Long-distance reconstruction of QED corrections to the hadronic vacuum polarization for the muon g-2

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The long-distance contribution of QED corrections to the hadronic vacuum polarization is particularly challenging to compute in lattice QCD+QED. Currently, it is one of the limiting factors towards matching the precision of the recent result by the Fermilab E989 experiment for the muon g-2. In this work, we present a method for obtaining high-precision results for this contribution by reconstructing exclusive finite-volume state contributions. We find relations between the pion-photon contributions of individual diagrams and demonstrate the reconstruction method with lattice QCD+QED data at a single lattice spacing of $a^{-1} \approx 1.73$ GeV and $m_\pi \approx 275$ MeV.

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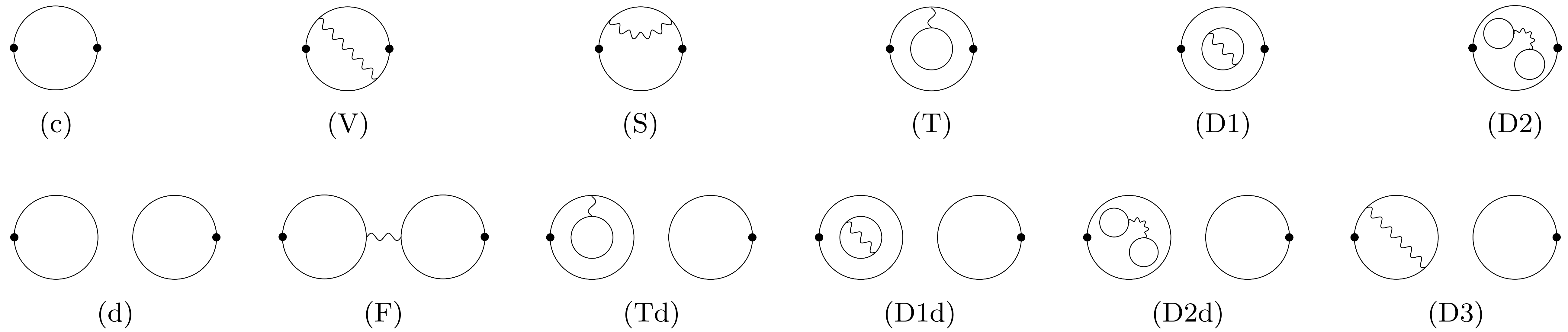
The vector-vector correlator including QED corrections at order e^2 is without SIB is

$$G(t) \equiv \frac{1}{3}\sum_i \langle V_i(t,\vec{p}=0)V_i^\dagger(t=0,\vec{p}=0)\rangle = G^{(0)}(t) + e^2 G^{(2)}(t)$$

$$V_j(t,\vec{p}) = \frac{1}{\sqrt{L^3}}\sum_{\vec{x}} e^{i\vec{x}\cdot\vec{p}}\left(i\frac{2}{3}\bar{u}(t,\vec{x})\gamma_j u(t,\vec{x}) - i\frac{1}{3}\bar{d}(t,\vec{x})\gamma_j d(t,\vec{x})\right)$$

which gives 12 diagrams



making the prefactors explicit

$$G = \frac{5}{9}(c) - \frac{1}{9}(d) + e^2\Big(-\frac{17}{81}((V)+2(S)) + \frac{25}{81}(F) + \frac{14}{81}((T)+(D3)) - \frac{10}{81}(Td) + \frac{25}{162}(D1) - \frac{5}{162}((D1d)+(D2)) + \frac{1}{162}(D2d)\Big)$$

Using a QED regulator with a finite-volume transfer matrix, a spectral representation is admitted

$$G(t) = \sum_n c_n e^{-E_n t} = \sum_n c_n^{(0)} e^{-E_n^{(0)} t} + e^2 \sum_n (c_n^{(2)} - tE_n^{(2)} c_n^{(0)}) e^{-E_n^{(0)} t}$$

$$E_n = E_n^{(0)} + e^2 E_n^{(2)}$$

$$c_n = c_n^{(0)} + e^2 c_n^{(2)} = |\langle 0|V_i|n\rangle|^2$$

Idea: study larger operator basis that includes operators that project to individual states

Minimal example: vector current and a separate operator for lowest-lying pion-photon state

$$C_{ij} = \langle O_i O_j^\dagger \rangle \qquad O_1 = V_i \qquad O_2 = O_{\pi\gamma}$$

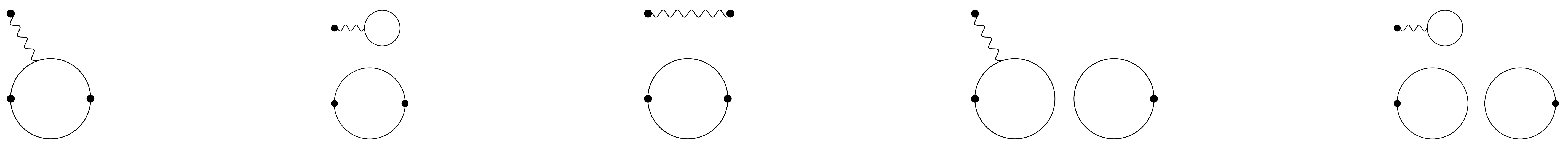
The correlation matrix

$$\begin{aligned} C(t) = &\begin{pmatrix} |V_{1,\rho}^{(0)}|^2 e^{-E_\rho^{(0)} t} & 0 \\ 0 & 0 \end{pmatrix} \\ &+ e^2 \begin{pmatrix} (2\mathrm{Re}(V_{1,\rho}^{(0)} V_{1,\rho}^{(2),*}) - t|V_{1,\rho}^{(0)}|^2 E_\rho^{(2)}) e^{-E_\rho^{(0)} t} + |V_{1,\pi\gamma}^{(1)}|^2 e^{-E_{\pi\gamma}^{(0)} t} & V_{1,\rho}^{(0)} V_{2,\rho}^{(2),*} e^{-E_\rho^{(0)} t} + V_{1,\pi\gamma}^{(1)} V_{2,\pi\gamma}^{(1),*} e^{-E_{\pi\gamma}^{(0)} t} \\ V_{2,\rho}^{(2)} V_{1,\rho}^{(0),*} e^{-E_\rho^{(0)} t} + V_{2,\pi\gamma}^{(1)} V_{1,\pi\gamma}^{(1),*} e^{-E_{\pi\gamma}^{(0)} t} & |V_{2,\pi\gamma}^{(1)}|^2 e^{-E_{\pi\gamma}^{(0)} t} \end{pmatrix} \end{aligned}$$

$$V = V^{(0)} + eV^{(1)} + e^2 V^{(2)}$$

$$V_{in} = \langle 0 | O_i | n \rangle$$

needs new diagrams



(L) (LT) (LR) (Ld) (LTd)

Full analysis allows for determination of individual state contributions to G(t).

Pion-photon operator in finite volume:

- Needs to transform in T_1^u irrep. of octahedral group
- Here: use Feynman gauge but only couple to transversal photons
- Photon and pion have fixed back-to-back momentum

Unique operator satisfying these conditions:

$$O_{\pi\gamma,\vec{p},i}(t) = \frac{e}{2} \sum_{\vec{q}\in H(\vec{p})} \Big(\big[\hat{q} \times \vec{A}^T(t,\vec{q})\big]_i O_\pi(t,-\vec{q}) \; - \; \big[\hat{q} \times \vec{A}^T(t,-\vec{q})\big]_i O_\pi(t,+\vec{q}) \Big)$$
$$= \frac{e}{2} \sum_{\vec{q}\in H(\vec{p})} \Big(\big[\hat{q} \times \vec{A}(t,\vec{q})\big]_i O_\pi(t,-\vec{q}) \; - \; \big[\hat{q} \times \vec{A}(t,-\vec{q})\big]_i O_\pi(t,+\vec{q}) \Big) \,.$$

$$A_i(t,\vec{p}) = \frac{1}{\sqrt{L^3}} \sum_{\vec{x}} e^{i\vec{x}\vec{p}} \tilde{A}_i(x)$$

$$A_j^T(t,\vec{p}) = \Big(\delta_{jm} - \hat{p}_j \hat{p}_m\Big) A_m(t,\vec{p}) \,,$$

$$O_\pi(t,\vec{p}) = \frac{i}{\sqrt{L^3}} \sum_{\vec{x}} e^{i\vec{x}\vec{p}} \bar{u}(x)\gamma_5 d(x)$$

$$\hat{p} = \frac{\vec{p}}{|\vec{p}|} \,.$$

with $H(\vec{p})$ being the orbit under chiral octahedral group (24 elements) of vector $\vec{p}$

