pi$

$$W^{\mu\nu} = \sum_{i=1}^{5} T_i^{\mu\nu} A_i(s,t,u)$$



➡ Pole and rescattering terms contribute to dispersion relations



➡ Roy-Steiner equation for partial waves and Omnès solution

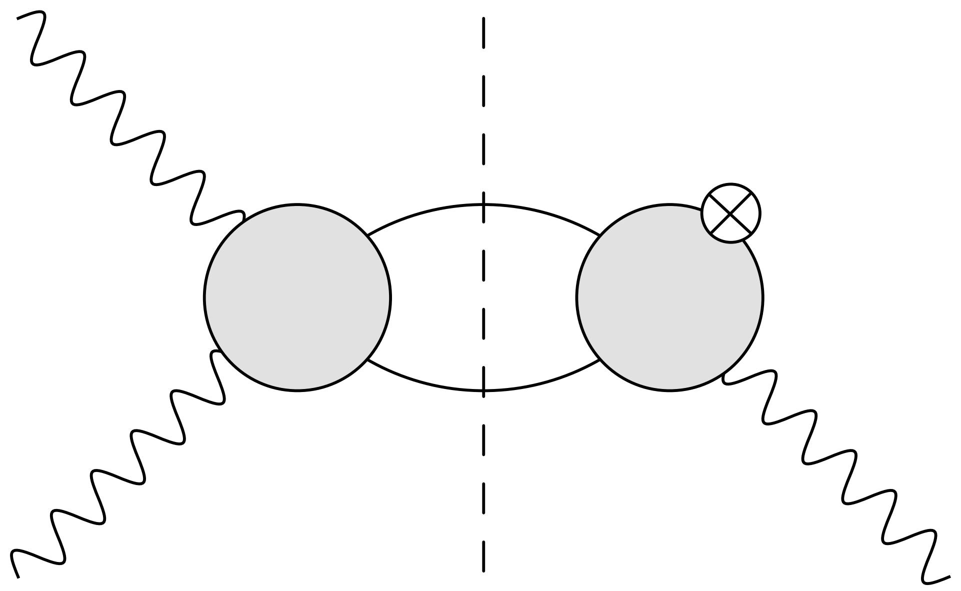
[Hoferichter, Stoffer 2023]

$$h_J(s) = \underline{\Delta_J(s)} + \frac{1}{\pi}\sum_{J'\geq J}\int_{4m_\pi^2}^{\infty} ds' K_{JJ'}(s,s') \star \text{Im } h_{J'}(s')$$

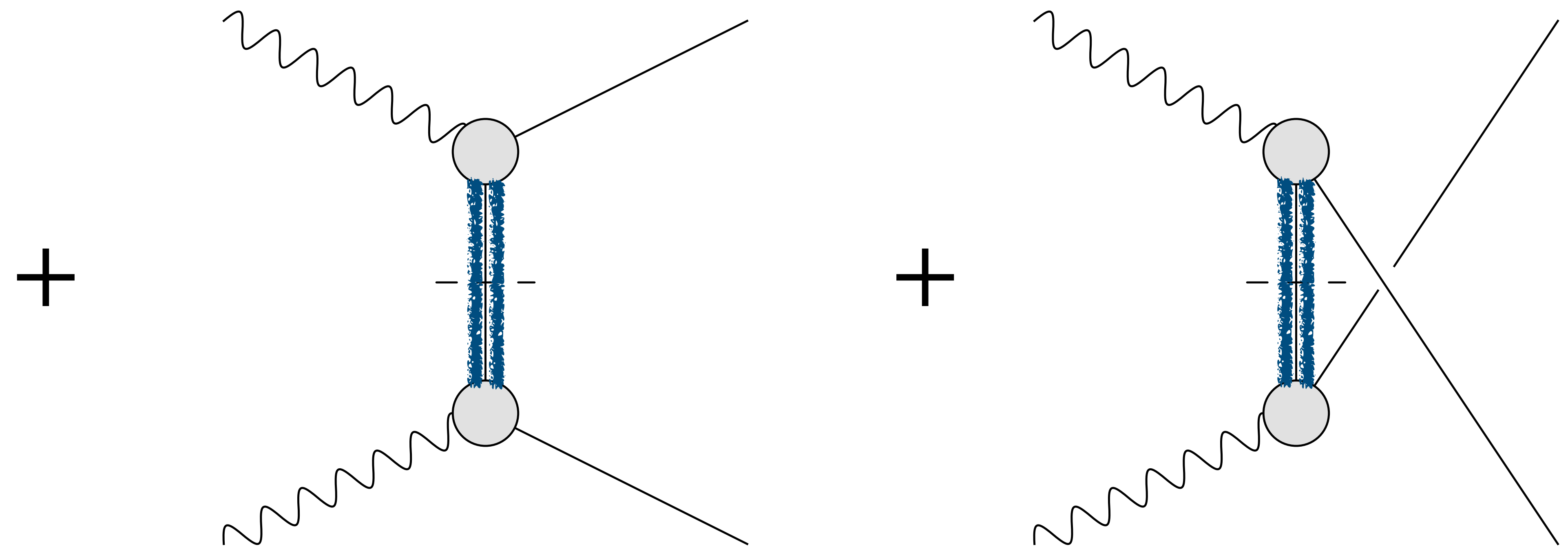
$$h_J(s) = \Delta_J(s) + \Omega_J(s)\,\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\Delta_J(s')\sin\delta_J(s')}{|\Omega_J(s')|\, s'(s'-s)}\,, \qquad \Omega_J(s) = \exp\left(\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\delta_J(s')}{s'(s'-s)}\right)$$

