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Isospin-breaking corrections to a_μ^{HVP} with the CCS Method

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a_μ^{HVP} from CCS [1706.01139]

$$a_\mu^{\text{HVP}} = \int_z H_{\lambda\sigma}(z) G_{\lambda\sigma}(z), \qquad H_{\lambda\sigma}(z) = -\delta_{\lambda\sigma}\mathcal{H}_1(|z|) + \frac{z_\lambda z_\sigma}{|z|^2}\mathcal{H}_2(|z|) + \partial_\lambda\left(z_\sigma f(|z|)\right)$$

$H_{\lambda\sigma}(z)$: Covariant coordinate-space (CCS) kernel (not unique)
$G_{\lambda\sigma}(z) = \langle j_\lambda^{em}(z) j_\sigma^{em}(0)\rangle_{\mathrm{QCD+QED}}$: vector-vector correlator of the e.m. vector current

Expand correlator in α around isosymmetric QCD:

$$\begin{aligned} G_{\lambda\sigma}(z) = & \left\langle j_\lambda^{em}(z) j_\sigma^{em}(0)\right\rangle_{\mathrm{QCD}} \\ & - \frac{e^2}{2}\lim_{\Lambda\to\infty}\left\{\int_{x,y} \delta_{\nu\rho}[\mathcal{G}(x-y)]_\Lambda \langle j_\lambda^{em}(z) j_\nu^{em}(y) j_\rho^{em}(x) j_\sigma^{em}(0)\rangle_{\mathrm{QCD}} + C_T(\Lambda)\right\} \\ & + O(\alpha^2) \end{aligned}$$

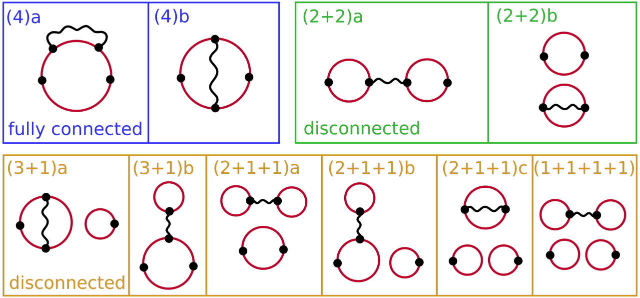
QED corrections to the HVP

$$a_\mu^{\mathrm{HVP,NLO}}(\Lambda) = -\frac{e^2}{2}\int_{x,y,z} H_{\lambda\sigma}(z)\delta_{\nu\rho}\left[\mathcal{G}(x-y)\right]_\Lambda \langle j_\lambda^{em}(z) j_\nu^{em}(x) j_\rho^{em}(y) j_\sigma^{em}(0)\rangle + C_T(\Lambda)$$

Pauli-Villars (PV) regulated photon propagator [2209.02149] :

$$[\mathcal{G}(y)]_\Lambda = \frac{1}{4\pi^2|y|^2} - \frac{\Lambda K_1\left(\Lambda\frac{|y|}{\sqrt{2}}\right)}{2\sqrt{2}\pi^2|y|} + \frac{\Lambda K_1(\Lambda|y|)}{4\pi^2|y|}$$

[2505.24344] [2501.03192]



The connected isospin-violating part [2505.24344]

$$a_\mu^{\text{HVP,NLO,38}}(\Lambda) = -\frac{e^2}{2}\int_{x,y,z} H_{\lambda\sigma}(z)\delta_{\nu\rho}\left[\mathcal{G}(x-y)\right]_\Lambda \langle j_\lambda^3(z) j_\nu^{em}(x) j_\rho^{em}(y) j_\sigma^8(0)\rangle_{\text{QCD}} + C_T(\Lambda)$$

$j_\mu^3(z)$: Isovector current, $j_\mu^8(z)$: Isoscalar current

$\langle j_\lambda^3(z) j_\mu^{em}(x) j_\nu^{em}(y) j_\sigma^8(0)\rangle_{\text{QCD}}^{(4)} = -2f_{\mathcal{Q}}^{(4)}$ Re[2Tr() + Tr()]

$$C_T(\Lambda) = \frac{\Delta M_K^{phys} - \Delta M_K^{em}(\Lambda)}{\langle K_{\vec{0}}^+ | \bar{u}u - \bar{d}d | K_{\vec{0}}^+\rangle} \left.\frac{\partial a_\mu^{\text{HVP,38}}}{\partial(m_u - m_d)}\right|_{m_u+m_d, m_s, g_0; \alpha=0}$$

