Multi-Hadron Decays on the Lattice

Felix Erben

Taming hadronic uncertainties in and beyond the Standard Model
ICJLab Orsay
23 Oct 2025

Introduction & Motivation

$K^*(892)$ & $\rho(770)$ at m_π^{phys}

Multi-Hadron Decays on the Lattice

Outlook & Conclusions

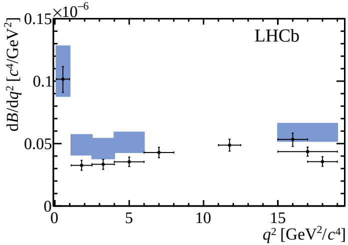
Introduction & Motivation

Scope of this Talk: Multihadron Decays on the Lattice

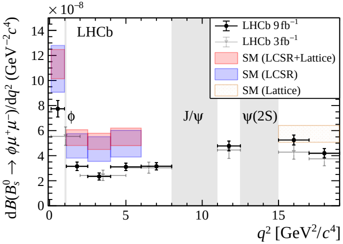
- **Focus: exclusive** semileptonic decays computed on the lattice
  - e.g. $B \to K^*(\to K\pi)\,\ell^+\ell^-$
- Two key challenges to control:
  - Heavy–light system: b quark and light quarks simultaneously
  - Resonant final states: need the full $K\pi \to K^* \to K\pi$ dynamics
- *Not covered here (but happy to discuss over coffee):*
  - Inclusive decays and spectral reconstruction approaches

Analogous issues arise for $B \to \rho(\to \pi\pi)\,\ell\nu$ from a lattice perspective.

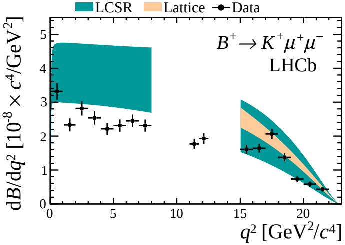
SOME EXTRA MOTIVATION: B-ANOMALIES



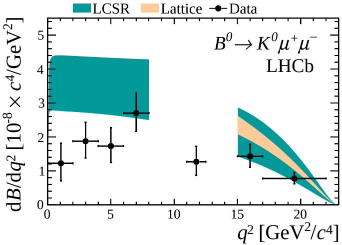
$B^0 \to K^{*0}$ [LHCb, JHEP 16]



$B_s \to \phi$ [LHCb, PRL 21]



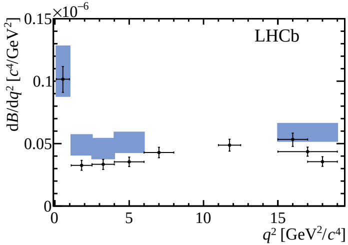
$B^+ \to K^+$ [LHCb, JHEP 14]



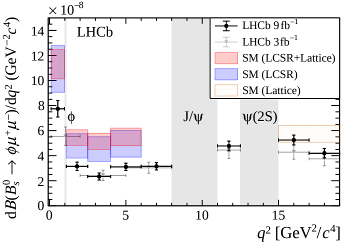
$B^0 \to K^0$ [LHCb, JHEP 14]

- most recent LHCb measurements of $b \to s\ell^+\ell^-$ branching fractions
- low-q^2: LCSR
- high-q^2: lattice
- experiment consitently lower than theory, particularly at low q^2
- tension reinforced by HPQCD $B \to K\ell^+\ell^-$
- can lattice contribute to $B \to K^*\ell^+\ell^-$?

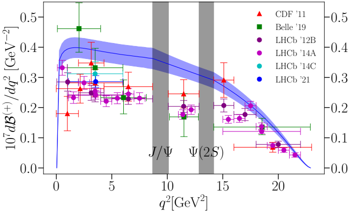
SOME EXTRA MOTIVATION: B-ANOMALIES



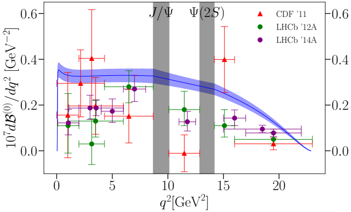
$\mathrm{B}^0 \to \mathrm{K}^{*0}$ [LHCb, JHEP 16]



$\mathrm{B}_s \to \phi$ [LHCb, PRL 21]



$\mathrm{B}^+ \to \mathrm{K}^+$ [Parrot et al.; PRD 23]



$\mathrm{B}^0 \to \mathrm{K}^0$ [Parrot et al.; PRD 23]

- most recent LHCb measurements of $\mathrm{b} \to s\ell^+\ell^-$ branching fractions
- low-q^2: LCSR
- high-q^2: lattice
- experiment consitently lower than theory, particularly at low q^2
- tension reinforced by HPQCD $\mathrm{B} \to \mathrm{K}\ell^+\ell^-$
- can lattice contribute to $\mathrm{B} \to \mathrm{K}^*\ell^+\ell^-$?