It is useful to consider I=0, I=1, and several QED charge assignments separately with

$$O_{\pi(I_3=-1)} = i\bar{u}\gamma_5 d\,, \qquad O_{\pi(I_3=0)} = \frac{i}{\sqrt{2}}\left(\bar{u}\gamma_5 u - \bar{d}\gamma_5 d\right)\,, \qquad O_{\pi(I_3=1)} = i\bar{d}\gamma_5 u\,,$$

$$V^j_{I=1,I_3=-1} = i\bar{u}\gamma_j d\,, \qquad V^j_{I=1,I_3=0} = \frac{i}{\sqrt{2}}\left(\bar{u}\gamma_j u - \bar{d}\gamma_j d\right)\,, \qquad V^j_{I=1,I_3=1} = i\bar{d}\gamma_j u\,,$$

$$V^j_{I=0} = \frac{i}{\sqrt{2}}\left(\bar{u}\gamma_j u + \bar{d}\gamma_j d\right)\,.$$

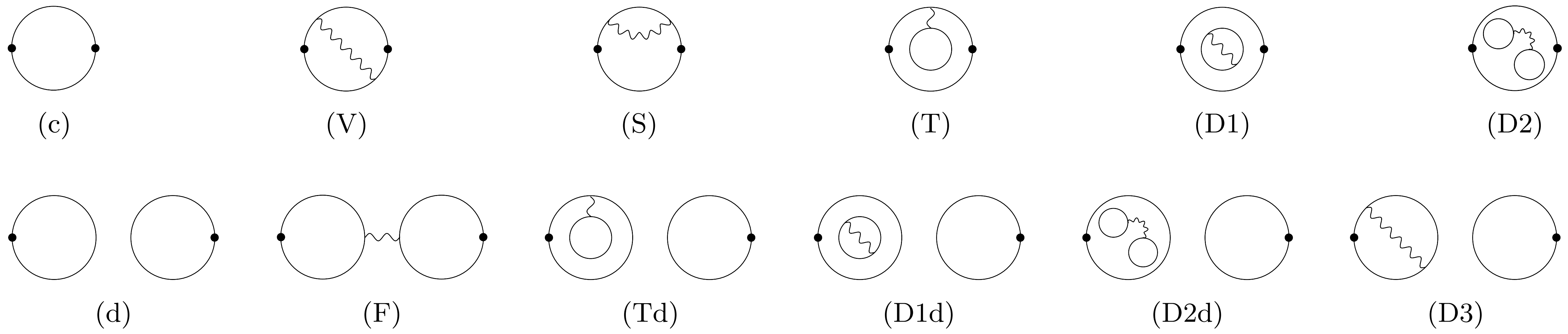
We show in the paper that various versions have the identical pion-photon contribution, which allows us to derive relations of the pion-photon contributions between individual diagrams:

$$(F) = (V)\,, \qquad 2(S) - (D1) = (V)\,, \qquad 2(T) - 4(Td) - (D1d) + 2(D3) + 2(D2d) - (D2) = 0\,.$$

Collecting pion-photon contributions to HVP from diagrams (S), (V), (F), (D1) then gives

$$-e^2\frac{9}{162}((V) + 2(S))$$

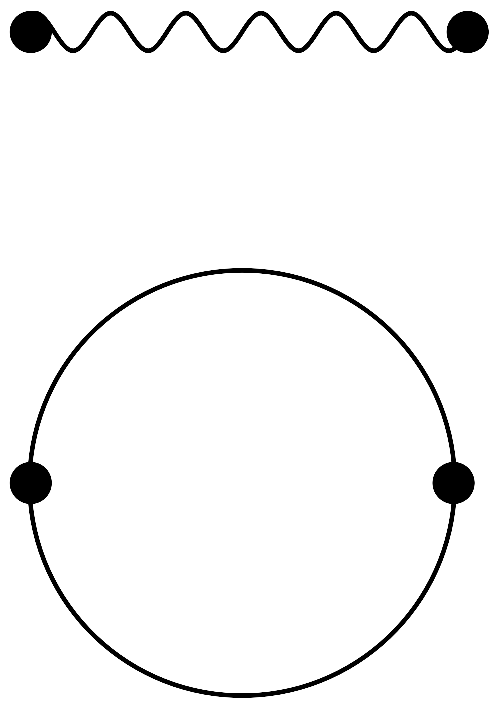
with a small QED charge pre-factor of 9/162.



First results for $a^{-1} = 1.7312(28)$ GeV, $m_\pi = 274.8(2.5)$ MeV, $m_K = 530.1(3.1)$ MeV with $m_\pi L = 3.8$

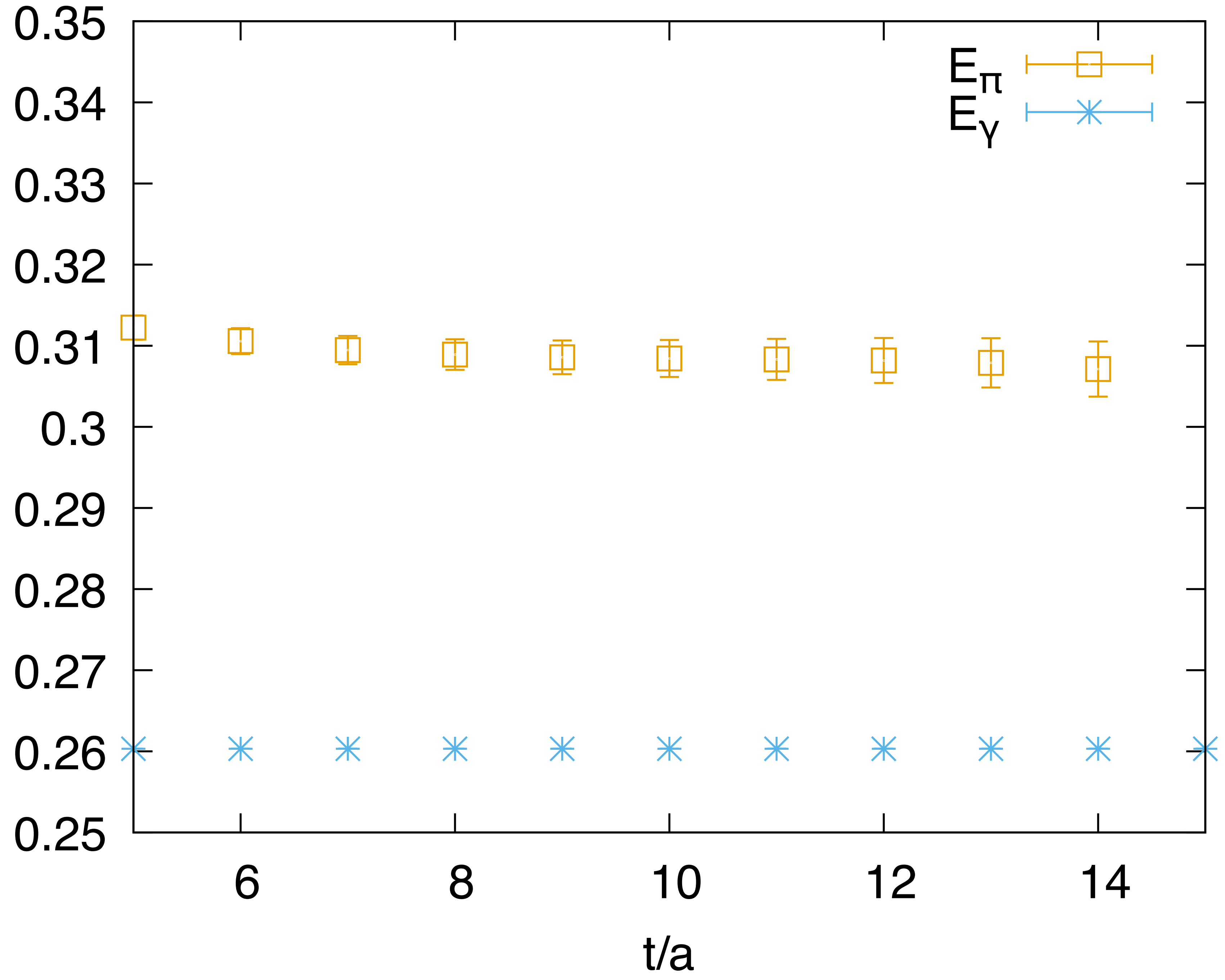
$$V_{in} = \langle 0|O_i|n\rangle \qquad V = V^{(0)} + eV^{(1)} + e^2 V^{(2)}$$

