

Light Cone Sum Rules techniques for heavy-meson decays

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Physics Siegen



Introduction

- Non-perturbative hadronic parameters (like form factors) : main source of uncertainty.
- QCD based methods: Sum Rules (analytical) and Lattice QCD (numerical). **See talk by Benoit Blossier for Lattice techniques**

QCD sum rules: "The method of expansion of the correlation functions in the vacuum condensates with the subsequent matching via the dispersion relations", Shifman

[M.A. Shifman, hep-ph/9802214]

[M.A. Shifman, A.I. Vainshtein, V. I. Zakharov [1979]]

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- Why do we still care for LCSR?



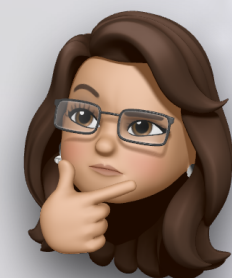
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- Why do we still care for LCSR?



- Large recoil region, resonance final states, etc are not yet accessible to lattice QCD $\implies$ LCSRs in several cases are the only available QCD-based method.
- LCSR allows to calculate the so called soft contribution (not accessible in purely pQCD-based methods) to a form factor.

a review by A. Khodjamirian, B. Melić, Y. M. Wang, [2311.08700]

Understanding main idea of QCD sum rules

- Correlation function (CF): Matrix element of quark-gluon interpolating operators taken between hadronic states.

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 - At $q^2 \rightarrow -\infty$, short distance quark-anti-quark fluctuations $\Rightarrow$ Can be treated in pQCD.
 - At positive q^2 , decompose in terms of hadronic states $\Rightarrow$ Dispersion relation

Dispersion relation

- Uses unitarity & analyticity of correlation functions.
- Written directly in terms of hadronic states.

Ways to calculate CF

Perturbative QCD

- Use of quarks & gluons dof.
- Treated in framework of operator product expansion (OPE).

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Matching the two gives estimates for the hadronic objects

Light cone sum rule (LCSR) in a Nutshell

Idea: To compute hadronic parameters using the analytic properties of the correlation function (treated in the framework of light cone OPE).

TOOLS TO DERIVE LCSR

Light cone OPE

(Enables one to write correlation function as a product of Hard scattering kernel and DAs)

Dispersion Relation

(Enables to relate the real part of the hadronic correlation function to its imaginary part)

Quark Hadron Duality

(Relates the non-perturbative spectral function to the perturbatively calculated correlation function)

Borel Transformation

(To suppress effect of continuum & higher resonances to reduce uncertainty due to duality approximation)

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	SVZ Sum Rules	LCSR
Correlator	Vacuum to Vacuum	Vacuum to hadron
Universal Non-Pert. inputs	QCD condensates	Distribution amplitudes
OPE expansion	Short distance OPE	Light-Cone OPE
Expansion parameter	Dimension of local operators	Twist
Used to compute	Decay constants, threshold parameters, condensates, etc	Form Factors, non-local effects, strong couplings, threshold parameters, etc

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LCSR applications

LCSR techniques have **very broad spectrum of applications**: from FF calculations involved in various radiative and weak decays of heavy and light mesons and baryons to determination of strong coupling constants, amplitude of non-leptonic decays, etc.

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In this talk:

- LCSR for Heavy to Light form factors .
- LCSRs for $B \rightarrow PP$ form factors
- LCSRs for non-local form factors in rare FCNCs.

Majorly based on A. Khodjamirian, B. Melić, Y. M. Wang, [2311.08700]



LCSR for Heavy to Light Form Factors



Heavy to Light Form Factors

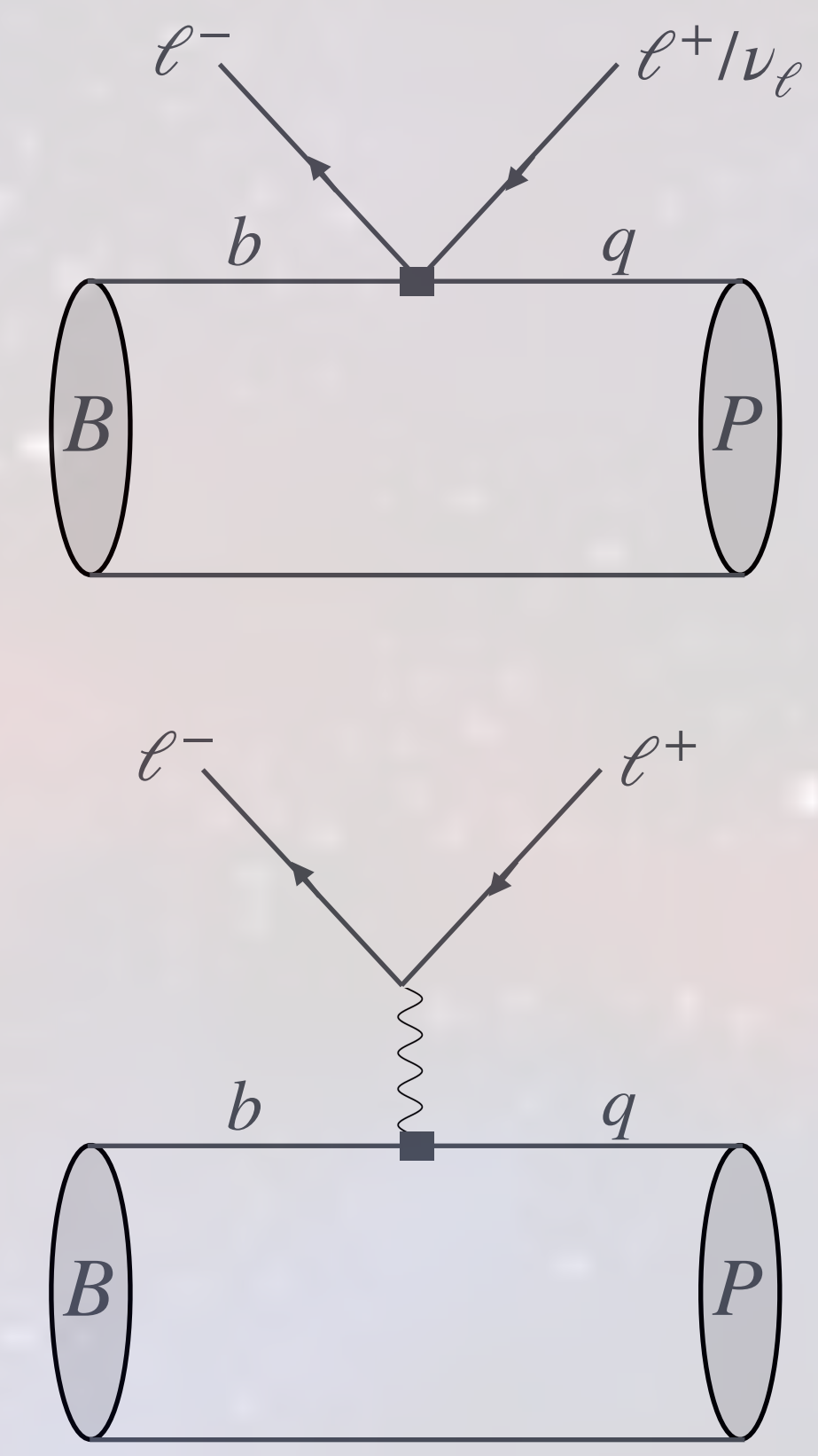
- Exclusive determination of $V_{ub} : B \rightarrow \pi, B_s \rightarrow K, B \rightarrow \rho$, etc
- Rare FCNC decays : $B \rightarrow K, B \rightarrow K^*, D \rightarrow \pi$, etc.

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- $B \rightarrow P$: Three types of local FFs: f_{BP}^+ (Vector), f_{BP}^0 (Scalar), f_{BP}^T (Tensor)

$$\langle P(k) | \bar{q} \gamma^\mu b | B(p) \rangle = \left[(p + k)^\mu - \frac{m_B^2 - m_P^2}{q^2} q^\mu \right] f_{BP}^+ + \frac{m_B^2 - m_P^2}{q^2} q^\mu f_{BP}^0$$

$$\langle P(k) | \bar{q} \sigma^{\mu\nu} q_\nu b | B(p) \rangle = \frac{i f_{BP}^T}{m_B + m_P} [q^2 (p + k)^\mu - (m_B^2 - m_P^2) q^\mu]$$



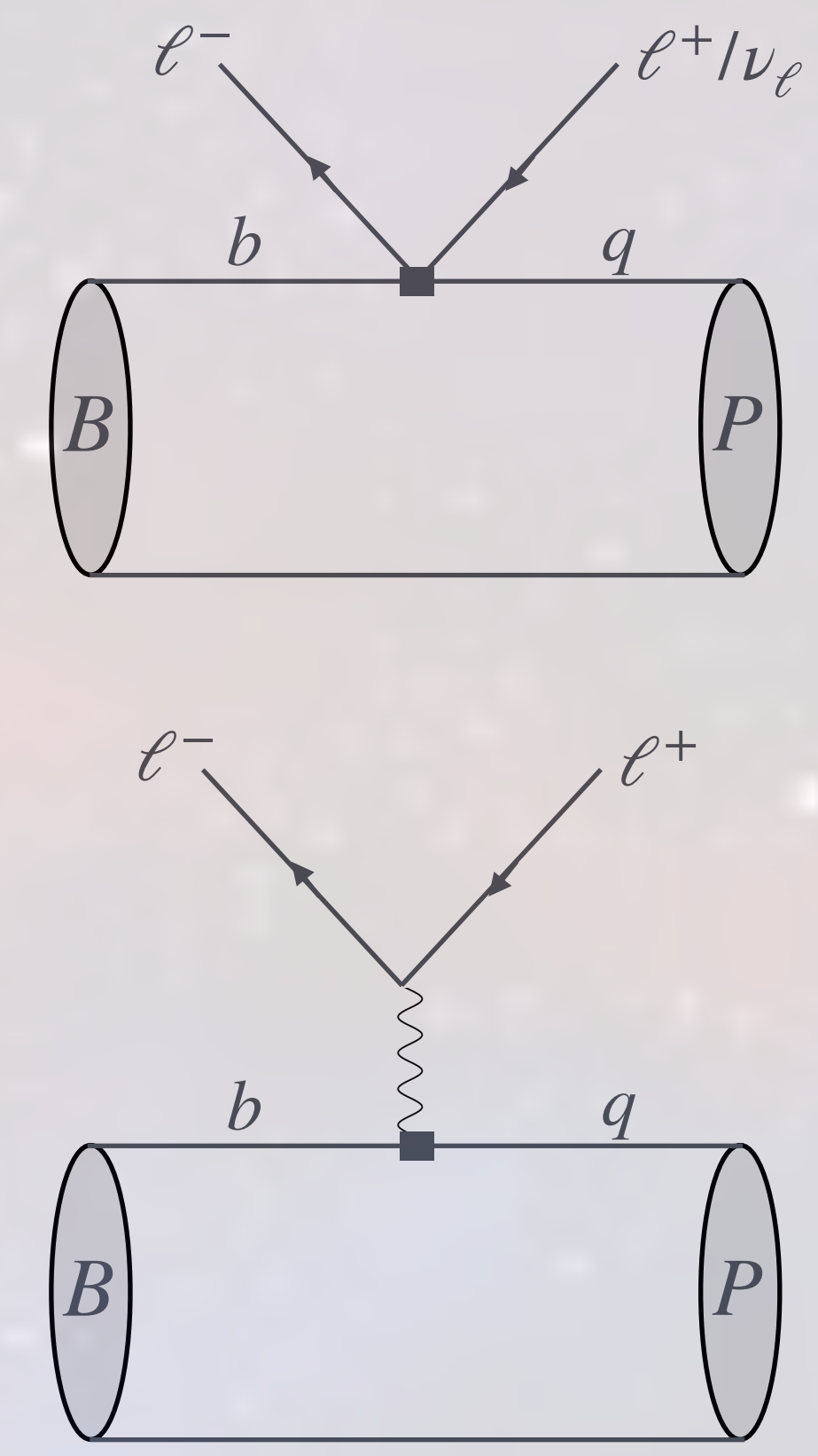
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- For $B \rightarrow V$, the number of local form factors increases to seven.



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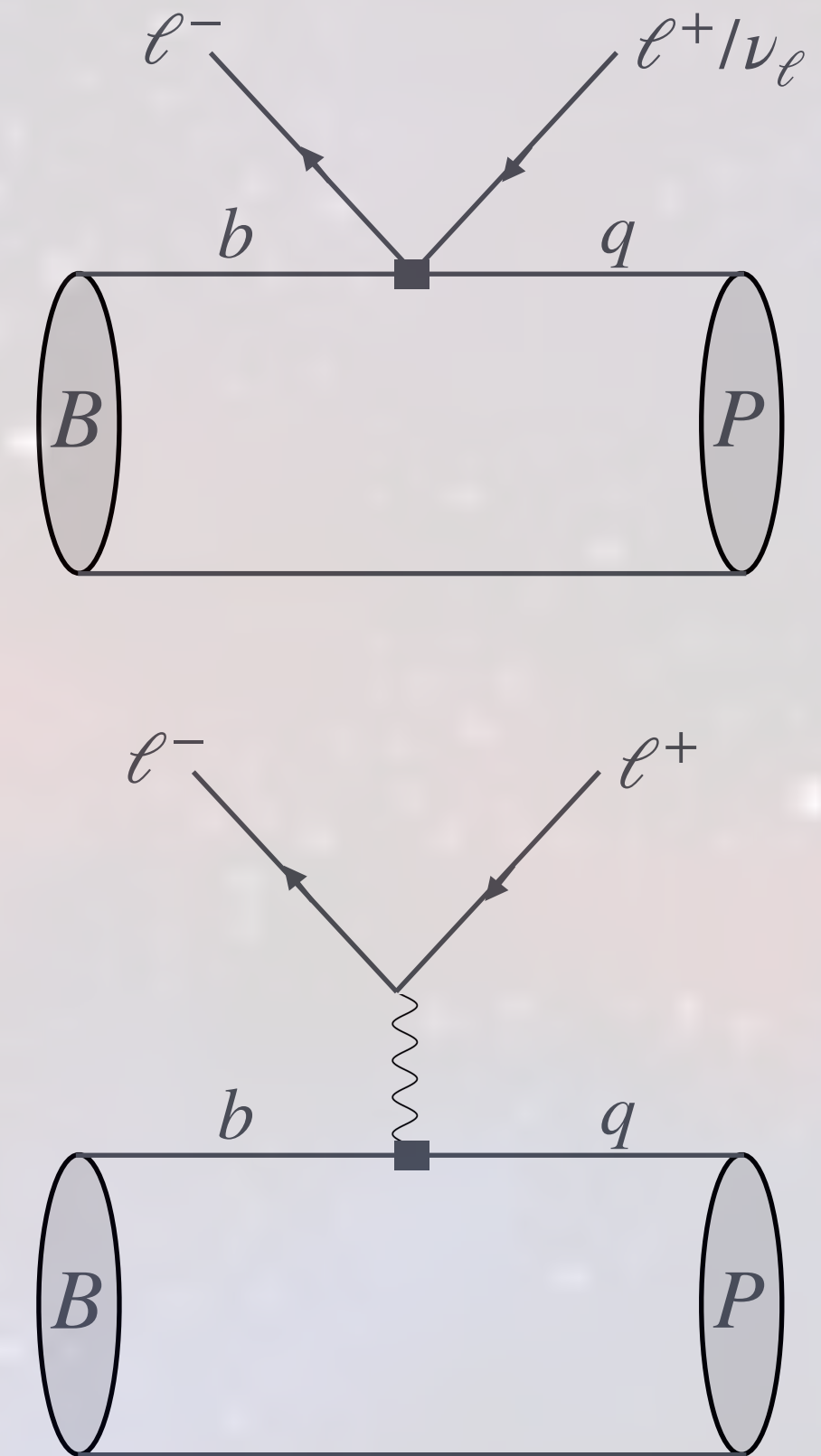
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Two ways of deriving LCSR:

1. Using light meson distribution amplitudes : Quark hadron duality in B -meson channel.
2. Using B distribution amplitudes : Quark hadron duality in light-meson channel.



An example: $B \rightarrow \pi$ Vector FF using Pion DAs

- Correlation function for $f_{B\pi}^+$:

Invariant amplitude important for $f_{B\pi}^+$

$$F_{B\pi}^\mu(p, q) = i \int d^4x e^{iq \cdot x} \langle \pi(p) | T \{ \bar{u}(x) \gamma^\mu b(x), m_b \bar{b}(0) i \gamma_5 d(0) \} | 0 \rangle = F_{B\pi}(q^2, (p+q)^2) p^\mu + \tilde{F}_{B\pi}(q^2, (p+q)^2) q^\mu$$

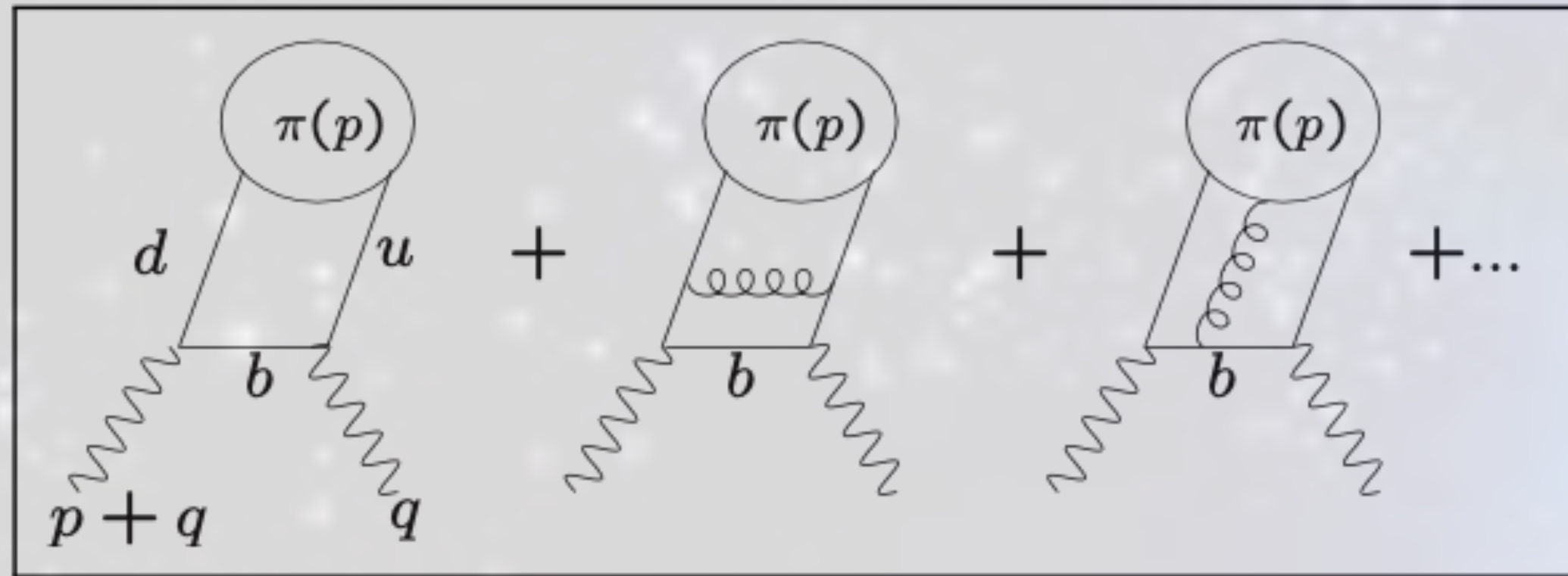
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LC-OPE



At $q^2 \ll m_b^2$ and $(p+q)^2 \ll m_b^2$

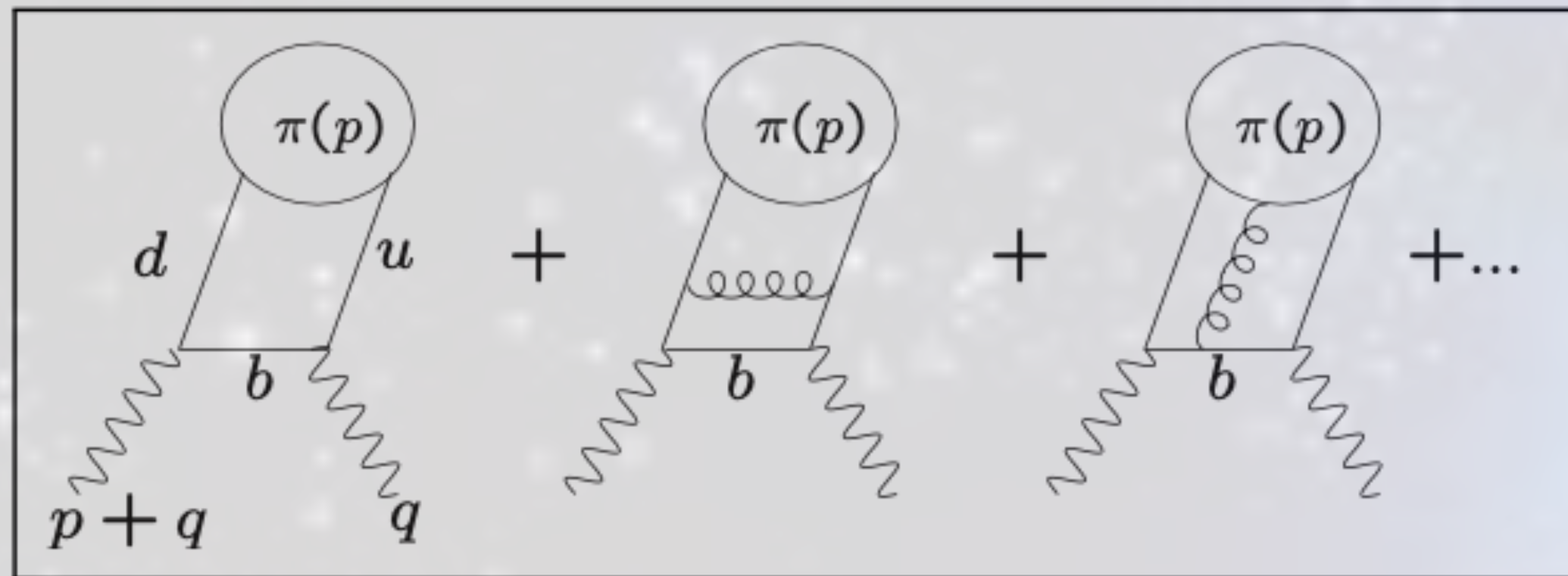
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Hadronic dispersion Relation

$$F(q^2, (p+q)^2) = \text{Diagram 1} + \sum_h \text{Diagram 2}$$

The first diagram shows a pion (π) at the top, a B meson (B) at the bottom, and a gluon line connecting them. The second diagram shows a pion (π) at the top, a B_h meson at the bottom, and a gluon line connecting them. The diagrams are summed together with ellipses indicating higher-order terms.

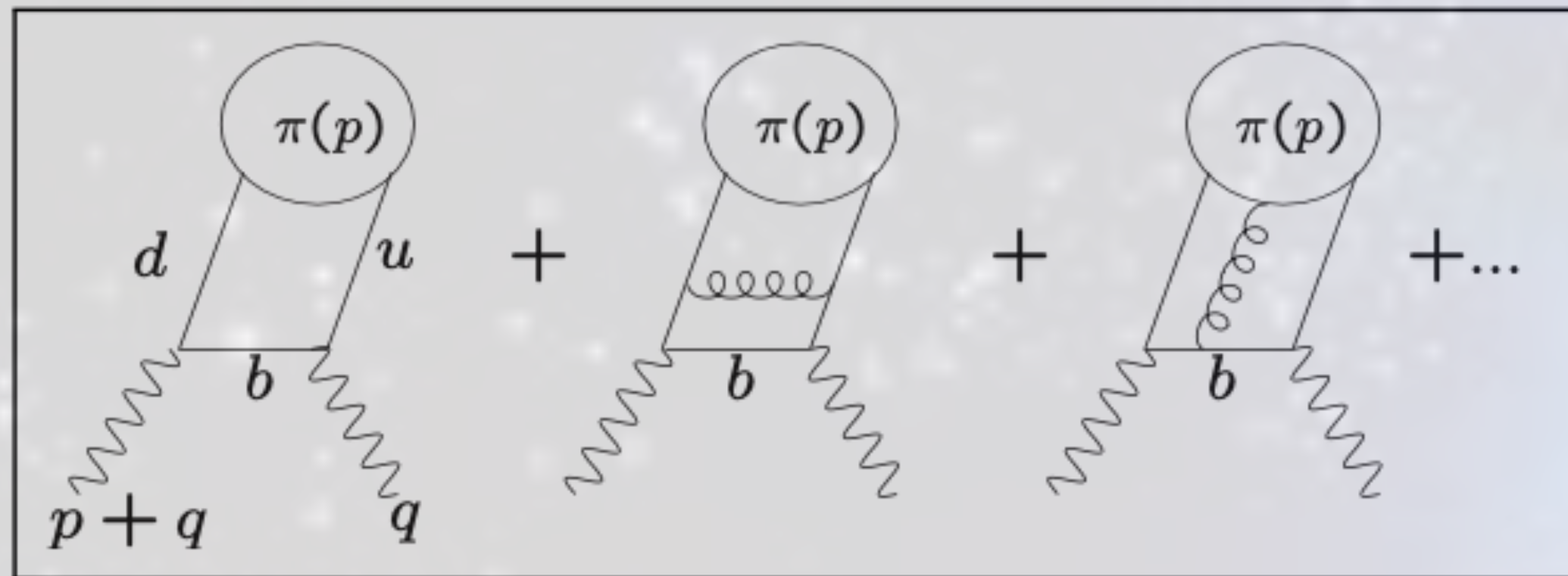
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The first diagram shows a pion (pi) connected to a B meson (B) via a quark line (u or b). The B meson is connected to a gluon (wavy line) which then splits into a quark-antiquark pair (p+q and q). The second diagram shows a pion (pi) connected to a B_h meson (B_h) via a quark line (u or b). The B_h meson is connected to a gluon (wavy line) which then splits into a quark-antiquark pair (p+q and q).