How Lattice QCD works (in the stable-hadron world)

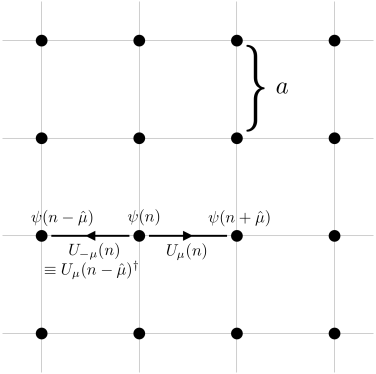
A recap from Benoit's talk yesterday:

Nonperturbative definition: introduce explicit IR and UV cutoffs

- finite spatial volume: $L < \infty$ (infrared)
- finite lattice spacing: $a > 0$ (ultraviolet)

Make it numerically tractable:

- Wick rotate to Euclidean time: $\tau = it$
- $e^{iS} \to e^{-S_E}$
- Fermion fields integrated out: $\int \mathcal{D}\Psi\mathcal{D}\bar{\Psi}\, e^{-S_F} = \prod_f \det(D_f)$



Typically $M_\pi > M_\pi^{\text{phys}}$, though direct simulations at physical masses are now common

$\Rightarrow$ Heavy pion masses can also be used as a tool

From bare parameters to physical predictions

We extract observables at fixed:

- lattice spacing a
- finite volume L
- bare quark masses m_q

To compare with experiment, we take the following limits (in a controlled way):

$$a \to 0 \qquad \text{(continuum limit)}$$

$$L \to \infty \qquad \text{(infinite volume)}$$

$$m_q \to m_q^{\text{phys}} \qquad \text{(physical quark masses)}$$

Ideally using several lattice discretizations with different:

- scaling to the continuum
- chiral properties
- computational cost

So far, this framework describes stable hadrons.

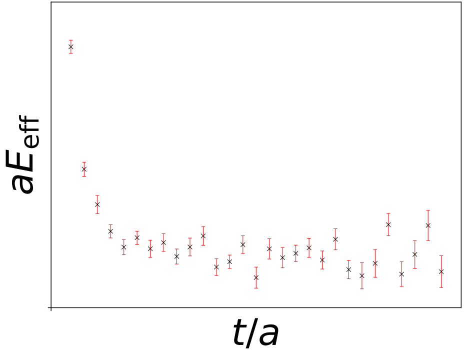
Example: Meson two-point function

$$C_2^D(\mathbf{p},t) = \sum_x e^{i\mathbf{p}\cdot\mathbf{x}} C_2^D(\mathbf{x},t)$$
$$= Z_D^2 e^{-E_D t}\left[1 + O(e^{-\Delta E\, t})\right]$$

A fit yields Z_D^2 and E_D. Effective mass:

$$E_{\text{eff}}(t) = \ln \frac{C_2^D(\mathbf{p},t)}{C_2^D(\mathbf{p},t+1)}$$

Signal-to-noise deteriorates for large t/a.

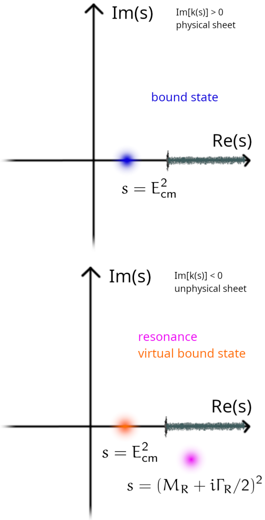


BUT: No lattice interpolator has a true K^* or ρ as it's stable lowest state!

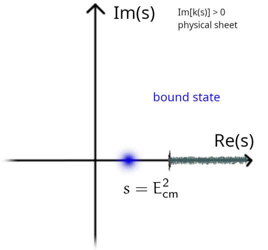
In infinite volume, unstable states are identified through the analytic structure of scattering amplitudes.

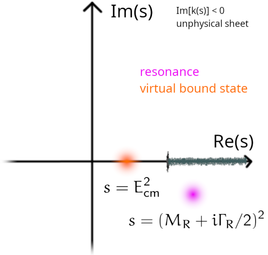
- Example: $\pi\pi$ scattering, with $k^2 = s/4 - M_\pi^2$
- Physical sheet ($\mathrm{Im}\, k(s) > 0$):
    - bound states appear as poles on the real axis below threshold
- Unphysical sheet ($\mathrm{Im}\, k(s) < 0$):
    - virtual states on the real axis
    - resonances as poles at $s = (m_R - i\Gamma_R/2)^2$

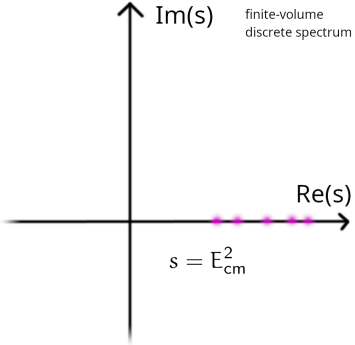
$\Rightarrow$ Resonances are not asymptotic states, but poles in infinite-volume amplitudes.