Here the charge factor $f_{\mathcal{Q}}^{(4)} = 1/36$ and $\Delta M_K^{phys} = -3.934$ MeV

All calculations are done at the SU(3) symmetric point
→ For final result only (2+2)a diagram needs to be added

The connected isospin-violating part [2505.24344]

$$a_{\mu,(4)}^{\text{HVP},38} = 2f_{\mathcal{Q}}^{(4)} e^2 \pi^2 \int_0^\infty d|y|\ |y|^3 \sum_{x,z} H_{\lambda\sigma}(z)\delta_{\nu\rho}\left[\mathcal{G}(x-y)\right]_\Lambda \times$$

$\times$ Re[2Tr() + Tr()]

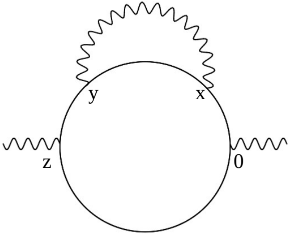


Fig. 2.a)
Self-energy diagram

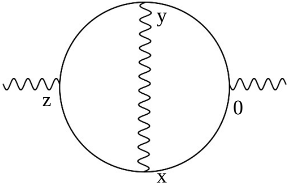


Fig. 2.b)
2-Loop diagram

The currents at the x- and z-vertices can be either local or conserved.

Gluonless crosscheck

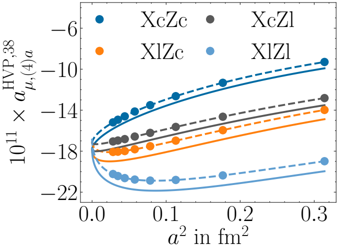


Fig. 3.a)
Self-energy

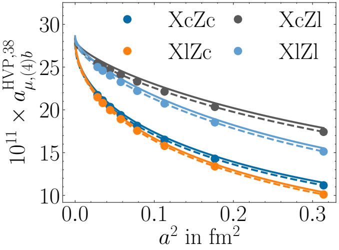


Fig. 3.b)
2-Loop

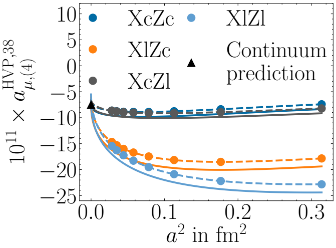


Fig. 3.c)
Total=2×SE+2Loop

Fit function: $f_{fit}(a) = c_0 + c_1\, a + c_2\, a^2 + c_3\, a^3$, PV-mass: $\Lambda = 3\, m_\mu$

Table: The values are given in units of 10^{-11}. The expected value is -7.5×10^{-11}.

	XlZl	XcZl	XlZc	XcZc
Total	-6.90(15)	-7.36(16)	-7.44(18)	-7.56(18)

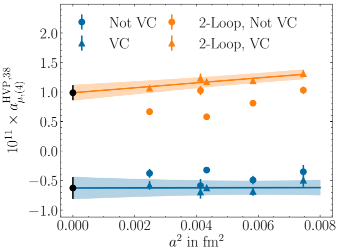


Fig. 4.a)
Self-energy and 2-Loop

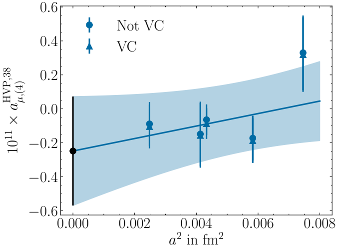


Fig. 4.b)
Total=2×SE+2Loop

Fit function: $f_{fit}(a, m_\pi L) = c_0 + c_1\, a^2 + c_2\, e^{-\frac{m_\pi L}{2}}$, PV-mass: $\Lambda = 16\, m_\mu$

Continuum values from fit:

Self-Energy:	2-Loop:	Total:
$-0.63(19) \times 10^{-11}$	$0.99(13) \times 10^{-11}$	$-0.25(33) \times 10^{-11}$

Results for isospin-violating part at the $SU(3)_f$ point

$a_{\mu,(4)}^{\mathrm{HVP},38}(\Lambda)$	$-0.249(321)_{\mathrm{st.}}$
$a_{\mu,(2+2)a}^{\mathrm{HVP},38}$	$-0.53(17)_{\mathrm{st.}}$
$C_T(\Lambda)$	$11.27(1.24)_{\mathrm{st.}}(66)_{\mathrm{sy.}}$
$a_{\mu}^{\mathrm{HVP},38}(\Lambda)$	$10.49(1.29)_{\mathrm{st.}}(66)_{\mathrm{sy.}}$

Table: Results for $\Lambda = 16\, m_\mu$.
All values are given in units of 10^{-11}.