$$f_B f_{B\pi}^+(q^2)$$

From 2-point sum rules

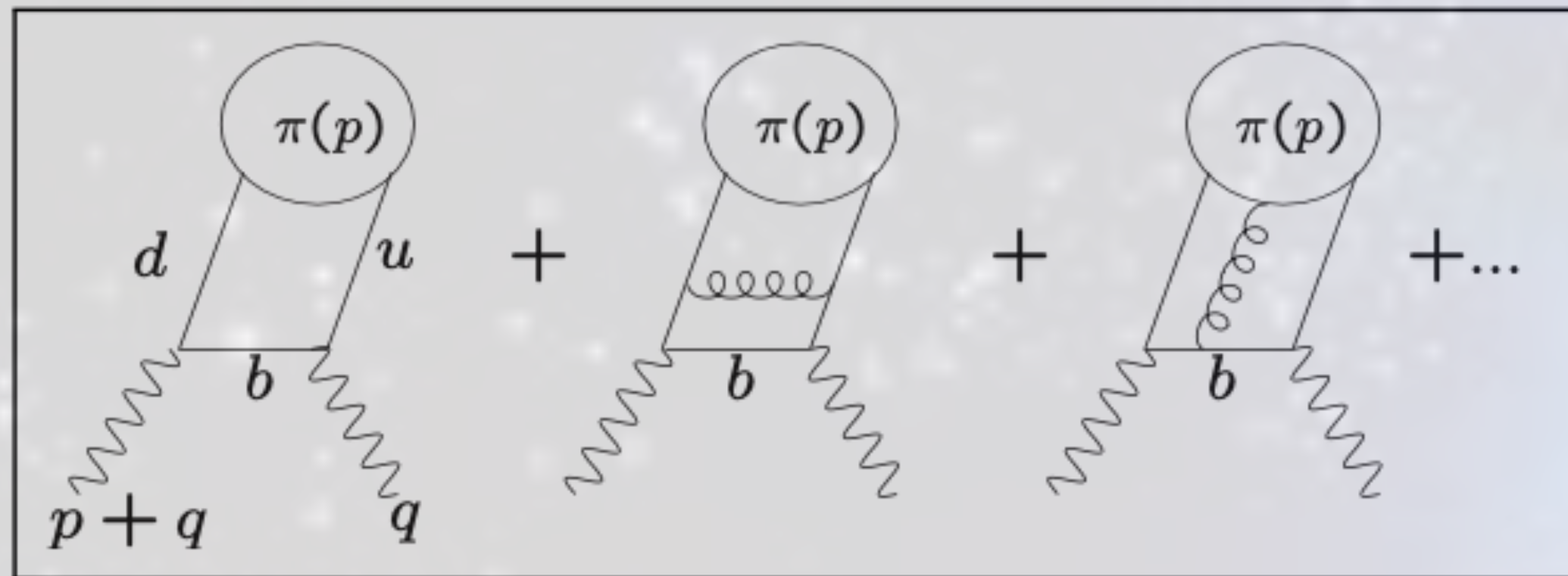
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The first diagram shows a pion (pi) and a B meson (B) connected by a quark line (u to b) and a gluon line (q). The second diagram shows a pion (pi) and a B_h meson (B_h) connected by a quark line (u to b) and a gluon line (q). The diagrams are summed together with ellipses indicating higher-order terms.

$$f_B f_{B\pi}^+(q^2)$$

From 2-point sum rules

$$\sum_{B_h} \Rightarrow \int_{s_0^B} ds \frac{\text{Im} F(q^2, s)_{OPE}}{s - (p+q)^2}$$

Quark-hadron duality

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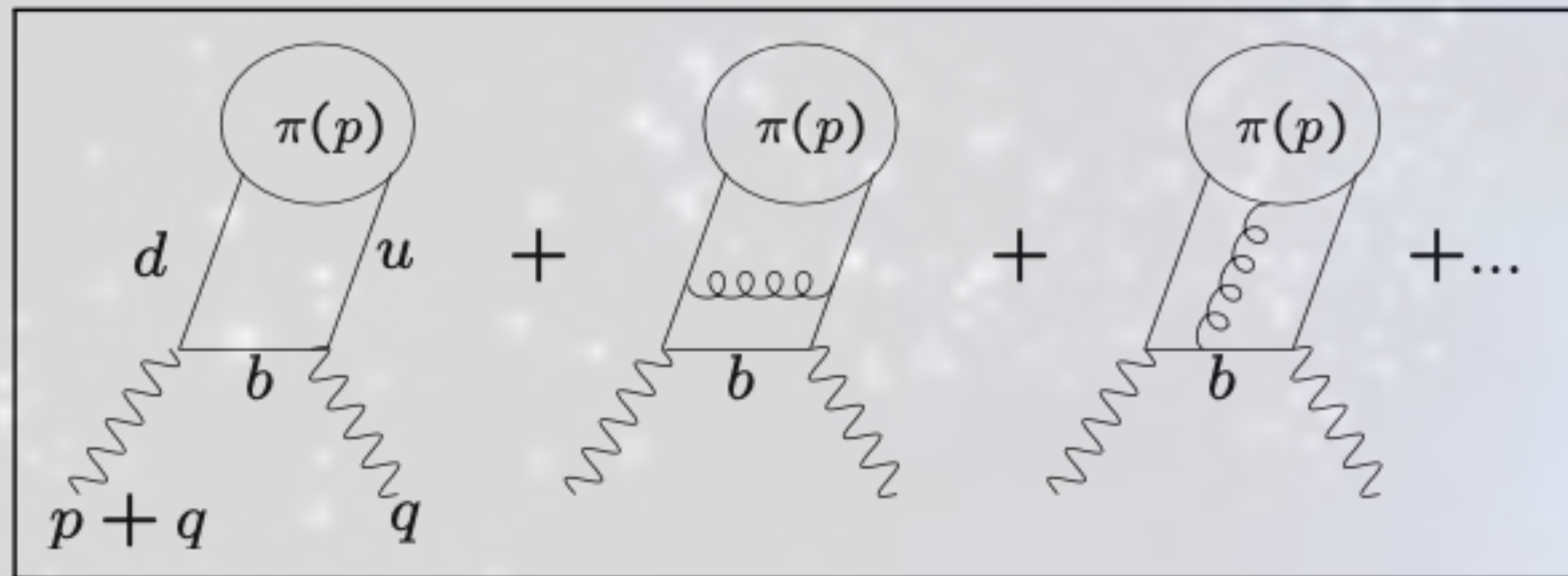
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Invariant amplitude important for $f_{B\pi}^+$

$\sigma^{\mu\nu}$ for $f_{B\pi}^T$

Required for LCSR for $f_{B\pi}^0$

LC-OPE



At $q^2 \ll m_b^2$ and $(p+q)^2 \ll m_b^2$

LCSR methods are flexible and universal:
Access to different FFs by adopting right
currents, DAs and constant parameters.

Hadronic dispersion Relation

$$F(q^2, (p+q)^2) = \text{Diagram 1} + \sum_h \text{Diagram 2}$$

The diagram on the left shows a pion (pi) connected to a quark line (u and b) which is connected to a B meson (B) which is then connected to a quark line (b and q). The diagram on the right shows a pion (pi) connected to a quark line (u and b) which is connected to a B_h meson (B_h) which is then connected to a quark line (b and q). The diagrams are summed together with ellipses indicating higher-order terms.

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From 2-point sum rules

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Quark-hadron duality

Status of OPE calculation

- Three layers of expansion in play:
 - Expansion in α_s : coming from gluonic corrections to perturbative kernel.
 - Expansion in twist “ t ”
 - Expansion in quark gluon multiplicities of DA:

Current status of OPE

multiplicity of DA	order in α_s	twist	Ref.
$m = 2$	LO	$t = 2, 3, 4$	[A. Khodjamirian, et. al’ 93,94]
	NLO	$t = 2$	[A. Khodjamirian, et. al’ 97, P. Ball et. al’ 97]
		$t = 3$	[A. Khodjamirian et. al’ 08, P. Ball, et. al’ 01]
	partial NNLO	$t = 2$	[A. Bharucha, 1203.1359]
$m = 3$	LO	$t = 3, 4$	[A. Khodjamirian, et. al’ 93,94]
$m = 4$	LO	$t = 5, 6$	[A. Rusov, 1705.01929]

Table 1 The calculated terms in the OPE expansion (5). The multiplicities $m = 2, 3$ and $m = 4$ correspond, respectively to the quark-antiquark, quark-antiquark-gluon DAs and to the diquark-antidiquark DAs in the factorized approximation (quark condensate $\otimes$ two-particle DA).

A. Khodjamirian, B. Melić, Y. M. Wang, [2311.08700]

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- Dependence of each DA on fractions of meson momenta shared between light degrees of freedom brings another level of expansion : **Conformal partial wave expansion**

Twist-2 π -DA: Expansion in Gegenbauer polynomials

V. M. Braun, G. P. Korchemsky, D. Müller, [hep-ph/0306057]

$$\phi_\pi(u,\mu) = 6u(1-u)\big(1 + a_2^\pi(\mu)C_2^{3/2}(2u-1) + a_4^\pi(\mu)C_4^{3/2}(2u-1) + \dots\big)$$

Usually taken up to $a_4^\pi(u)$.

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A. Khodjamirian, B. Melić, Y. M. Wang, [2311.08700]

Final form of LCSR:

$$m_B^2 f_B^2 f_{B\pi}^+(q^2) e^{-m_B^2/M^2} = \int_{m_b^2}^{s_0^B} ds \, e^{-s/M^2} \left[Im F(s, q^2) \right]_{OPE}$$

Inputs and accuracy

1. m_b, m_c, m_s, α_s from PGD : mainly provided by lattice QCD (Known with good precision).
2. LCSRs combined with 2-point QCD sum rules are “self-sufficient”: but the accuracy of certain input parameters (e.g. f_B, f_D) is limited: Nowadays using FLAG averages
3. Gegenbauer moments $a_n^{(2)}$ of π , and K DAs calculated in lattice QCD (still large uncertainties).
eg. V.M. Braun, et al., [1503.03656]
4. Duality Threshold s_0^B : using a daughter (derivative) sum rule.
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Important: to check that higher/continuum states contribution is $\leq 30 - 40 \%$

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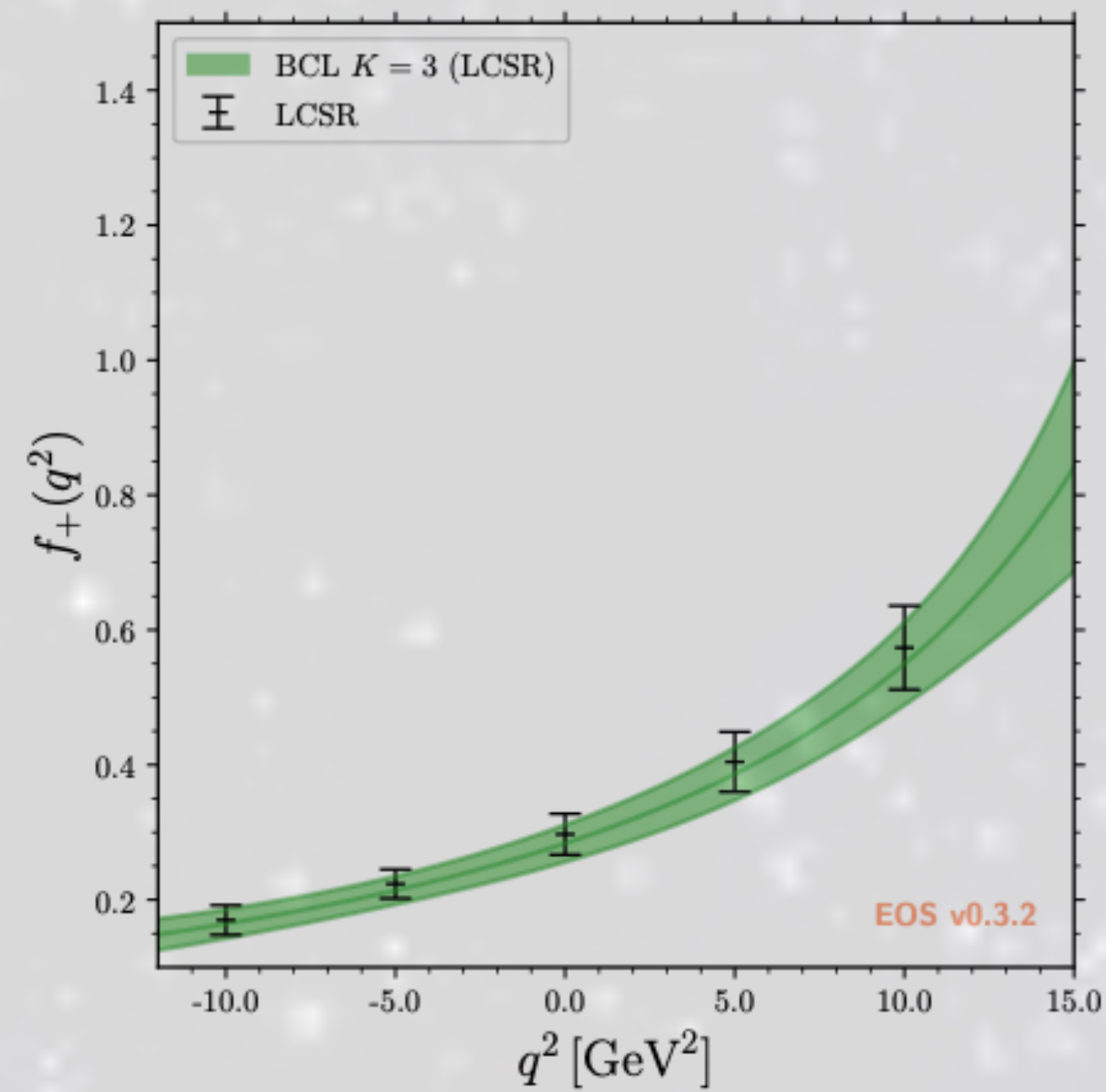
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4. Duality Threshold s_0^B : using a daughter (derivative) sum rule.
5. Borel Parameter: usually chosen in a window where FFs are more or less independent of it

Important: to check that higher/continuum states contribution is $\leq 30 - 40 \%$

The systematic uncertainties related to quark hadron dualities remain the main challenge:

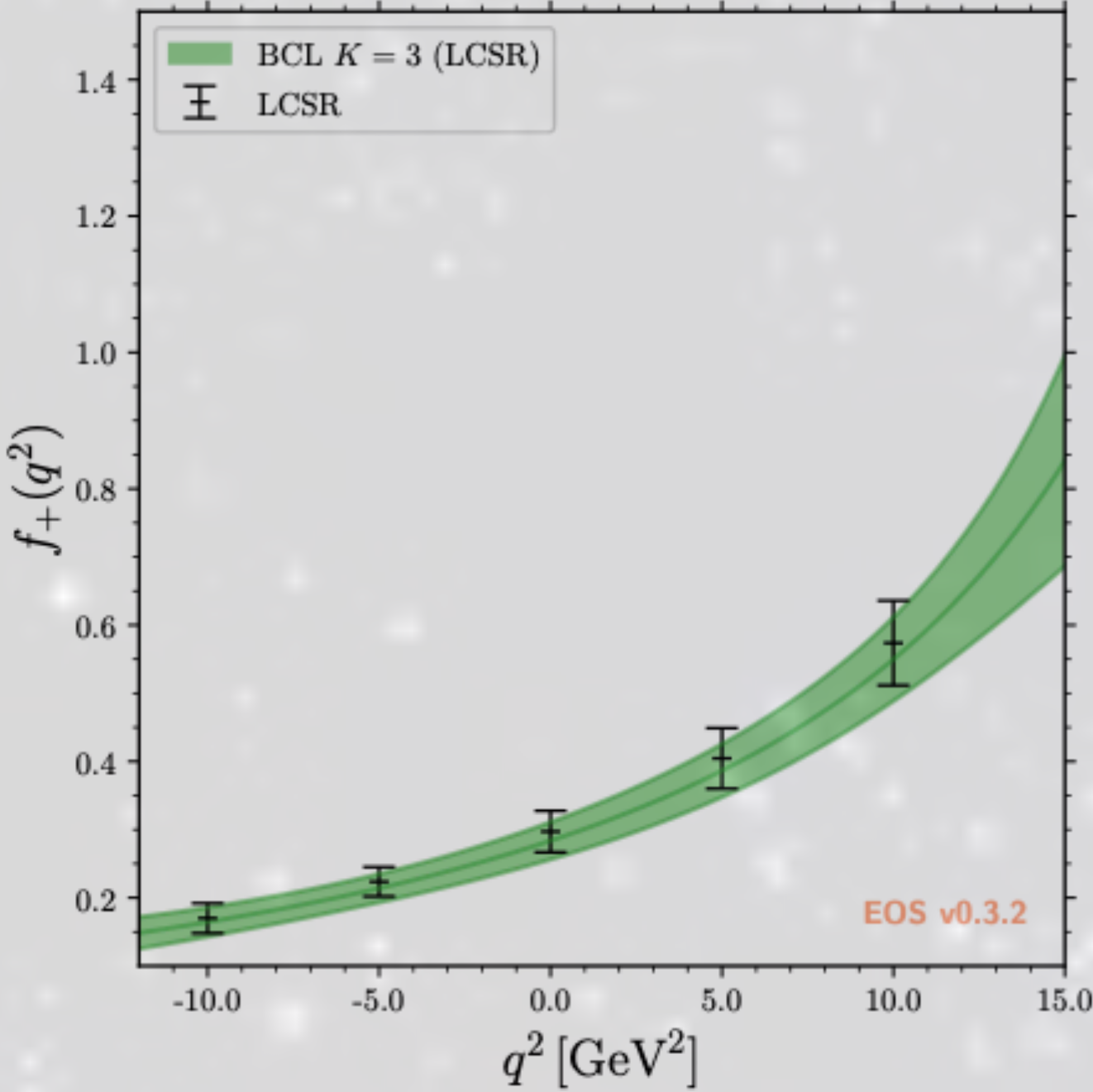
Can not be fully captured or reduced

Results using light meson DAs



D. Leljak, B. Melić, D. Van Dyk, [2102.07233]

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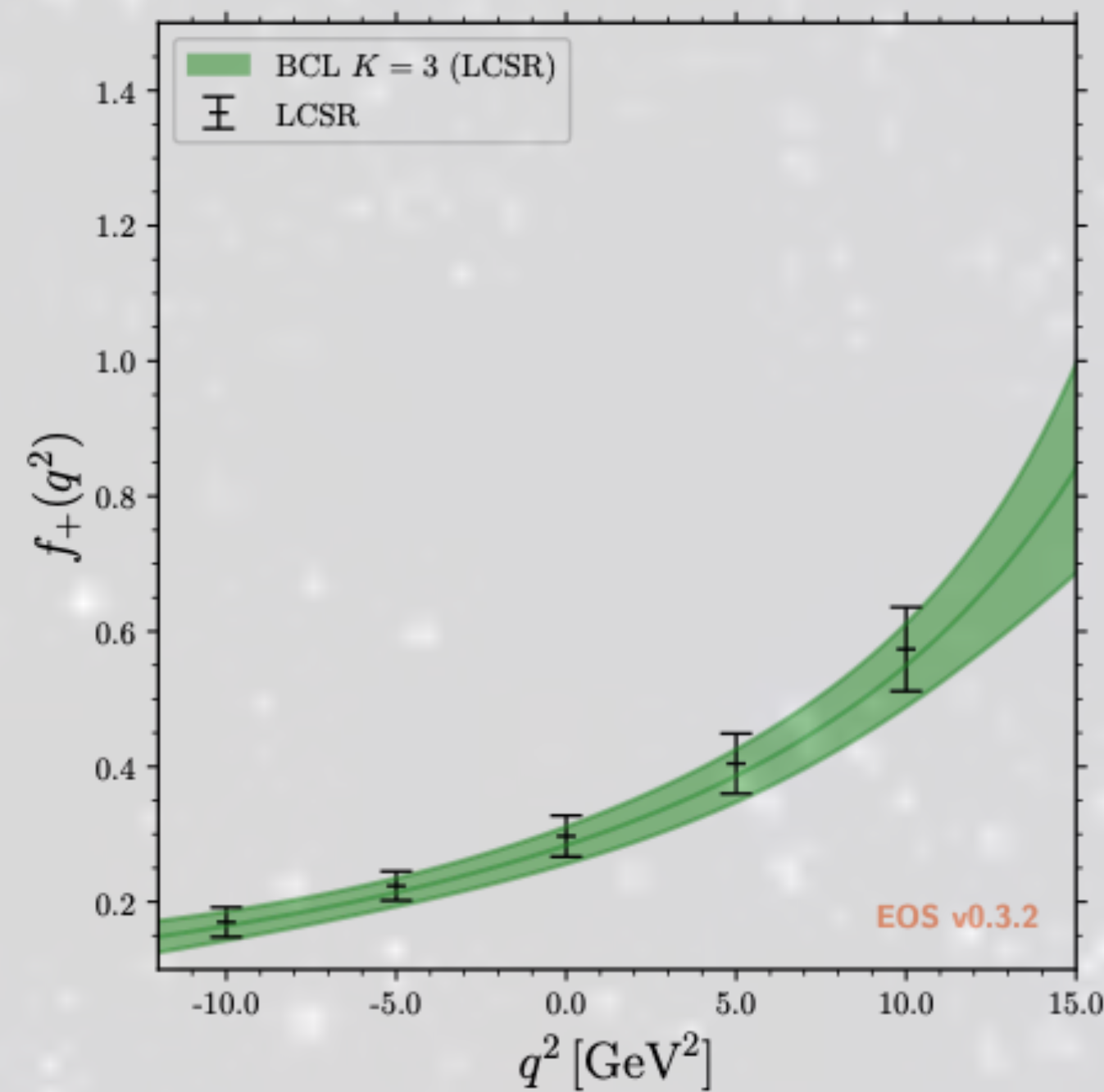
- Same analytical results can be easily extended to other local FFs with appropriate changes in quark content, input parameters and distribution amplitudes.

FF	$f^+(0) = f^0(0)$	$f^T(0)$	Ref.
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Table 3 The most recent LCSR results for $B_{(s)} \rightarrow P$ form factors at $q^2 = 0$. Their full q^2 dependence is given in the corresponding papers.

A. Khodjamirian, B. Melić, Y. M. Wang, [2311.08700]

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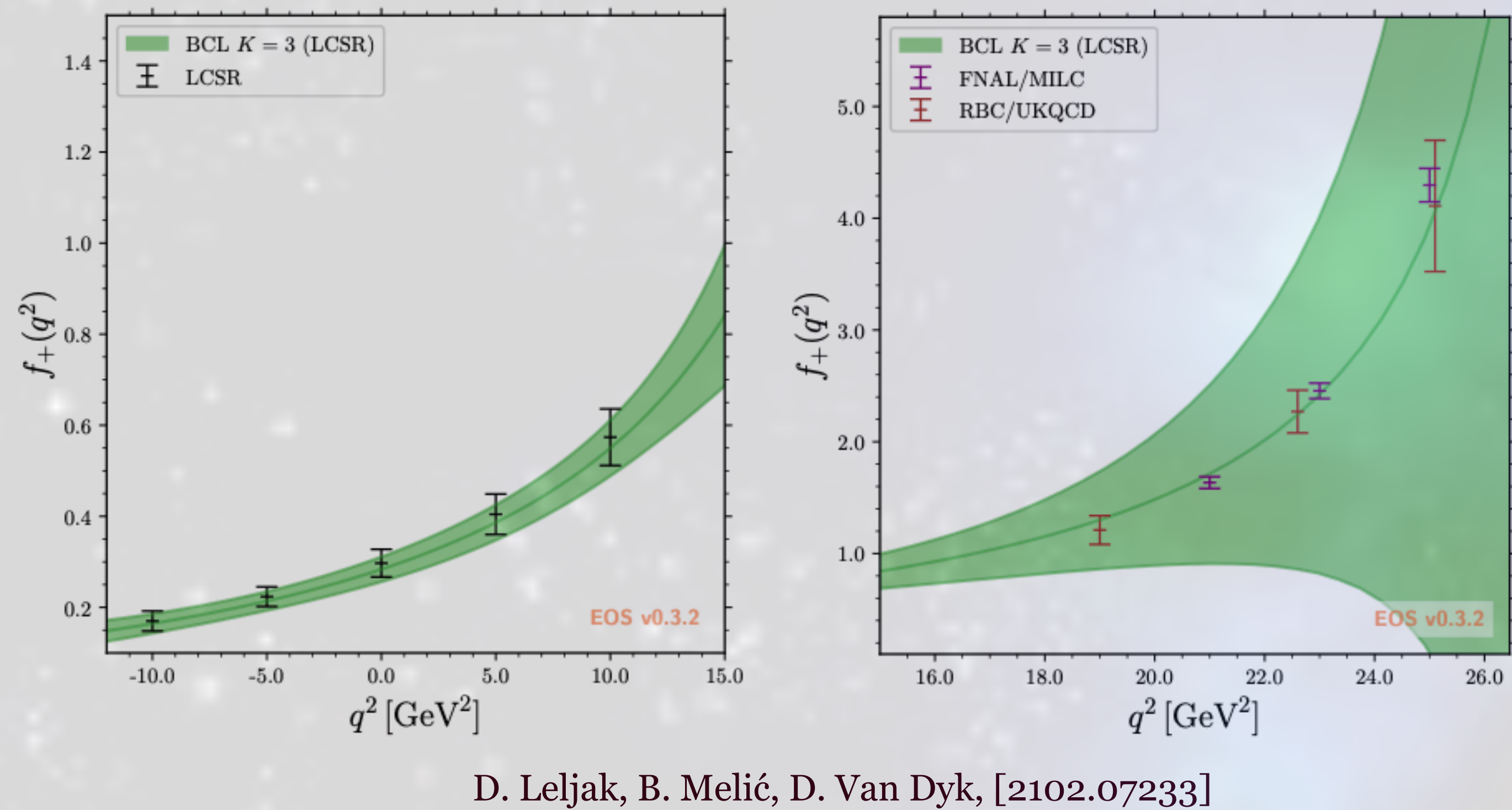
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Form Factor Extrapolation

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$$f_{\lambda}(q^2) = \frac{1}{1 - q^2/m_{\mathcal{F}}^2} \left[\sum_{n=0}^{\infty} a_{\lambda,n} (z(q^2; s_{\Gamma}, s_0)^n - z(0; s_{\Gamma}, s_0)^n) \right]$$