Resonances in the complex energy plane





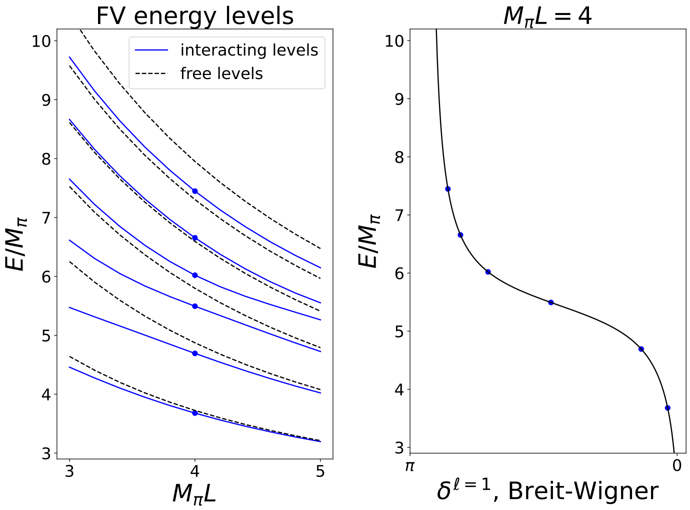


finite volume leads to

- discrete spectrum
- no branch cuts, sheets
- no exposed resonance poles

Resonances in Lattice QCD

- for resonances (vast majority of QCD states):
- **no simple correspondence** between finite-volume energies and resonances
- free levels $E = 2\sqrt{M_\pi^2 + \left(\frac{2\pi}{L}|p|\right)^2}$
- resonances leave their marks as **imprints on the finite-volume energy spectrum**
- these power-like FV effects are different to $e^{-M_\pi L}$ ones



RESONANCES IN QCD WEAK DECAYS

Resonances appear also in computations of

- hadronic decays
  - $K \to \pi\pi$
  - $D \to \pi\pi$, $D \to KK$, $D \to K\pi$
- on-shell intermediate states in e.g. rare decays and mixing
  - $K \to \pi\ell^+\ell^-$
  - $\Sigma \to p\ell^+\ell^-$
  - $K_L \to \mu^+\mu^-$
  - ϵ_K in $K - \bar{K}$ mixing
- muon $g - 2$
- ...

Recipe for resonances in LQCD ($\rho \to \pi\pi$)

relevant interpolator basis

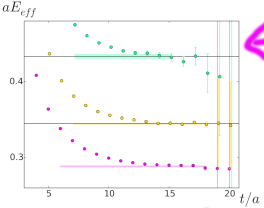
$$\rho(\vec{P}, t) = \sum_{\vec{x}} e^{-i\vec{P}\cdot\vec{x}}(\bar{u}\gamma_i u - \bar{d}\gamma_i d)$$

$$(\pi\pi)(\vec{P}, t) = \pi^+(\vec{p}_1, t)\pi^-(\vec{p}_2, t) - \pi^-(\vec{p}_1, t)\pi^+(\vec{p}_2, t)$$

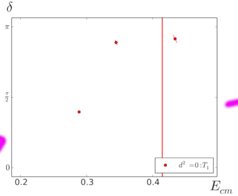
complete correlator matrix

$$\begin{pmatrix} \langle\rho(t)\rho^\dagger(t_0)\rangle & \langle\rho(t)(\pi\pi)_1^\dagger(t_0)\rangle & \langle\rho(t)(\pi\pi)_2^\dagger(t_0)\rangle & \cdots \\ \langle(\pi\pi)_1(t)\rho^\dagger(t_0)\rangle & \langle(\pi\pi)_1(t)(\pi\pi)_1^\dagger(t_0)\rangle & \langle(\pi\pi)_1(t)(\pi\pi)_2^\dagger(t_0)\rangle & \cdots \\ \langle(\pi\pi)_2(t)\rho^\dagger(t_0)\rangle & \langle(\pi\pi)_2(t)(\pi\pi)_1^\dagger(t_0)\rangle & \langle(\pi\pi)_2(t)(\pi\pi)_2^\dagger(t_0)\rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

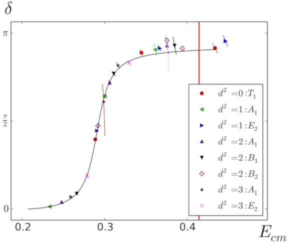
GEVP



finite-volume formalism



repeat for momenta, irreps



GEVP, FV formalism: complete (hopefully!) citations in backup slide. Plots from [FE et al.; PRD 20]

RECIPE FOR RESONANCES IN LQCD ($\rho \to \pi\pi$)

relevant interpolator basis

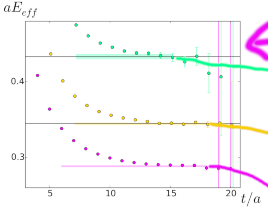
$$\rho(\vec{P}, t) = \sum_{\vec{x}} e^{-i\vec{P}\cdot\vec{x}}(\bar{u}\gamma_i u - \bar{d}\gamma_i d)$$

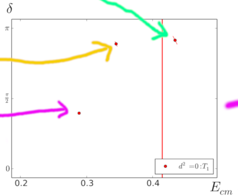
$$(\pi\pi)(\vec{P}, t) = \pi^+(\vec{p}_1, t)\pi^-(\vec{p}_2, t) - \pi^-(\vec{p}_1, t)\pi^+(\vec{p}_2, t)$$