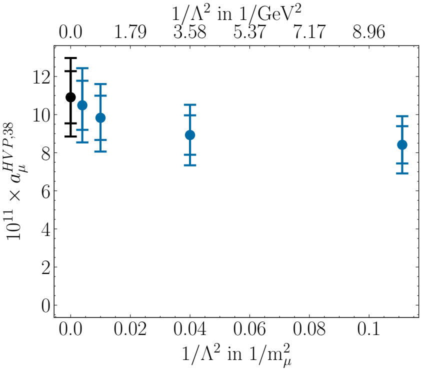
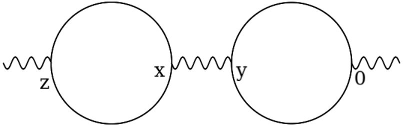


Fig. 5.a)
Λ extrapolation.

$$2\, a_\mu^{\mathrm{HVP},38} = 21.8(2.8)_{\mathrm{stat}}(1.4)_{\mathrm{syst}} \times 10^{-11} \qquad (M_\pi = M_K \simeq 416\,\mathrm{MeV})$$

The (2+2)a contribution [2501.03192]

The (2+2)a contribution is UV finite
$\Rightarrow$ Can take $\Lambda \to \infty$
$\Rightarrow$ No additional counterterms



$$a_{\mu,(2+2)a}^{\text{HVP,NLO}} = -\frac{e^2}{2} \int_{x,y,z} H_{\lambda\sigma}(z)\delta_{\nu\rho}\mathcal{G}_0(x-y)\langle j_\lambda^{em}(z) j_\nu^{em}(x) j_\rho^{em}(y) j_\sigma^{em}(0)\rangle_{\text{QCD}}^{(2+2)a}$$

$$\langle j_\lambda^{em}(z) j_\nu^{em}(x) j_\rho^{em}(y) j_\sigma^{em}(0)\rangle_{\text{QCD}}^{(2+2)a} = \sum_{f,f' \in \{l,s\}} C^{(f,f')} \langle \hat{\Pi}_{\lambda\nu}^{f}(z,x) \hat{\Pi}_{\rho\sigma}^{f'}(y,0)\rangle_{\text{QCD}}$$

$$\hat{\Pi}_{\lambda\sigma}^{f}(x,y) = \Pi_{\lambda\sigma}^{f}(x,y) - \langle \Pi_{\lambda\sigma}^{f}(x,y)\rangle_{\text{QCD}}$$

$$\Pi_{\lambda\sigma}^{f}(x,y) = -\text{Re}\left[\text{Tr}\left(D_f^{-1}(y,x)\gamma_\lambda D_f^{-1}(x,y)\gamma_\sigma\right)\right]$$

Phenomenological description

$$a_\mu^{(2+2)a-ll,\text{model}} = -\frac{25}{9} a_\mu^{\pi^0}(m_\pi, m_{V,\pi}, F_\pi) + \hat{c}_\eta^{(ll)}(m_\pi, m_K, m_\eta, m_{\eta'}, \theta) a_\mu^{\eta}(m_\eta, m_{V,\eta}, F_\eta) + \frac{50}{81} a_\mu^{\pi^+\pi^-}(m_\pi, m_{V,\pi}) + \hat{c}_{\eta'}^{(ll)}(m_\pi, m_K, m_\eta, m_{\eta'}, \theta) a_\mu^{\eta'}(m_{\eta'}, m_{V,\eta'}, F_{\eta'})$$

- PME shows only mild chiral dependence
- Charged pion loop increases drastically when approaching the physical point

$$a_\mu^{(2+2)a,\pi^+\pi^-} \propto m_\pi^{-3}$$

$$\text{for } m_\pi^{\text{phys}} \leq m_\pi \leq m_\pi^{\text{SU(3)}}$$

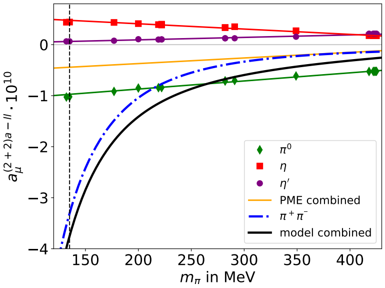


Fig. 6.a)
m_π dependence of model

Approximation of the tail

- Tail of the integrand is reconstructed from model $f^{\pi^0,\eta,\eta'}(|x|) + f^{\pi^+\pi^-}(|x|)$ for $|x| > x_{\text{cut}}$
- Integrand for π^0, η, η' (PME) is calculated analytically with VMD forr factor
- Shape of $\pi^+\pi^-$ integrand is obtained in scalar QED
- Assign 50% of tail contribution as systematic error