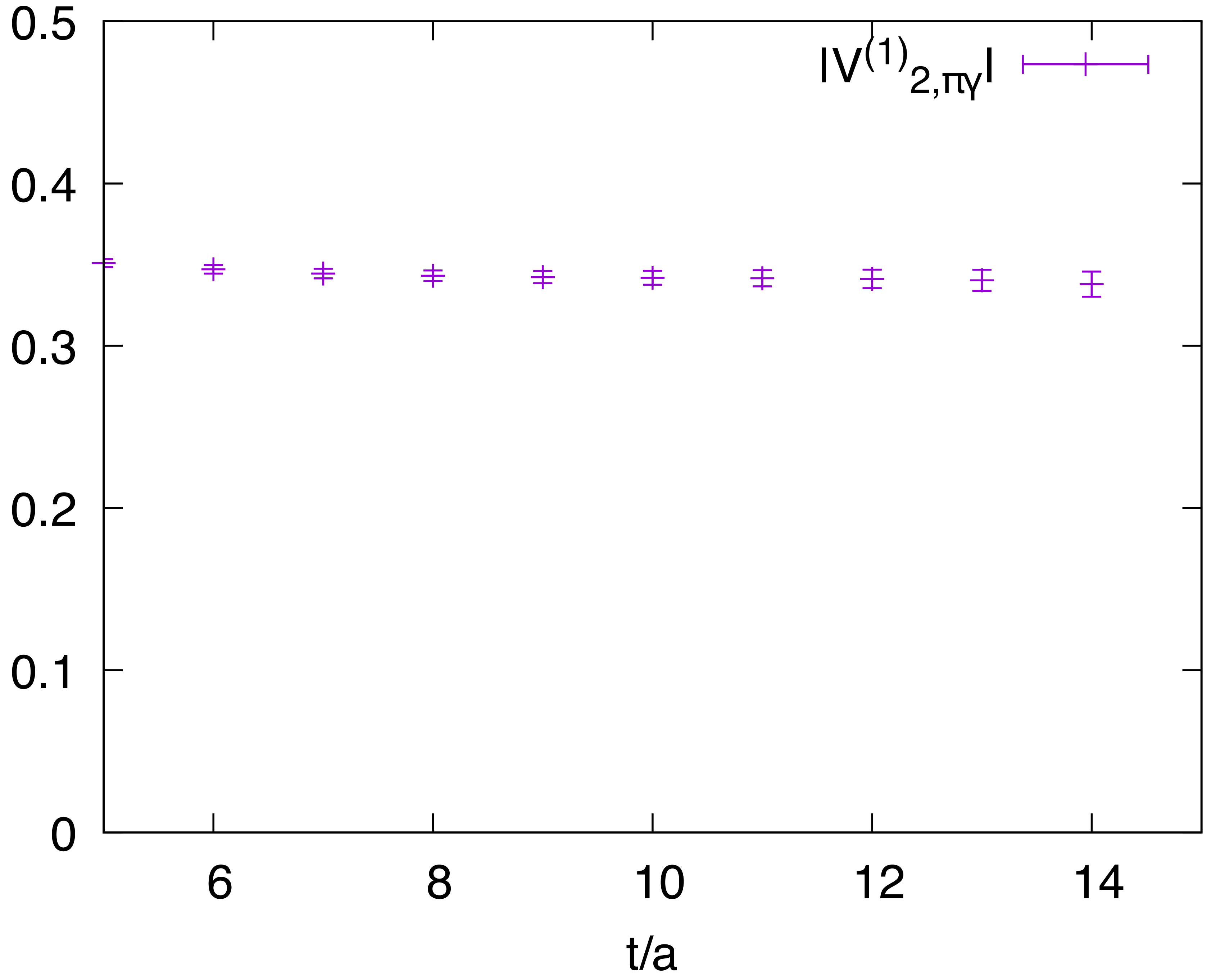
$$O_1 = V_i \qquad O_2 = O_{\pi\gamma}$$

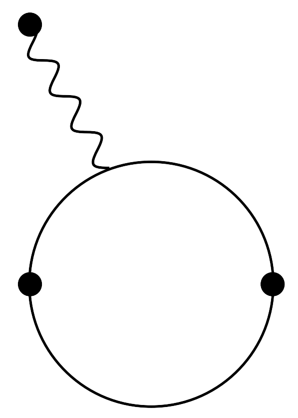


(LR)

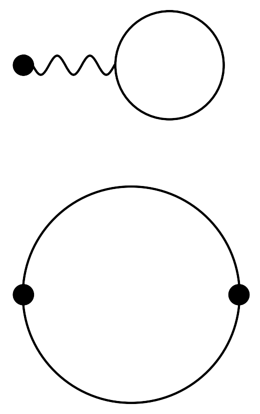
$$C(t) = \begin{pmatrix} |V^{(0)}_{1,\rho}|^2 e^{-E^{(0)}_\rho t} & 0 \\ 0 & 0 \end{pmatrix} + e^2 \begin{pmatrix} (2\mathrm{Re}(V^{(0)}_{1,\rho} V^{(2),*}_{1,\rho}) - t|V^{(0)}_{1,\rho}|^2 E^{(2)}_\rho) e^{-E^{(0)}_\rho t} + |V^{(1)}_{1,\pi\gamma}|^2 e^{-E^{(0)}_{\pi\gamma} t} & V^{(0)}_{1,\rho} V^{(2),*}_{2,\rho} e^{-E^{(0)}_\rho t} + V^{(1)}_{1,\pi\gamma} V^{(1),*}_{2,\pi\gamma} e^{-E^{(0)}_{\pi\gamma} t} \\ V^{(2)}_{2,\rho} V^{(0),*}_{1,\rho} e^{-E^{(0)}_\rho t} + V^{(1)}_{2,\pi\gamma} V^{(1),*}_{1,\pi\gamma} e^{-E^{(0)}_{\pi\gamma} t} & |V^{(1)}_{2,\pi\gamma}|^2 e^{-E^{(0)}_{\pi\gamma} t} \end{pmatrix}$$







(L)