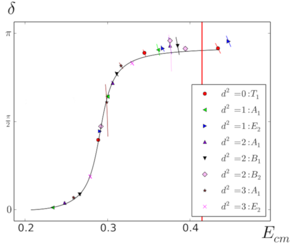
complete correlator matrix

$$\begin{pmatrix} \langle\rho(t)\rho^\dagger(t_0)\rangle & \langle\rho(t)(\pi\pi)_1^\dagger(t_0)\rangle & \langle\rho(t)(\pi\pi)_2^\dagger(t_0)\rangle & \cdots \\ \langle(\pi\pi)_1(t)\rho^\dagger(t_0)\rangle & \langle(\pi\pi)_1(t)(\pi\pi)_1^\dagger(t_0)\rangle & \langle(\pi\pi)_1(t)(\pi\pi)_2^\dagger(t_0)\rangle & \cdots \\ \langle(\pi\pi)_2(t)\rho^\dagger(t_0)\rangle & \langle(\pi\pi)_2(t)(\pi\pi)_1^\dagger(t_0)\rangle & \langle(\pi\pi)_2(t)(\pi\pi)_2^\dagger(t_0)\rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

GEVP





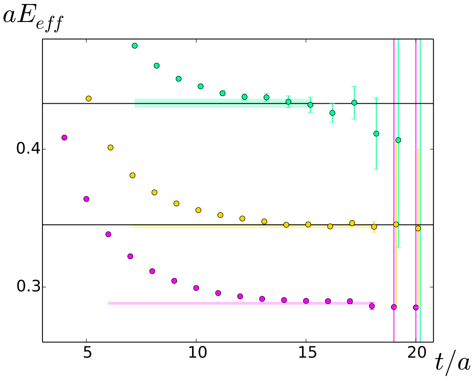


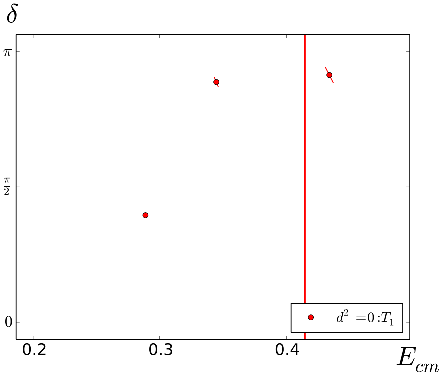
finite-volume formalism

repeat for momenta, irreps

GEVP, FV formalism: complete (hopefully!) citations in backup slide. Plots from [FE et al.; PRD 20]

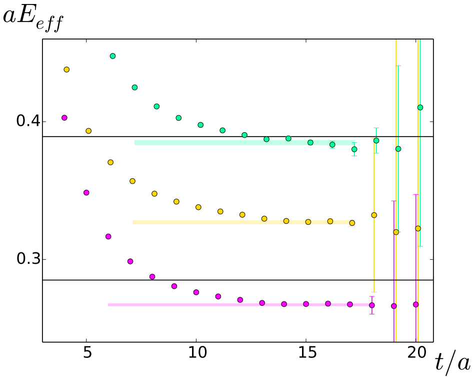
RECIPE FOR RESONANCES IN LQCD ($\rho \to \pi\pi$)

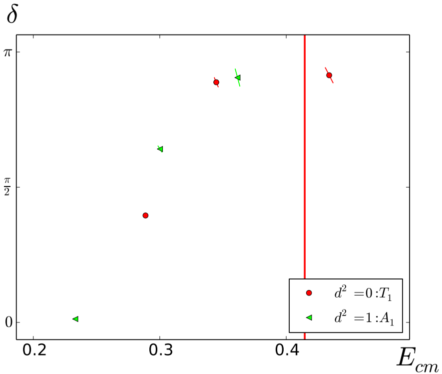




$$\left|\frac{L}{2\pi}\vec{P}\right|^2 = 0; \quad \Lambda = T_1$$

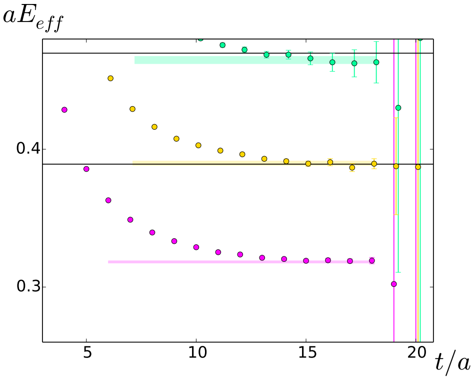
Recipe for resonances in LQCD ($\rho \to \pi\pi$)