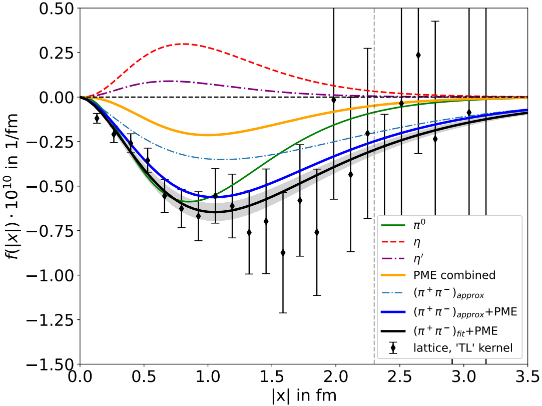


Fig. 7.a)
Model and lattice integrand on D450

Results of the (2+2)a contribution at the physical point

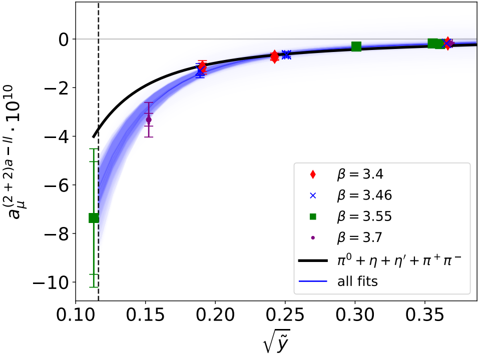


Fig. 8.a)
Chiral extrapolation of the (2+2)a contribution.

$$a_\mu^{(2+2)a-ll} = -6.42(89)_{\rm stat}(81)_{\rm tail}(51)_{\rm finite-size}(30)_{\rm extra} \times 10^{-10}$$

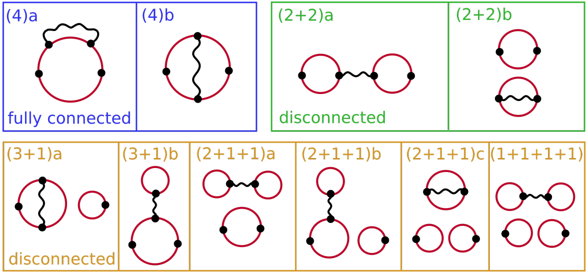
Summary and Outlook

So far:

- Successful calculation of connected diagrams at SU(3) point with CCS method
- Successful calculation of (2+2)a diagram at physical point
- Exploratory calculations of (2+2)b diagram at physical point [2501.03192]

Outlook:

- Calculations of connected diagrams towards the physical point
- Calculations of additional disconnected diagrams



Additional Slides

The connected isospin-violating Part [2505.24344]

$$a_\mu^{HVP,NLO,38}(\Lambda) = -\frac{e^2}{2}\int_{x,y,z} H_{\lambda\sigma}(z)\delta_{\nu\rho}\,[\mathcal{G}(x-y)]_\Lambda\,\langle j^3_\lambda(z) j^{em}_\nu(x) j^{em}_\rho(y) j^8_\sigma(0)\rangle + C_T(\Lambda)$$

$j^3_\mu(z)$: Isovector current, $j^8_\mu(z)$: Isoscalar current

$\langle j^3_\lambda(z) j^{em}_\mu(x) j^{em}_\nu(y) j^8_\sigma(0)\rangle = -\,2 f^{(4)}_{\mathcal{Q}}$ Re[2Tr() + Tr()]

$$C_T(\Lambda) = \frac{\Delta M_K^{phys} - \Delta M_K^{em}(\Lambda)}{\langle K^+_{\vec{0}}|\bar{u}u - \bar{d}d|K^+_{\vec{0}}\rangle}\;\frac{\partial a_\mu^{HVP,38}}{\partial(m_u - m_d)}\bigg|_{m_u+m_d,m_s,g_0;\alpha=0}$$

Here the charge factor $f^{(4)}_{\mathcal{Q}} = 1/36$ and $\Delta M_K^{phys} = -3.934$ MeV

