ETH zürich

Institute for Theoretical Physics, D-PHYS

The Dependence of a_μ^{HVP} on the Muon Mass

and how we can use it to reduce noise in lattice computations

Leo Filipovic*, Letizia Parato†, Marina Krstic Marinkovic‡

*lfilipovic@student.ethz.ch, †lparato@phys.ethz.ch, ‡marinama@ethz.ch

Motivation

One way of computing the leading order hadronic vacuum polarization (HVP) contribution to $(g-2)_\mu$ is by its Mellin-Barnes representation [1]:

$$a_\mu^{\mathrm{HVP}}(m_\mu) = \frac{1}{2\pi i}\cdot\frac{\alpha}{\pi}\int_{c-i\infty}^{c+i\infty} ds \left(\frac{m_\mu^2}{t_0}\right)^{1-s} \mathcal{F}(s)\mathcal{M}^{\mathrm{HVP}}(s) \quad (1)$$

$$\mathcal{F}(s) = -\Gamma(3-2s)\Gamma(1+s)\Gamma(s-3) \quad (2)$$

$$\mathcal{M}^{\mathrm{HVP}}(s) = \int_{t_0=4m_\pi^2}^{\infty} \frac{dt}{t}\left(\frac{t}{t_0}\right)^{s-1}\frac{1}{\pi}\Im\Pi(t) \qquad \Re s < 1 \quad (3)$$

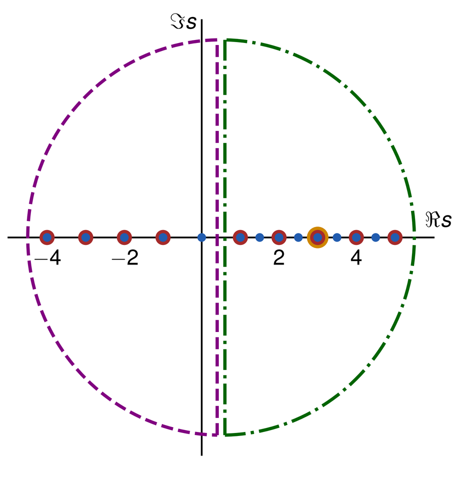

Figure 1: Poles of the integrand of (1) in the complex plane and the two contours to evaluate it by Cauchy's residue theorem: The purple (green) contour is used for small (large) m_μ.

The poles of the integrand in (1) stem from both $\mathcal{F}(s)$ and $\mathcal{M}^{\mathrm{HVP}}(s)$ and lie on the real axis of the complex s-plane (see fig. 1). Closing the contour at infinity to the right or left and using Cauchy's residue theorem allows for the derivation of asymptotic expansions of $a_\mu^{\mathrm{HVP}}(m_\mu)$ in the limits $m_\mu \to 0$ and $m_\mu \to \infty$.

In their recent work [2], D. Greynat and E. de Rafael show that $a_\mu^{\mathrm{HVP}}(m_\mu)$ can be reconstructed on its entire domain based on its behavior in a limited range of m_μ. To achieve this, they apply the FO-transfer-theorem[1] by P. Flajolet and A. Odlyzko [3, 4] to these asymptotic expansions of $a_\mu^{\mathrm{HVP}}(m_\mu)$.

To adapt to the notation in [2], we define

$$\frac{\alpha^2}{\pi^2}A(\omega) = a_\mu^{\mathrm{HVP}}(m_\mu) \quad \text{and} \quad \omega = \frac{\sqrt{z}-1}{\sqrt{z}+1}, \qquad z = \frac{m_\mu^2}{m_\pi^2}. \quad (4)$$

The authors suggest the following [2]: $A(\omega)$ can be computed in an optimal ω-region for lattice QCD and then reconstructed at $\omega(m_\mu^{\mathrm{phys}}) = -0.12184(1)$ [5] via the approximants:

$$A^{JK}(\omega) = \frac{\sum_{n=0}^{J} p_n(1+\omega)^n}{1+\sum_{n=1}^{K} q_n(1+\omega)^n} + A^{sing}(\omega) \quad (5)$$

where $A^{sing}(\omega)$ is given by the FO-transfer-theorem[1], and the paramters $p_{0,\dots,J}$ and $q_{1,\dots,K}$ are to be fixed by a curve fit.

The goal of our work is to investigate the feasibility of this suggestion.