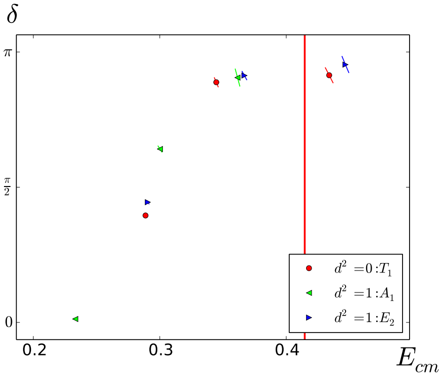




$$\left|\frac{L}{2\pi}\vec{P}\right|^2 = 1; \quad \Lambda = A_1$$

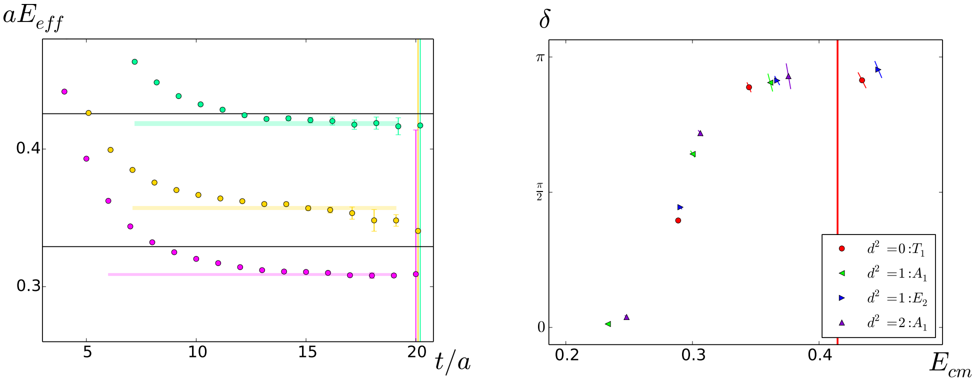
Recipe for resonances in LQCD ($\rho \to \pi\pi$)





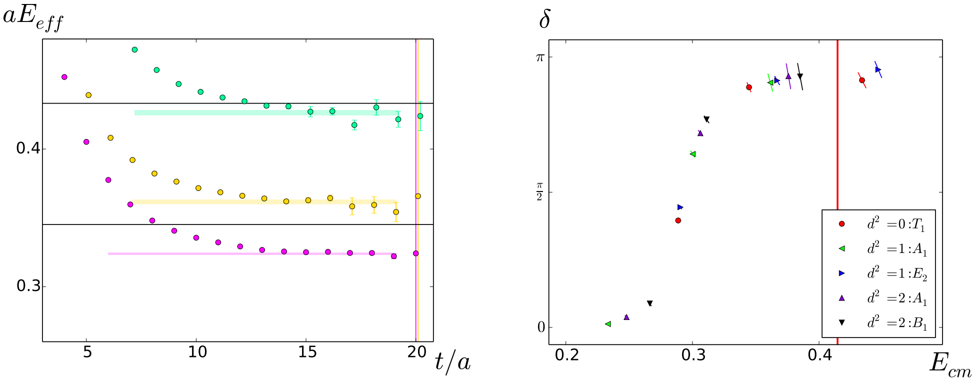
$$\left|\frac{L}{2\pi}\vec{P}\right|^2 = 1; \quad \Lambda = E_2$$

Recipe for resonances in LQCD ($\rho \to \pi\pi$)



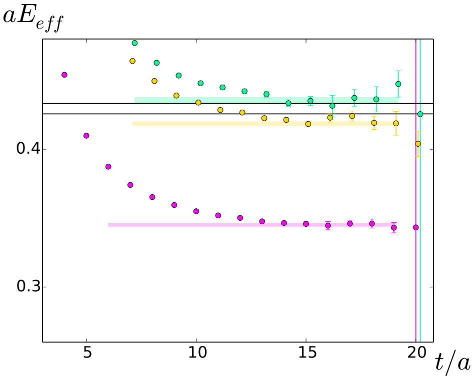
$$\left|\frac{L}{2\pi}\vec{P}\right|^2 = 2; \quad \Lambda = A_1$$

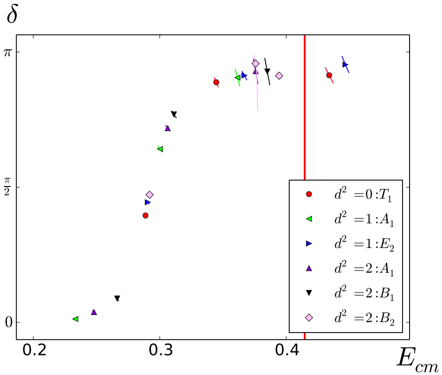
Recipe for resonances in LQCD ($\rho \to \pi\pi$)



$$\left|\frac{L}{2\pi}\vec{P}\right|^2 = 2; \quad \Lambda = B_1$$

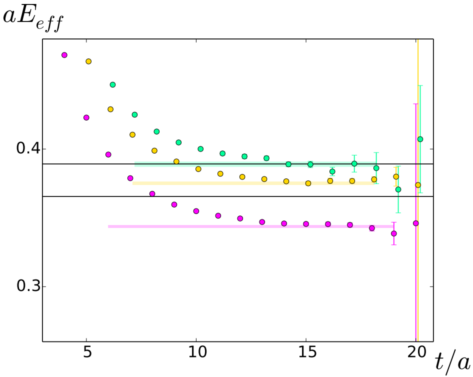
Recipe for resonances in LQCD ($\rho \to \pi\pi$)