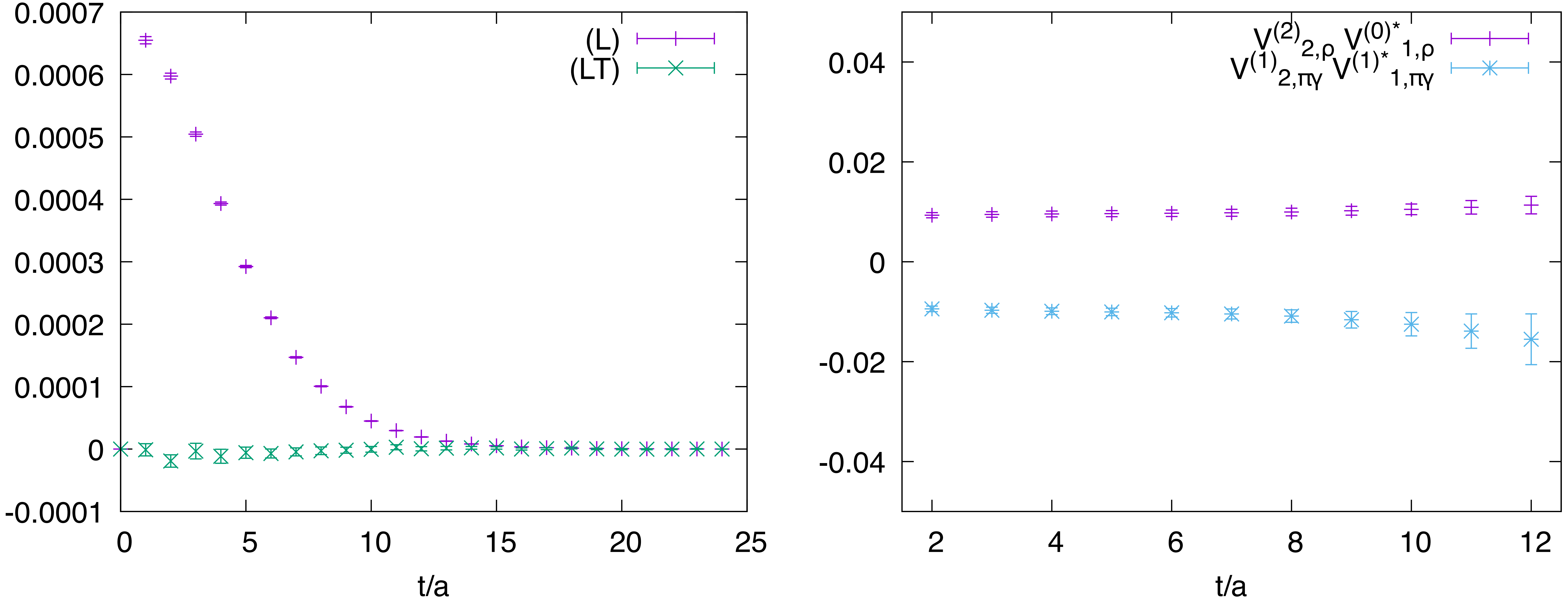


(LT)

$$V_{in} = \langle 0|O_i|n\rangle \qquad V = V^{(0)} + eV^{(1)} + e^2 V^{(2)}$$

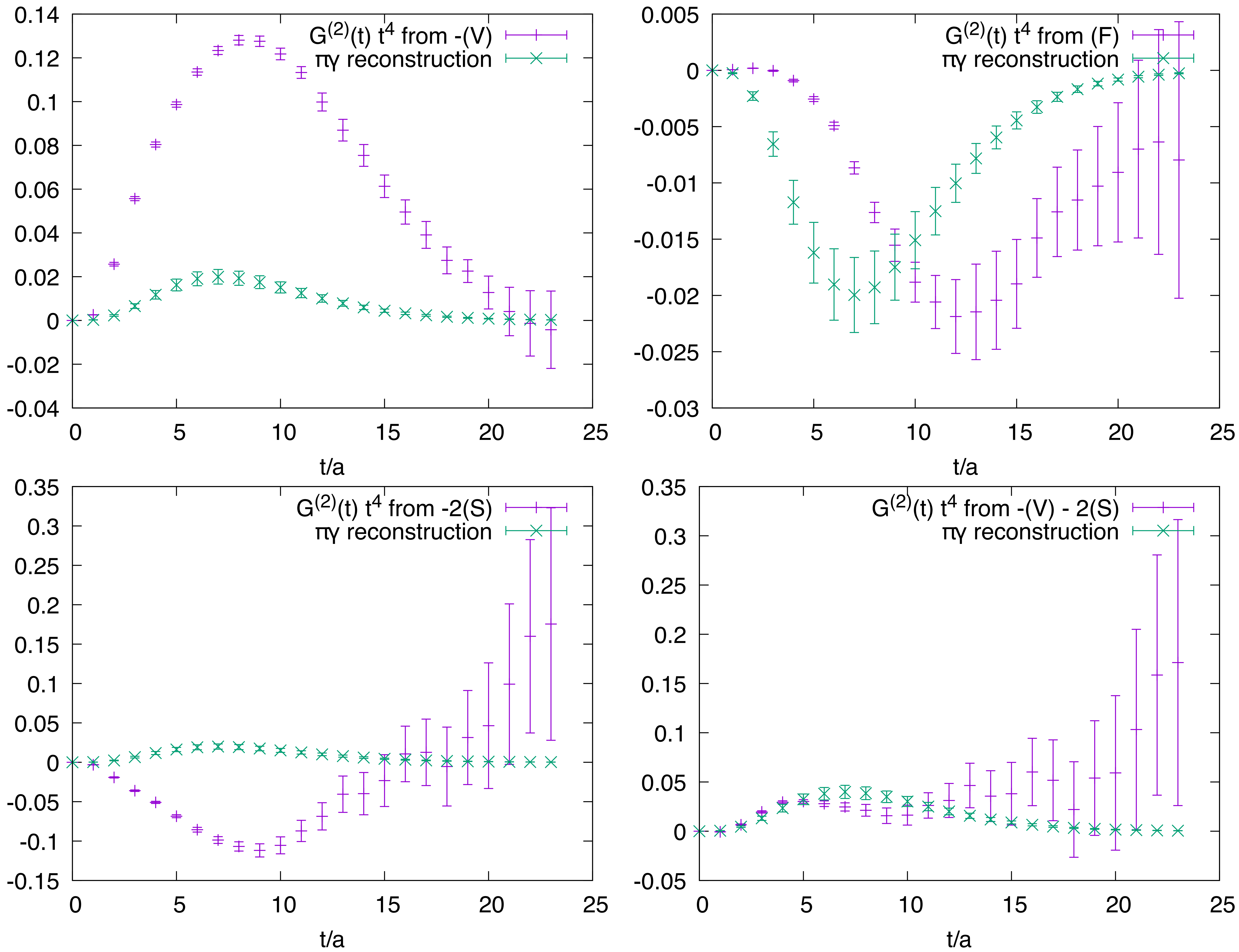
$$O_1 = V_i \qquad O_2 = O_{\pi\gamma}$$