[1] $A(\omega)$ admits a Taylor expansion $\sum_n g_n\omega^n$ for $\omega \in (-1,1)$. The FO-transfer-theorem tells us that the behavior of $A(\omega)$ near its dominant singularities at the boundary of the convergence disk ($\omega = \pm 1$) governs the asymptotic growth of the coefficients g_n as $n \to \infty$. By subtracting from $A(\omega)$ a series $\sum_n g_n^{AS}\omega^n = A^{sing}(\omega)$ with matching large-n behavior, the remainder has rapidly decaying coefficients and can be effectively approximated by a rational function. Note: the coefficients g_n^{AS} also depend on the HVP moments $\mathcal{M}^{\mathrm{HVP}}(-n)$.

Phenomenological model

We model $\Im\Pi(t)$ by a single Breit-Wigner peak (upper left of fig. 2) meant to mimic the $I=1$ channel of $\sigma(e^+e^- \to \text{hadrons})$ and transform it to a model of the electromagnetic correlator in Euclidean time by the Laplace transform [6]:

$$G^{\rho\rho}(x_0) = \frac{1}{2}\int_0^\infty ds\,\sqrt{s}\,\frac{\Im\Pi(s)}{4\pi^2\alpha}\,e^{-\sqrt{s}|x_0|}. \quad (6)$$

We generate mock lattice data for $G^{\rho\rho}(x_0)$ by assuming $var(G^{\rho\rho}(x_0)) \propto e^{-2m_\pi x_0}$ and $cov(G^{\rho\rho}(x_0), G^{\rho\rho}(x_0')) \propto e^{-m_\mu|x_0'-x_0|}$ (upper right of fig. 2).

Noise from the long-distance tail dominates the uncertainty of $a_\mu^{\mathrm{HVP},I=1}(m_\mu)$; increasing m_μ reduces sensitivity to this noise, making **large m_μ preferable for lattice calculations.**

