Describing three-body decays using Khuri–Treiman formalism

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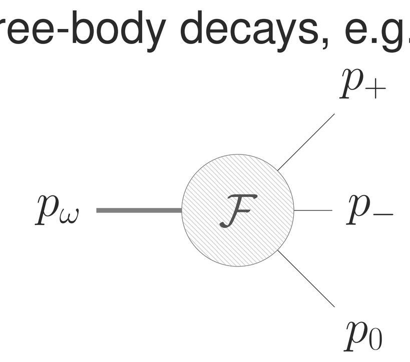
Introduction

Description of three-body decays, e.g. $\omega \to \pi^+\pi^-\pi^0$,



at low energies using Khuri-Treiman equations [1, 2, 3]:

- Dispersive construction of amplitudes
- Correct analytic and unitary properties
- Pion FSI handled non-perturbatively
- Phase shifts as phenomenological input

Kinematics

Define **Mandelstam variables**

$$s = (p_+ + p_-)^2\,,\ t = (p_+ + p_0)^2\,,\ u = (p_- + p_0)^2\,,$$

which fulfil $s + t + u = M_\omega^2 + 3M_\pi^2 \equiv s_0$,

$\pi\pi$ threshold: $s_{\text{th}} = 4M_\pi^2$

pseudo-threshold: $p_{\text{th}} = (M_\omega - M_\pi)^2$

regular threshold: $r_{\text{th}} = (M_\omega + M_\pi)^2$

$\to$ Decay region limited by s_{th} and p_{th}.

$$t(s,z) = (s_0 - s + \kappa(s)z)/2$$
$$u(s,z) = (s_0 - s - \kappa(s)z)/2$$

with $\kappa(s) = \lambda^{1/2}(s, M_\omega^2, M_\pi^2)\sqrt{1 - 4M_\pi^2/s}$ and $z = \cos\theta$, θ is scattering angle in s-channel centre-of-mass system.

Discontinuity Equation

Initial assumptions:

- Only elastic rescattering
- Partial waves higher than P-wave may be neglected

Single-Variable Amplitude (SVA):
Decomposition into SVAs with right-hand cut [1]:

$$\mathcal{F}(s,t,u) = \mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u)$$

Partial-wave expansion

$$\mathcal{F}(s,t,u) = \sum_J f_J(s) P_J'(z)$$

leads to

$$f_1(s) = \mathcal{F}(s) + \hat{\mathcal{F}}(s)\,,$$

where

$$\hat{\mathcal{F}}(s) = 3\langle(1 - z^2)\mathcal{F}\rangle$$
$$\langle z^n \mathcal{F}\rangle = \frac{1}{2}\int_{-1}^{+1} \mathrm{d}z\, z^n \mathcal{F}(t(s,z))\,.$$

$\to$ Since $\hat{\mathcal{F}}$ possesses only left-hand cut,

$$\mathrm{disc} f_1(s) = \mathrm{disc}\left(\mathcal{F}(s) + \hat{\mathcal{F}}(s)\right) = \mathrm{disc}\mathcal{F}(s)\,.$$

In elastic approximation,

$$\mathrm{disc} f_1(s) = f_1 2i \sin\delta(s) e^{-i\delta(s)}$$

with $\pi\pi$ P-wave phase shift δ.

Discontinuity Equation

$$\mathrm{disc}\mathcal{F}(s) = 2i\sin\delta(s)e^{-i\delta(s)}\left(\mathcal{F}(s) + \hat{\mathcal{F}}(s)\right)$$

Solution

Inhomogeneous Omnès solution [4]

$$\mathcal{F}(s) = \Omega(s)\left(P_{m-1}(s) + \frac{s^m}{\pi}\int_{s_{\text{th}}}^{\infty}\frac{\mathrm{d}x}{x^m}\frac{\sin\delta(x)/|\Omega(x)|}{x - s - i\epsilon}\hat{\mathcal{F}}(x)\right)$$

with

$$\Omega(s) = \exp\left(\frac{s}{\pi}\int_{s_{\text{th}}}^{\infty}\frac{\mathrm{d}x}{x}\frac{\delta(x)}{x - s - i\epsilon}\right)\,.$$