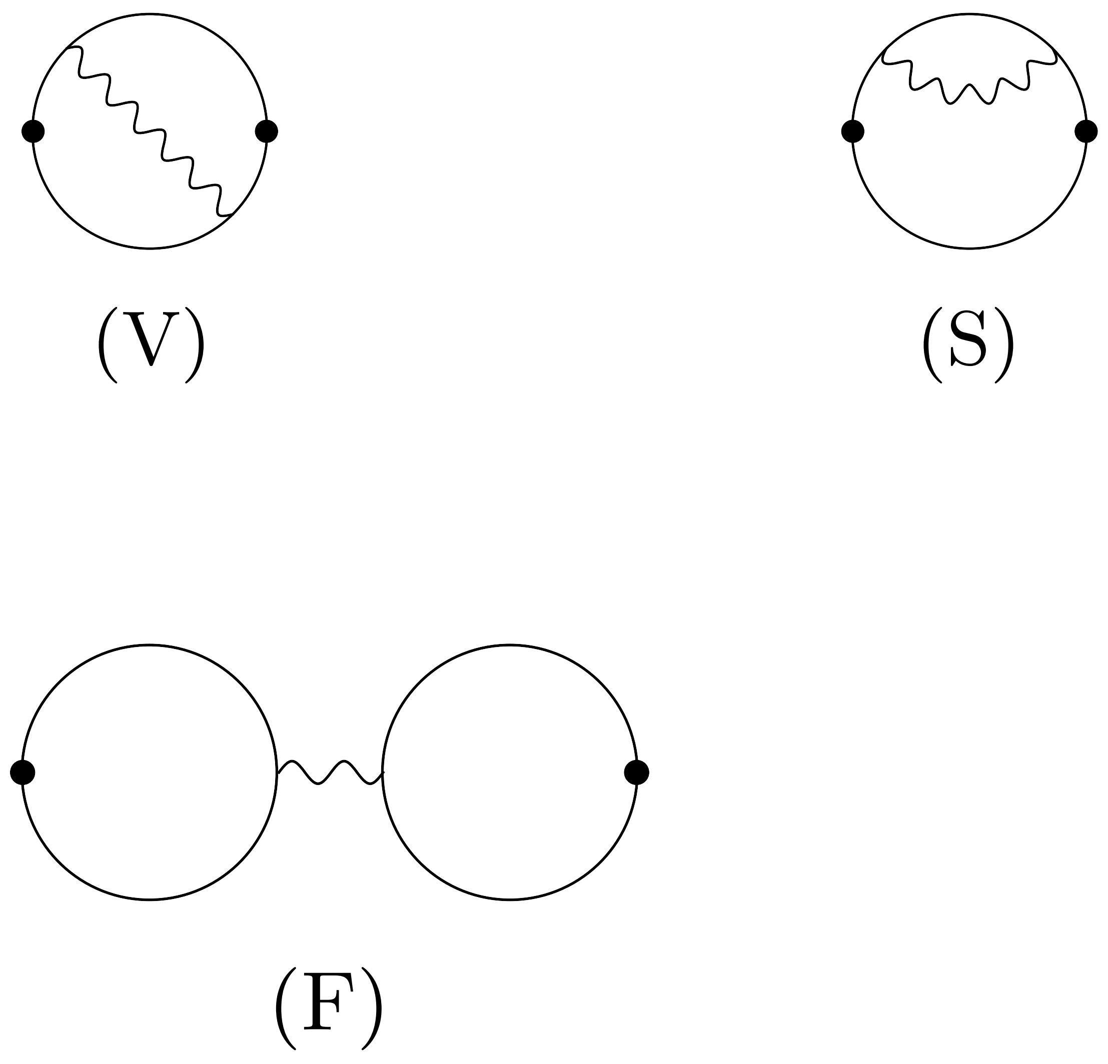
$$C(t) = \begin{pmatrix} |V^{(0)}_{1,\rho}|^2 e^{-E^{(0)}_\rho t} & 0 \\ 0 & 0 \end{pmatrix} + e^2 \begin{pmatrix} (2\mathrm{Re}(V^{(0)}_{1,\rho} V^{(2),*}_{1,\rho}) - t|V^{(0)}_{1,\rho}|^2 E^{(2)}_\rho) e^{-E^{(0)}_\rho t} + |V^{(1)}_{1,\pi\gamma}|^2 e^{-E^{(0)}_{\pi\gamma} t} & V^{(0)}_{1,\rho} V^{(2),*}_{2,\rho} e^{-E^{(0)}_\rho t} + V^{(1)}_{1,\pi\gamma} V^{(1),*}_{2,\pi\gamma} e^{-E^{(0)}_{\pi\gamma} t} \\ V^{(2)}_{2,\rho} V^{(0),*}_{1,\rho} e^{-E^{(0)}_\rho t} + V^{(1)}_{2,\pi\gamma} V^{(1),*}_{1,\pi\gamma} e^{-E^{(0)}_{\pi\gamma} t} & |V^{(1)}_{2,\pi\gamma}|^2 e^{-E^{(0)}_{\pi\gamma} t} \end{pmatrix}$$



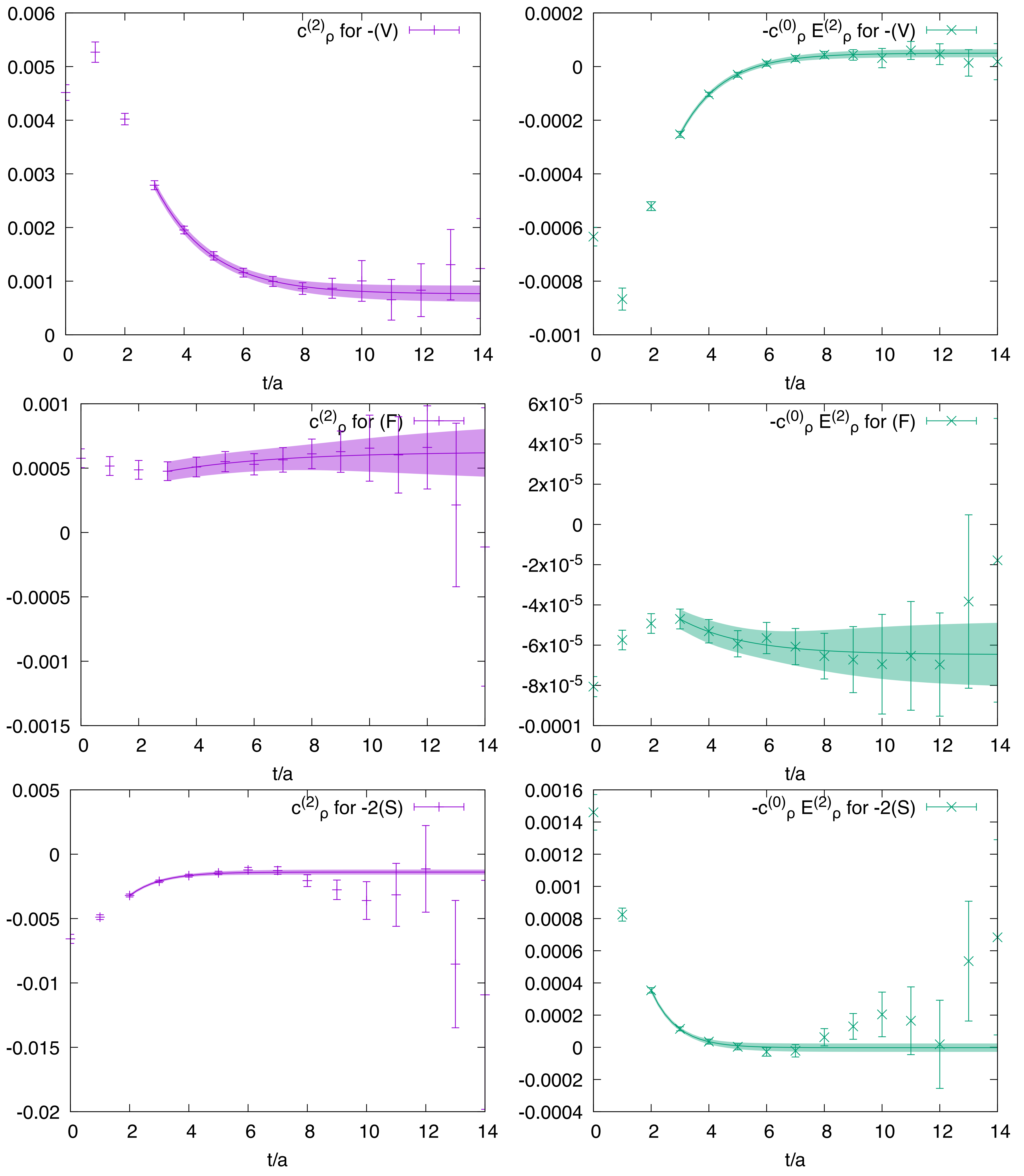
Fit to matrix elements V (right panel) is stable versus fit range [t/a,24]

For now, only study $G(t)t^4$ which mimics behavior of HVP integrand (blinded study)





Next: subtract reconstructed pion-photon contribution from G(t) and determine QED corrections to ρ