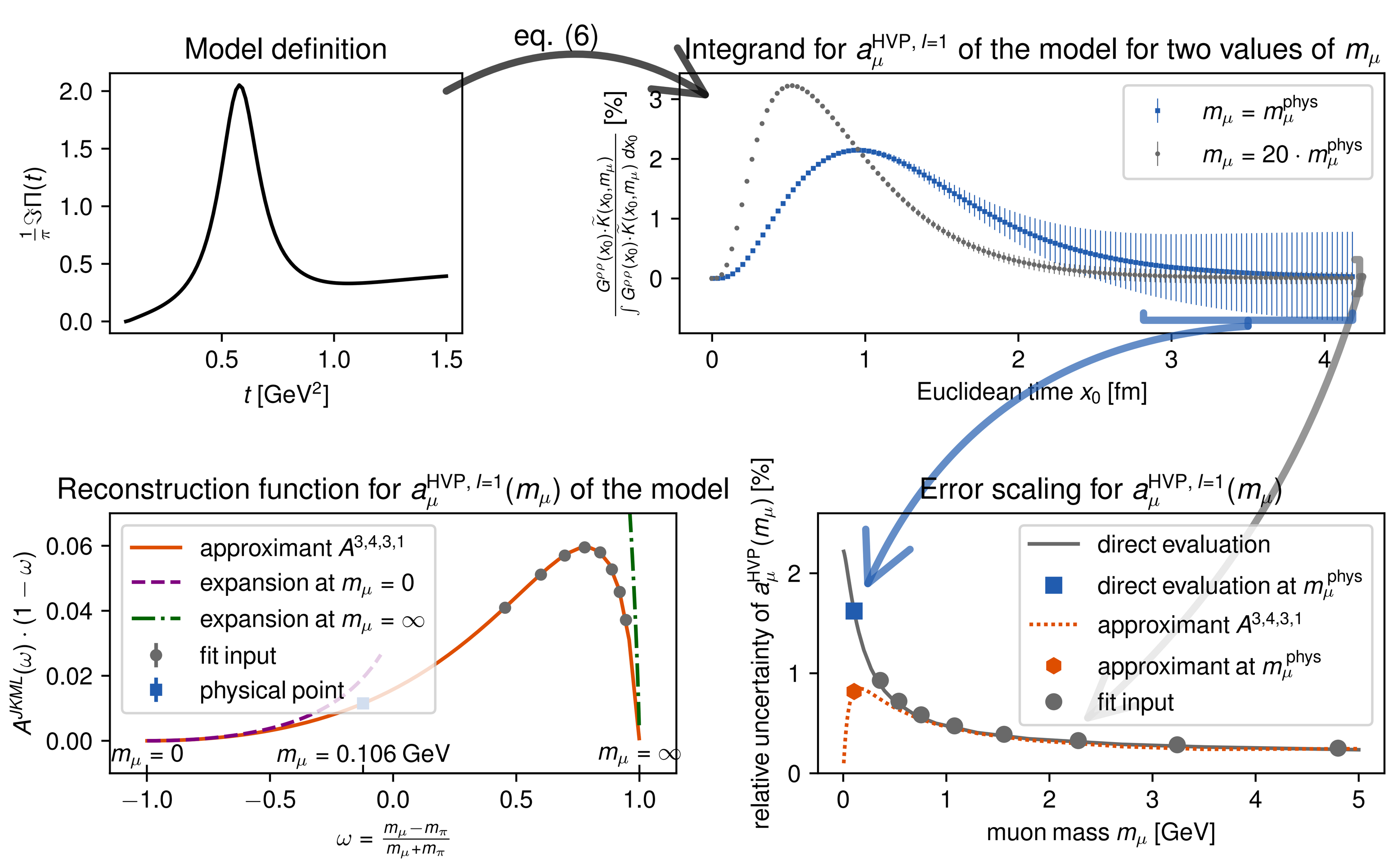

Figure 2: Upper left: The model of the HVP function[2] from [2]. Upper Right: Integrand of $a_\mu^{\mathrm{HVP},I=1}(m_\mu)$ (↔ fig. 3, right panel). Lower left: Fit[3] of $A^{J=3,K=4,M=3,L=1}(\omega)$ with input data at unphysically large m_μ. Lower right: Relative uncertainty of $a_\mu^{\mathrm{HVP},I=1}(m_\mu)$ for direct evaluation (gray) vs the described reconstruction from input data (orange).

We compute $A(\omega)$ from the mock data for eight values of $m_\mu \in [0.36, 4.8]$ GeV and use this input to reconstruct $A(\omega)$ in the region $-1 < \omega < 1$ using functions (5). One resulting function[3] and its uncertainty are displayed in fig. 2.

[2] To maximize the analogy with the next section, we remove the kaon contribution from $\Im\Pi(s)$ since G8 has $N_f = 2$. This requires a negligible modification to the model in [2].

[3] We fit the input data to a variety of models $A^{JKML}(\omega)$ where labels J and K refers to the number of parameters in (5). In addition, we also test different, asymptotically equivalent singular functions $A^{sing}(\omega)$, which we label by M and L.

Application to ensemble w/ O(a)-improved Wilson action

Next, we apply the method for the same eight values of $m_\mu \in [0.36, 4.8]$ GeV as before on measurements of $G^{\rho\rho}(x_0) = -\frac{1}{3}\sum_k \int d^3x\,\langle J_k^\rho(x)J_k^\rho(0)\rangle$ with (unimproved) $I=1$ current $J_k^\rho(x) = \frac{1}{2}\left(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\right)$ performed on the ensemble G8 by CLS [7]:

N_f	V	β	a [fm]	m_π [MeV]	$m_\pi \cdot L$	N_{cnfg}	N_{src}/N_{cnfg}	BC
2	128×64^3	5.3	0.068	185	4.2	173	256	periodic