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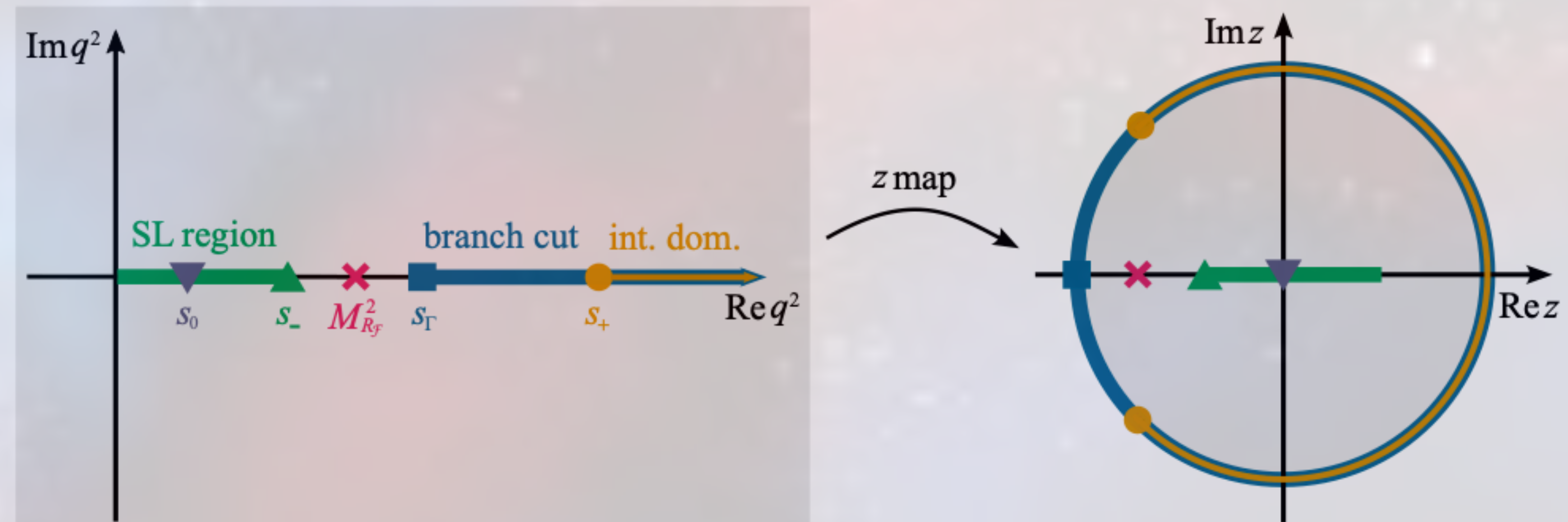
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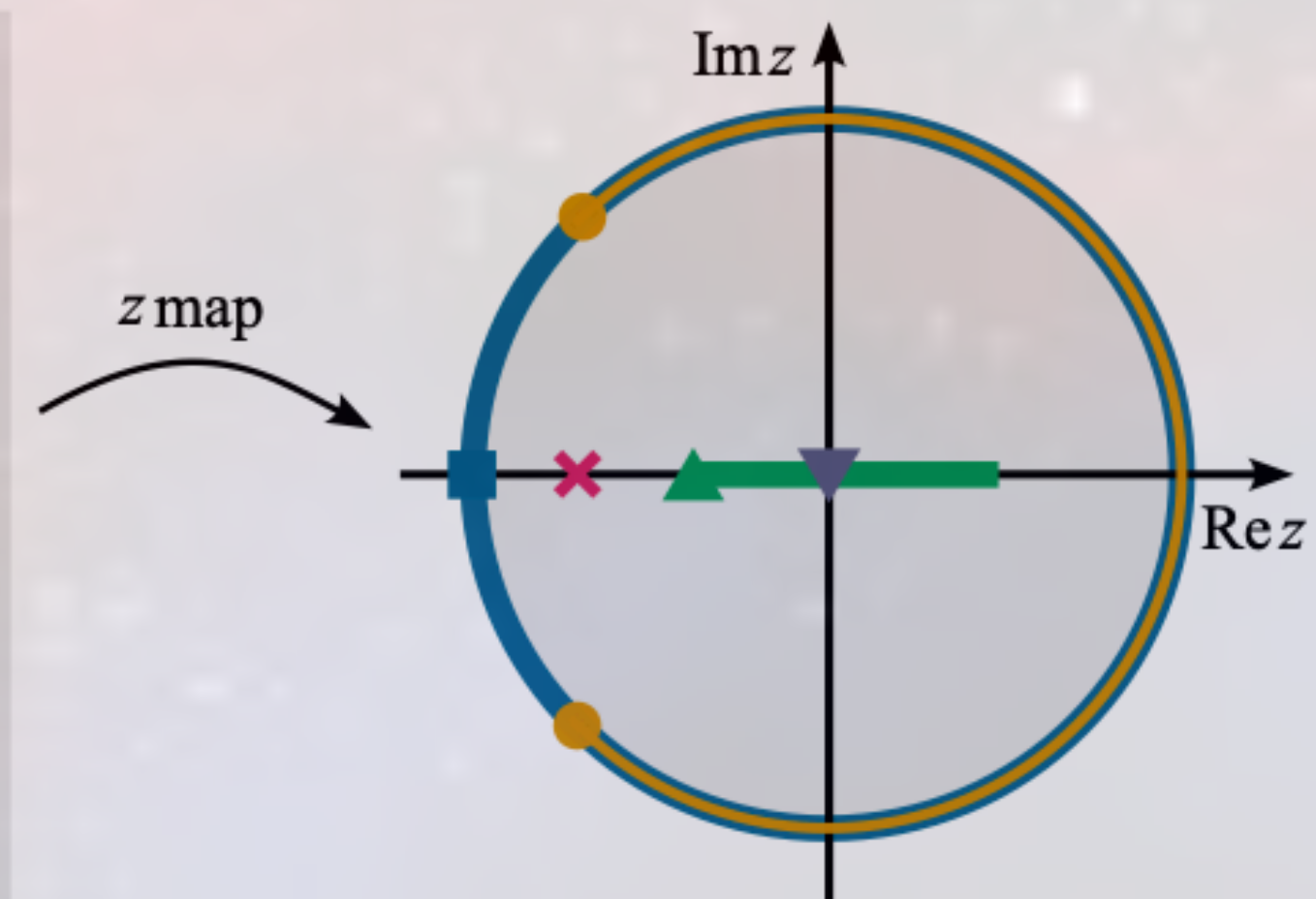
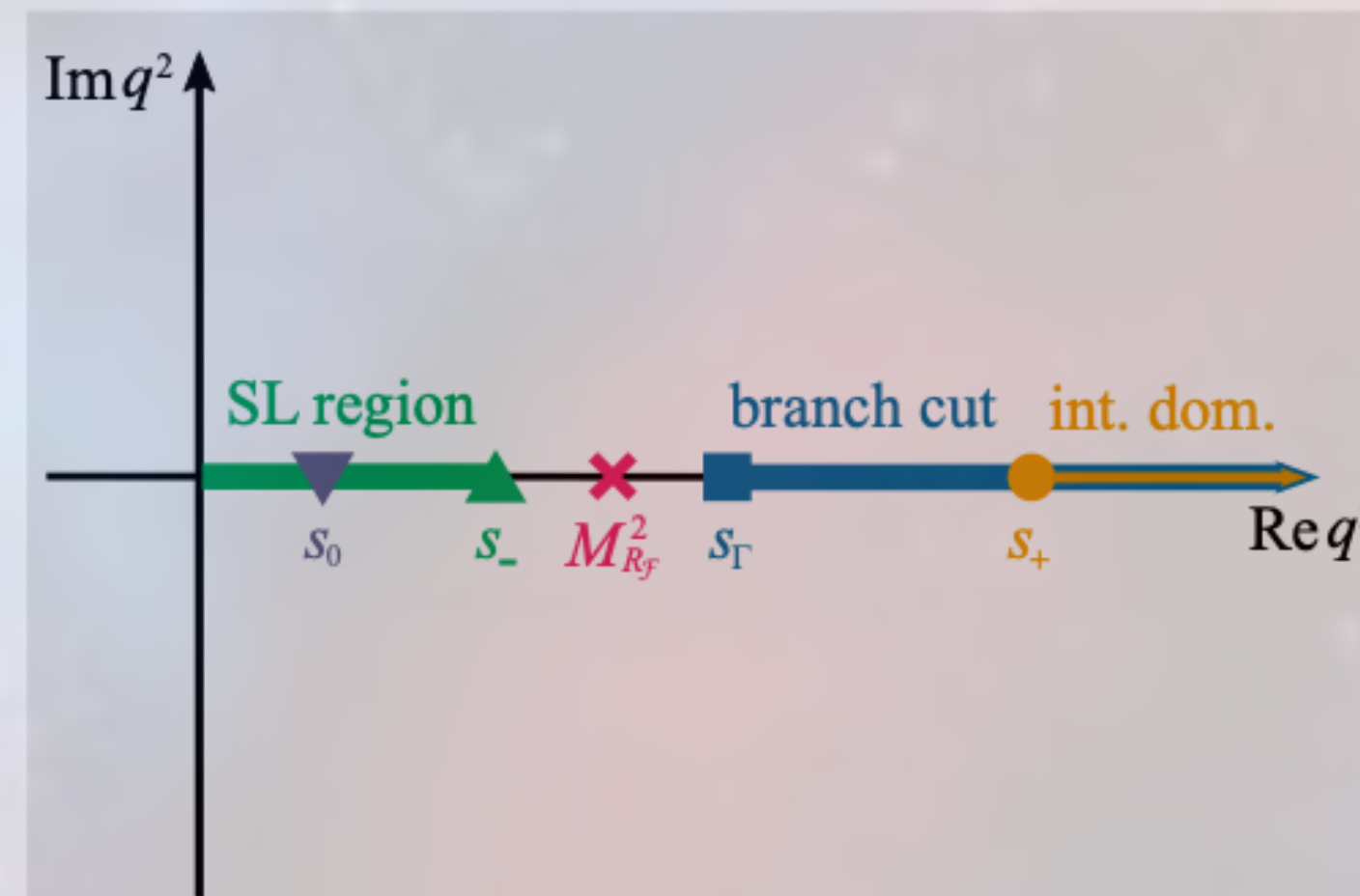
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K=2: usually enough to provide excellent description of data :
Truncation error?

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- Result into a bound on parametrization coefficients: $\sum_{n=0}^{\infty} |a_{\lambda,n}|^2 < 1$: Useful in estimating truncation error

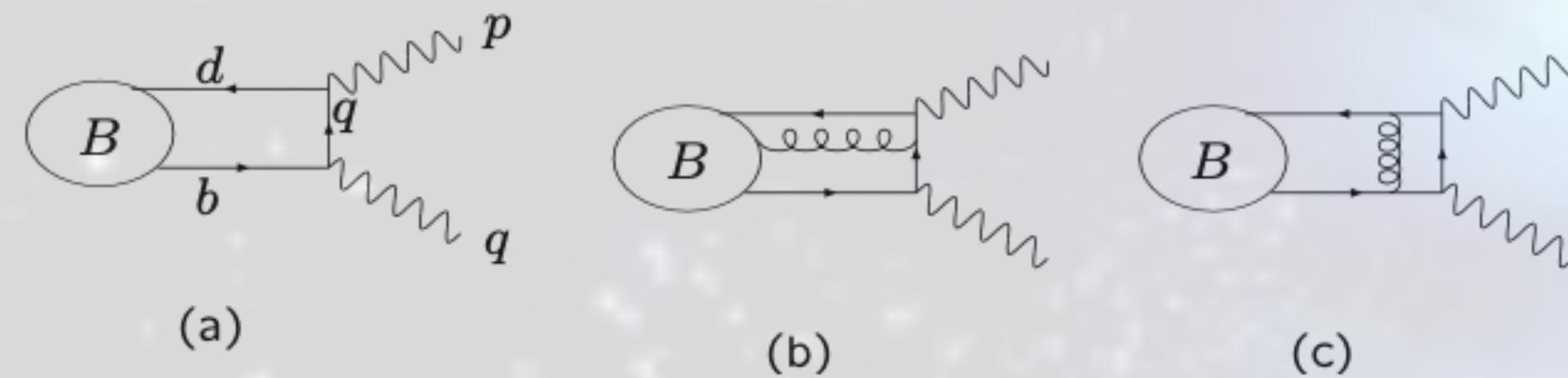
A. Khodjamirian, T. Mannel, D. Van Dyk [1409.7816]

An alternative: $B \rightarrow \pi$ Vector FF using B-meson DAs

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OPE



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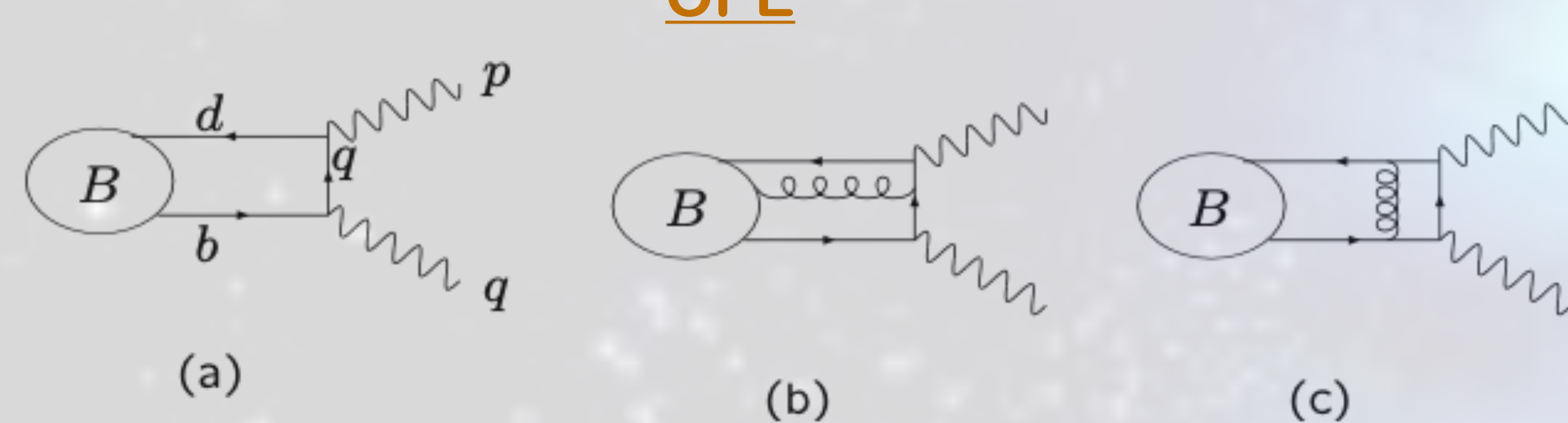
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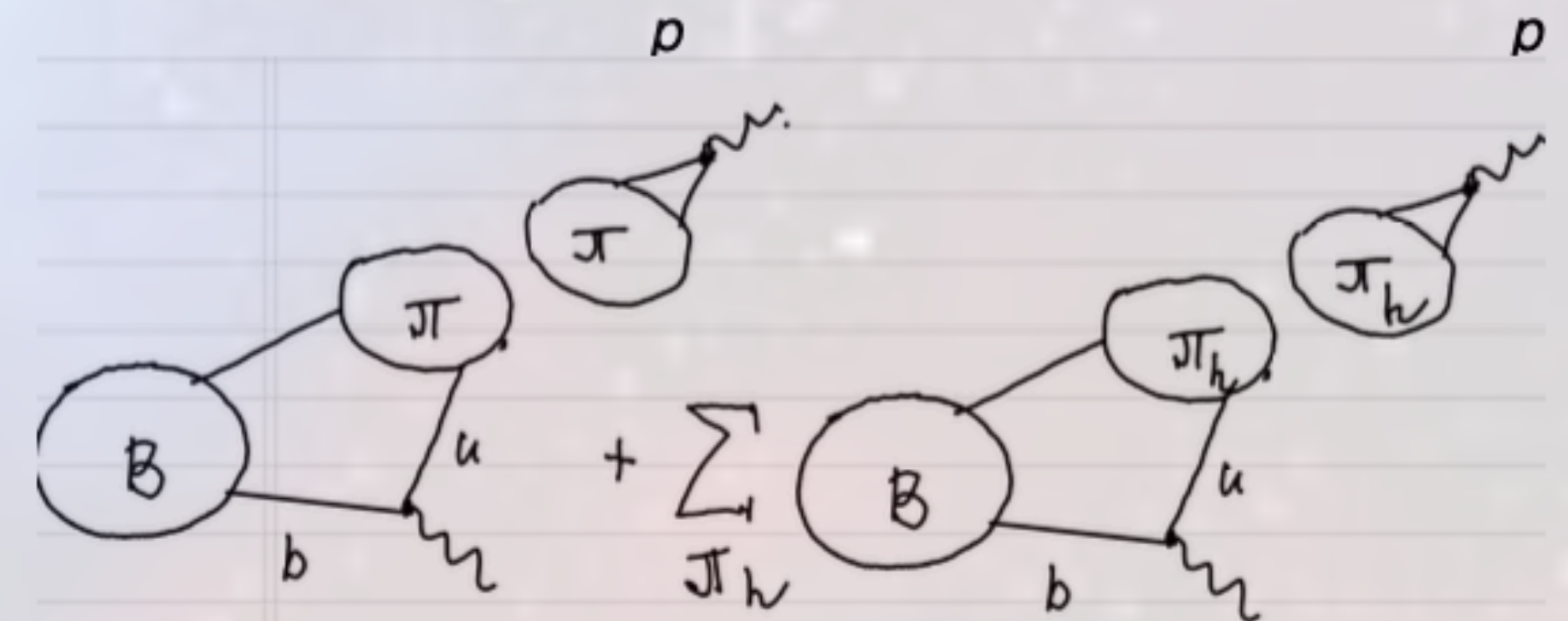
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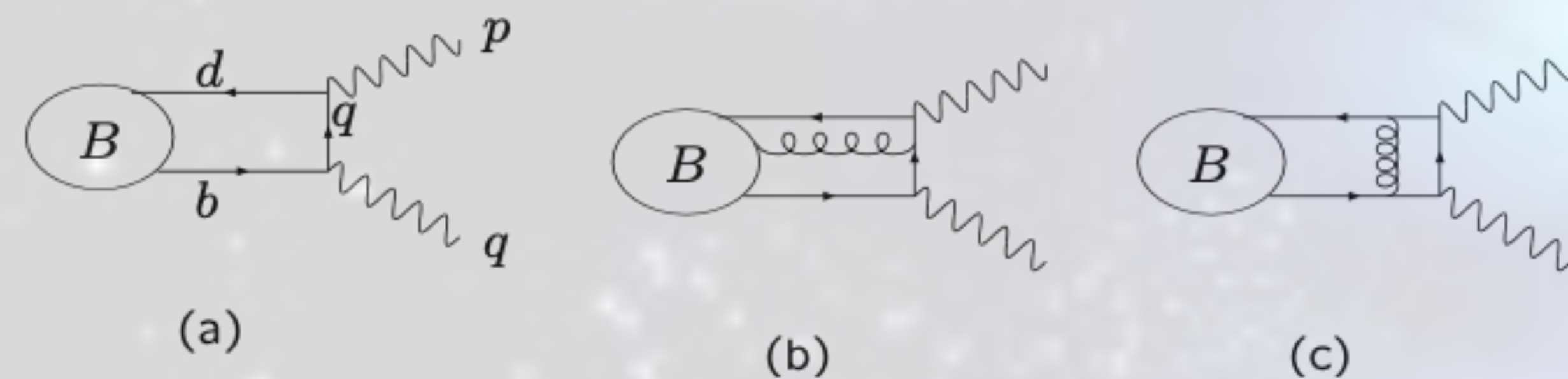
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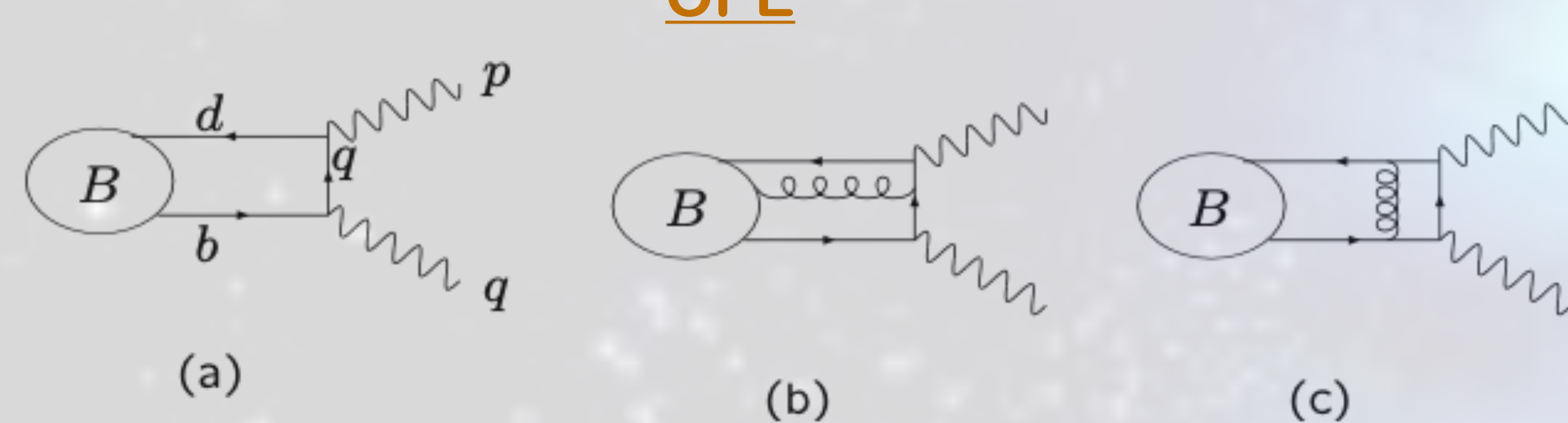
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Advantages:

- Additional independent results to compare & combine with light-meson LCSR.
- Heavy to heavy FFs are only accessible using Heavy DAs.

Status of results with B-meson DAs

- Key input parameter: the inverse moment

$$\frac{1}{\lambda_B(\mu)} = \int_0^\infty d\omega \frac{\phi_+^B(\omega, \mu)}{\omega}$$

- Possible extraction: $B \rightarrow \ell \nu_\ell \gamma$
- Current limit from Belle II : $\lambda_B > 240 \text{ MeV}$
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Y. M. Wang, L. Y. Shen, [1506.00667]

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Table 5 The LCSR results for $B_{(s)} \rightarrow P$ transition form factors at $q^2 = 0$ from LCSRs with B -meson DAs. (LCSRs at LO + SCET SRs at NLO)

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- Current limit from Belle II : $\lambda_B > 240$ MeV
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See talk by Martino Borsato for experimental updates

- Higher twist DA parameters: λ_B & λ_H (large uncertainties from two point QCD sum rules)

Two important questions:

1. Size of $1/m_b$ effects? :Expected to be small.
2. Size of NLO ($\mathcal{O}(\alpha_s)$) : Using SCET sum rules

Y. M. Wang, L. Y. Shen, [1506.00667]

- Challenges related to Quark hadron duality & Borel parameters is same as in light DA case.

FF	$f^+(0) = f^0(0)$	$f^T(0)$	Ref.
$B \rightarrow \pi$	0.191 ± 0.073	0.222 ± 0.078	[99]
	0.21 ± 0.07	0.19 ± 0.06	[81]
$B \rightarrow K$	0.325 ± 0.085	0.381 ± 0.097	[99]
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Table 5 The LCSR results for $B_{(s)} \rightarrow P$ transition form factors at $q^2 = 0$ from LCSR with B -meson DAs. (LCSRs at LO + SCET SRs at NLO)

A. Khodjamirian, B. Melić, Y. M. Wang, [2311.08700]

Status of results with B-meson DAs

- Key input parameter: the inverse moment

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An alternative method bypassing use of semi-local quark hadron duality is proposed to mitigate the systematic errors associated to LCSR in trade of the need of going higher in LCOPE

A. Carvunis, F. Mahmoudi, Y. Monceaux [2404.01290]

see talk by Alexandre Carvunis

$B \rightarrow V$ Form Factors in Narrow Width Approximation

- Results for $B \rightarrow P$ FFs can be simply extended to $B \rightarrow V$ with proper modifications in input parameters & DAs.

FF	$V(0)$	$A_0(0)$ $A_1(0)$ $A_{12}(0)$	$T_1(0) = T_2(0)$ $T_{23}(0)$
$B \rightarrow \rho$	0.327 ± 0.031	0.356 ± 0.042 0.262 ± 0.026 0.297 ± 0.035	0.272 ± 0.026 0.747 ± 0.076
$B \rightarrow \omega$	0.304 ± 0.038	0.328 ± 0.048 0.243 ± 0.031 0.270 ± 0.040	0.272 ± 0.026 0.683 ± 0.090
$B \rightarrow K^*$	0.341 ± 0.036	0.356 ± 0.046 0.269 ± 0.029 0.256 ± 0.033	0.282 ± 0.031 0.668 ± 0.083
$B_s \rightarrow \phi$	0.387 ± 0.033	0.389 ± 0.045 0.296 ± 0.027 0.246 ± 0.029	0.309 ± 0.027 0.676 ± 0.071
$B_s \rightarrow K^*$	0.296 ± 0.030	0.314 ± 0.048 0.230 ± 0.025 0.229 ± 0.035	0.239 ± 0.024 0.597 ± 0.076

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A. Khodjamirian, B. Melić, Y. M. Wang, [2311.08700]

FF (Ref. [83])	$V(0)$	$A_0(0)$ $A_1(0)$ $A_{12}(0)$	$T_1(0) = T_2(0)$ $T_{23}(0)$
$B \rightarrow \rho$	$0.327^{+0.204}_{-0.135}$	$0.317^{+0.129}_{-0.102}$ $0.249^{+0.155}_{-0.103}$ $0.265^{+0.107}_{-0.086}$	$0.287^{+0.180}_{-0.118}$ $0.711^{+0.356}_{-0.250}$
$B \rightarrow \omega$	$0.357^{+0.223}_{-0.148}$	$0.344^{+0.142}_{-0.107}$ $0.270^{+0.170}_{-0.111}$ $0.284^{+0.116}_{-0.087}$	$0.312^{+0.197}_{-0.129}$ $0.767^{+0.407}_{-0.266}$
$B \rightarrow K^*$	$0.419^{+0.245}_{-0.157}$	$0.382^{+0.154}_{-0.109}$ $0.306^{+0.180}_{-0.115}$ $0.273^{+0.112}_{-0.077}$	$0.361^{+0.211}_{-0.135}$ $0.793^{+0.402}_{-0.258}$

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- Question:** Is narrow width approximation really a good approximation for broad resonances like ρ -meson, etc?