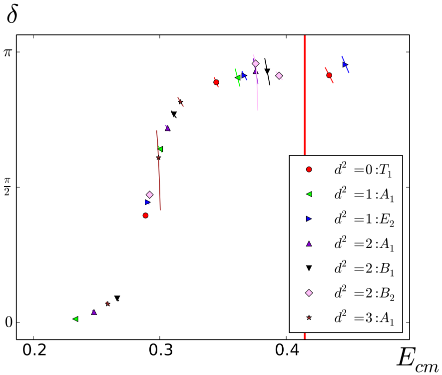




$$\left|\frac{L}{2\pi}\vec{P}\right|^2 = 2; \quad \Lambda = B_2$$

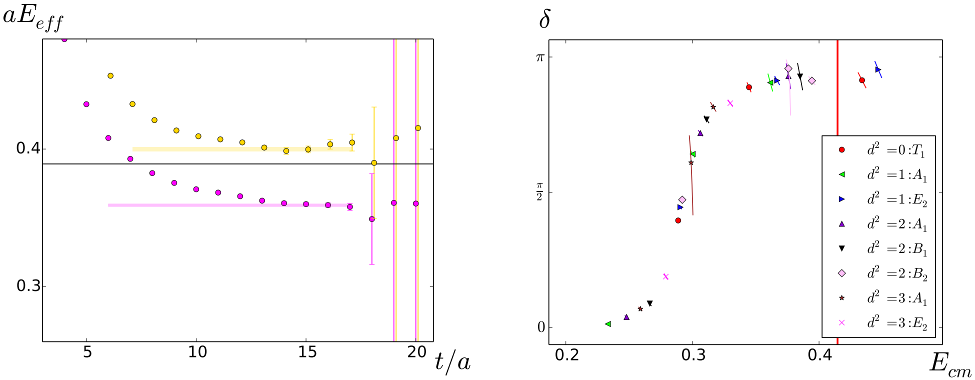
Recipe for resonances in LQCD ($\rho \to \pi\pi$)





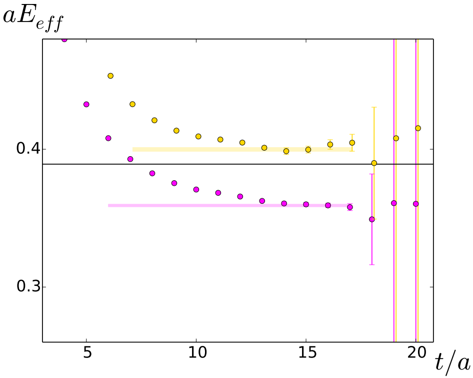
$$\left|\frac{L}{2\pi}\vec{P}\right|^2 = 3; \quad \Lambda = A_1$$

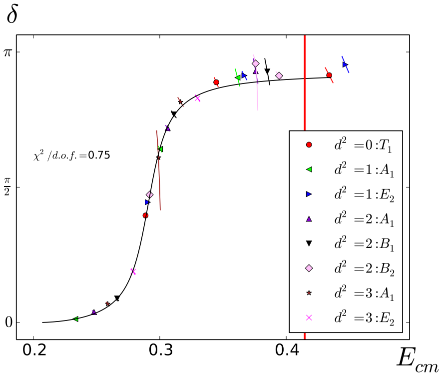
RECIPE FOR RESONANCES IN LQCD ($\rho \to \pi\pi$)



$$\left|\frac{L}{2\pi}\vec{P}\right|^2 = 3; \quad \Lambda = E_2$$

RECIPE FOR RESONANCES IN LQCD ($\rho \to \pi\pi$)

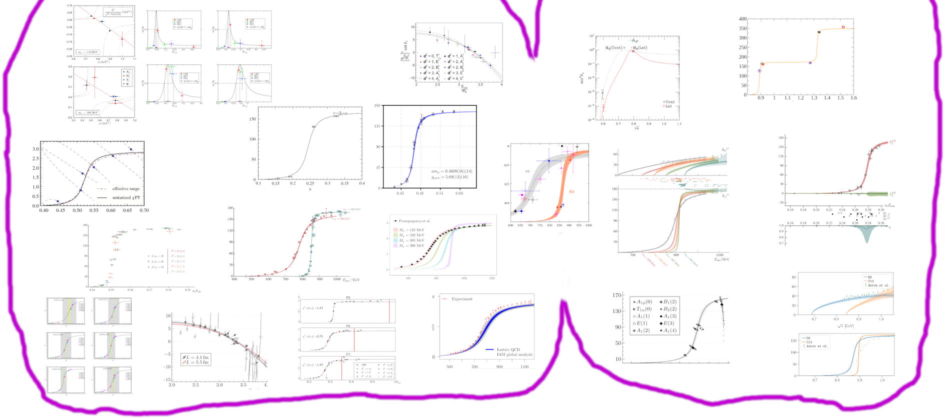




[FE et al., PRD 20] $m_\pi = 311$MeV

$K^*(892)$ & $\rho(770)$ at m_π^{phys}

$\pi\pi \to \rho \to \pi\pi$ $\quad$ $K\pi \to K^* \to K\pi$



[Aoki et al.; 07] [Aoki et al.; 11] [Feng et al.; 11] [Lang et al.; 11] [Fu, Fu; 12] [Pelissier, Alexandrou; 13] [Prelovsek et al.; 13] [Dudek et al.; 14]

[Wilson et al.; 15] [Wilson et al.; 15] [Bali et al.; 16] [Bulava et al.; 16] [Fu et al.; 16] [Alexandrou et al.; 17] [Andersen et al.; 18] [Brett et al.; 18]

[Wilson et al.; 19] [FE et al.; 20] [Werner et al.; 20] [Fischer et al.; 20] [Rendon et al.; 20]

$K^*(892)$ & $\rho(770)$ AT m_π^{phys} [BOYLE, FE, GÜLPERS, HANSEN, JOSWIG, LACHINI, MARSHALL, PORTELLI; 2024]