Subprocess $\gamma^*\gamma \to \pi\pi$

$$W^{\mu\nu} = \sum_{i=1}^{5} T_i^{\mu\nu} A_i(s,t,u)$$



➡ Consideration of left-hand cuts (LHCs), mediation of vector meson resonances



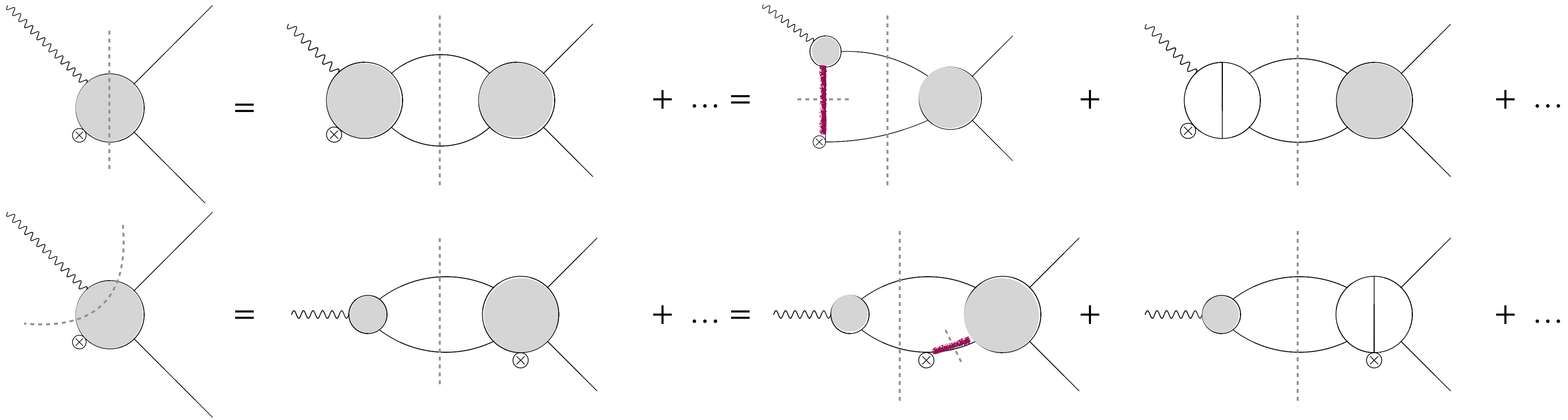
➡ Leads to a modification the Roy-Steiner equations and the Omnès solution

$$h_J(s) = \Delta_J(s) + \boxed{h_J^{\mathrm{V,fixed-s}}(s)} + \frac{1}{\pi}\sum_{J'\geq J}\int_{4m_\pi^2}^{\infty} ds' K_{JJ'}(s,s') \star \mathrm{Im}\ h_{J'}(s')$$

$$h_J(s) = \Delta_J(s) + \underline{h_J^{\mathrm{V,fixed-s}}}(s) + \Omega_J(s)\,\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{(\Delta_J + \underline{h_J^{\mathrm{V,fixed-s}}})(s')\sin\delta_J(s')}{|\Omega_J(s')|\,s'(s'-s)}$$