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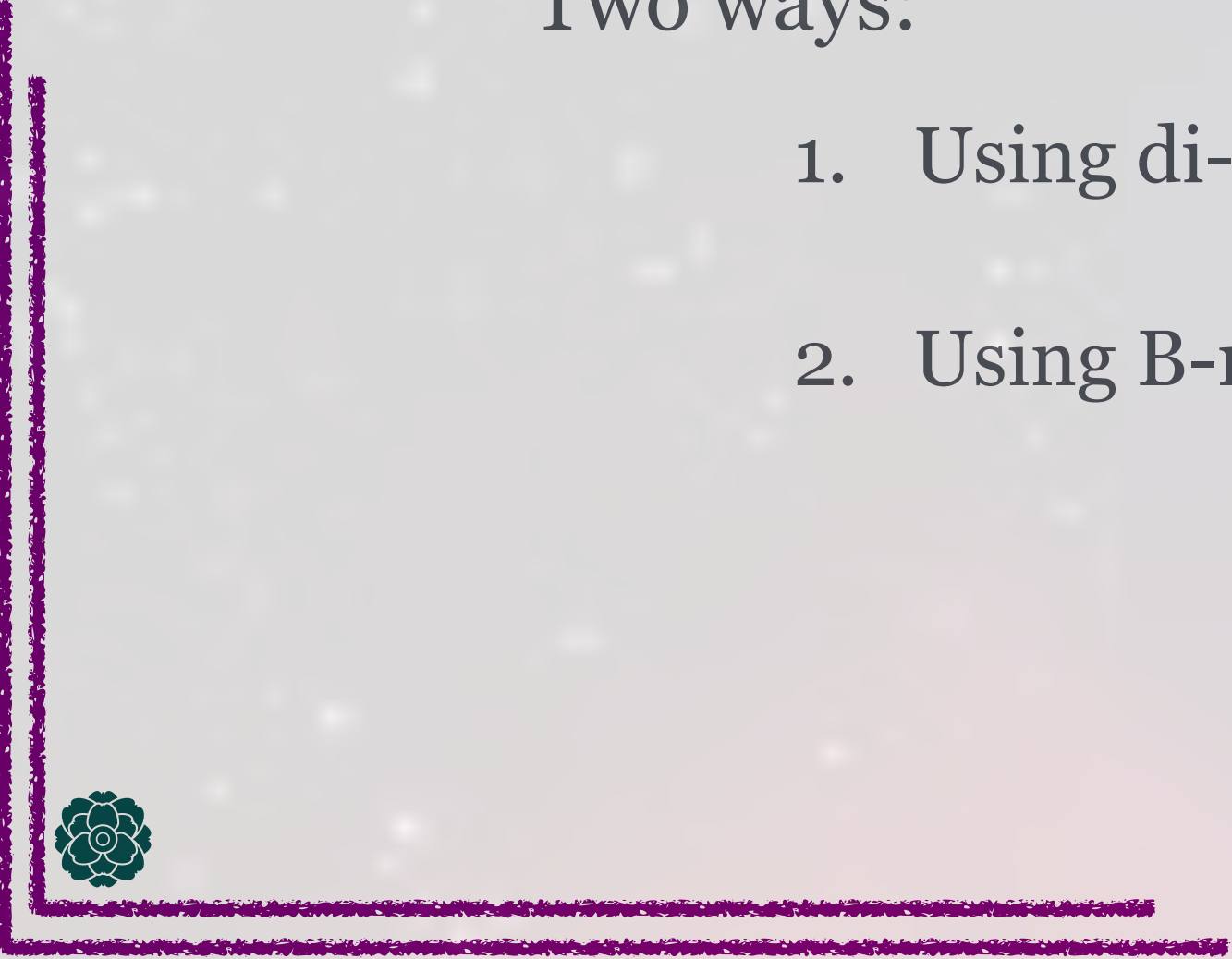
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A. Khodjamirian, B. Melić, Y. M. Wang, [2311.08700]



Going beyond Narrow Width Approximation: LCSR for $B \rightarrow PP$ Form Factors

Two ways:

1. Using di-meson Distribution amplitudes.
 2. Using B-meson Distribution amplitudes.
- 

$\bar{B}^0 \rightarrow \pi^+ \pi^0 \ell \nu$ decays and LCSR

See also talk my Florian Herren

- ρ resonance is the dominant part of dipion $\implies B \rightarrow \rho$ FFs are resonance part of $\bar{B}^0 \rightarrow \pi^+ \pi^0$ FFs (isospin 1).
- Form Factor:

$$\langle \pi^+(k_1) \pi^0(k_2) | \bar{u} \gamma^\mu (1 - \gamma_5) b | \bar{B}^0(p) \rangle = -F_\perp(q^2, k^2, \zeta) \frac{4}{\sqrt{k^2 \lambda_B}} i \epsilon^{\mu\alpha\beta\gamma} q_\alpha k_{1\beta} k_{2\gamma} \oplus \text{axial current form factors}$$

Expansion in partial waves ($\ell = 1, 3, 5, \dots$): Series in Legendre Polynomials

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Expansion in partial waves ($\ell = 1, 3, 5, \dots$): Series in Legendre Polynomials

- For ρ resonance, $\ell = 1$ component, $F_\perp^{(\ell=1)}(q^2, k^2)$ is required:

Slight model dependence is unavoidable

$$\frac{\sqrt{3} F_\perp^{(l=1)}(q^2, k^2)}{\sqrt{k^2} \sqrt{\lambda_B}} = \frac{g_{\rho\pi\pi}}{m_\rho^2 - k^2 - \underbrace{im_\rho \Gamma_\rho(k^2)}_{\text{Width effect}}} \frac{V^{B \rightarrow \rho}(q^2)}{m_B + m_\rho} + \dots$$

Width effect

Captures non-resonant effect

$\bar{B}^0 \rightarrow \pi^+ \pi^0$ form factors: Using dipion DAs

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M. Diehl, T. Gousset, B. Pire, O. Teryaev' 98

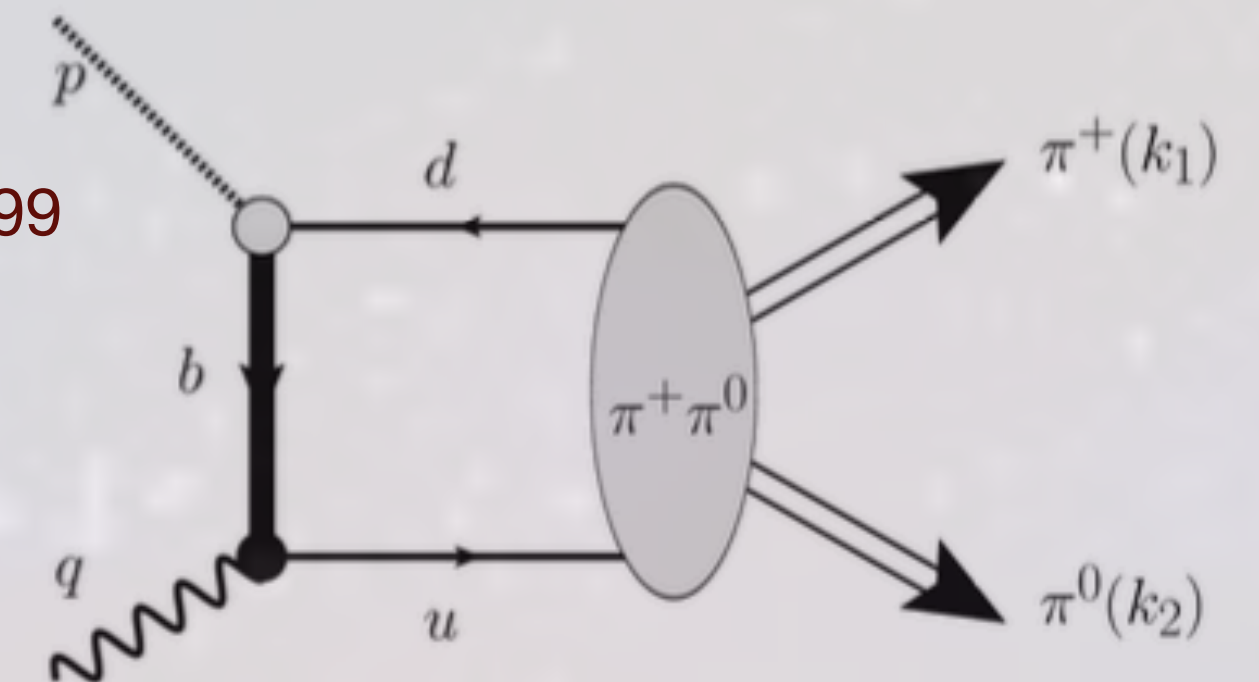
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D. Müller, D. Robaschik, B. Geyer, F.M. Dittles, J. Horejsi' 94, M.V. Polyakov' 99

$$\langle \pi^+(k_1) \pi^0(k_2) | \bar{u}(x) \gamma_\mu [x,0] d(0) | 0 \rangle = -\sqrt{2} k_\mu \int_0^1 du e^{iuk.x} \Phi_{\parallel}^{l=1}(u, \zeta, k^2)$$

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Normalised using pion timelike FFs.

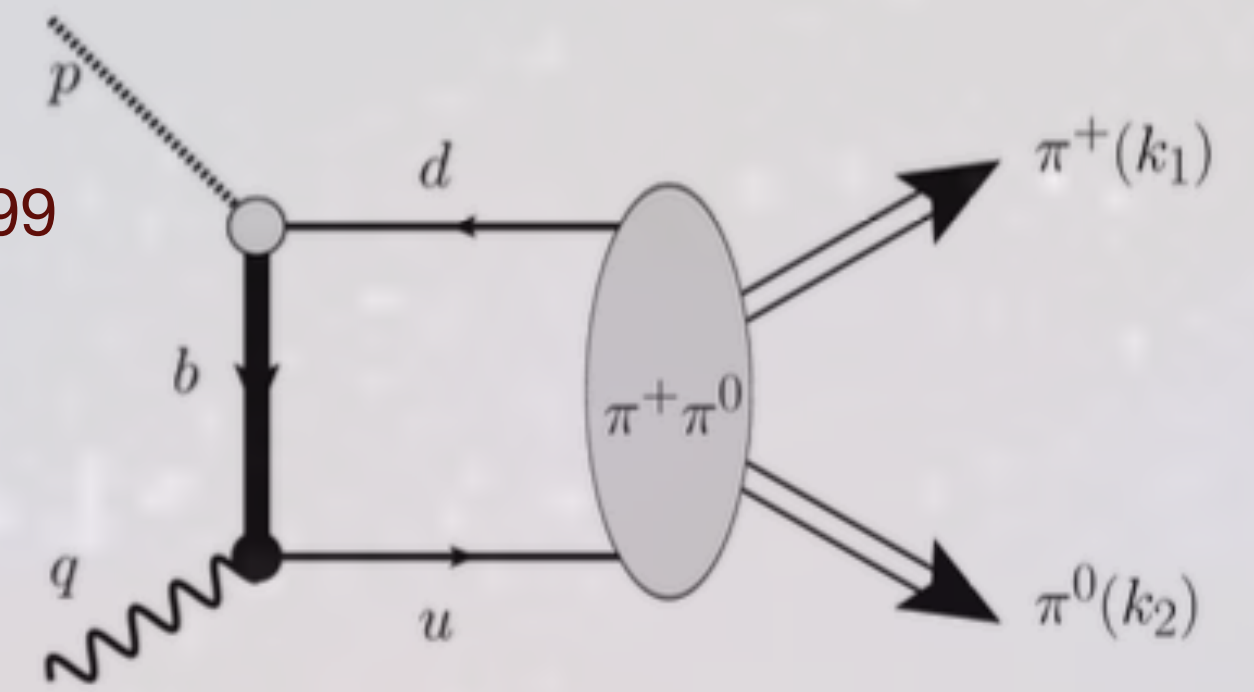


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Known for only small k^2

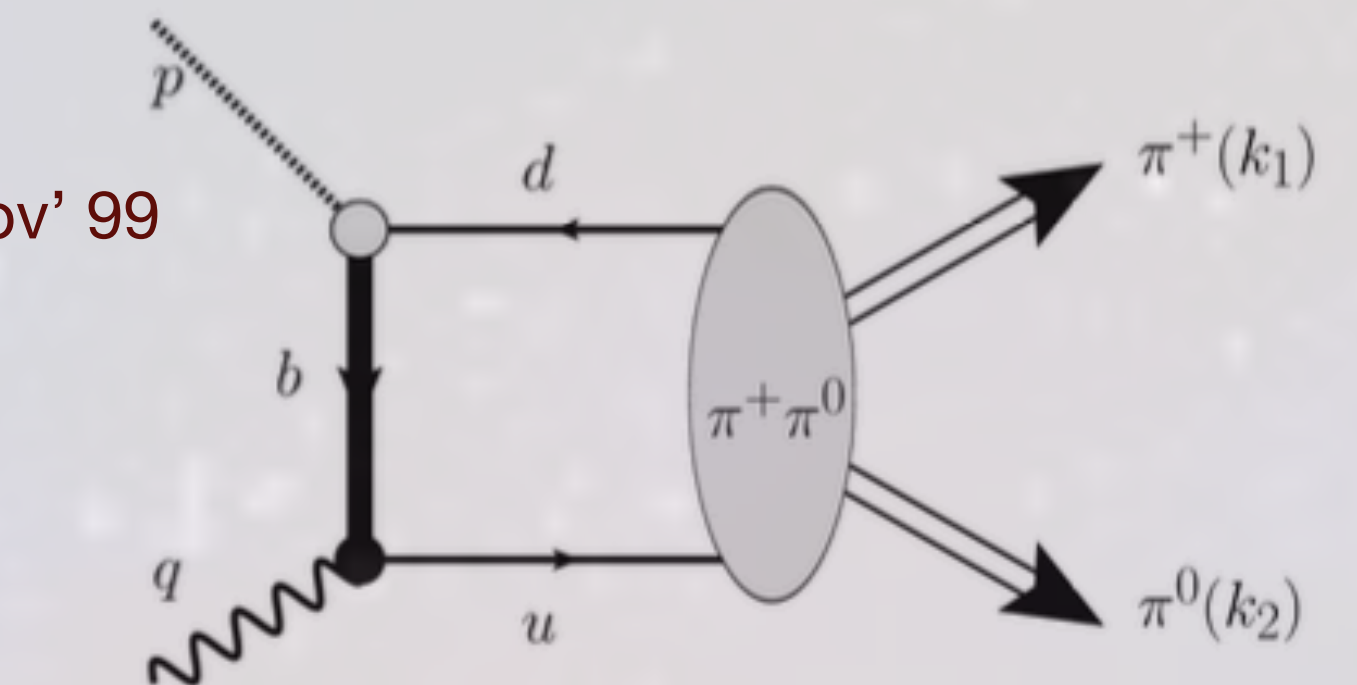
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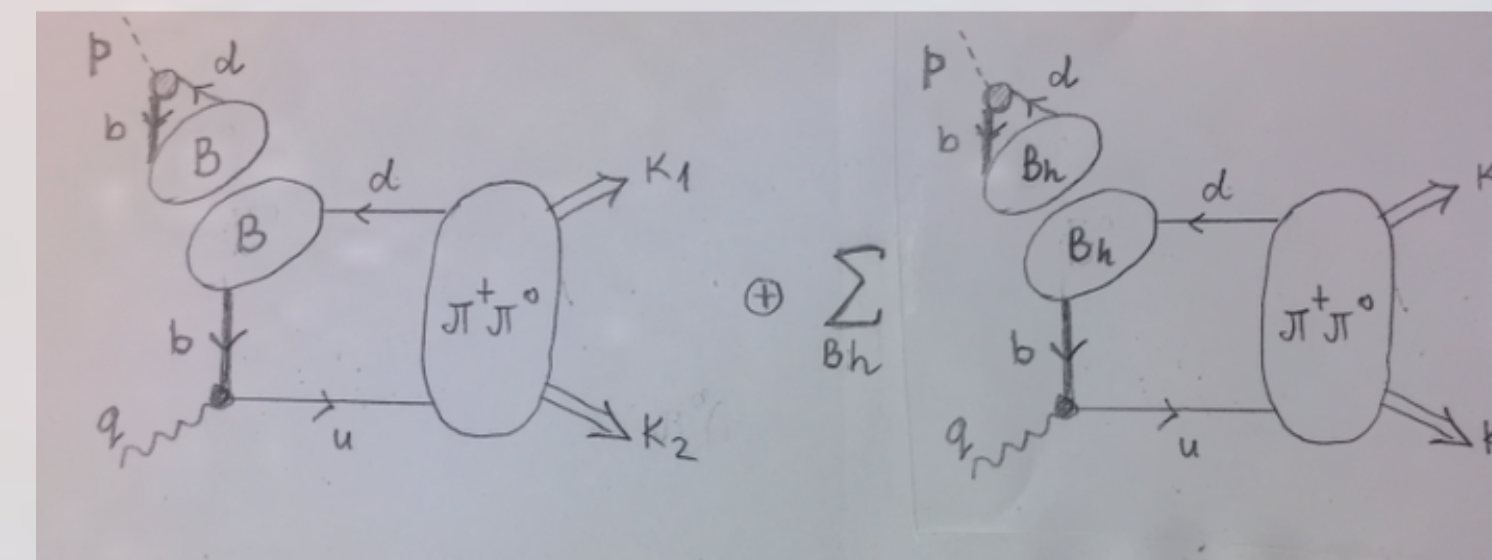
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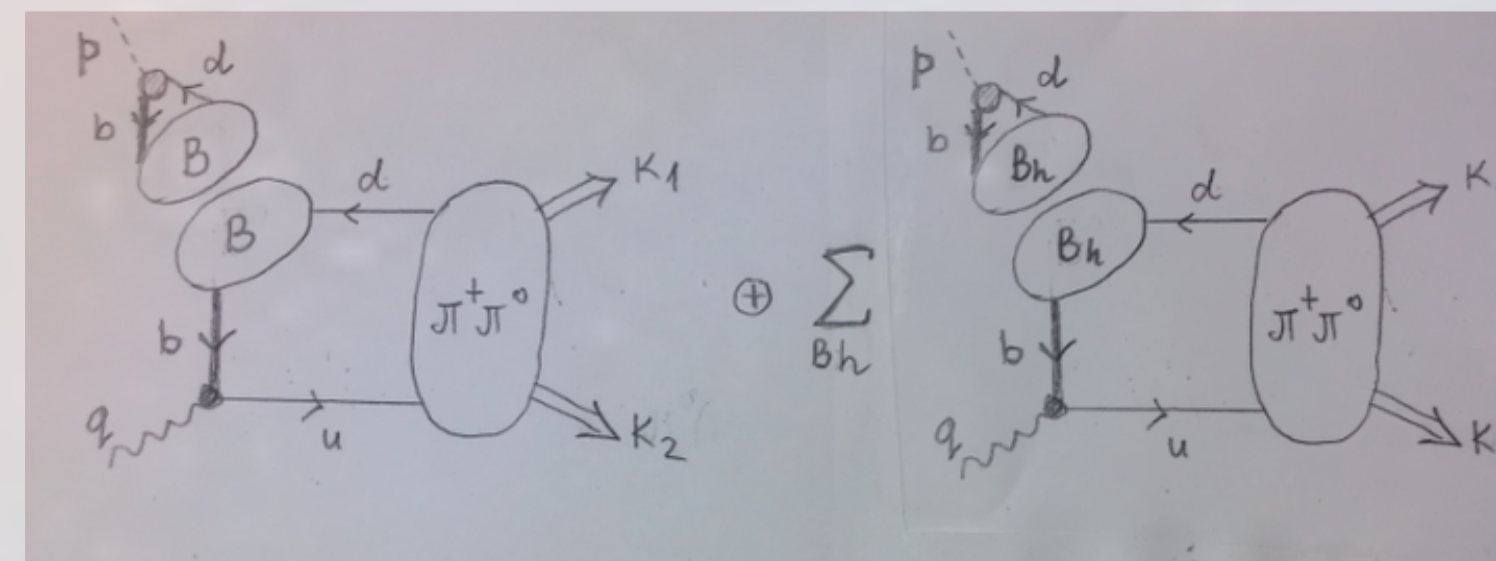
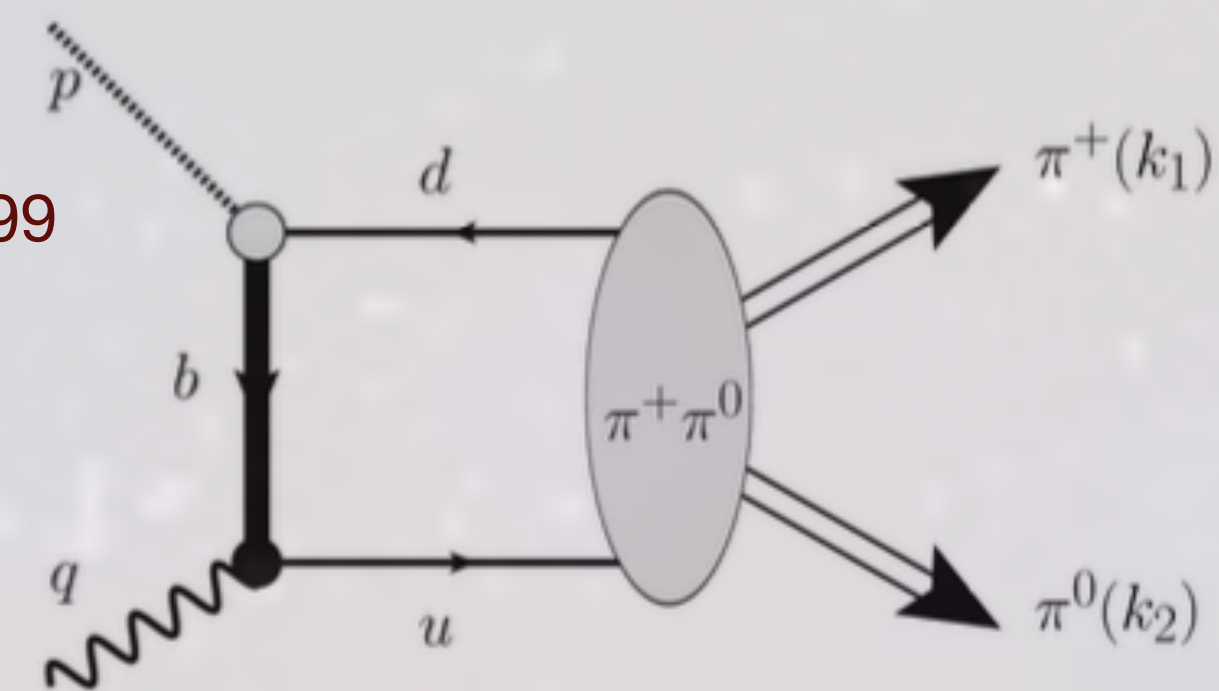
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Quark hadron duality $\Rightarrow$

$$\int_{s_0^B}^\infty ds \frac{\rho_\mu^{OPE,(V)}(s)}{s - p^2}$$

Region of validity:
 $k^2 \ll m_b^2, q^2 \ll m_b^2$



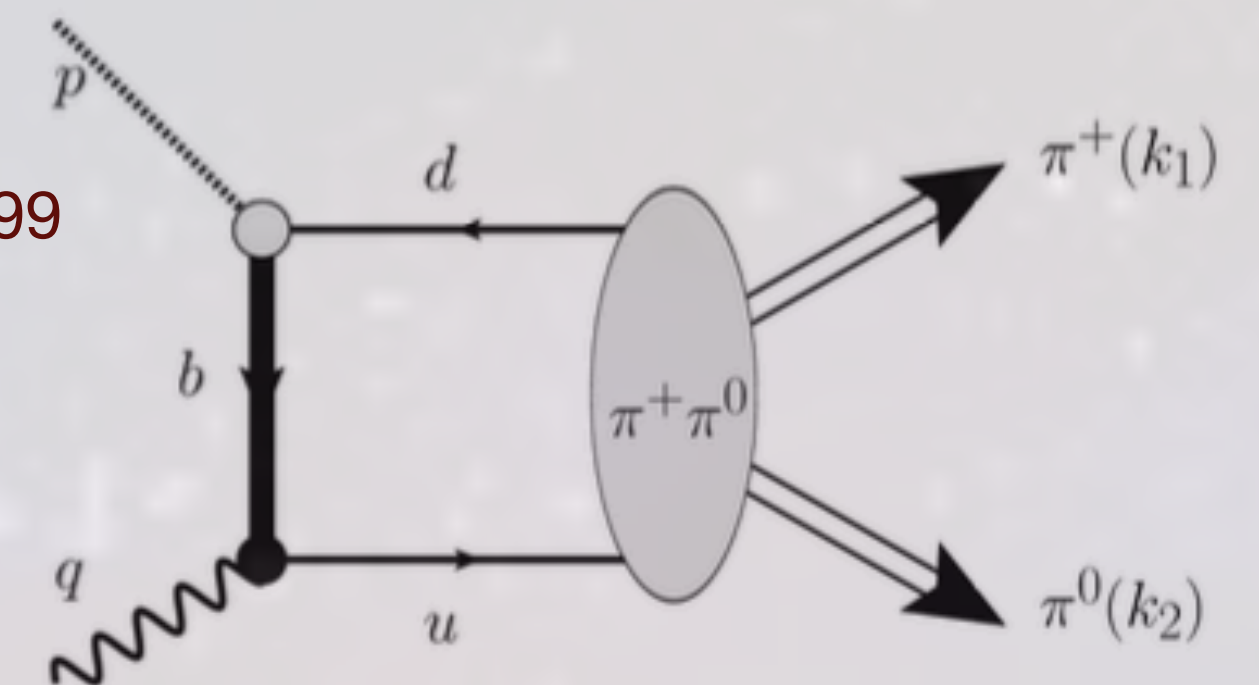
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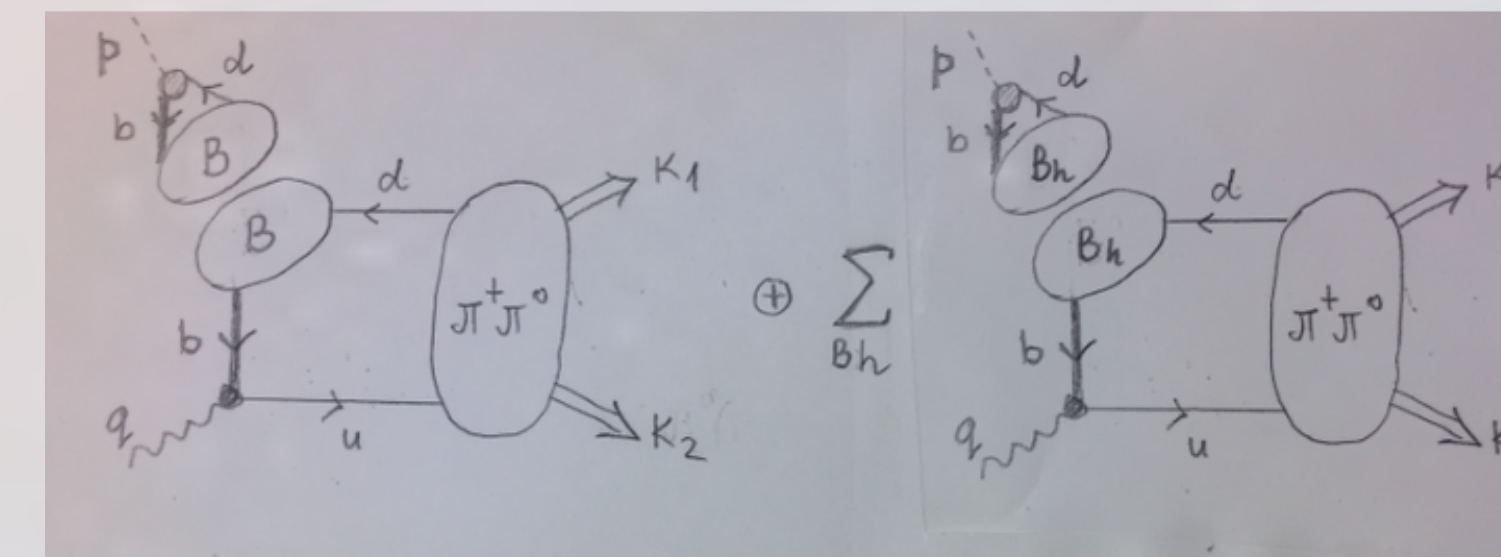
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Credit: A. Khodjamirian

- Limited by limited knowledge of di-meson DAs: More work needed.

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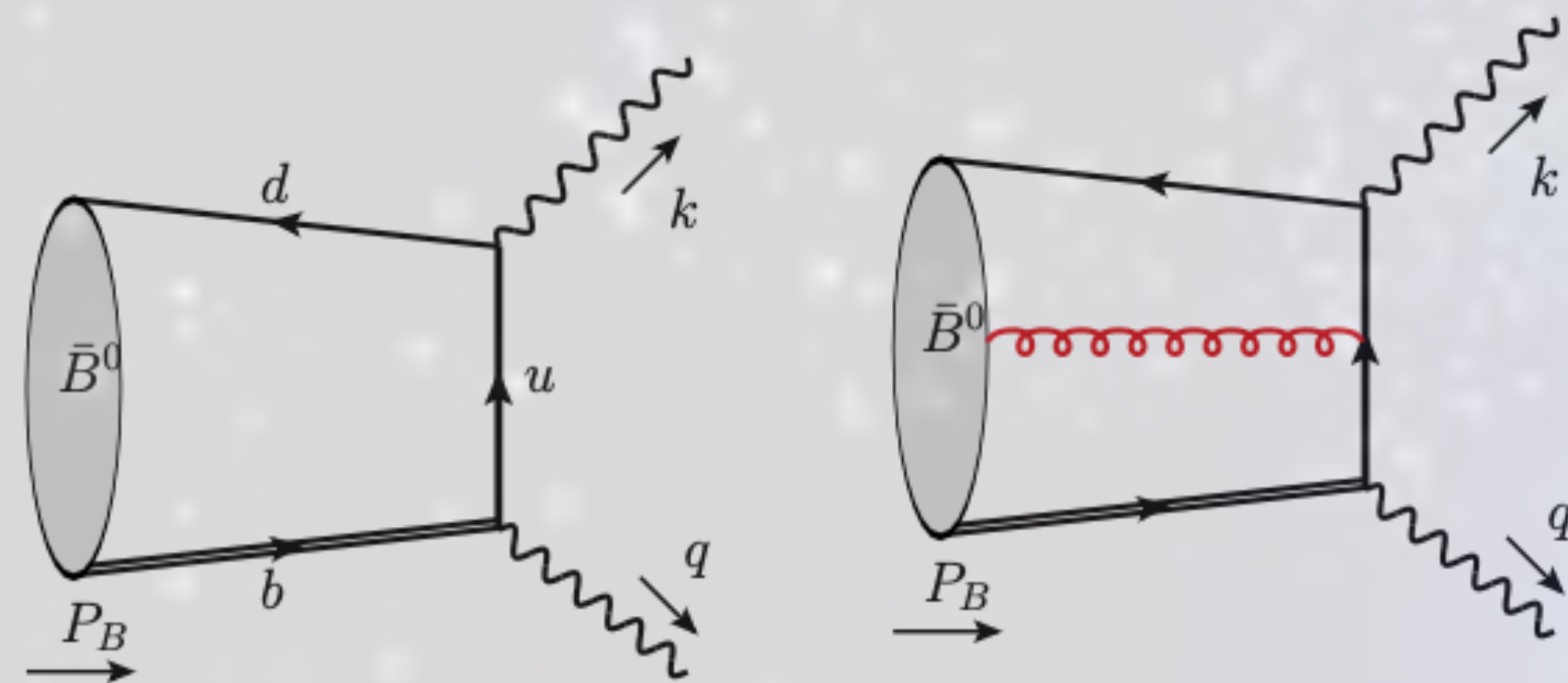
S. Cheng, A. Khodjamirian, J. Virto, [1701.01633]

- LCSR with **B meson DA** and $\bar{u}\gamma_\mu d$ interpolating current.