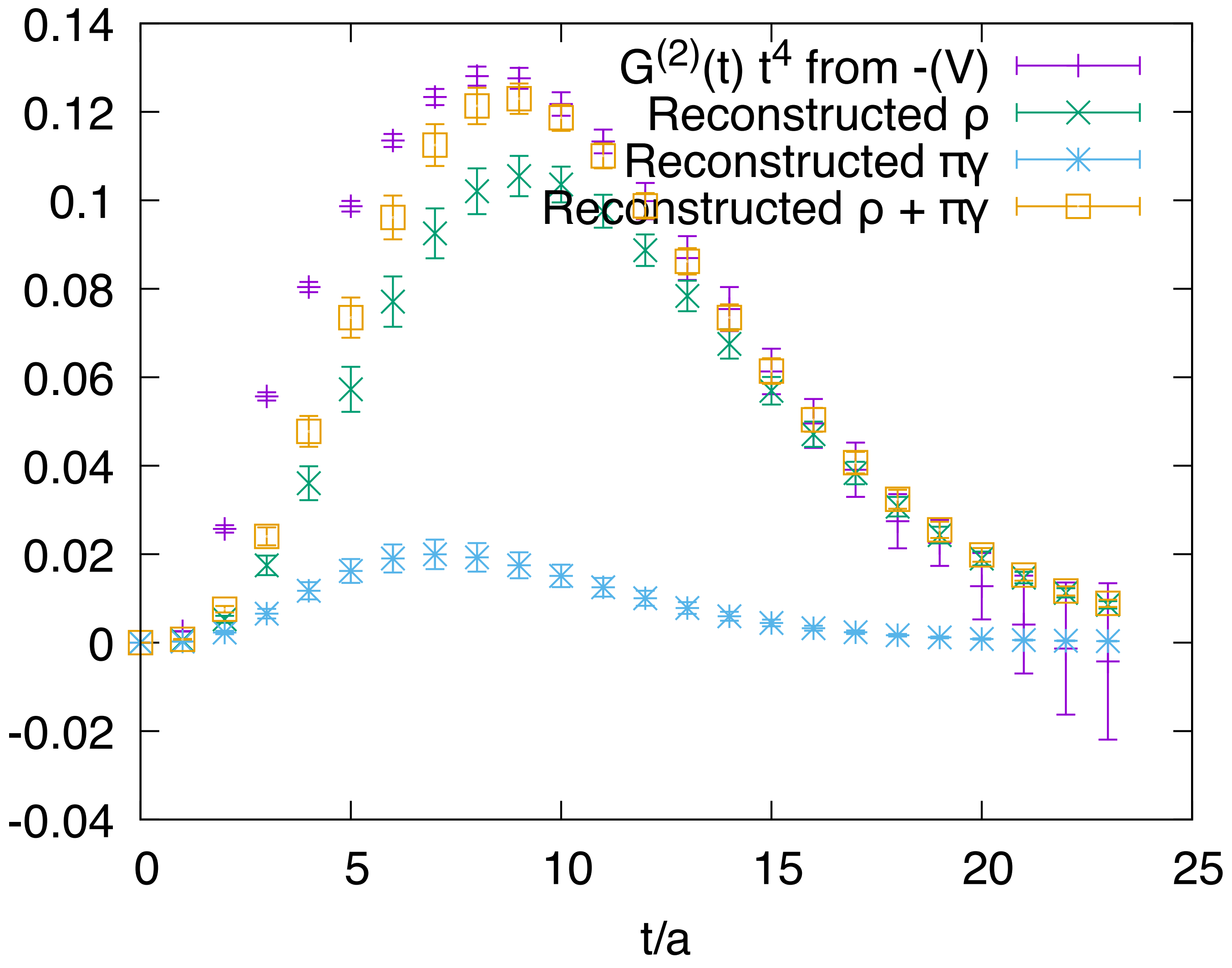
Fits are stable over fit ranges [t/a,24]

$$V_{in} = \langle 0|O_i|n\rangle \qquad V = V^{(0)} + eV^{(1)} + e^2 V^{(2)}$$

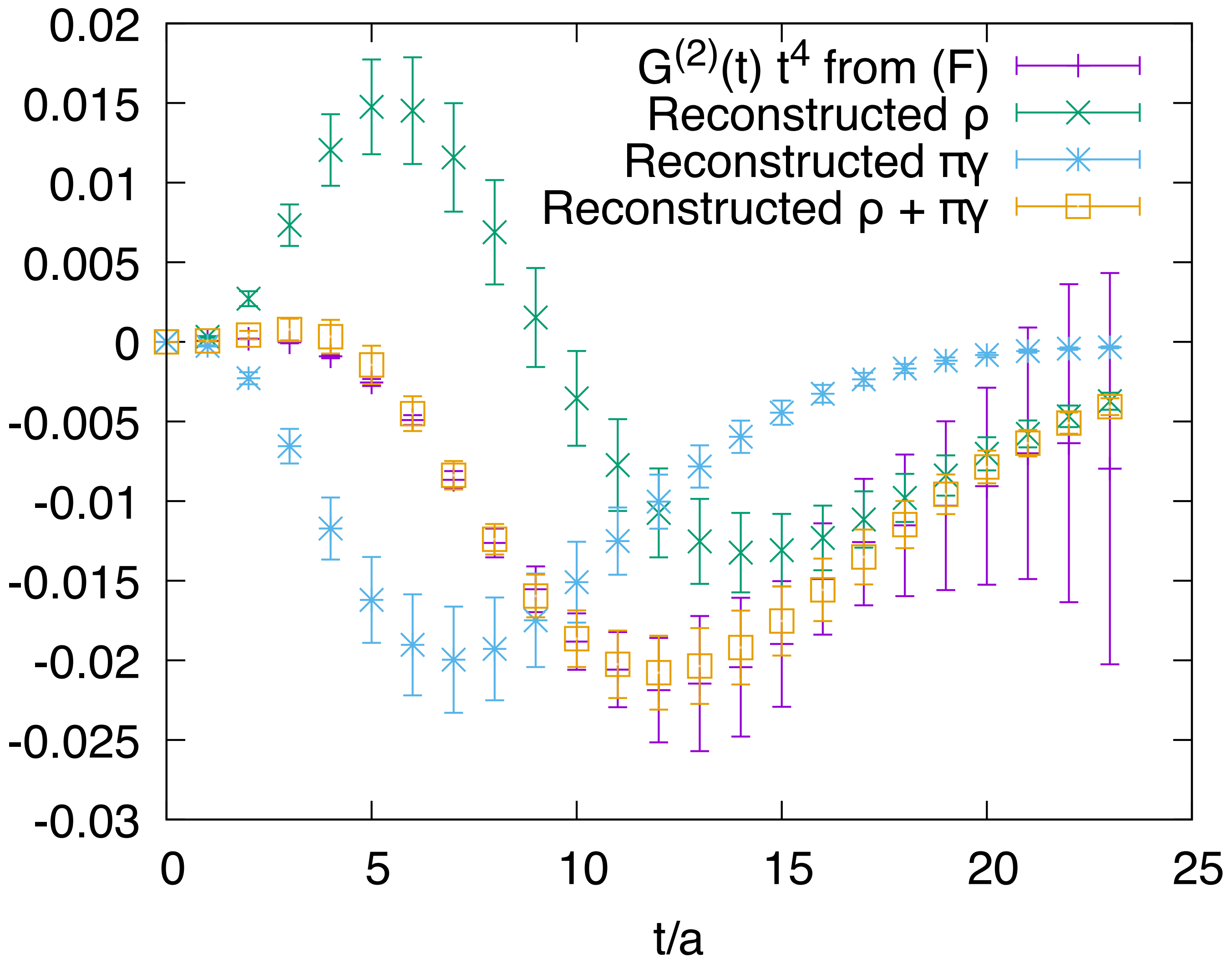
$$O_1 = V_i \qquad O_2 = O_{\pi\gamma}$$

$$C(t) = \begin{pmatrix} |V^{(0)}_{1,\rho}|^2 e^{-E^{(0)}_\rho t} & 0 \\ 0 & 0 \end{pmatrix}$$
$$+ e^2 \begin{pmatrix} (2\mathrm{Re}(V^{(0)}_{1,\rho}V^{(2),*}_{1,\rho}) - t|V^{(0)}_{1,\rho}|^2 E^{(2)}_\rho) e^{-E^{(0)}_\rho t} + |V^{(1)}_{1,\pi\gamma}|^2 e^{-E^{(0)}_{\pi\gamma} t} & V^{(0)}_{1,\rho}V^{(2),*}_{2,\rho} e^{-E^{(0)}_\rho t} + V^{(1)}_{1,\pi\gamma}V^{(1),*}_{2,\pi\gamma} e^{-E^{(0)}_{\pi\gamma} t} \\ V^{(2)}_{2,\rho}V^{(0),*}_{1,\rho} e^{-E^{(0)}_\rho t} + V^{(1)}_{2,\pi\gamma}V^{(1),*}_{1,\pi\gamma} e^{-E^{(0)}_{\pi\gamma} t} & |V^{(1)}_{2,\pi\gamma}|^2 e^{-E^{(0)}_{\pi\gamma} t} \end{pmatrix}$$