Subprocess $\gamma^*\gamma \to \pi\pi$ in soft kinematics

- Strategy: Derive Roy-Steiner equations, solve the Omnès problem



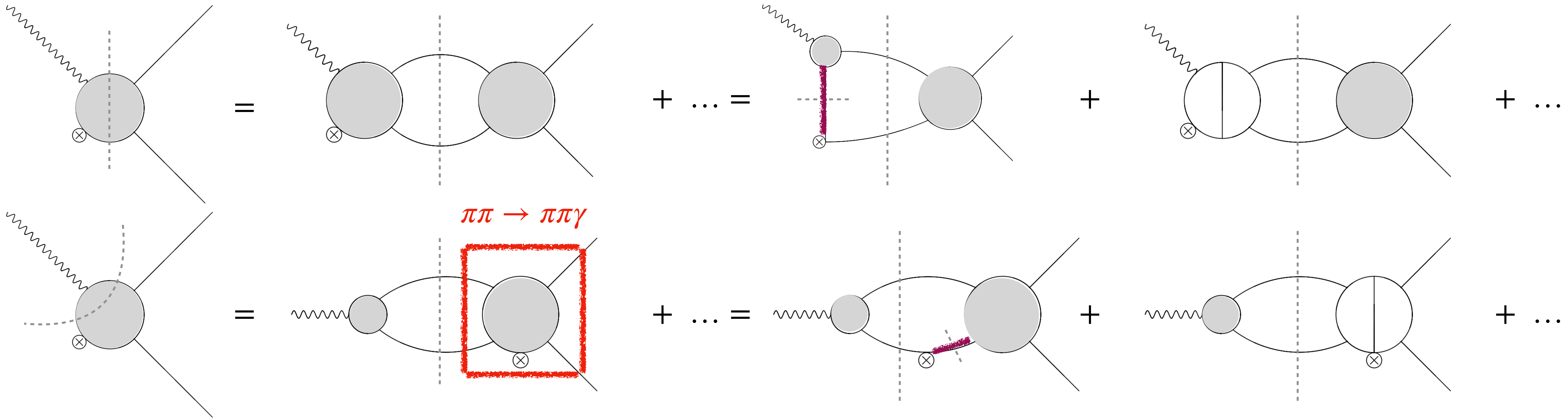
- General feature: only S- and D-waves are required for the reconstruction

$$A^V_{1,\text{soft}} = \frac{2(q_1^2 - 4m_\pi^2)}{M_V^2 - m_\pi^2} = \sum_{i=1}^{5} \left(c_{1i}(s,z) h^V_{0,i}(s) + c_{2i}(s,z) h^V_{2,i}(s) \right) , \qquad h_{J,i}(s) \propto \lambda^{J/2-1}(s, q_1^2, q_2^2) \tilde{h}_{J,i}(s)$$