- Correlation function:

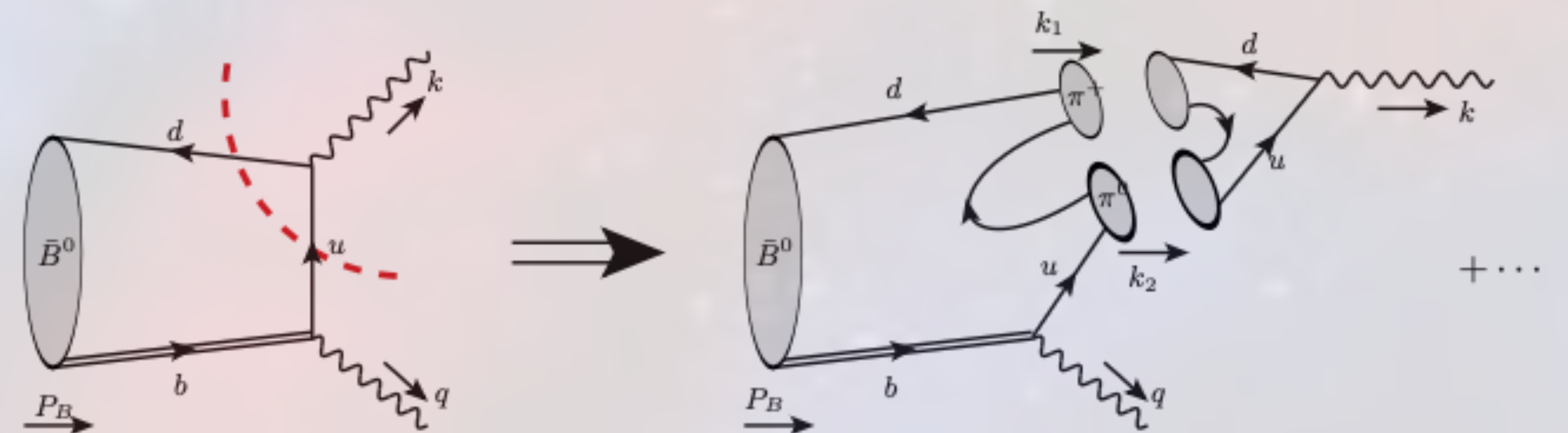
$$F_{\mu\nu}(k, q) = i \int d^4x e^{ik \cdot x} \langle 0 | T \{ \bar{d}(x) \gamma_\mu u(x), \bar{u}(0) \gamma_\nu (1 - \gamma_5) b(0) \} | \bar{B}^0(q + k) \rangle$$

OPE



Dispersion Relation

$$F_{(\epsilon)}(k^2, q^2) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{\text{Im} F_{(\epsilon)}(s, q^2)}{s - k^2}$$



$$2\text{Im} F_{\mu\nu}(k, q) = \int d\tau_{2\pi} \underbrace{\langle 0 | \bar{d} \gamma_\mu u | \pi^+ \pi^0 \rangle}_{F_\pi(s)} \underbrace{\langle \pi^+ \pi^0 | \bar{u} \gamma_\nu (1 - \gamma_5) b | \bar{B}^0(q + k) \rangle}_{B \rightarrow 2\pi (\ell = 1) \text{ form factors}} + \dots$$

$\bar{B}^0 \rightarrow \pi^+ \pi^0$ form factors: Using B-meson DAs

- Resulting sum rule: (eg. for $F_{\perp}^{(\ell=1)}$ of vector current)

$$\int_{4m_{\pi}^2}^{s_0^{2\pi}} ds e^{-s/M^2} \frac{\sqrt{s} [\beta_{\pi}(s)]^3}{4\sqrt{6}\pi^2\sqrt{\lambda}} F_{\pi}^*(s) F_{\perp}^{(\ell=1)}(s, q^2) = f_B m_B \left[\int_0^{\sigma_0^{2\pi}} d\sigma e^{-s(\sigma, q^2)/M^2} \frac{\phi_+^B(\sigma m_B)}{\bar{\sigma}} + 3\text{-particle DA contri.} \right]$$

- Similar sum rules for other form factors.
- The complex phases of $B \rightarrow 2\pi$ FFs and $F_{\pi}(s)$ equal at low s : a usual Watson theorem.
- Not a direct calculation, given the shape of the $B \rightarrow 2\pi$ FFs, these sum rules can provide normalisation.

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- Necessary inputs in these sum rules:
 1. B-meson DAs.
 2. Timelike vector form factors of pion : experimentally well known

Probing ρ –resonance model

S. Cheng, A. Khodjamirian, J. Virto, [1701.01633]

◦ Ansatz for $B \rightarrow \pi\pi$ FF : inspired by experimental fit of $F_\pi(s)$

$$\frac{\sqrt{3}F_{\perp}^{(\ell=1)}(q^2,k^2)}{\sqrt{k^2}\sqrt{\lambda_B}} = \sum_{R \in \rho, \rho', \rho''} \frac{g_{R\pi\pi}}{m_R^2 - k^2 - im_R\Gamma_R(k^2)} \frac{V^{B \rightarrow R}(q^2)}{m_B + m_R}$$

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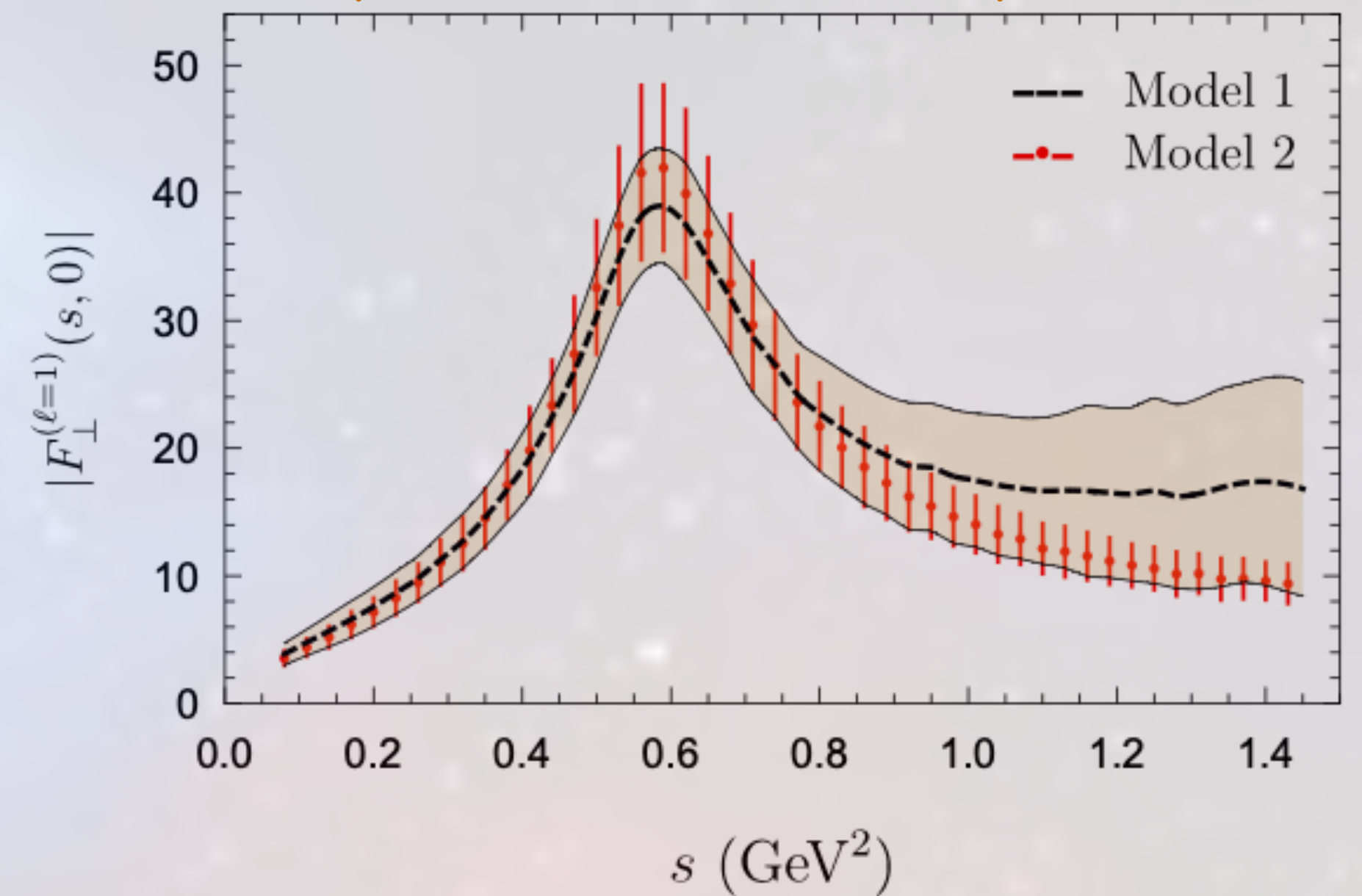
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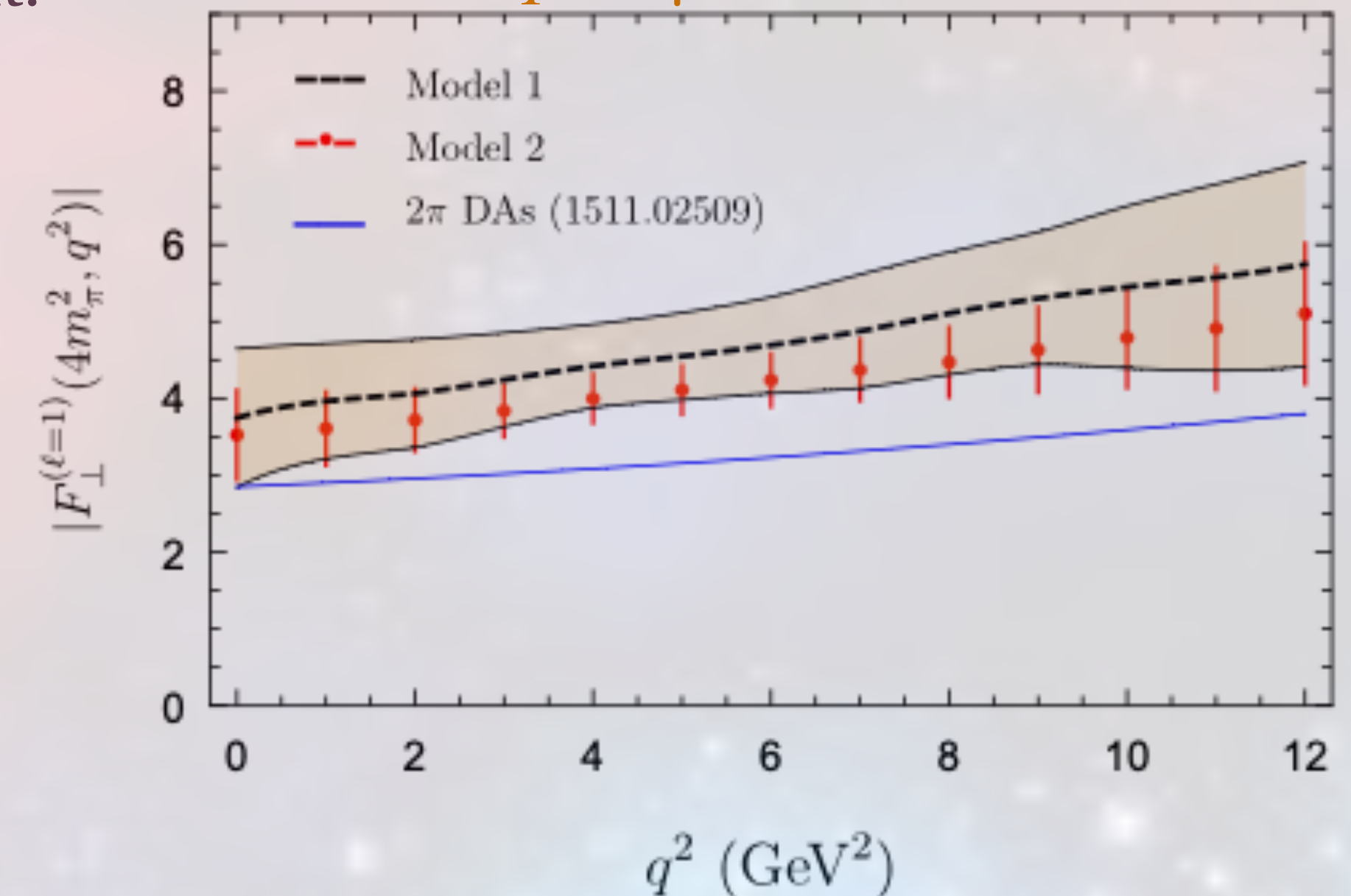
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- Model-2:**
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Dipion invariant mass dependence



q^2 dependence



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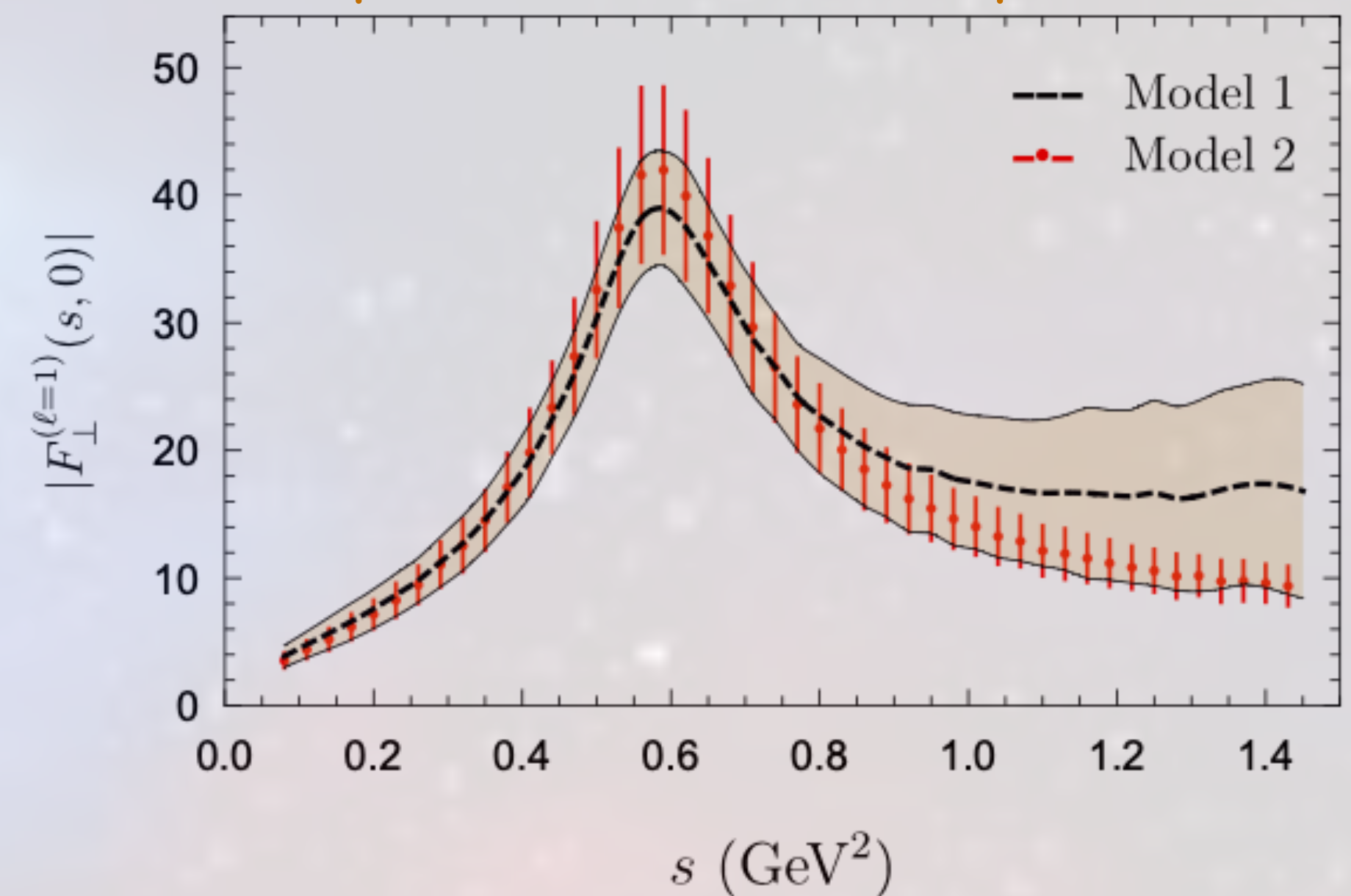
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- Applications to $B \rightarrow K\pi\ell^+\ell^-$ form factors with $K\pi$ in P and S waves.

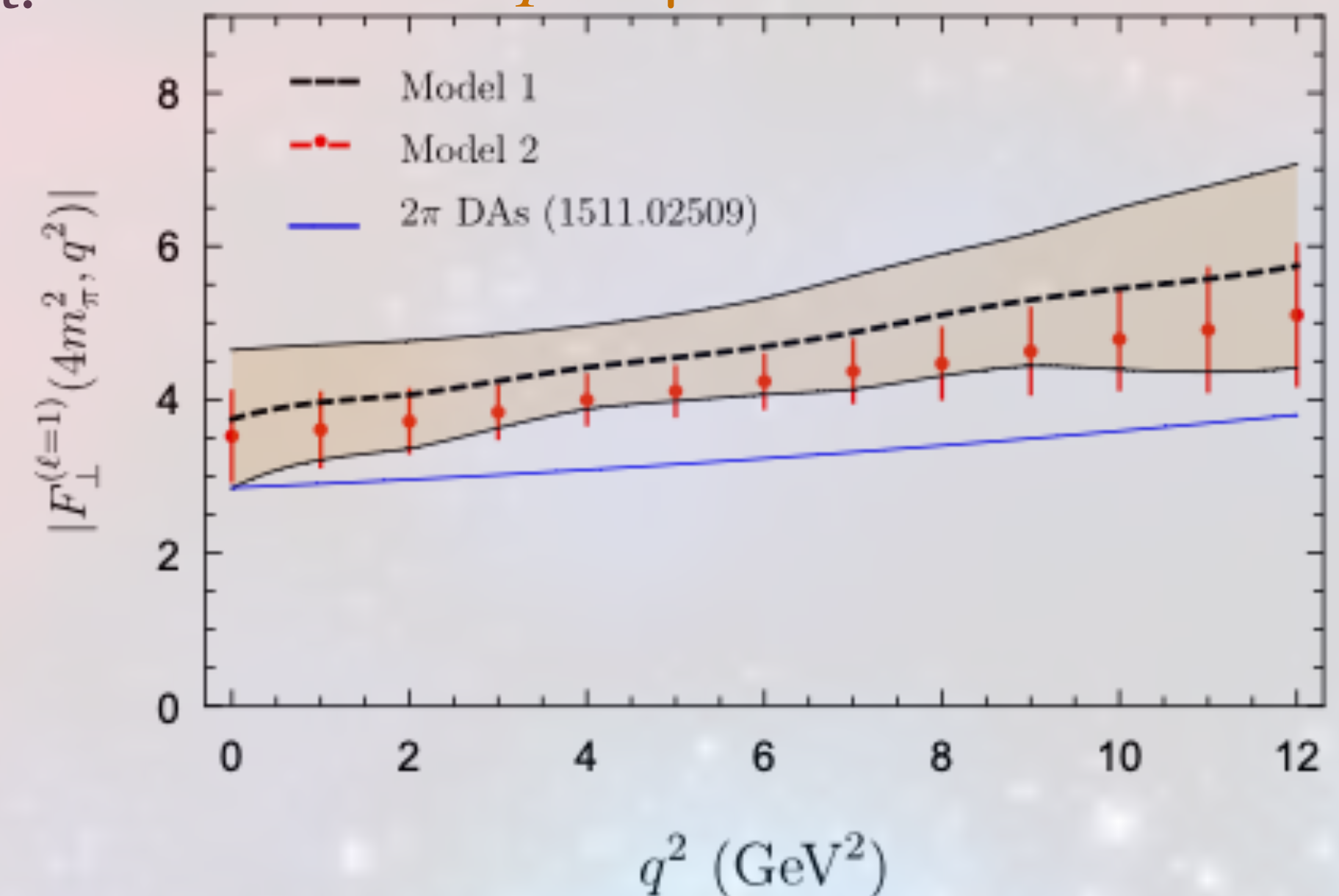
S. Descotes-Genon, A. Khodjamirian, J. Virto, [1908.02267]

S. Descotes-Genon, A. Khodjamirian, J. Virto, K. K. Vos, [2304.02973]

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Non-local FFs in rare FCNCs



Non-Local Form Factors in $b \rightarrow s \ell^+ \ell^-$ decays

$$\mathcal{A}_\lambda^{L,R}(B \rightarrow M_\lambda \ell \ell) = \mathcal{N}_\lambda \left\{ (C_9 \mp C_{10}) \mathcal{F}_\lambda(q^2) + \frac{2m_b M_B}{q^2} \left[C_7 \mathcal{F}_\lambda^T(q^2) - 16\pi^2 \frac{M_B}{m_b} \mathcal{H}_\lambda(q^2) \right] \right\}$$

See also talks by Simone Mutke
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- Leading contribution: Local form factors

$f_{BK}^+(q^2 = 0)$	method	Ref.
0.395 ± 0.033	K DAs	AK, A.Rusov , 1703.04765
0.27 ± 0.08	B DAs	N. Gubernari, A,Kokulu, D.van Dyk, 1811.00983
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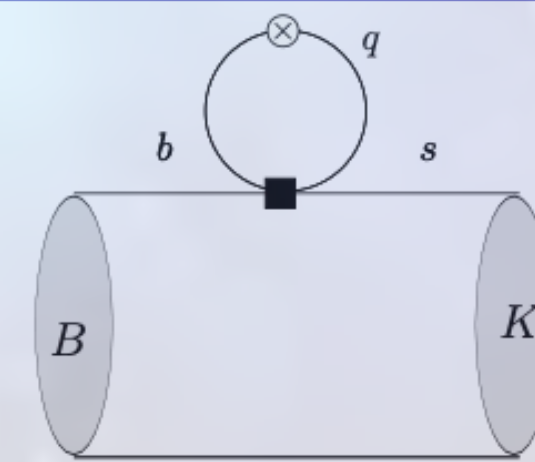
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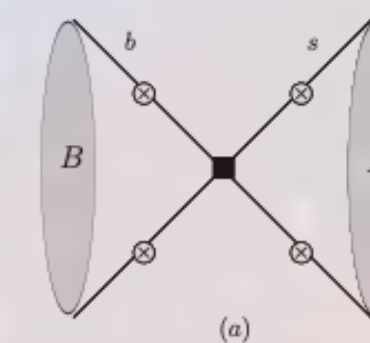
Main contributions: the ‘charm-loops’

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Anatomy of nonlocal matrix elements in $B \rightarrow K \ell \ell$

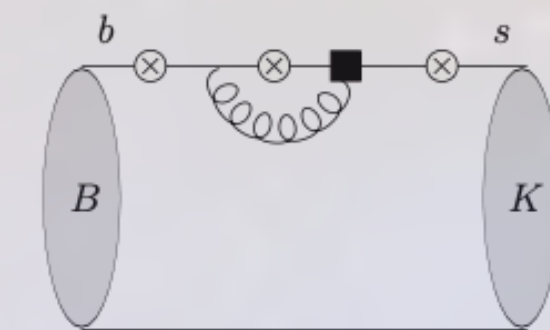
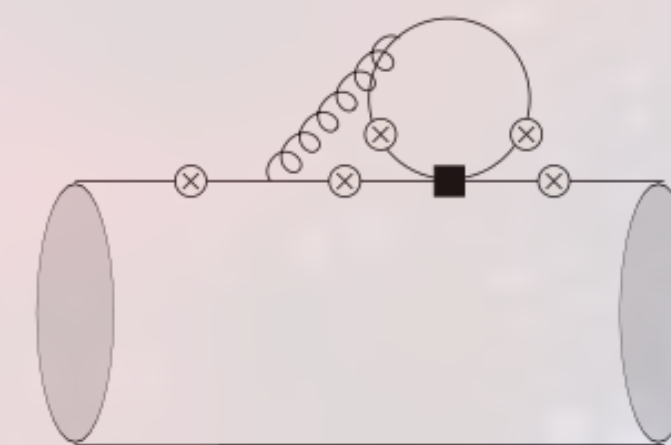
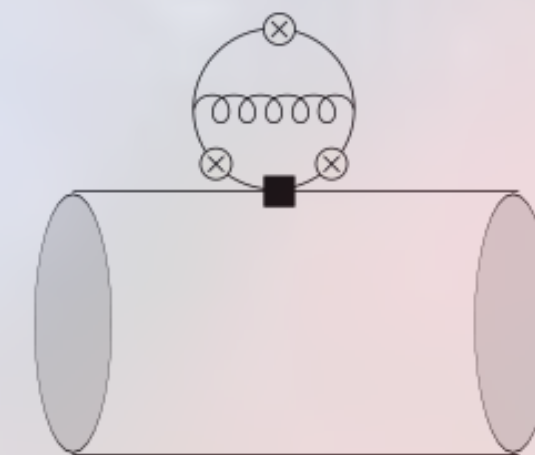


LO (factor.)