Basis functions:
Solution linear in subtraction constants.
For $P_{m-1}(s) = \sum_{k=0}^{m-1} a_k s^k$ one may write

$$\mathcal{F}(s) = \sum_{k=0}^{m-1} a_k \mathcal{F}_k(s)$$

with the basis functions

$$\mathcal{F}_k(s) = \Omega(s)\left(s^k + \frac{s^m}{\pi}\int_{s_{\text{th}}}^{\infty}\frac{\mathrm{d}x}{x^m}\frac{\sin\delta(x)/|\Omega(x)|}{x - s - i\epsilon}\hat{\mathcal{F}}_k(x)\right)\,.$$

Algorithms

Iterative solution strategy [5]:
Initial guess $\mathcal{F}_k^{(0)}(s) = \Omega(s)s^k$ yields first estimate for $\hat{\mathcal{F}}_k^{(1)}$, which is used to compute $\mathcal{F}_k^{(1)}, \ldots$
$\to$ no convergence guaranteed: problems observed for large decay masses or many subtractions.

Solution via matrix inversion [6]:
Idea: solve for $\tilde{\mathcal{F}} = \kappa^3\hat{\mathcal{F}}$.
Rewrite equation as

$$A_k(s) = \int_{s_{textth}}^{\infty}\mathrm{d}x\left(\delta(x - s) - \frac{K(s,x)}{\pi}\right)\tilde{\mathcal{F}}_k(x)$$

with

$$A_k(s) = \int_{t_-(s)}^{t_+(s)}\mathrm{d}t\,\xi(s,t,k)\,,$$
$$K(s,x) = \frac{\sin\delta(x)}{x^m|\Omega(x)|}\int_{t_-(s)}^{t_+(s)}\mathrm{d}t\frac{\xi(s,t,m)}{x - t}\,,$$
$$\xi(s,t,k) = 3\left(\kappa^2(s) - (\kappa(s)z(s,t))^2\right)\Omega(t)t^k\,.$$

Discretisation allows solving for $\tilde{\mathcal{F}}_k$.
$\to$ Solution follows immediately, Cauchy kernel in angular average needs to be handled carefully.

Pinocchio Integral

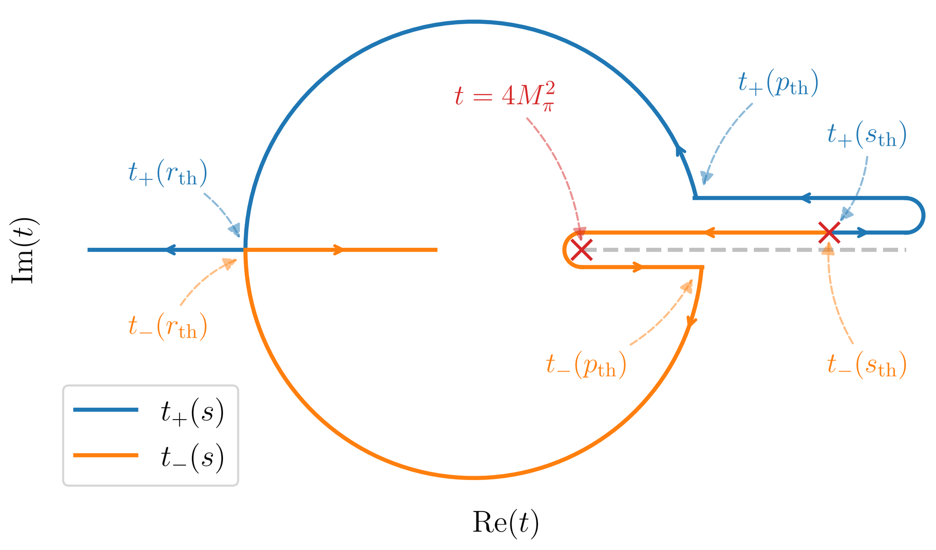
Computation of angular average:

$$\tilde{\mathcal{F}}(s) = 3\int_{t_-(s)}^{t_+(s)}\mathrm{d}t\left(\kappa^2(s) - (\kappa(s)z(s,t))^2\right)\mathcal{F}(t)$$

with s-dependent integration bounds

$$t_\pm(s) = \frac{1}{2}(s_0 - s \pm \kappa(s))\,.$$

$\to$ Integration path chosen, such that cut never crossed.