- To be done: explicit calculation of LHC and pion poles, soft-finite terms

Subprocess $\gamma^*\gamma \to \pi\pi$ in soft kinematics

➡ Strategy: Derive Roy-Steiner equations, solve the Omnès problem



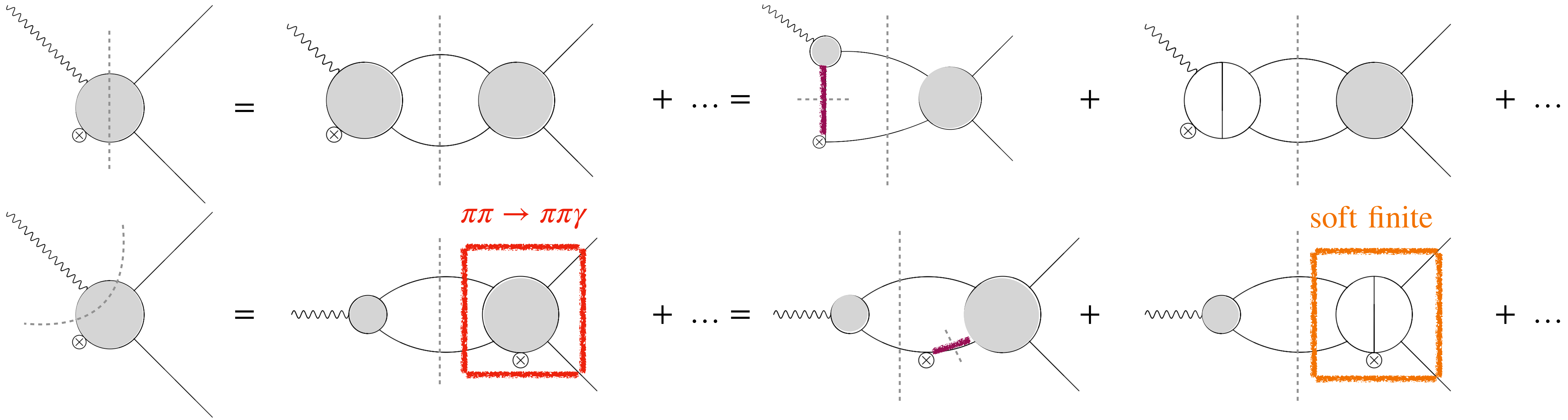
➡ General feature: only S- and D-waves are required for the reconstruction

$$A^V_{1,\text{soft}} = \frac{2(q_1^2 - 4m_\pi^2)}{M_V^2 - m_\pi^2} = \sum_{i=1}^{5} \left(c_{1i}(s,z) h^V_{0,i}(s) + c_{2i}(s,z) h^V_{2,i}(s) \right) , \qquad h_{J,i}(s) \propto \lambda^{J/2-1}(s, q_1^2, q_2^2) \tilde{h}_{J,i}(s)$$

➡ To be done: explicit calculation of LHC and pion poles, soft-finite terms

Subprocess $\gamma^*\gamma \to \pi\pi$ in soft kinematics

➡ Strategy: Derive Roy-Steiner equations, solve the Omnès problem



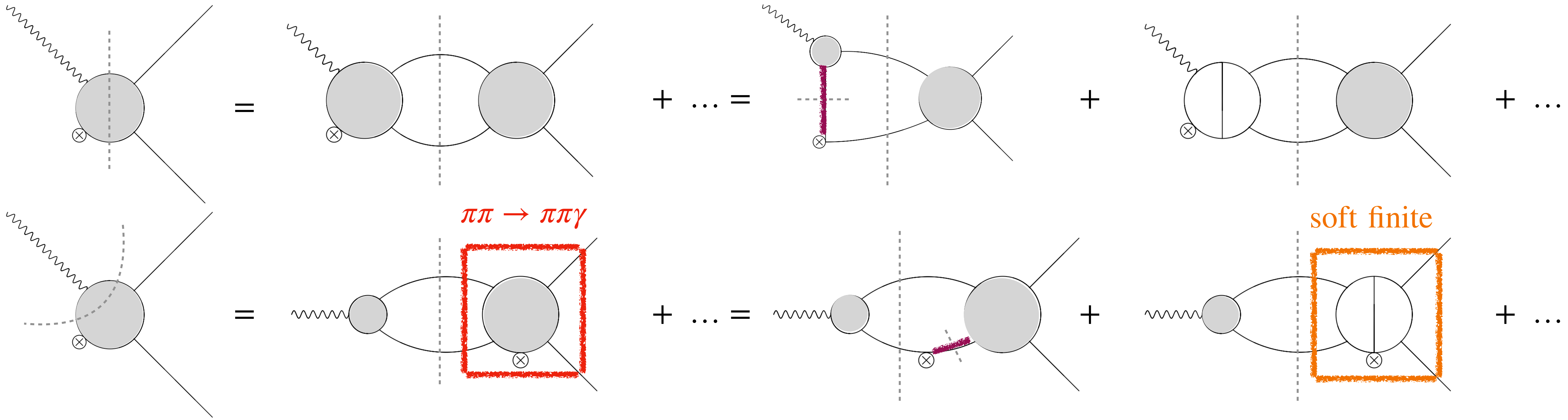
➡ General feature: only S- and D-waves are required for the reconstruction