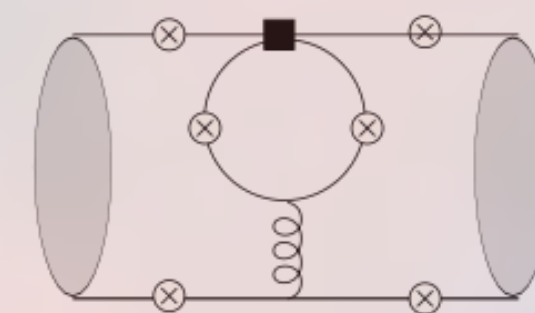


weak annihilation

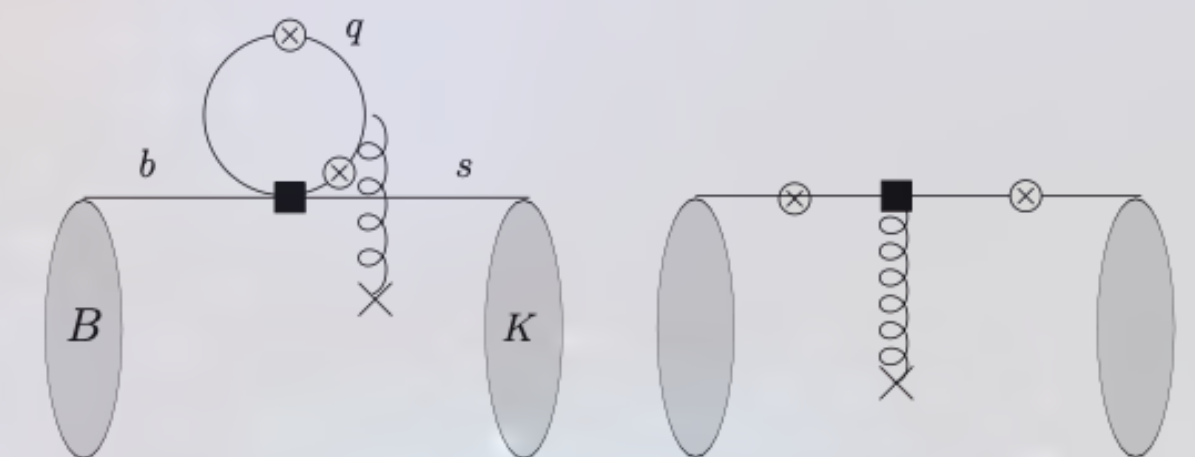
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NLO (factor.) H. Asatryan, C. Greub , J.Virto. [hep-ph/0109140]



spectator (nonfactor.)



soft (low virtuality) gluons

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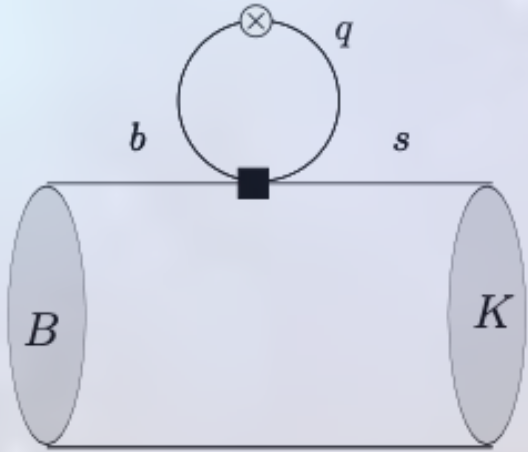
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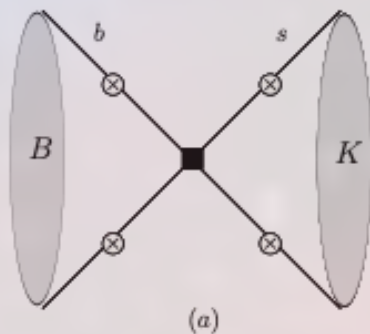
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- Small but important as it mimic new physics by shifting C_9
- No predictions from LQCD so far.

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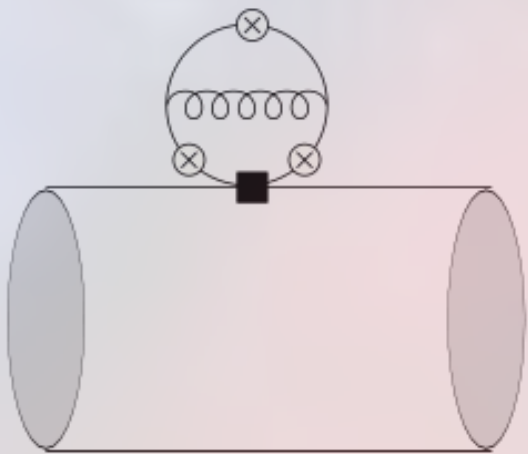


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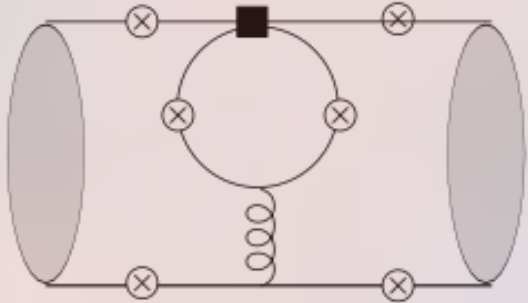
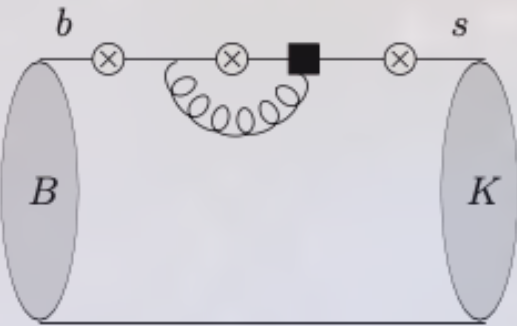


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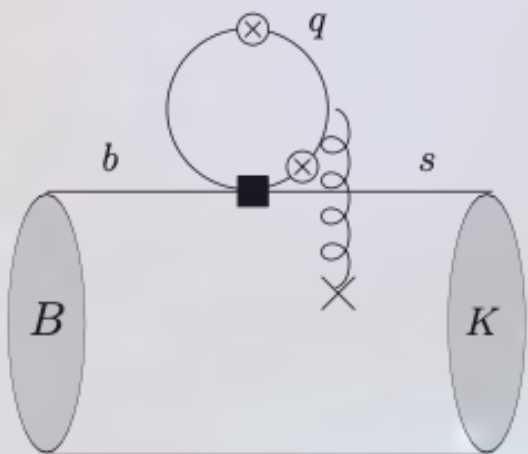
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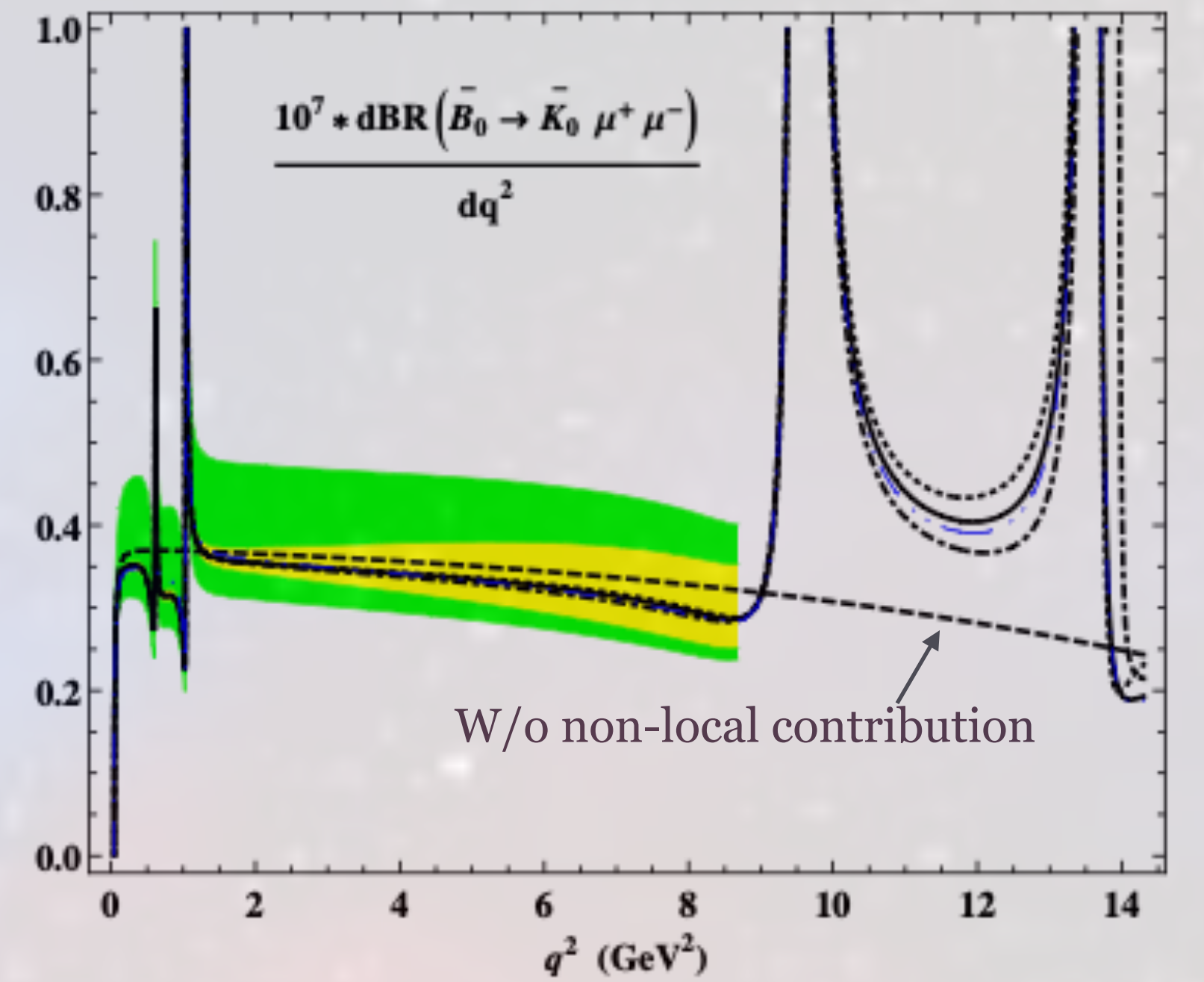
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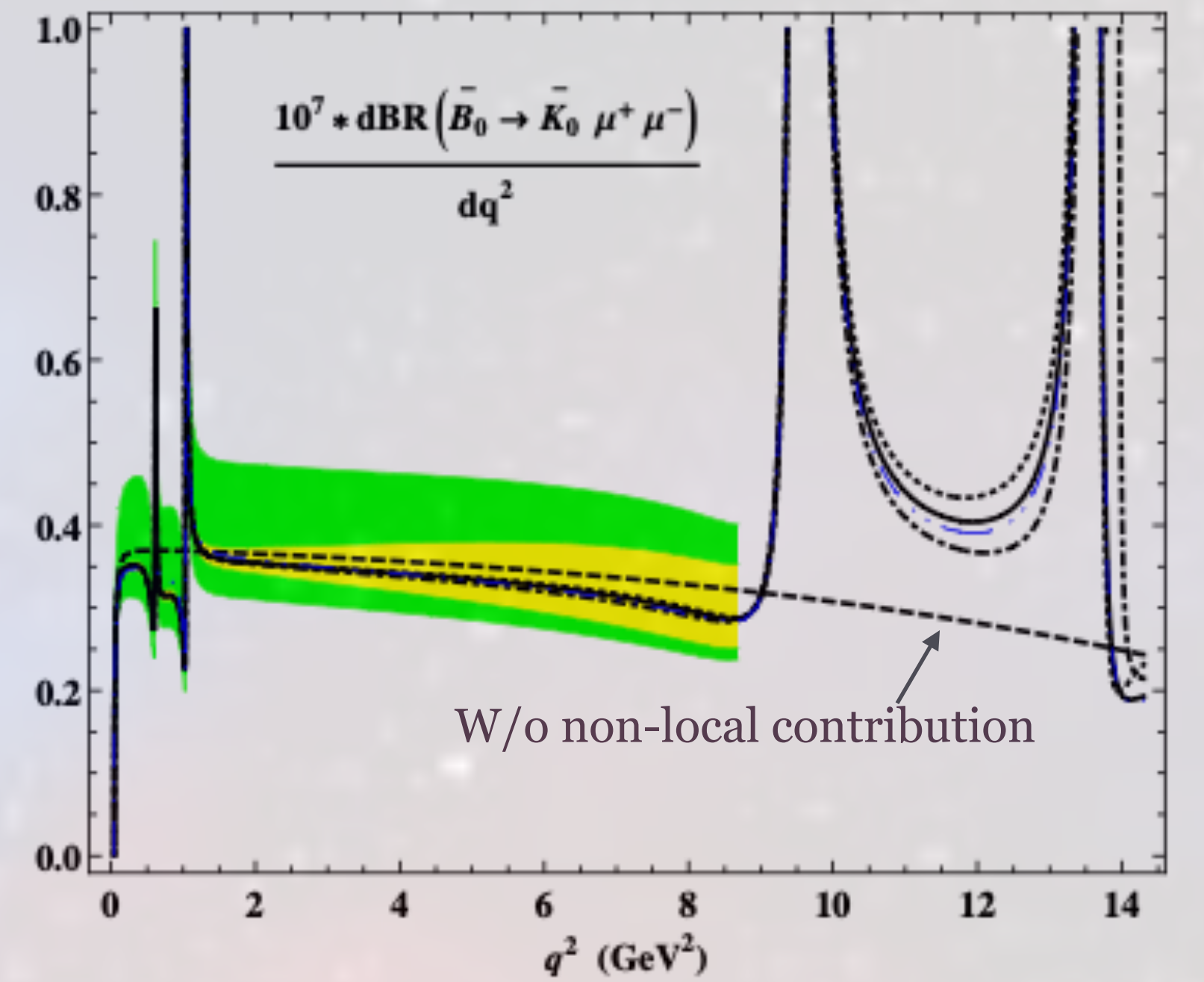
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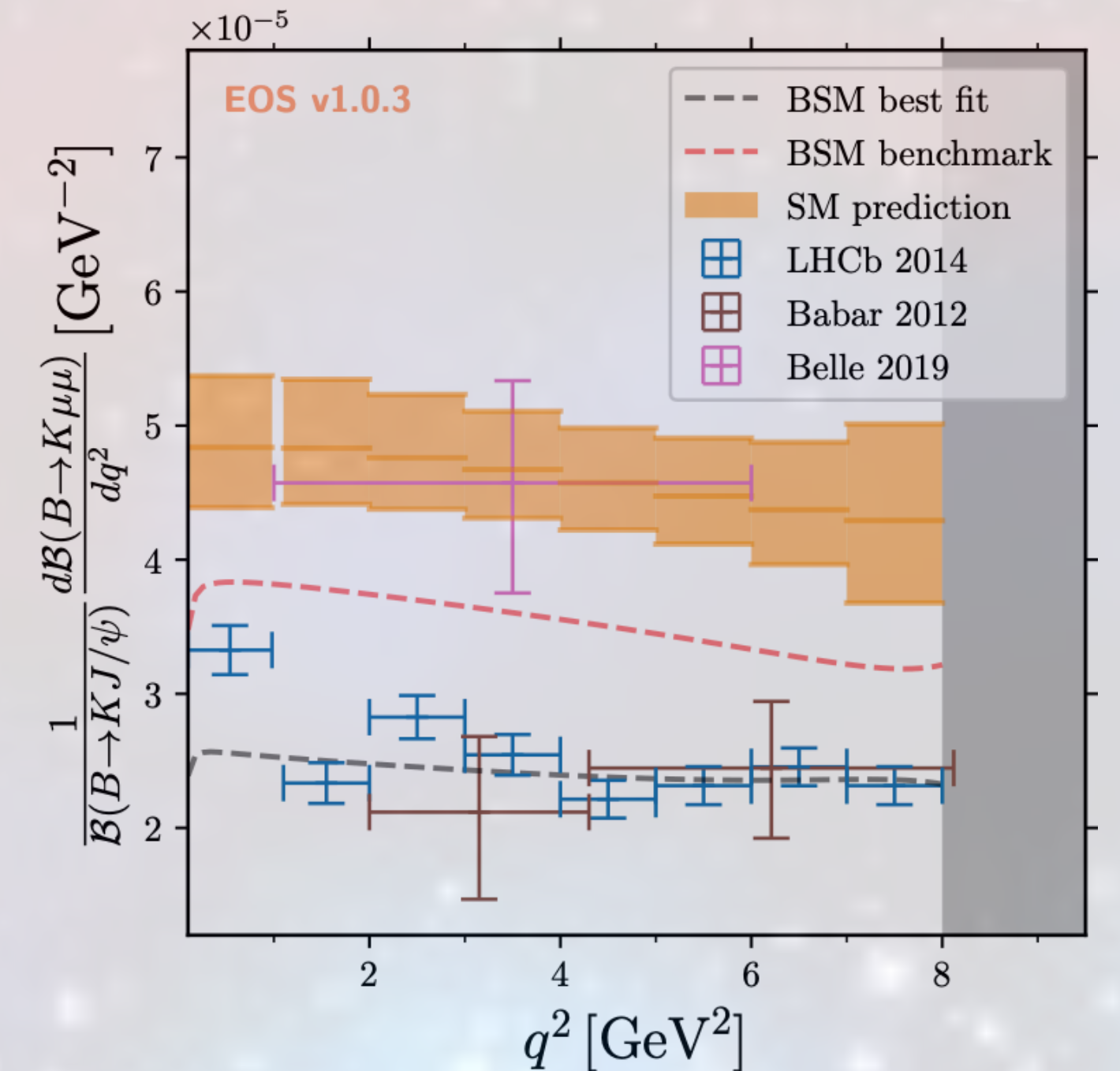
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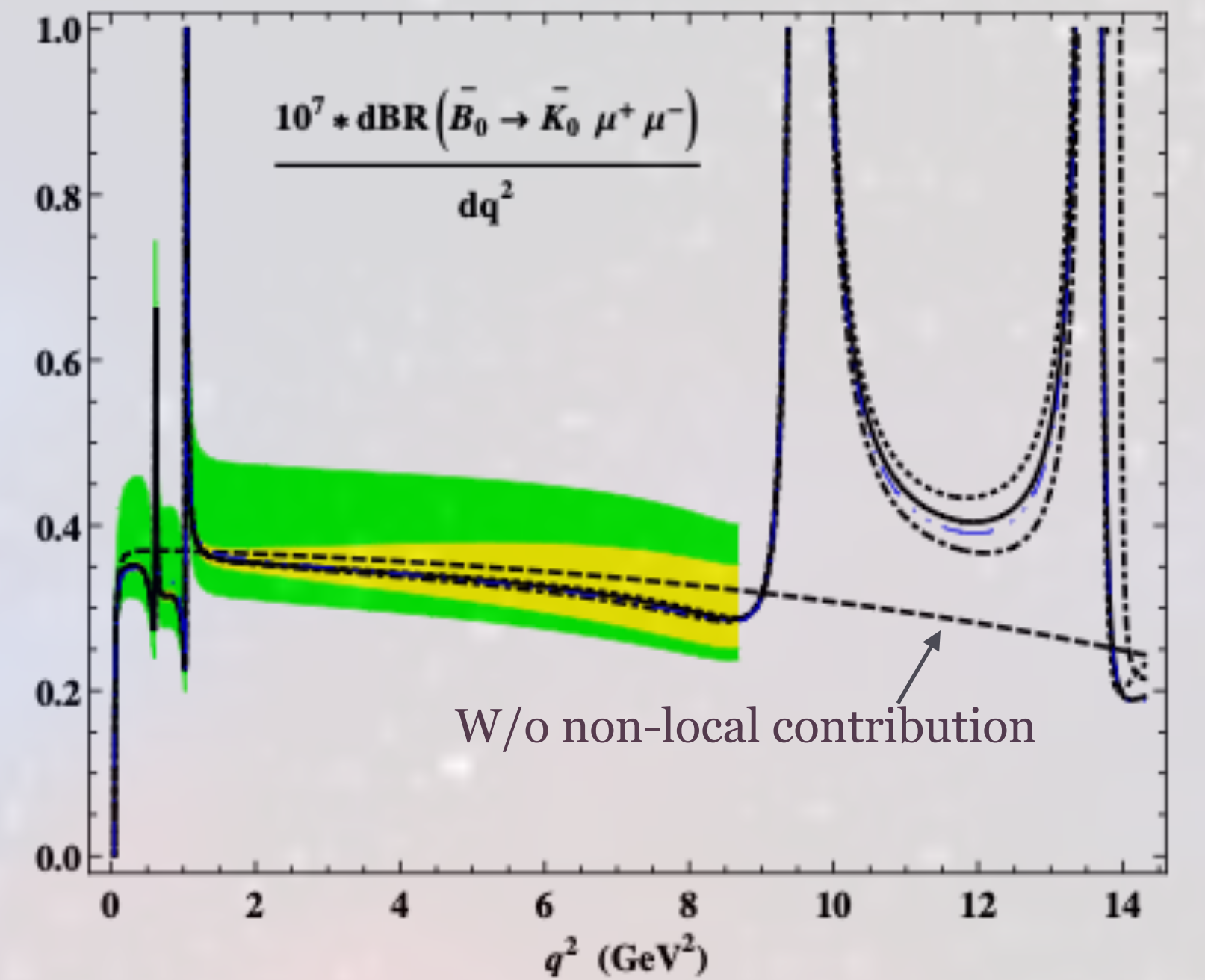
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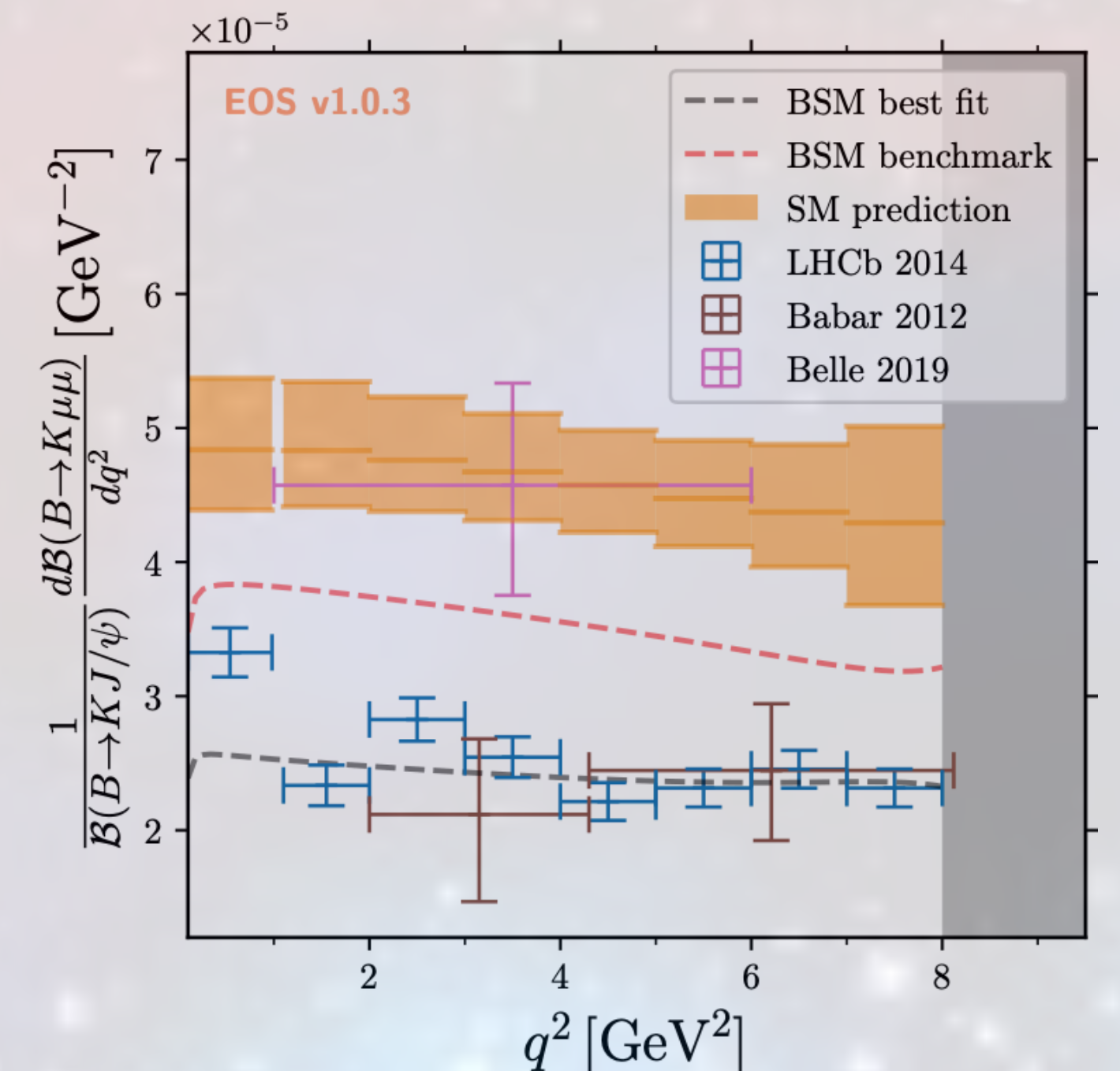
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- Charm loop effects still under investigation.