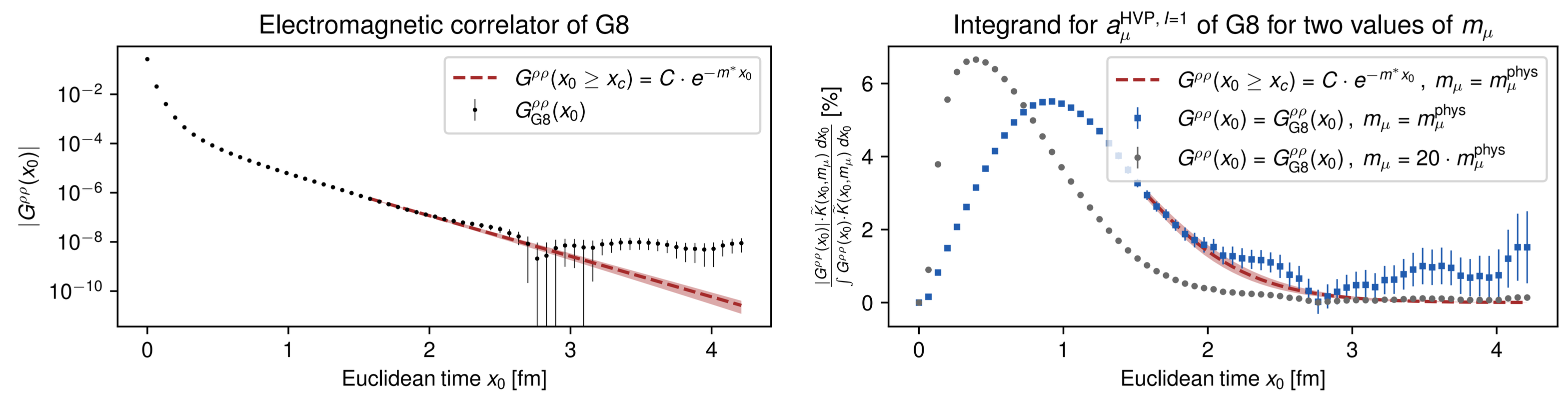

Figure 3: Left: The electromagnetic correlator of the ensemble G8 by CLS. In the long distance region, the noisy raw correlator can be replaced by a single exponential $G^{\rho\rho}(x_0 > 1.6\,\mathrm{fm}) \propto e^{-m^* x_0}$. Right: The integrand of $a_\mu^{\mathrm{HVP},I=1}(m_\mu)$ for G8 at physical and unphysical m_μ.

To combine the many possible fit functions[3], we apply the Akaike Information Criterion (AIC) and weight the results by $w = e^{-\frac{1}{2}\mathrm{AIC}}$. To propagate statistical uncertainties, we use the python library `pyerrors` [8]. Results at m_μ^{phys} are displayed in fig. 4.