The Kaon Mass Splitting with a PV Cutoff Λ (Counterterm)

$$\Delta m_K^{em}(\Lambda) = (m_{K^+} - m_{K^0})(\Lambda)$$

At SU(3) point only two diagrams contribute:

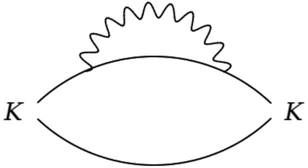


Fig. 10.a)
K1 diagram

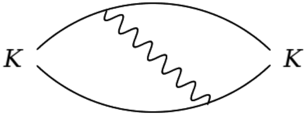


Fig. 10.b)
K2 diagram

Known analytic large distance (elastic) behavior
→ Use lattice data only for short distance part

Computational Strategy

1. Compute diagrams for different source-sink separation times
2. Restrict lattice data to short distance part
3. Extrapolate to infinite separation times and zero lattice spacing
4. Add long-distance part using the kaon e.m. form factor
5. Repeat for different PV-masses ($\Lambda/m_\mu \in [3, 5, 8, 10, 16, 20, 32, 50, 64]$)

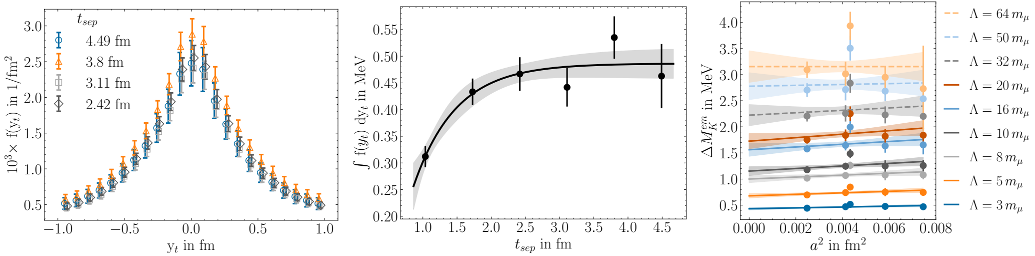


Fig. 11.a)
Data limited to $y_t < 1$ fm

Fig. 11.b)
Extrapolation of left data

Fig. 11.c)
Continuum extrapolation

Large PV-Mass Behavior

Use Operator Product Expansion [2209.02149] to predict divergent terms for $\Lambda \to \infty$:

$$(m_{K^+} - m_{K^0})_{QED}(\Lambda) \approx \frac{3\alpha}{2\pi} \log\left(\frac{\Lambda}{\mu_{IR}}\right)(Q_u^2 - Q_d^2) m_l \frac{\partial m_K}{\partial m_l} \quad (1)$$

$$= \mathcal{C} \log\left(\frac{\Lambda}{\mu_{IR}}\right), \ \mathcal{C} \approx 0.12 \text{ MeV}$$

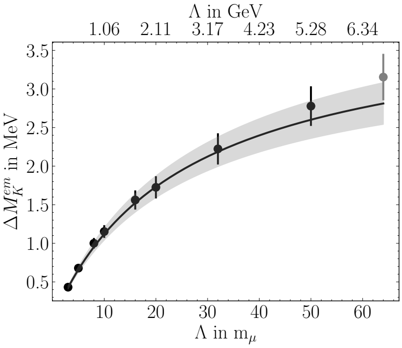


Fig. 12.a)
PV-mass extrapolation

Fit function:

$$f_{fit}(\Lambda) = c_0 \frac{\Lambda}{\Lambda + c_1} + \mathcal{C} \log\left(\frac{\Lambda + c_2}{c_2}\right) \quad (2)$$

Reproduces expected behavior for $\Lambda \to \infty$ and $\Lambda \to 0$
χ^2/DOF of 0.92

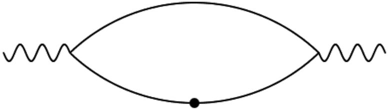


Fig. 13.a)
HVP mass insertion

$$\frac{\partial a_\mu^{HVP}}{\partial m_l} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_t \; w(x_t) G(x_t)$$

With the TMR kernel $w(x_t)$

This calculation uses stochastic wall sources

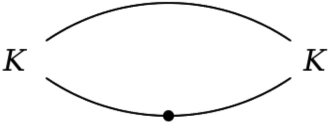


Fig. 14.a)
Kaon propagator mass insertion

Get matrix element $\langle K^+ | \bar{u}u - \bar{d}d | K^+ \rangle$ with constant fit method