D. Mishra, N. Mahajan, [2409.00181]



A. Khodjamirian, T. Mannel, Y.M. Wang [1211.0234]



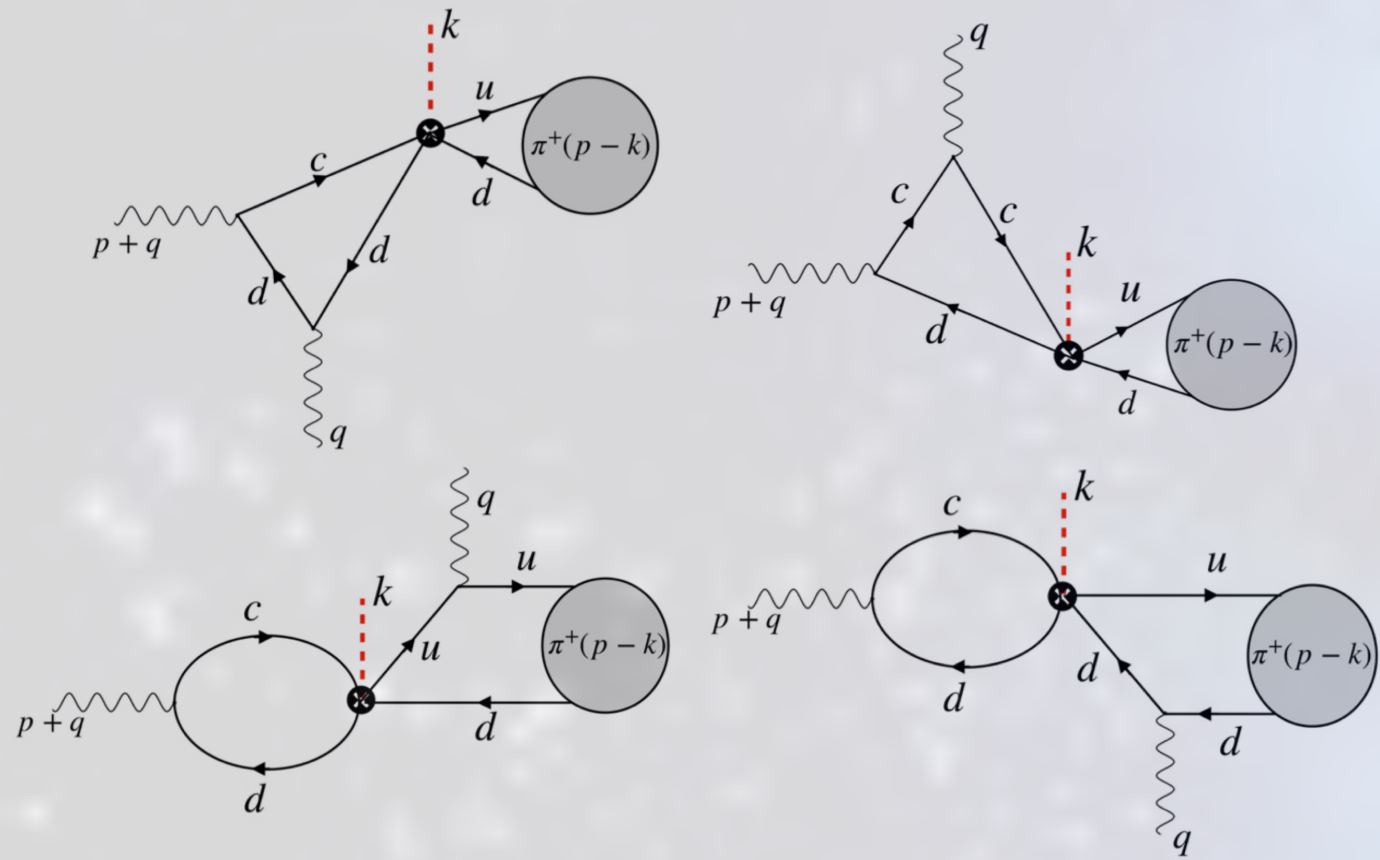
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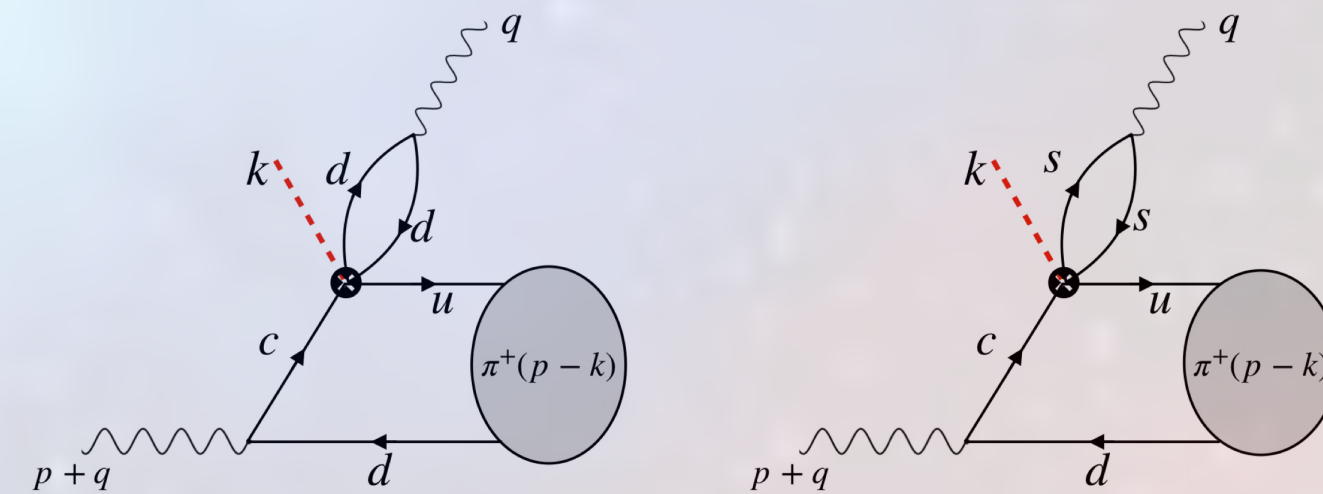
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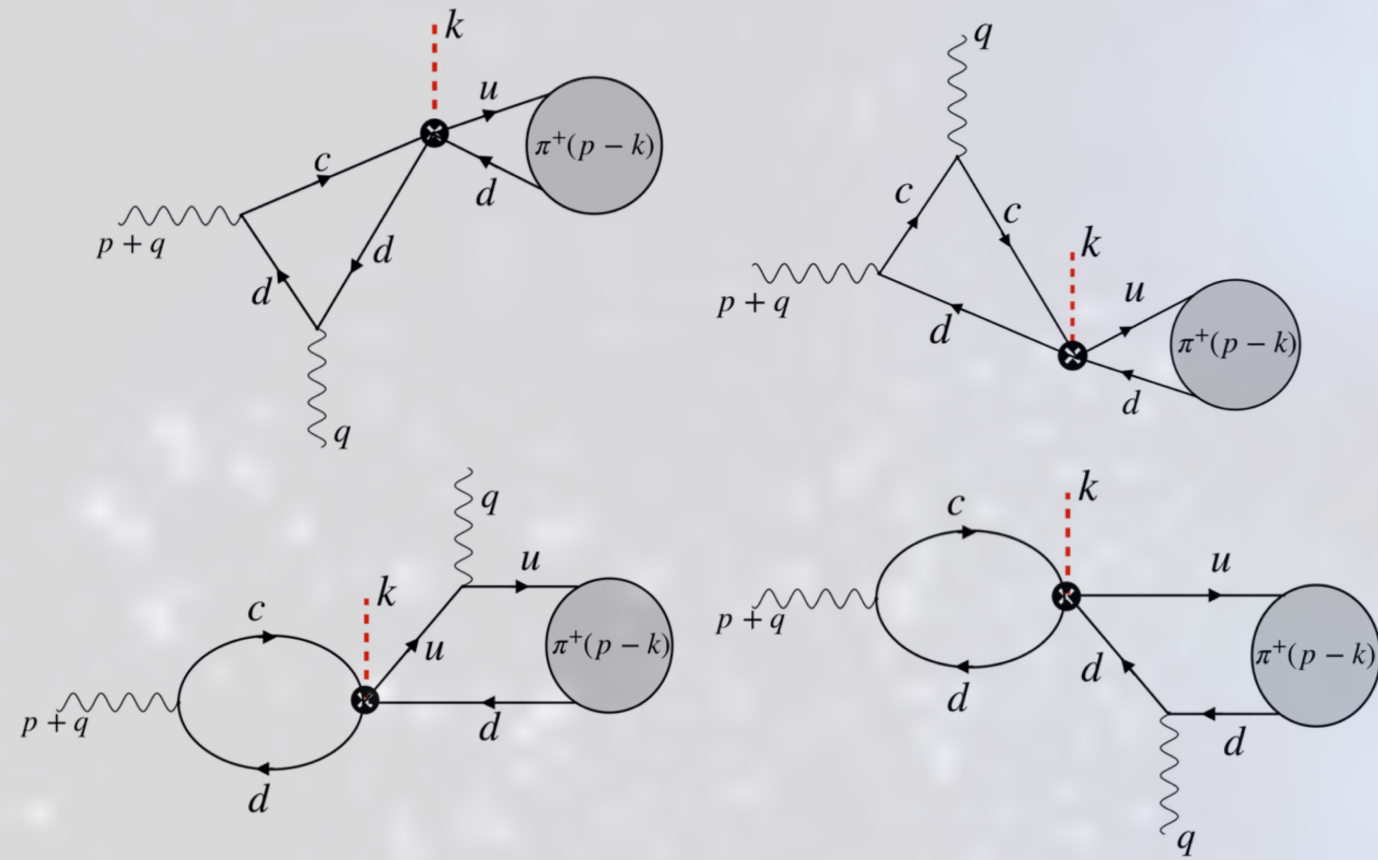
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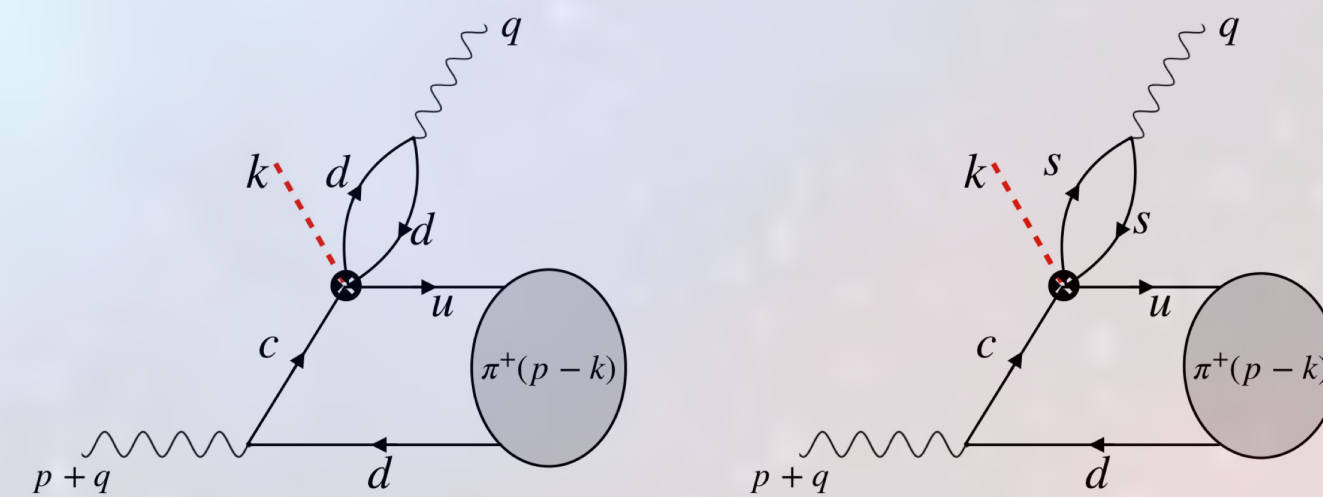
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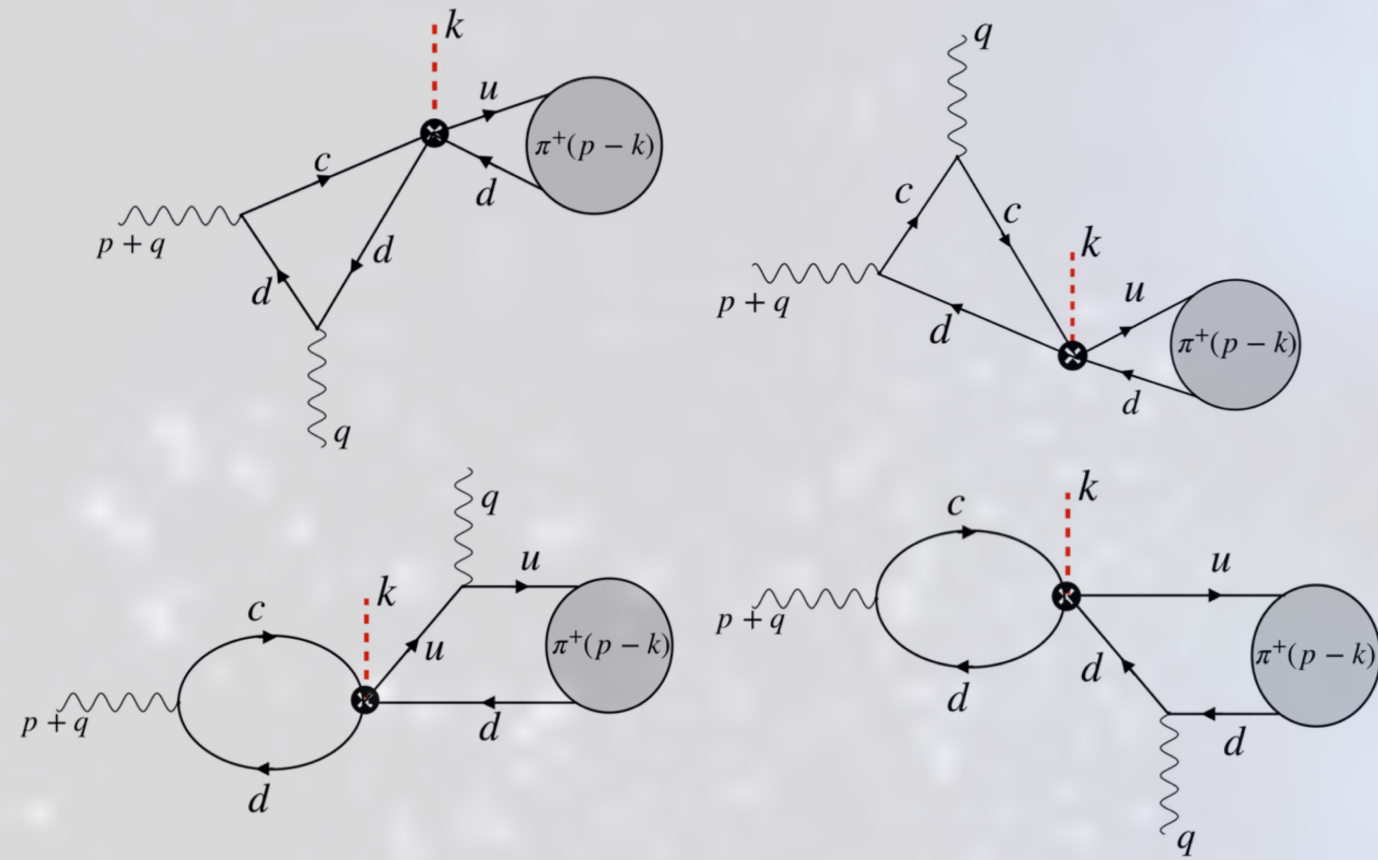
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A. Khodjamirian et. al., hep-ph/0012271, hep-ph/0304179, hep-ph/0509049, 1706.07780

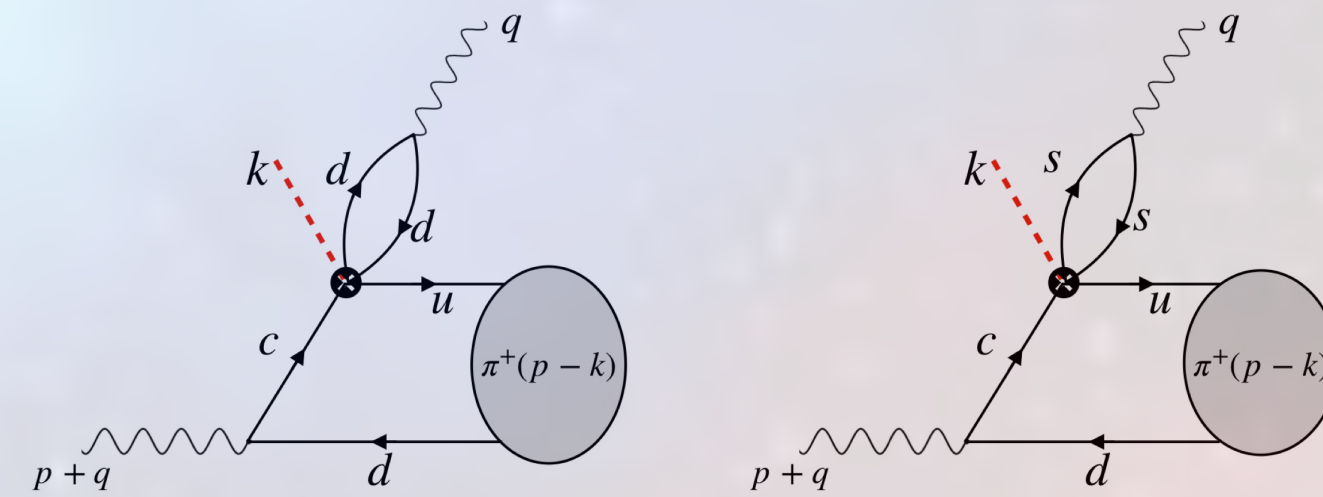
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Light Cone OPE (LO twist-2)

Dispersion relation in $(p + q)^2$

$$\frac{1}{\pi} \int_{m_c^2}^{\infty} ds \frac{\text{Im} F^{(OPE)}(s, q^2, P^2 = m_D^2)}{s - (p + q)^2} = \underbrace{\frac{m_D^2 f_D A^{(D^+ \rightarrow \pi^+ \gamma^*)}(q^2)}{m_D^2 - (p + q)^2}}_{\text{contribution from D-meson state}} + \underbrace{\int_{s_{hd}}^{\infty} ds \frac{\rho_{hd}(s, q^2, P^2 = m_D^2)}{s - (p + q)^2}}_{\text{Quark Hadron duality}}$$

$$\approx \frac{1}{\pi} \int_{s_0^D}^{\infty} ds \frac{\text{Im} F^{(OPE)}(s, q^2, P^2 = m_D^2)}{s - (p + q)^2}$$

Results for $D^+ \rightarrow \pi^+ \ell^+ \ell^-$ using LCSR assisted dispersion relation

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AB, A. Khodjamirian, T. Mannel, [2505.21369]

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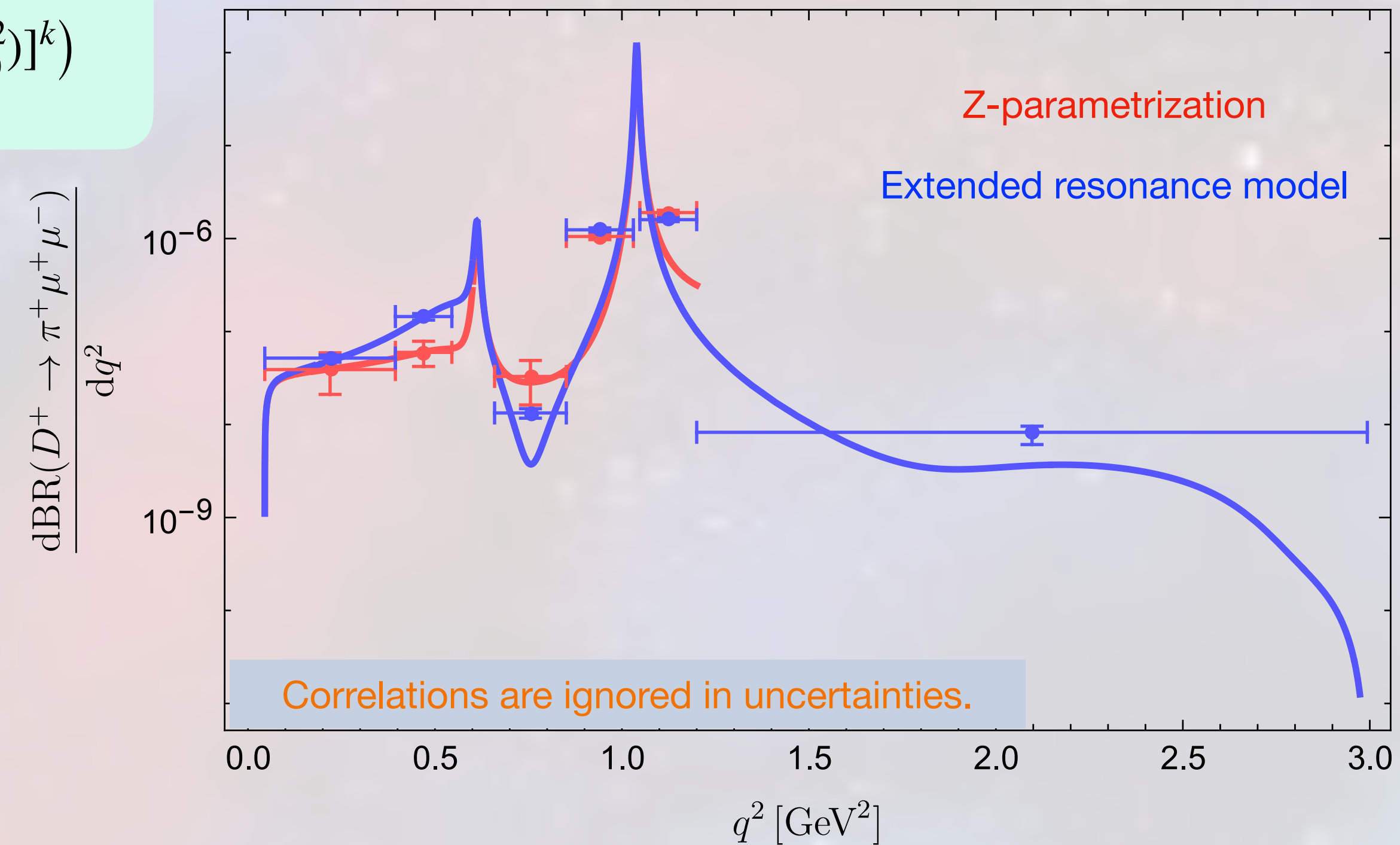
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$$\int_{s_0^h}^{\infty} ds \frac{\rho_h(s)}{(s - q^2 - i\epsilon)} = \sum_{V'=\rho', \phi'} \frac{r_{V'} e^{i\varphi_{V'}}}{(m_{V'}^2 - q^2 - i\sqrt{q^2} \Gamma_{V'}(q^2))} + \frac{r_{eff} e^{i\varphi_{eff}}}{m_{eff}^2 - q^2 - i\sqrt{q^2} \Gamma_{eff}(q^2)}$$



Results for $D^+ \rightarrow \pi^+ \ell^+ \ell^-$ using LCSR assisted dispersion relation

Models for Hadronic dispersion relation:

AB, A. Khodjamirian, T. Mannel, [2505.21369]

Z-parametrization model

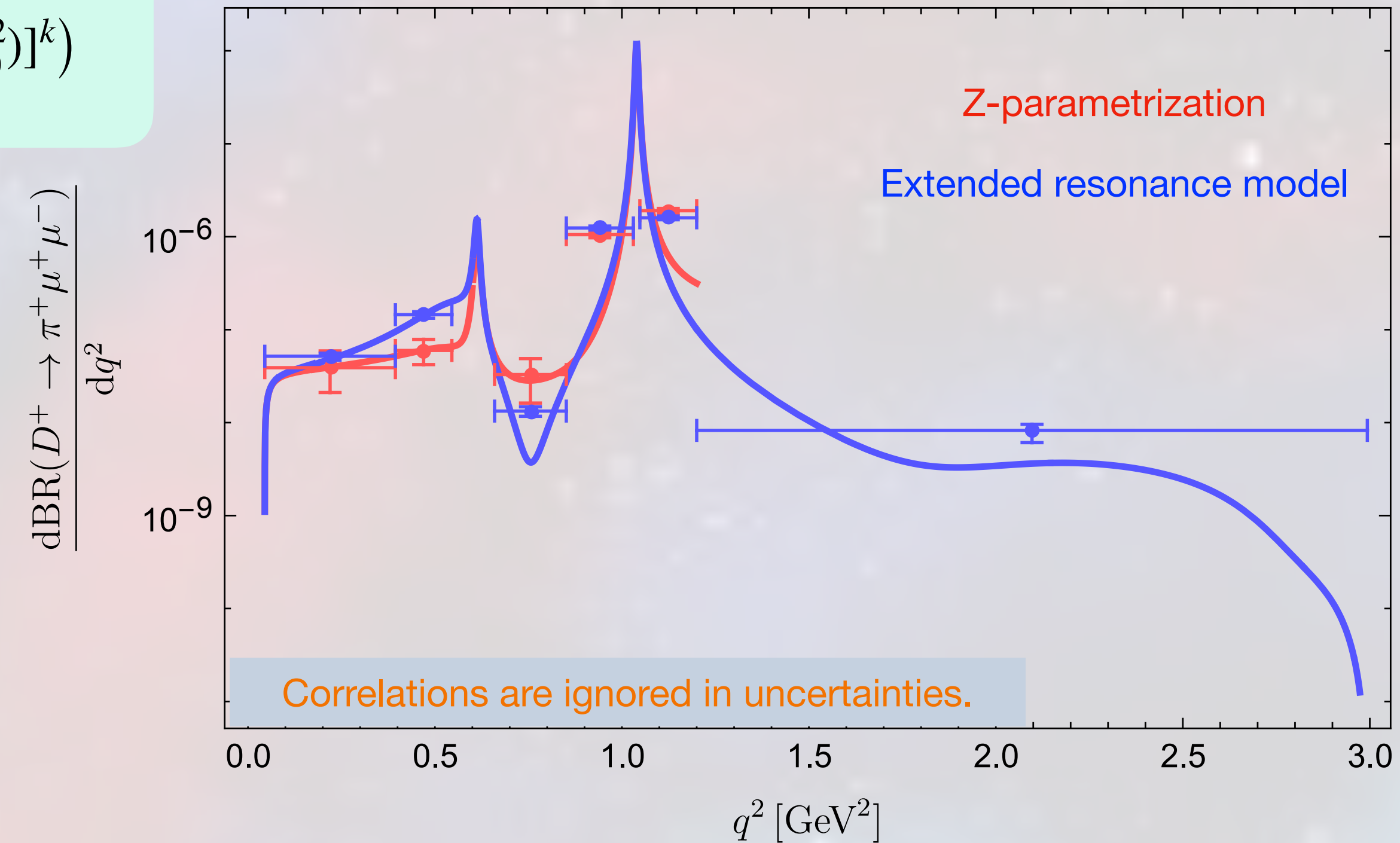
$$\mathcal{A}^{(D^+ \rightarrow \pi^+ \gamma^*)}(q^2) = \mathcal{A}^{(D^+ \rightarrow \pi^+ \gamma^*)}(q_0^2) + \left[(q^2 - q_0^2) \sum_{V=\rho, \omega, \phi} \frac{k_V f_V A_{D\pi V} e^{i\varphi_V}}{(m_V^2 - q_0^2)(m_V^2 - q^2 - im_V \Gamma_V^{tot})} + \int_{s_{th}}^{\infty} ds \frac{(q^2 - q_0^2) \rho_h(s)}{(s - q_0^2)(s - q^2 - i\epsilon)} \right]$$

$$\int_{s_{th}}^{\infty} ds \frac{(q^2 - q_0^2) \rho_h(s)}{(s - q_0^2)(s - q^2 - i\epsilon)} = \sum_{k=1}^K a_k ([z(q^2)]^k - [z(q_0^2)]^k)$$

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◦ A more sophisticated fit procedure (bayesian methods) may help in reducing the uncertainties.

Summary and Outlook

- ❖ LCSR have a very broad spectrum of applications : provides approximate analytic expressions for hadronic quantities.
- ❖ LCSRs have been successfully complimenting LQCD so far.
- ❖ Presently, resonance final states , non-local effects, etc are only accessible using LCSR techniques.
- ❖ Non-leptonic amplitudes also studied in LCSR: requires better understanding of final state interactions.
 - A. Khodjamirian et. al., hep-ph/0012271, hep-ph/0304179, hep-ph/0509049, 1706.07780 ,..
 - B. Melić, [hep-ph/0303250], Y.M. Wang, et. al, [2202.08073], M.L. Piscopo et. al., 2307.07594
- ❖ Unknown systematic uncertainties from quark hadron duality approximations: need for alternative methods
- ❖ Major sources of parametric uncertainties lies in poor knowledge of LCDA parameters: input from Lattice will be useful.
- ❖ Recent applications of LCSR to study dark matter models
 - A. Khodjamirian, M. Wald [2206.11601], A. Boushchmeleva, M. Wald, [2311.13482]

Thank you !!!



Thank you !!!



Back Up

Light Cone dominance

- For $Q^2 = -q^2 \rightarrow \infty$ and $|(p - q)^2| \rightarrow \infty$: The correlation functions are dominated by regions near light cone ($x^2 \rightarrow 0$).

$$|\nu| = |p \cdot q| \sim |(p - q)^2| \sim Q^2 \gg \Lambda_{QCD}^2 \quad \Rightarrow \quad \xi = \frac{2\nu}{Q^2} \sim 1$$

- Consider a reference frame where $\vec{p}$ is small (non-zero) : $|\vec{p}| \sim \mu$, $|p_0| \sim \mu$ such that $\mu^2 \ll Q^2, \nu$.

$$\Rightarrow \quad q \cdot x \simeq \frac{Q^2 \xi}{4\mu} (x_0 - x_3) - \frac{2\mu}{\xi} x_3$$

Argument of the exponential in typical correlation function

- To avoid large oscillations: $x_0 - x_3 \sim \frac{4\mu}{(Q^2 \xi)}$ and $x_3 \sim \frac{\xi}{(2\mu)}$

$$x_0^2 \simeq x_3^2 + \frac{4}{Q^2} + O\left(\frac{\mu^2}{Q^4}\right) \Rightarrow x^2 \sim 1/Q^2 \rightarrow 0$$

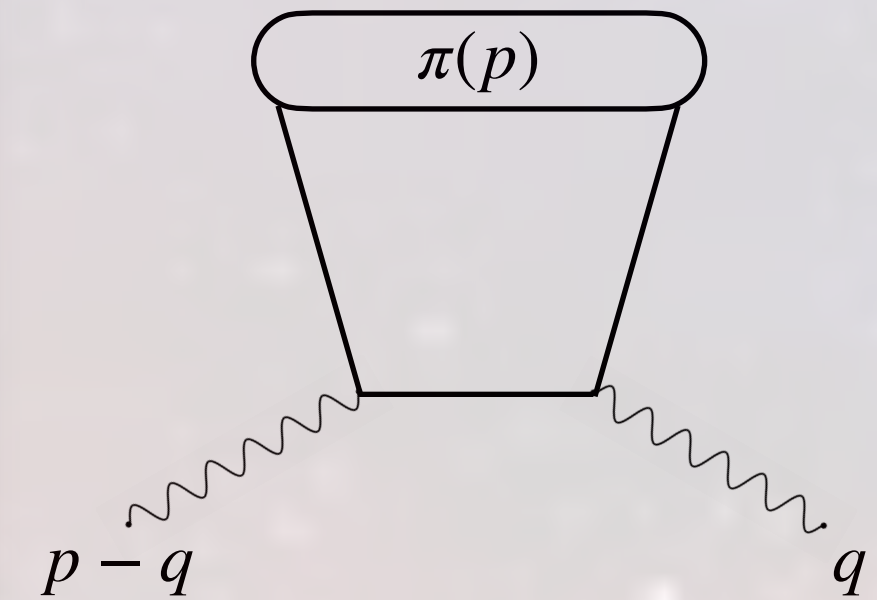


Fig. : Diagrammatic representation of a typical correlation function of two currents between the initial and the final state containing pion and vacuum.