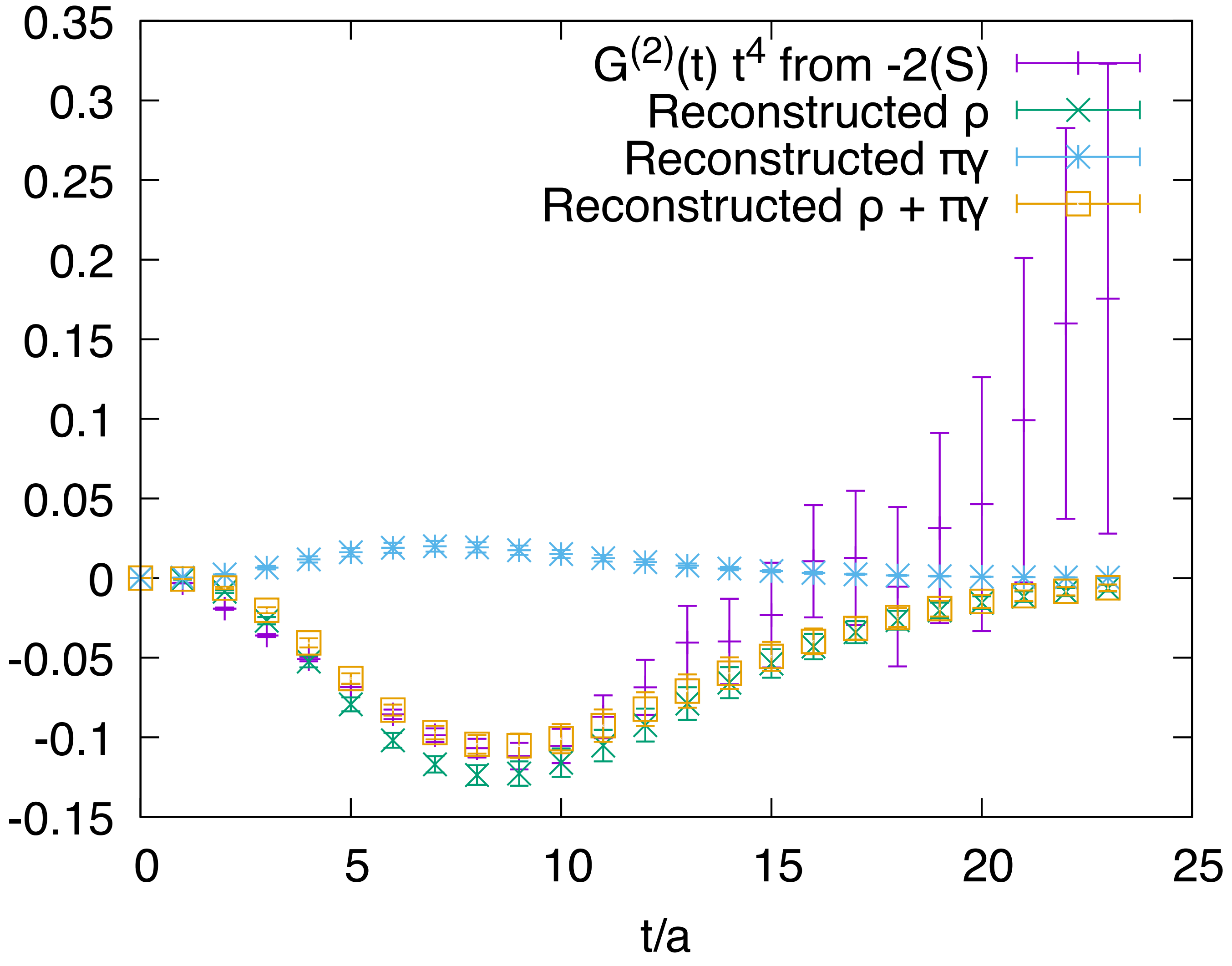
Compare reconstruction with pion-photon and rho compared to inclusive correlator:



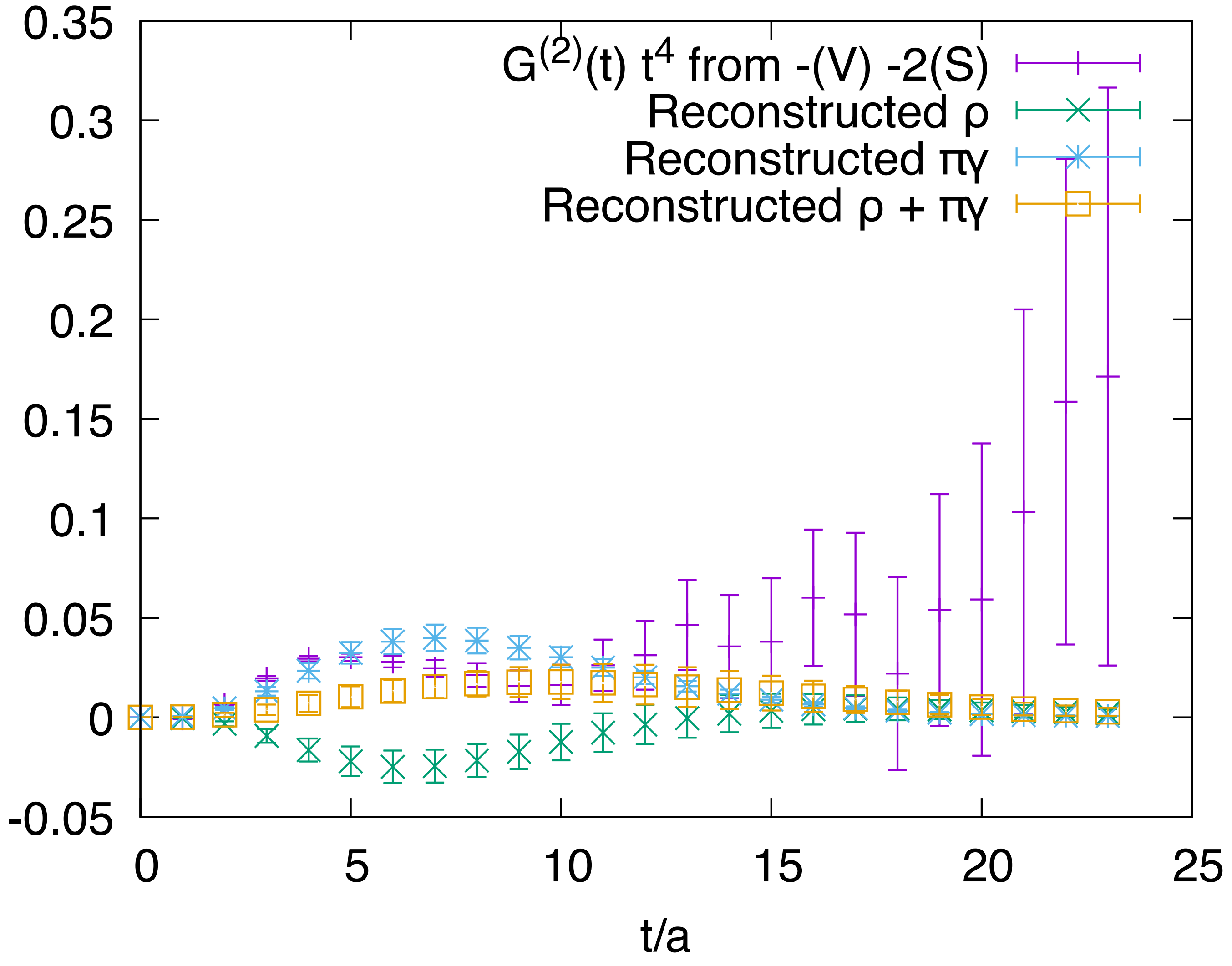
(V)



(L)



(V)