Ensembles connected contribution

Id	β	$(\frac{L}{a})^3 \times \frac{T}{a}$	a [fm]	m_π [MeV]	M_{VMD} [MeV]	$m_\pi L$	L[fm]	$\hat{Z}_V$
H101	3.4	$32^3 \times 96$	0.08636	416(4)	921(13)	5.8	2.8	0.71540
B450	3.46	$32^3 \times 64$	0.07634	415(4)	942(25)	5.1	2.4	0.72645
H200	3.55	$32^3 \times 96$	0.06426	416(5)	979(26)	4.3	2.1	0.74030
N202		$48^3 \times 128$		412(5)	952(15)	6.4	3.1	
N300	3.7	$48^3 \times 128$	0.04981	419(4)	1001(23)	5.1	2.4	0.75912

$16 \times m_\mu \times a_{H101} = 0.736$

Ensembles (2+2)a contribution

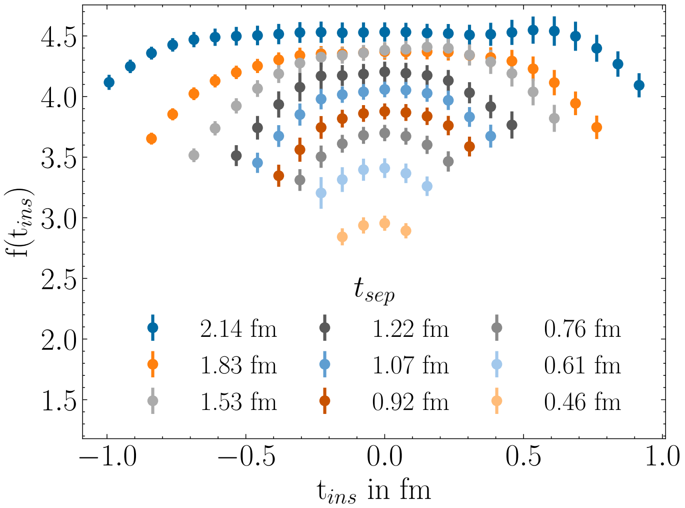
Id	β	$N_L^3 \times N_T$	a [fm]	m_π [MeV]	m_K [MeV]	$m_\pi L$	L [fm]	#sources	#confs
H101	3.4	$32^3 \times 96$	0.0849(9)	424(5)	424(5)	5.8	2.7	32	182
N101		$48^3 \times 128$		282(4)	468(5)	5.8	4.1	48	200
H105*		$32^3 \times 96$		283(4)	470(5)	3.9	2.7	64	180
C101		$48^3 \times 96$		222(3)	478(5)	4.6	4.1	96	200
S100*		$32^3 \times 128$		222(3)	478(5)	2.9	2.7	128	100
B450	3.46	$32^3 \times 96$	0.0751(8)	422(5)	422(5)	5.1	2.4	32	200
N451		$48^3 \times 128$		291(4)	468(5)	5.3	3.6	48	200
D450		$64^3 \times 128$		219(3)	483(5)	5.3	4.8	64	200
H200*	3.55	$32^3 \times 96$	0.0635(6)	423(5)	423(5)	4.4	2.0	32	200
N202		$48^3 \times 128$		418(5)	418(5)	6.5	3.0	96	200
N203		$48^3 \times 128$		349(4)	447(5)	5.4	3.0	48	200
E250		$96^3 \times 192$		132(2)	495(6)	4.1	6.1	192	200
N300	3.7	$48^3 \times 128$	0.0491(5)	425(5)	425(5)	5.1	2.4	48	200
E300		$96^3 \times 192$		177(2)	497(6)	4.2	4.7	192	200

Kaon mass splitting values

Λ/m_μ	ΔM_K^{em} [MeV]
3	0.432(21)
5	0.678(41)
8	1.001(69)

Λ/m_μ	ΔM_K^{em} [MeV]
10	1.153(83)
16	1.562(124)
20	1.726(144)

Λ/m_μ	ΔM_K^{em} [MeV]
32	2.224(202)
50	2.778(257)
64	3.154(302)



4.5
4.0
3.5
3.0
2.5
2.0
1.5
$f(t_{ins})$
-1.0
-0.5
0.0
0.5
1.0
t_{ins} in fm
t_{sep}
2.14 fm
1.83 fm
1.53 fm
1.22 fm
1.07 fm
0.92 fm
0.76 fm
0.61 fm
0.46 fm