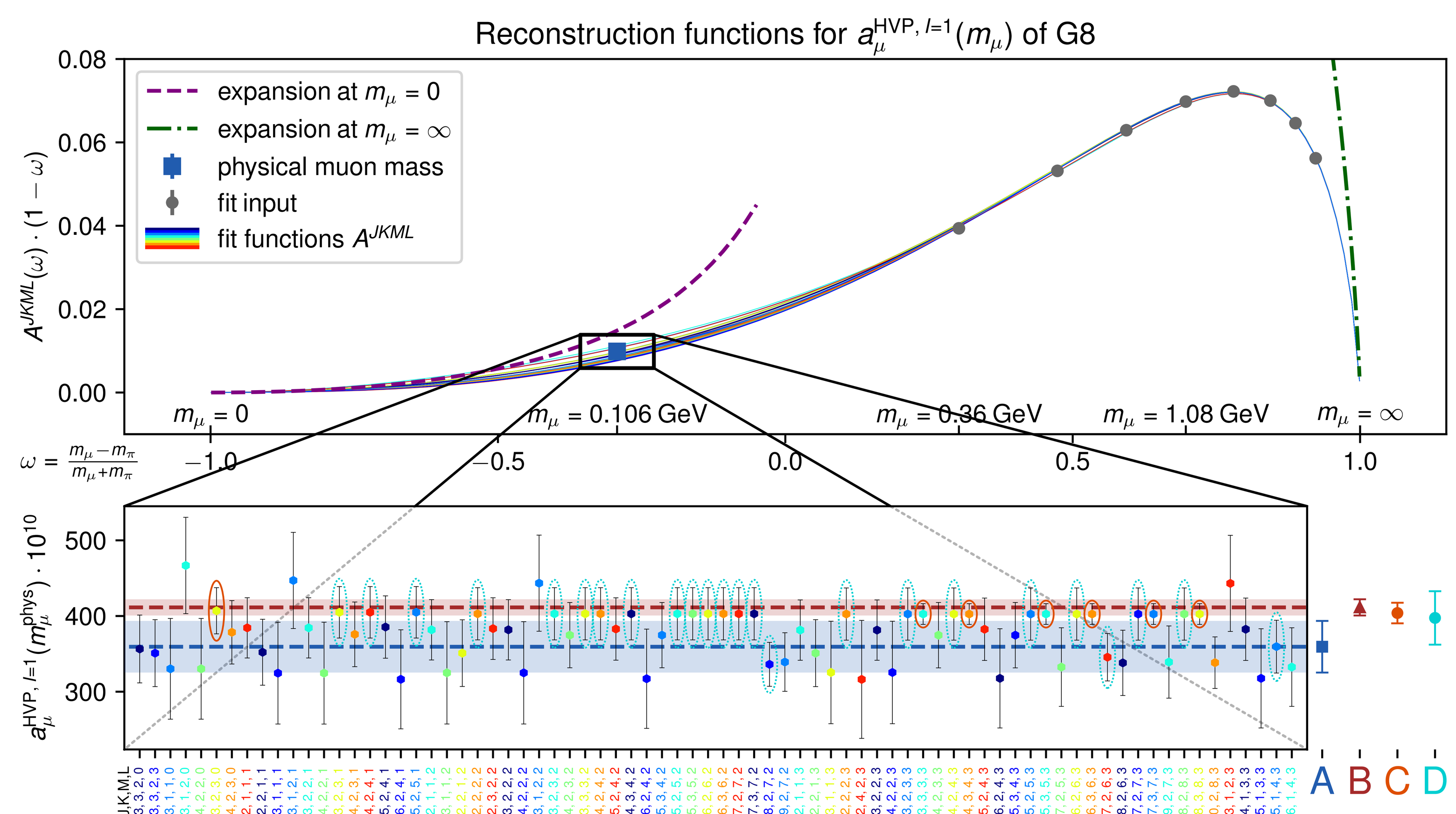

Figure 4: A number of functions $A^{JKML}(\omega)$ to reconstruct $a_\mu^{\mathrm{HVP},I=1}(m_\mu)$ (top) and their result at m_μ^{phys} (bottom). On the right their weighted mean is compared to integrating over the blue or brown data in fig. 3. For the orange value (C) only functions with a relative uncertainty below 8% were considered. For the turquoise result (D) functions with a relative uncertainty up to 10% were included.

Table 1: Comparison of different calculations of $a_\mu^{\mathrm{HVP},I=1}(m_\mu)$

A	direct evaluation at m_μ^{phys} - raw correlator	$(359 \pm 34_{\mathrm{stat}})\cdot 10^{-10}$
B	direct evaluation at m_μ^{phys} - single exponential tail	$(411 \pm 11_{\mathrm{stat}})\cdot 10^{-10}$
C	reconstruction - weighted mean of ()	$(404 \pm 13_{\mathrm{stat}} \pm 2_{\mathrm{sys}}\,[14_{\mathrm{tot}}])\cdot 10^{-10}$
D	reconstruction - weighted mean of () and ()	$(397 \pm 31_{\mathrm{stat}} \pm 17_{\mathrm{sys}}\,[35_{\mathrm{tot}}])\cdot 10^{-10}$

Remarks:

- $A^{sing}(\omega)$ depends on time moments $G_{2j} = \int dx_0 x_0^{2j} G(x_0)$ for each $1 < j \leq L+1$. These are long distance observables, regardless on m_μ. However, their precise determination does not seem to affect the picture in fig. 4 significantly. The above results are obtained using G_{2j} from $G_{\mathrm{G8}}^{\rho\rho}(x_0)$ with single exponential tail (−− in fig. 3).
- A full analysis of other sources of systematics (mainly the number and position of input points, the starting fit parameters and the minimization method) is still ongoing.

Conclusions and Outlook

- For the relatively low pion mass of 185 MeV, the dependency of $a_\mu^{\mathrm{HVP},I=1}$ on m_μ can be reconstructed with analytical functions without significantly increasing the uncertainty at the physical muon mass.
- The method can in principle be combined with other noise reduction techniques.
- Since A^{sing} depends linearly on the correlator, the proposed method can be used for window contributions to a_μ^{HVP}.
- Instead of at finite lattice spacing a, the method can be applied in the continuum to data of $a_\mu^{\mathrm{HVP}}(a=0, m_\mu \gg m_\mu^{\mathrm{phys}})$. This may be advantageous for computations with staggered fermions, for which discretization effects come from long distance and are thus suppressed at $m_\mu \gg m_\mu^{\mathrm{phys}}$.

References

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