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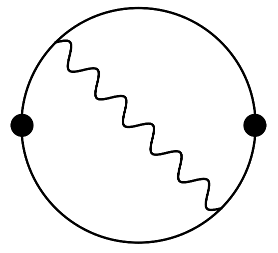
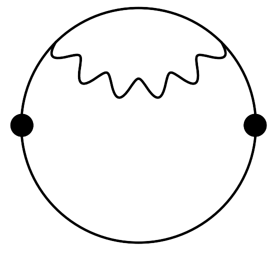
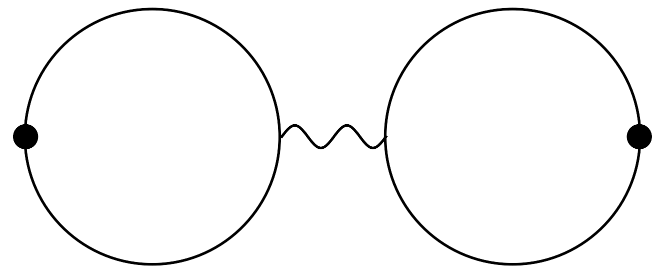


(V)



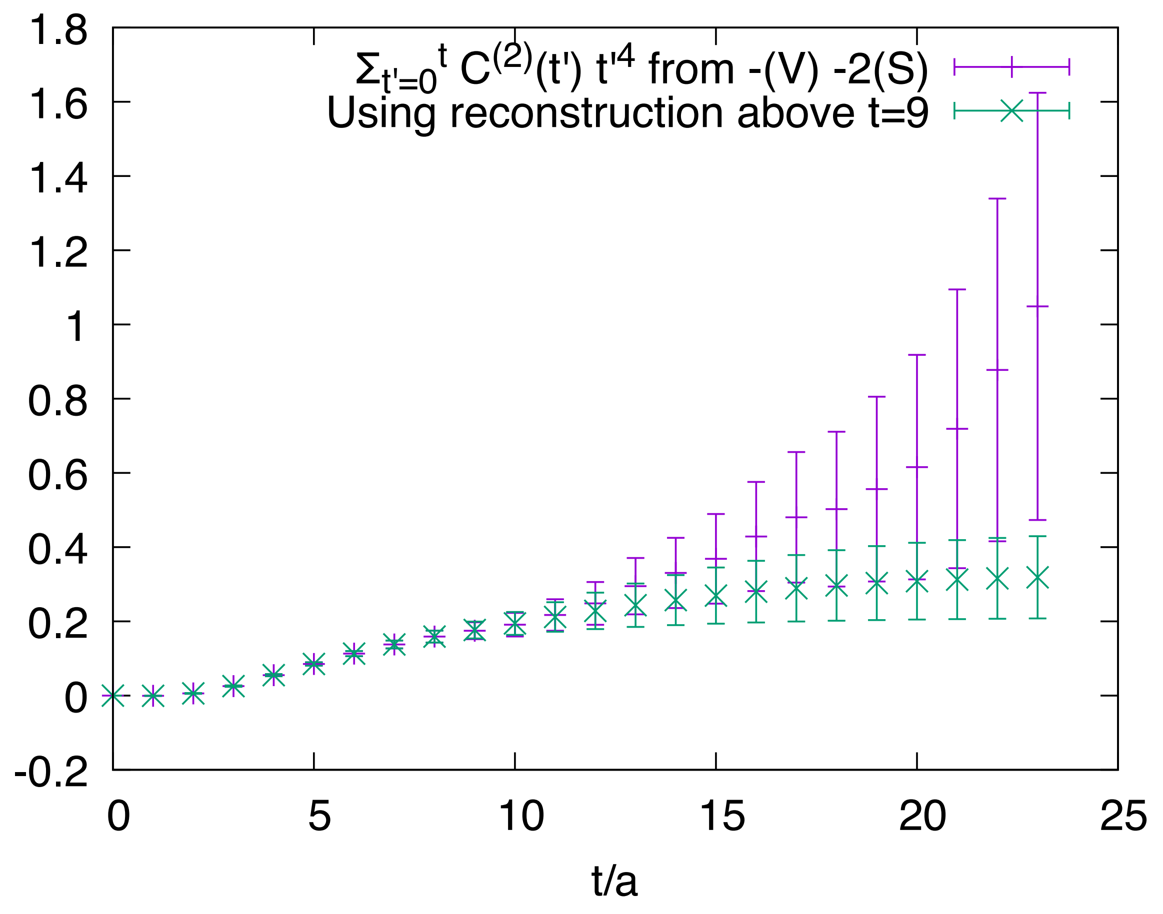
(S)



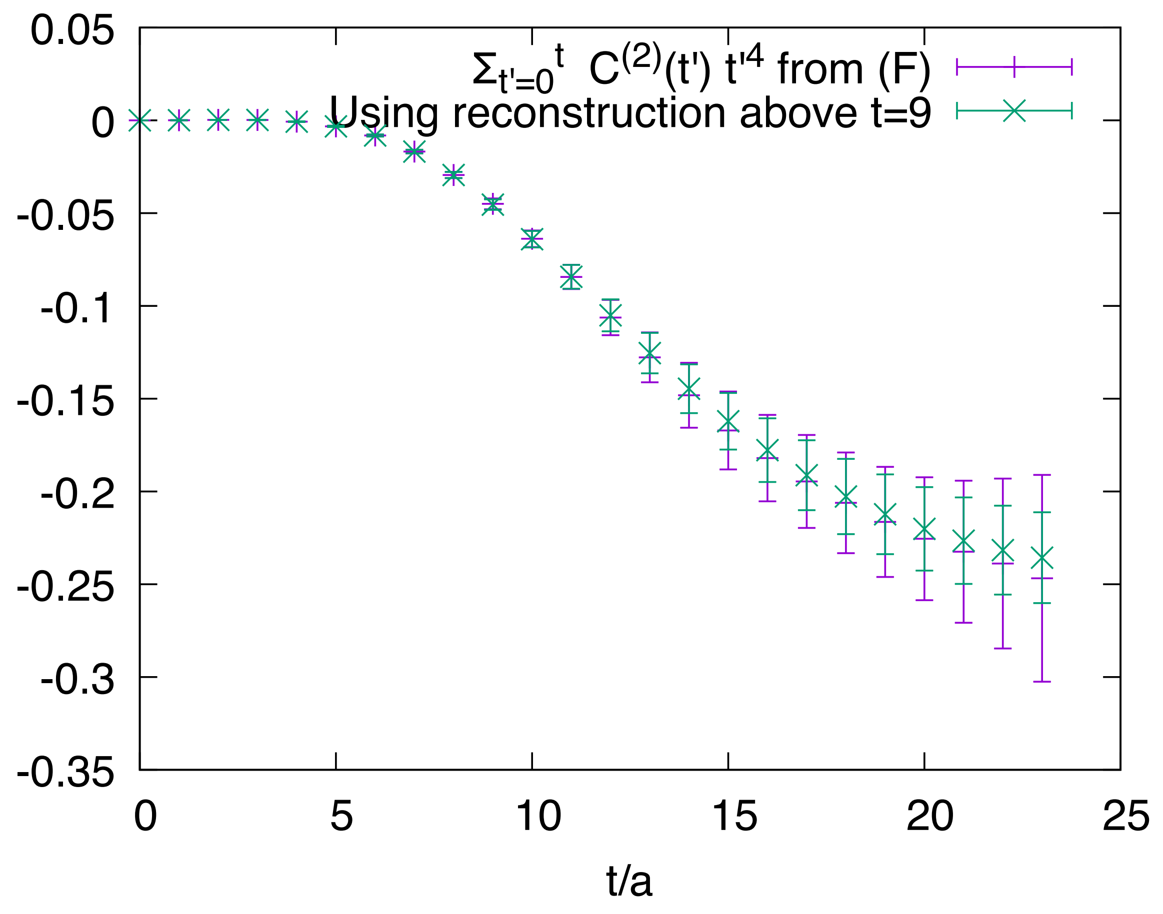
(F)

Compare noise of long-distance tail with reconstruction to original:

arXiv:2508.21685



(V)



(L)

Noise reduction by factor > 5 for noisy (S) diagram!

Next:

- extend this analysis to full list of ensembles, all diagrams, blinded analysis with multiple analysis groups
- combination of finite-volume QED with infinite-volume QED result for pion-photon contribution