$$A^V_{1,\text{soft}} = \frac{2(q_1^2 - 4m_\pi^2)}{M_V^2 - m_\pi^2} = \sum_{i=1}^{5} \left(c_{1i}(s,z) h^V_{0,i}(s) + c_{2i}(s,z) h^V_{2,i}(s) \right) , \qquad h_{J,i}(s) \propto \lambda^{J/2-1}(s, q_1^2, q_2^2) \tilde{h}_{J,i}(s)$$

➡ To be done: explicit calculation of LHC and pion poles, soft-finite terms

Subprocess $\gamma^*\gamma \to \pi\pi$ in soft kinematics

➡ Strategy: Derive Roy-Steiner equations, solve the Omnès problem



➡ General feature: only S- and D-waves are required for the reconstruction

$$A^V_{1,\text{soft}} = \frac{2(q_1^2 - 4m_\pi^2)}{M_V^2 - m_\pi^2} = \sum_{i=1}^{5} \left(c_{1i}(s,z) h^V_{0,i}(s) + c_{2i}(s,z) h^V_{2,i}(s) \right), \qquad h_{J,i}(s) \propto \lambda^{J/2-1}(s, q_1^2, q_2^2)\tilde{h}_{J,i}(s)$$

↳ 0, for $J > 2$

➡ To be done: explicit calculation of LHC and pion poles, soft-finite terms

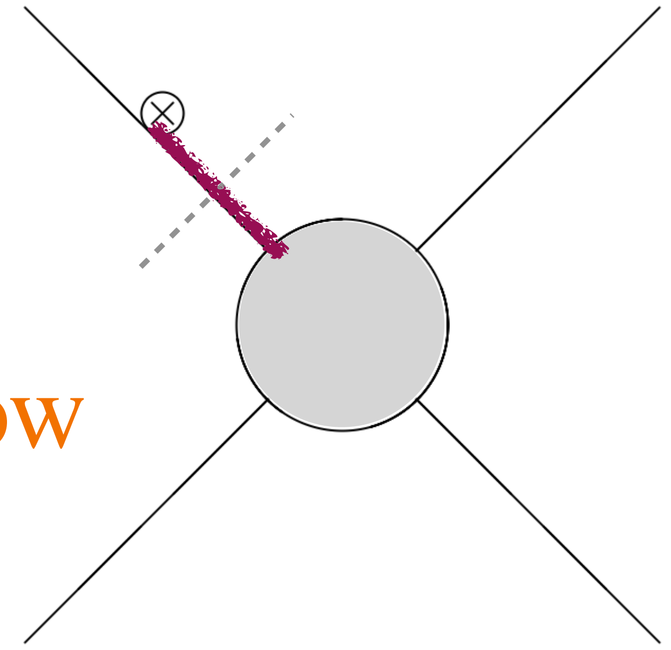
Reconstruction of $\pi\pi \to \pi\pi\gamma$

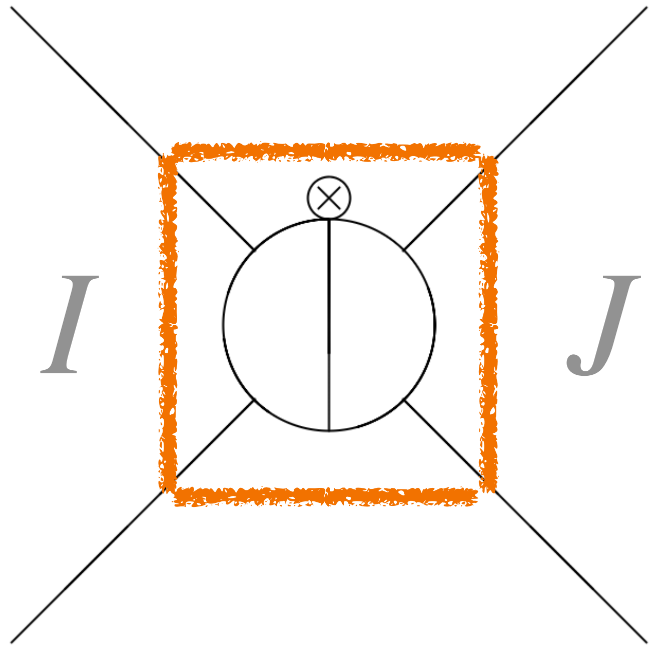
[Lüdtke, PhD thesis 2023] [Lüdtke, Procura, Stoffer, 2023][Lüdtke, Procura, Stoffer, JT, ~2025]