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- No short distance dominance as $x_0 \sim x_3 \sim \xi/(2\mu) \gg 1/\sqrt{Q^2}$

- An expansion in local operators around $x=0$ is not applicable. Rather an expansion around $x^2 = 0$ is possible.

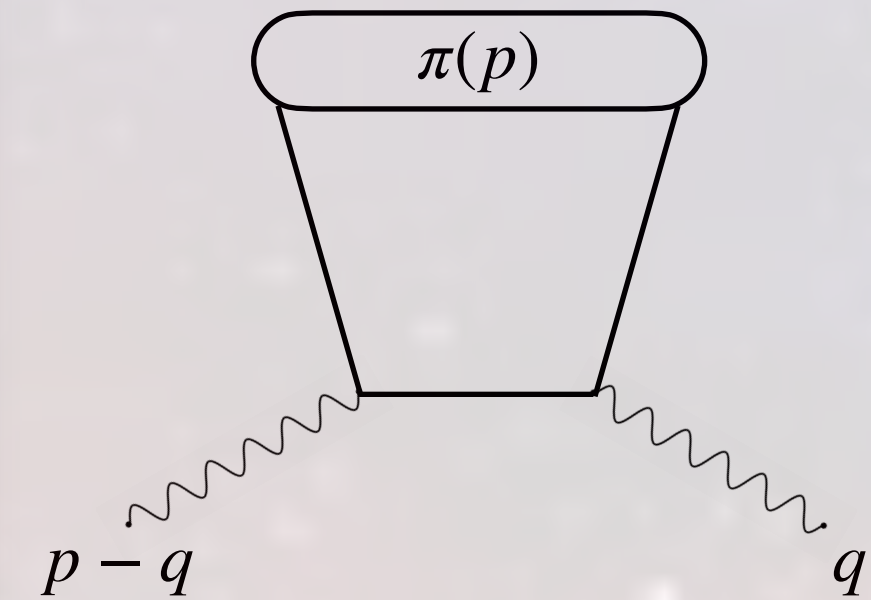


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Dispersion Relation: Analyticity & Unitarity

- The dispersion relation relates the imaginary part of these functions to the real part.
- According to Cauchy's integral and Schwartz reflection principle,

$$\Pi(q^2) = \frac{1}{2\pi i} \oint_C ds \frac{\Pi(s)}{s - q^2} = \frac{1}{\pi} \int_0^\infty ds \frac{\text{Im}\Pi(s)}{s - q^2}$$

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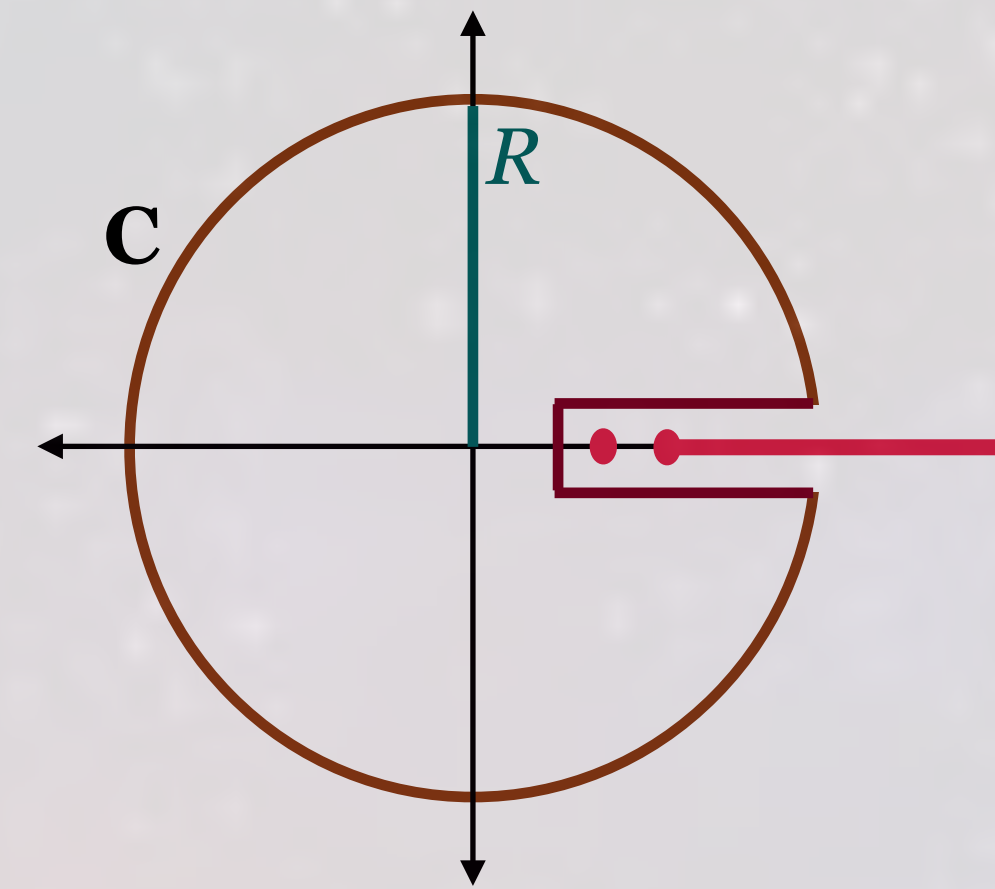


Fig. : Showing analytical properties of a typical correlation function in field theory.

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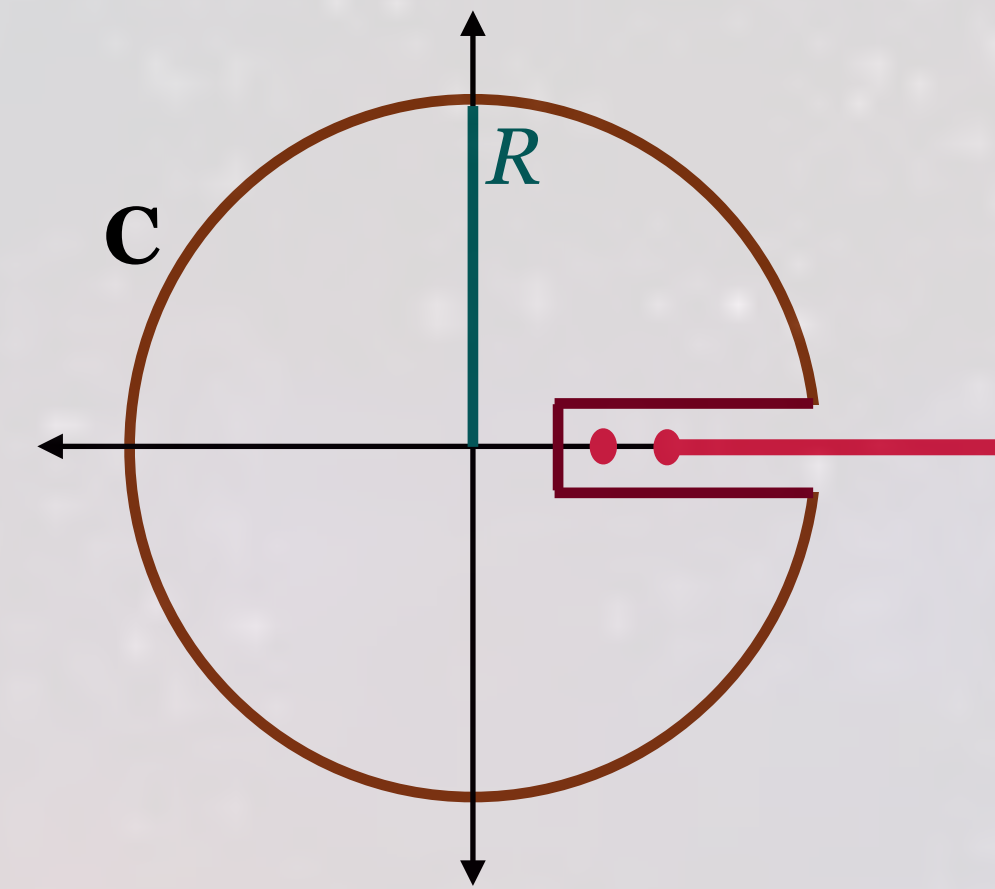


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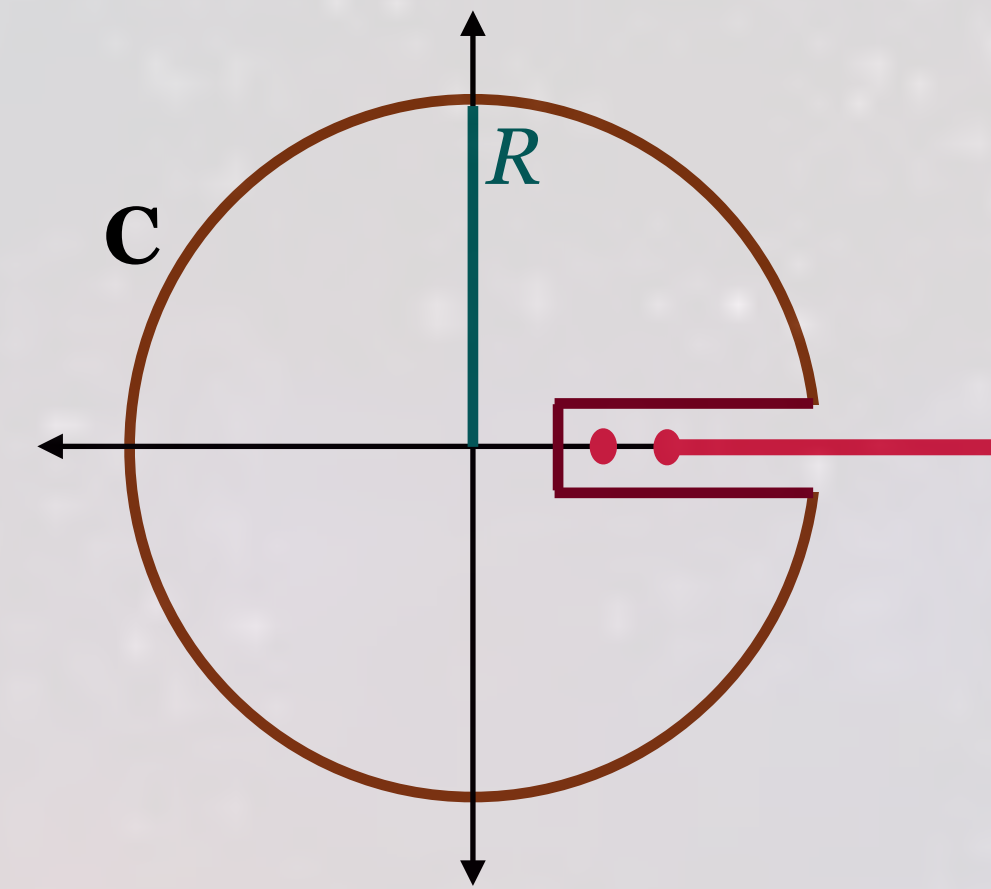


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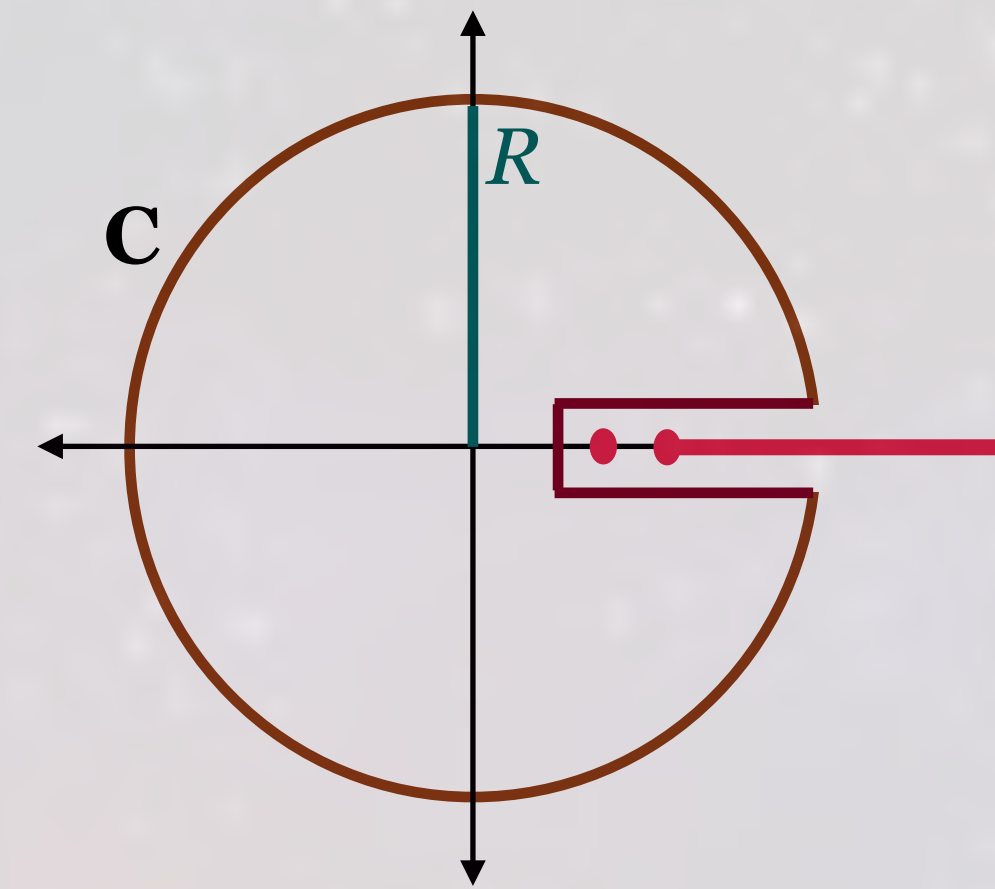


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The resonance contribution coming from the lowest energy state.

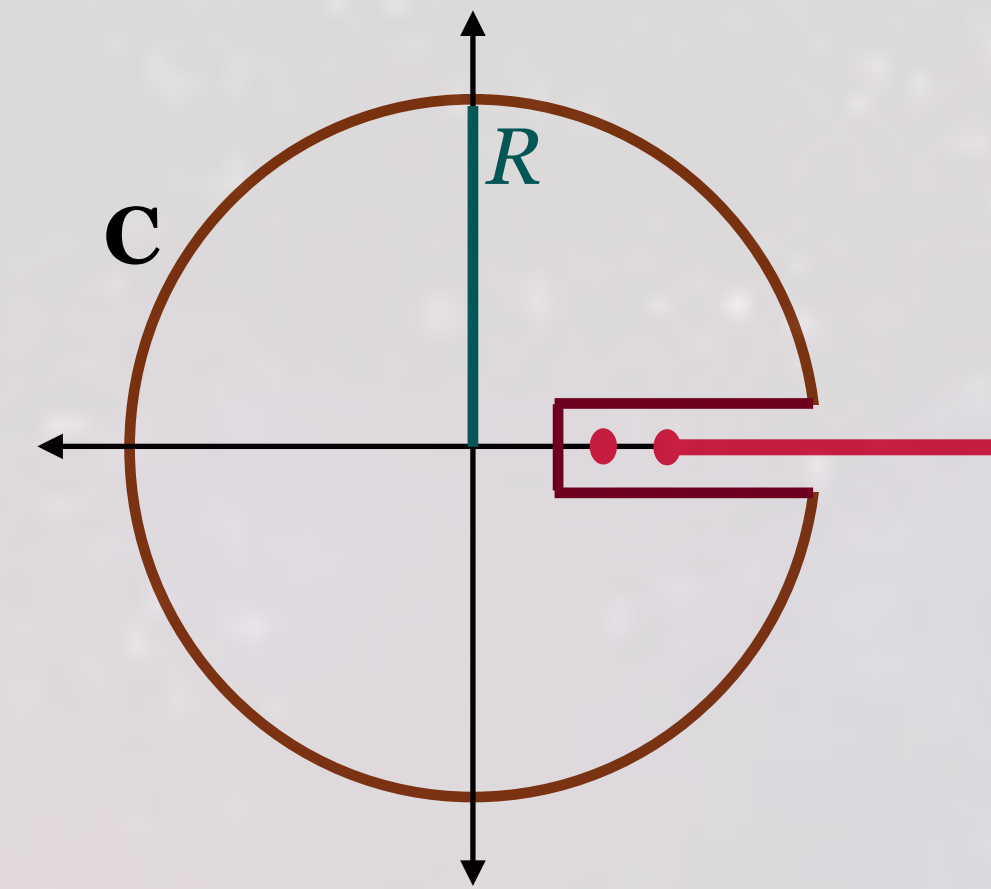


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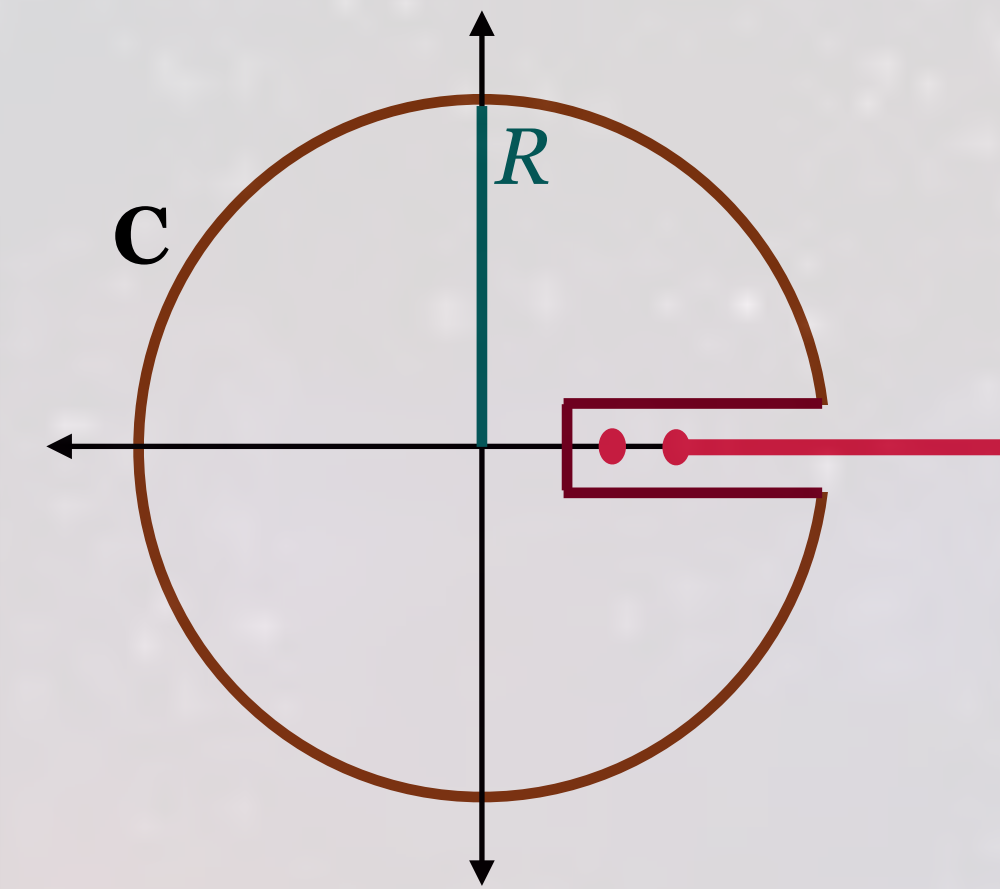


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Spectral density

$$\approx \frac{1}{\pi} \text{Im}(\Pi^h(s))$$

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Quark Hadron Duality

- Very less is known about the spectral density $\rho^h(s) \approx \frac{1}{\pi} \text{Im}(\Pi^h(s))$.

Two types

```
graph TD; A[Two types] --> B[Global quark-hadron duality]; A --> C[Local quark-hadron duality];
```

Global quark-hadron duality

$$\int_{t_{min}}^{\infty} ds \frac{\text{Im}\Pi(s)}{(s - q^2)} \simeq \int_{4m^2}^{\infty} ds \frac{\text{Im}(\Pi^{QCD}(s))}{s - q^2}$$

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Continuum threshold

s_0 needs to be fitted and is expected to be close to the mass square of the first excited state.

Borel Transformation

- Uncertainty due to quark hadron duality is to be reduced $\implies$ suppress the higher states contribution
- Borel transform is a mathematical operations which can be used for this purpose.

$$\Pi(M^2) \equiv \mathcal{B}_{M^2} \Pi(q^2) = \lim_{\substack{-q^2, n \rightarrow \infty \\ -q^2/n = M^2}} \frac{(-q^2)^{(n+1)}}{n!} \left(\frac{d}{dq^2} \right)^n \Pi(q^2)$$

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s_0 (the continuum threshold) and M (the Borel mass): Two independent parameters in sum rules

Operator Product Expansion (OPE)

- OPE enables one to **separate short and long distance physics**.
- The **short distance** physics can be calculated using **perturbative QCD** and **long distance** physics can be written in universal non-perturbative objects like **condensates or distribution amplitudes (DAs)**

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Types of OPE

Short Distance OPE

- $x \rightarrow 0$

$$\Pi^{QCD}(q^2) = \sum_d C_d(q^2) \langle 0 | \mathcal{O}_d | 0 \rangle$$

Wilson Coefficients
(Short distance effect)

Vacuum Condensates
(Long distance effect)

- It is an expansion in canonical dimension.

Light Cone OPE

- $x^2 \rightarrow 0$

$$\Pi^{QCD}(q^2, s) = \int_0^1 du \left(T(q^2, s, u, \mu) \Phi(u, \mu) \right)$$

Hard Scattering kernel
(Short distance effect)

DA
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LCSR uses the light cone OPE

Twist in QCD and Conformal symmetry

- For practical purpose: Conformal symmetry $\implies$ Scaling Symmetry (no dimension-ful parameter in the theory like)

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

- The full conformal algebra in 4 dimensions include 15 generator:

- * P_μ (4 translations)
- * $M_{\mu\nu}$ (6 Lorentz rotations)
- * D (dilatation/scaling)
- * K_μ (4 special conformal transformations)

- Collinear subgroup of full conformal group: (P_+, M_{-+}, D, K_-)

- $[E, \phi(\alpha)] = t\phi(\alpha)$ where, $E = \frac{i}{2}(D - M_{-+})$ and t is the twist.

- Twist (t)= Canonical dimension (d) - Spin (s)

- Spin projection operators

$$\begin{array}{l} \Pi_+ = \frac{1}{2}\gamma_- \gamma_+ \\ \swarrow \\ \text{Spin } +\frac{1}{2} \end{array} \quad \begin{array}{l} \Pi_- = \frac{1}{2}\gamma_+ \gamma_- \\ \swarrow \\ \text{Spin } -\frac{1}{2} \end{array}$$

Twist-2: $\mathcal{Q}_+ = \bar{\psi}_+ \gamma_+ \psi_+ \equiv \mathcal{Q}^{1,1}$

Twist-3: $\mathcal{Q}_\perp = \bar{\psi}_+ \gamma_\perp \psi_- + \bar{\psi}_- \gamma_\perp \psi_+ \equiv \mathcal{Q}^{1,1/2} + \mathcal{Q}^{1/2,1}$

Twist-4: $\mathcal{Q}_- = \bar{\psi}_- \gamma_- \psi_- \equiv \mathcal{Q}^{1/2,1/2}$

Light Cone OPE

● The matrix element of non-local operator,

$$\langle \pi^0 | \bar{u}(x) \gamma^\rho \gamma_5 u(0) | 0 \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} x_{\mu_1} x_{\mu_2} \dots x_{\mu_n} \underbrace{\langle \pi^0 | \bar{u}(0) \bar{D}^{\mu_1} \bar{D}^{\mu_2} \dots \bar{D}^{\mu_n} \gamma^\rho \gamma_5 u(0) | 0 \rangle}_{(-i)^{n+1} p^\rho p^{\mu_1} p^{\mu_2} \dots p^{\mu_n} \mathcal{M}_n^d - (-i)^{n+1} g^{\mu_1 \mu_2} p^\rho p^{\mu_3} \dots p^{\mu_n} \mathcal{M}_n^{d+2} + \dots}$$

Here, $\mathcal{M}_n^d$, and $\mathcal{M}_n^{d+2}$ are non-perturbative parameters of increasing twist.

● The coefficient of $n = 0$ implies,

$$\langle \pi^0 | \bar{u}(0) \gamma^\rho \gamma_5 u(0) | 0 \rangle = -ip^\rho \mathcal{M}_0^d \quad \Longrightarrow \quad \mathcal{M}_0^d \propto f_\pi$$

Light cone propagator

- The propagator : $S(x_1, x_2, m) = -i \langle 0 | T\{\bar{q}(x_2)q(x_1)\} | 0 \rangle$ is not gauge invariant.
- Gauge invariance requires the insertion of a wilson line ($W(x_1, x_2)$) between $\bar{q}(x_2)$ and $q(x_1)$ in a non-local operator.

$$W(x_1, x_2) = P \left\{ \exp \left(ig \int_{x_2}^{x_1} \mathcal{A}_\mu^a(z) T^a dz^\mu \right) \right\}$$

- Using the $\mathcal{A}_\mu x^\mu = 0$ gauge i.e. the fixed-point gauge, light cone expansion of the quark propagator is

$$S(x_1, x_2, m) = \int \frac{d^k}{(2\pi)^4} e^{-ik(x_1-x_2)} \left\{ \frac{\gamma_\beta k^\beta + m}{k^2 - m^2} - \int_0^1 dv G^{\mu\nu}(vx_1 + (1-v)x_2) \left[\frac{1}{2} \frac{k_\beta \gamma^\beta + m}{(k^2 - m^2)^2} \sigma_{\mu\nu} - \frac{1}{k^2 - m^2} v(x_1 - x_2)_\mu \gamma_\nu \right] \right\}$$

Heavy to Heavy Form Factors

FF	$f^+(0) = f^0(0)$	$V(0)$	$A_0(0)$ $A_1(0)$ $A_2(0)$	Ref.
$B \rightarrow D^{(*)}$	0.586 ± 0.103	0.703 ± 0.160	0.623 ± 0.112 0.704 ± 0.119 0.803 ± 0.186	[115]
	0.65 ± 0.08	0.69 ± 0.13	— 0.60 ± 0.09 0.51 ± 0.09	[81]
$B_s \rightarrow D_s^{(*)}$	0.572 ± 0.115	0.671 ± 0.168	0.582 ± 0.120 0.661 ± 0.117 0.765 ± 0.163	[115]

Table 7 $B_{(s)} \rightarrow D_{(s)}^{(*)}$ form factors at $q^2 = 0$ from the LCSRs with B -meson DAs.