Physical-mass calculation of $\rho(770)$ and $K^*(892)$ resonance parameters via $\pi\pi$ and $K\pi$ scattering amplitudes from lattice QCD

Peter Boyle,[1,2] Felix Erben,[3,2] Vera Gülpers,[2] Maxwell T. Hansen,[2]
Fabian Joswig,[2] Nelson Pitanga Lachini,[4,2,*] Michael Marshall,[2] and Antonin Portelli[2,3,5]
[1] Physics Department, Brookhaven National Laboratory, Upton NY 11973, USA
[2] School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3JZ, United Kingdom

[arXiv:2406.19193]

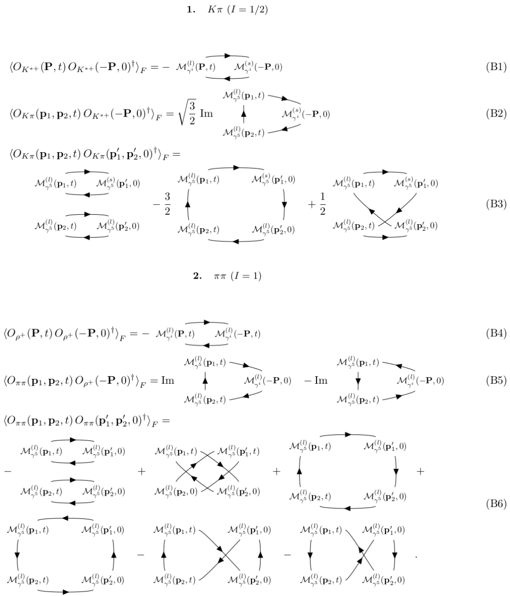
Light and strange vector resonances from lattice QCD at physical quark masses

Peter Boyle,[1,2] Felix Erben,[3,2] Vera Gülpers,[2] Maxwell T. Hansen,[2]
Fabian Joswig,[2] Nelson Pitanga Lachini,[4,2,*] Michael Marshall,[2] and Antonin Portelli[2,3,5]
[1] Physics Department, Brookhaven National Laboratory, Upton NY 11973, USA
[2] School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3JZ, United Kingdom
[3] CERN, Theoretical Physics Department

[arXiv:2406.19194]

- computation of two of the most fundamanetal mesonic QCD resonances, $K^*(892)$ & $\rho(770)$
- at physical quark masses
- Domain-Wall fermions
- lattice spacing $a^{-1} \sim 1.7$ GeV
- full error budget for *statistical* and *systematic* errors
- PhD project of **Nelson Lachini**, who has made most plots in this section

Distillation [Peardon et al.; PRD 09] [Morningstar et al.; PRD 11] code



GRID Hadrons

[github:paboyle/grid] [github:aportelli/hadrons]

- ***Distillation*** is an efficient technique to compute correlator matrix
- Distillation code written for Grid and Hadrons
    - free & open source
    - code documentation: [Hadrons/MDistil]
- Correlator data published on CERN Document Server [CDS, 2024]
    - easy mapping to paper notation

Return to the resonance recipe

relevant interpolator basis

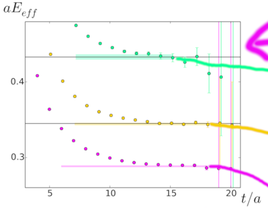
$$\rho(\vec{P},t) = \sum_{\vec{x}} e^{-i\vec{P}\cdot\vec{x}} (\bar{u}\gamma_i u - \bar{d}\gamma_i d)$$

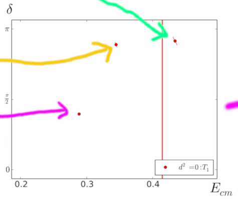
$$(\pi\pi)(\vec{P},t) = \pi^+(\vec{p}_1,t)\pi^-(\vec{p}_2,t) - \pi^-(\vec{p}_1,t)\pi^+(\vec{p}_2,t)$$

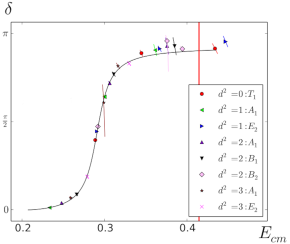
complete correlator matrix

$$\begin{pmatrix} \langle\rho(t)\rho^\dagger(t_0)\rangle & \langle\rho(t)(\pi\pi)_1^\dagger(t_0)\rangle & \langle\rho(t)(\pi\pi)_2^\dagger(t_0)\rangle & \cdots \\ \langle(\pi\pi)_1(t)\rho^\dagger(t_0)\rangle & \langle(\pi\pi)_1(t)(\pi\pi)_1^\dagger(t_0)\rangle & \langle(\pi\pi)_1(t)(\pi\pi)_2^\dagger(t_0)\rangle & \cdots \\ \langle(\pi\pi)_2(t)\rho^\dagger(t_0)\rangle & \langle(\pi\pi)_2(t)(\pi\pi)_1^\dagger(t_0)\rangle & \langle(\pi\pi)_2(t)(\pi\pi)_2^\dagger(t_0)\rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