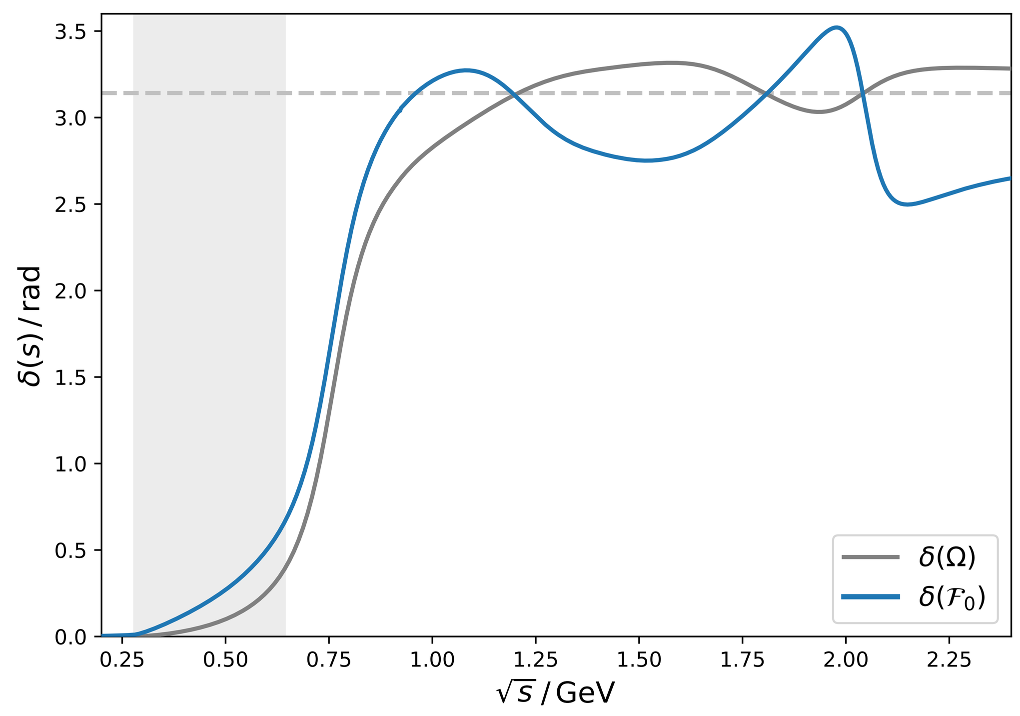


Application [1]