➡ BTT decomposition split into singular and regular contributions

$$\mathcal{M}^\mu = \sum_{i=1}^{6} \hat{T}_i^\mu \hat{\mathcal{M}}_i \,, \qquad \hat{\mathcal{M}}_i = \hat{\mathcal{M}}_i^{\text{pole}} + \hat{\mathcal{M}}_i^{\text{regular}}$$

done ↵ ↳ discussed now





➡ Soft-finite terms are dispersively reconstructed

$$\mathcal{M}_i^{(IJ)} \sim P_{n_i}^{(IJ)}(s) + \frac{s^3}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\operatorname{Im} f_i^{(IJ)}(s')}{s'^3(s'-s)(s'-4m_\pi^2)} + (t-\text{terms}) + (u-\text{terms})$$

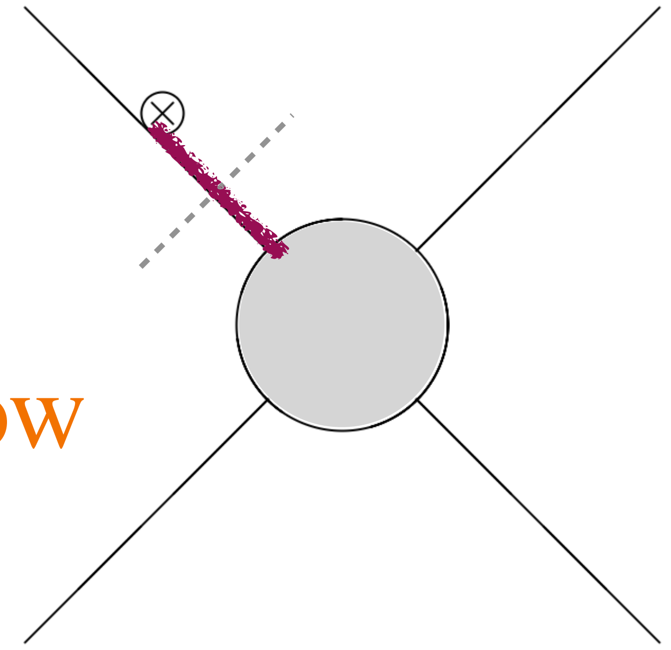
Reconstruction of $\pi\pi \to \pi\pi\gamma$

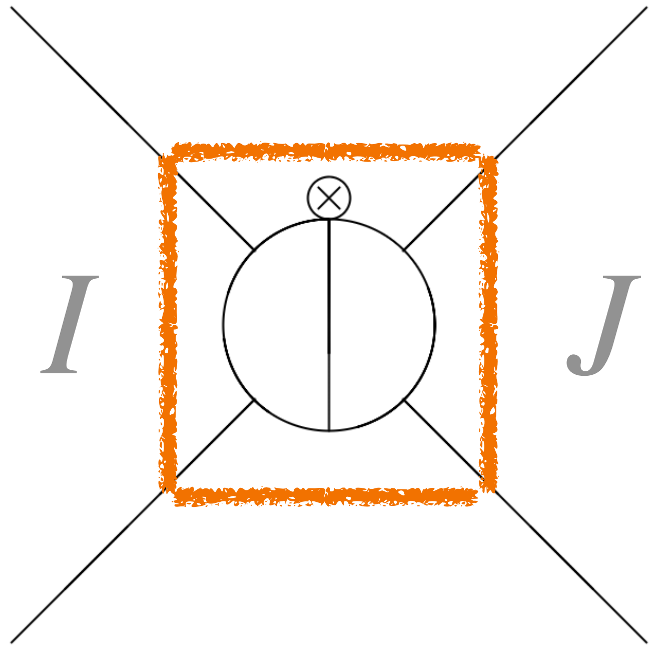
[Lüdtke, PhD thesis 2023] [Lüdtke, Procura, Stoffer, 2023][Lüdtke, Procura, Stoffer, JT, ~2025]