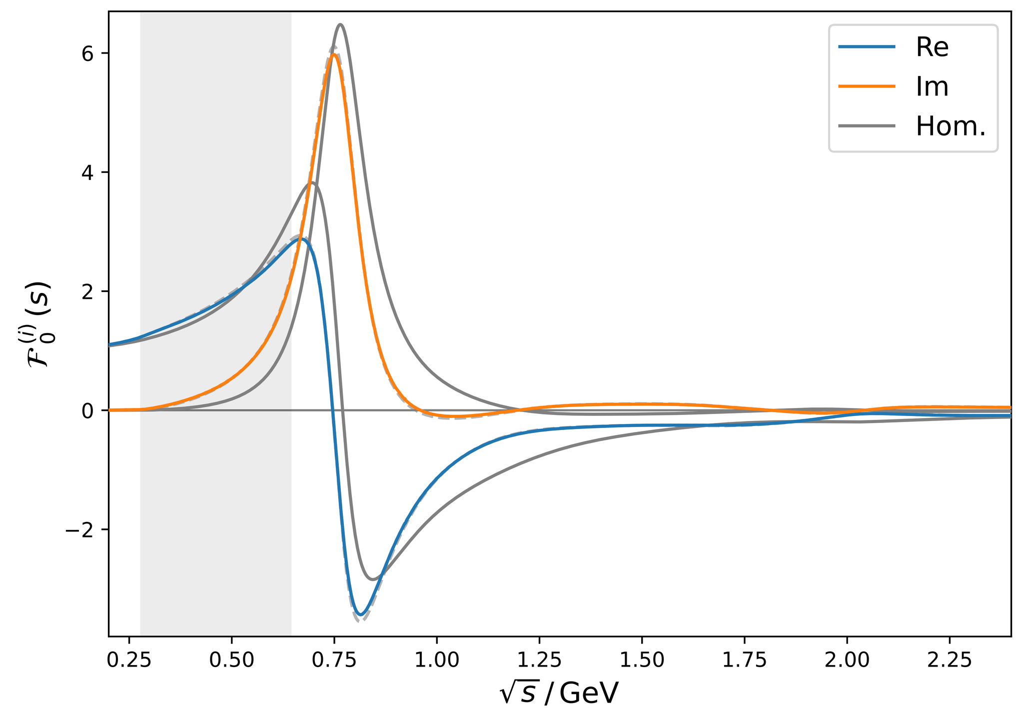
Solve once-subtracted case

$$\mathcal{F}_0(s) = \Omega(s)\left(1 + \frac{s}{\pi}\int_{s_{\text{th}}}^{\infty}\frac{\mathrm{d}x}{x}\frac{\sin\delta(x)/|\Omega(x)|}{x - s - i\epsilon}\hat{\mathcal{F}}_0(x)\right)$$

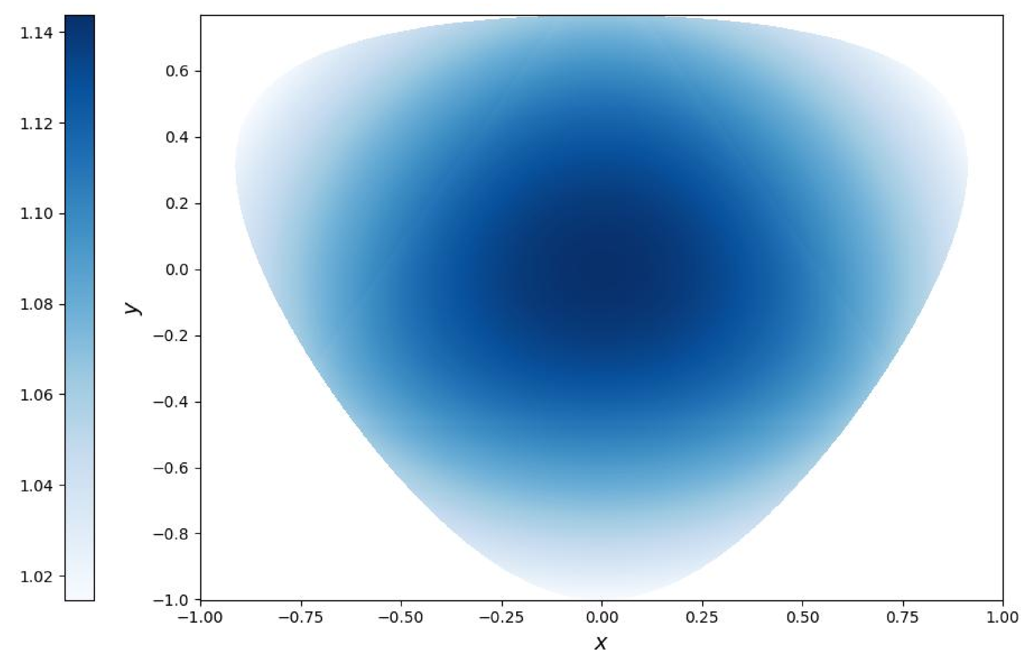
using $\pi\pi$ phase shift extracted from multi-channel unitary model [7]



$\to$ Resulting basis function



$\to$ Dalitz plot normalised with homogeneous solution



$$x = \frac{t - u}{\sqrt{3}R_\omega}\,,\quad y = \frac{s_0/3 - s}{R_\omega}\,,$$

with $R_\omega = \frac{2}{3}M_\omega(M_\omega - 3M_\pi)\,.$

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