➡ BTT decomposition split into singular and regular contributions

$$\mathcal{M}^\mu = \sum_{i=1}^{6} \hat{T}_i^\mu \hat{\mathcal{M}}_i\,, \qquad \hat{\mathcal{M}}_i = \hat{\mathcal{M}}_i^{\text{pole}} + \hat{\mathcal{M}}_i^{\text{regular}}$$

done ↵ ↳ discussed now





➡ Soft-finite terms are dispersively reconstructed

$$\mathcal{M}_i^{(IJ)} \sim P_{n_i}^{(IJ)}(s) + \frac{s^3}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\operatorname{Im} f_i^{(IJ)}(s')}{s'^3(s'-s)(s'-4m_\pi^2)} + (t-\text{terms}) + (u-\text{terms})$$

subtraction polynomial

Content

Subtraction constants

[Lüdtke, Procura, Stoffer, JT, ~2025]

$$\mathscr{M}_i^{(IJ)} \sim P_{n_i}^{(IJ)}(s) + \frac{s^3}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\mathrm{Im}\, f_i^{(IJ)}(s')}{s'^3(s'-s)(s'-4m_\pi^2)} + (t-\text{terms}) + (u-\text{terms})$$

➡ Set of amplitudes for definite isospins I and J, labelled by n, can be diagonalized

$$\mathscr{M}_D^n = \Omega_n(s)\left[P_n(s) + \frac{s^{p_n}}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^{p_n}} \frac{g_D^n(s)\sin\delta_n(s')}{|\Omega_n(s')|(s'-s)}\right], \qquad \Omega_n(s) = \exp\left(\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\delta_n(s')}{s'-s}\right)$$

➡ Without further assumptions, there are 27 free parameters

➡ We use asymptotic constraints to reduce their number to 3

$$\lim_{s\to s_0<\infty} \delta_n(s) = k_n\pi\,, \qquad \int_{4m_\pi^2}^{\infty} ds' \sin\delta_n(s') \to \int_{4m_\pi^2}^{s_0} ds' \sin\delta_n(s')$$

Subtraction constants

[Lüdtke, Procura, Stoffer, JT, ~2025]

$$\mathcal{M}_i^{(IJ)} \sim P_{n_i}^{(IJ)}(s) + \frac{s^3}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\mathrm{Im}\, f_i^{(IJ)}(s')}{s'^3(s'-s)(s'-4m_\pi^2)} + (t-\text{terms}) + (u-\text{terms})$$

➡ Set of amplitudes for definite isospins I and J, labelled by n, can be diagonalized

$$\mathcal{M}_D^n = \Omega_n(s)\left[P_n(s) + \frac{s^{p_n}}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^{p_n}} \frac{g_D^n(s)\sin\delta_n(s')}{|\Omega_n(s')|(s'-s)}\right], \qquad \Omega_n(s) = \exp\left(\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\delta_n(s')}{s'-s}\right)$$

➡ Without further assumptions, there are 27 free parameters

➡ We use asymptotic constraints to reduce their number to 3

$$\lim_{s\to s_0<\infty} \delta_n(s) = k_n\pi\,, \qquad \int_{4m_\pi^2}^{\infty} ds' \sin\delta_n(s') \to \underbrace{\int_{4m_\pi^2}^{s_0}}_{\text{finite}} ds' \sin\delta_n(s')$$

Conclusion

- ➡ Work in progress for HLbL in soft kinematics
- ➡ New insights from individual processes

  - S- and D-waves reconstruct the whole amplitude for $\gamma^*\gamma \to \pi\pi$ in the soft limit

  - Tailored strategies to handle singular and regular terms
- ➡ Assumptions on asymptotic behavior for the soft-finite terms need to be tested
- ➡ Goal: Compare general and soft kinematics for $\gamma^*\gamma \to \pi\pi$

Thank you for the attention!