GEVP







finite-volume formalism

repeat for momenta, irreps

GEVP

$$C(t) = \begin{pmatrix} \langle K^*(t)K^{*\dagger}(t_0)\rangle & \langle K^*(t)(K\pi)_1^\dagger(t_0)\rangle & \langle K^*(t)(K\pi)_2^\dagger(t_0)\rangle & \cdots \\ \langle (K\pi)_1(t)K^{*\dagger}(t_0)\rangle & \langle (K\pi)_1(t)(K\pi)_1^\dagger(t_0)\rangle & \langle (K\pi)_1(t)(K\pi)_2^\dagger(t_0)\rangle & \cdots \\ \langle (K\pi)_2(t)K^{*\dagger}(t_0)\rangle & \langle (K\pi)_2(t)(K\pi)_1^\dagger(t_0)\rangle & \langle (K\pi)_2(t)(K\pi)_2^\dagger(t_0)\rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

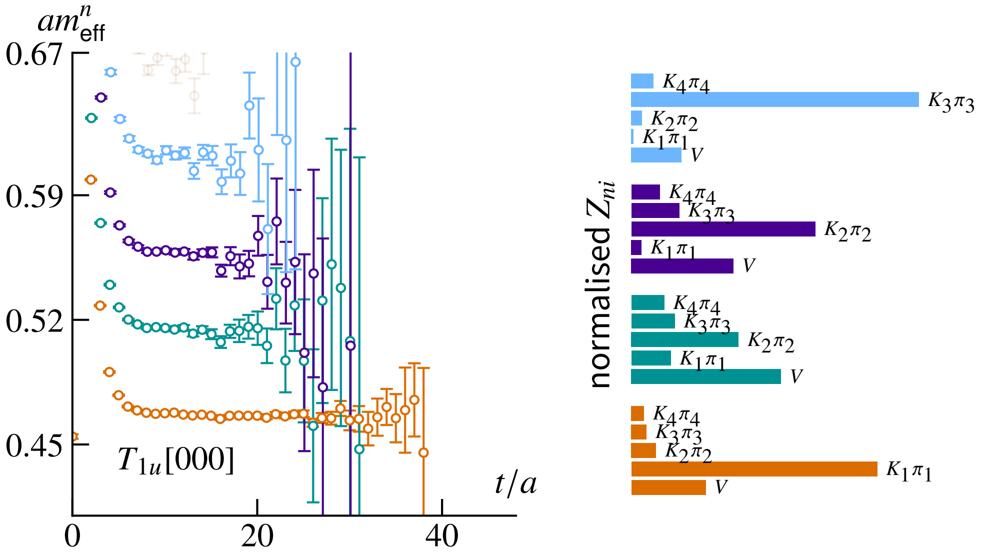
by solving the GEVP

$$C(t)v_n(t, t_0) = \lambda_n(t, t_0)C(t_0)v_n(t, t_0)$$

we get

- eigenvalues $\lambda_n(t, t_0)$ optimized correlators
- eigenvectors $v_n(t, t_0)$

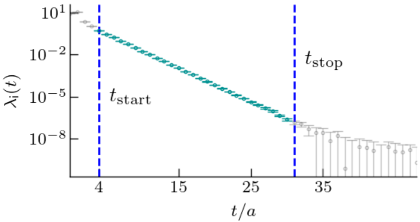
$\Rightarrow$ optimized interpolators $X_n(t) = \sum_i v_{ni}O_i(t)$

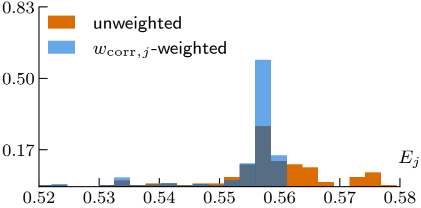


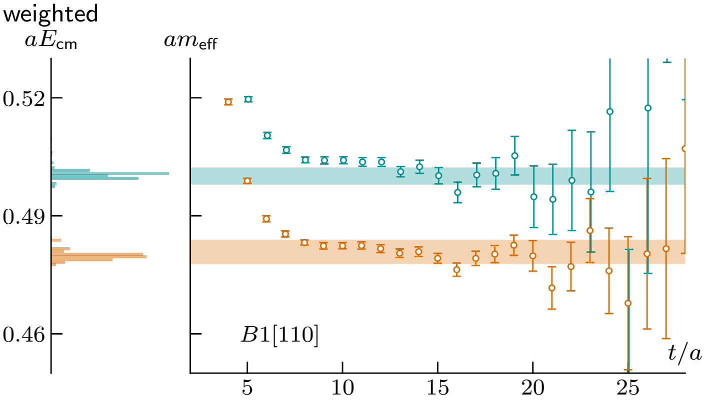
am^n_{eff}
0.67
0.59
0.52
0.45
$T_{1u}[000]$
t/a
0
20
40
normalised Z_{ni}
$K_4\pi_4$
$K_3\pi_3$
$K_2\pi_2$
$K_1\pi_1$
V

- data-driven method:
  1) for a given model, perform all possible fits to the data: **single-exponential** $C(t) = Ae^{-Et}$
  2) assign a goodness-of-fit criterion: $AIC = \chi^2 + 2n^{\text{par}} - n^{\text{data}}$
  3) create a histogram weighted by $e^{-\frac{1}{2}AIC}$
- tests which fits are compatible with the model
- width of histogram $\Rightarrow$ systematic error

Figures from Nelson's Lattice 24 talk [Lachini: Lattice 24]

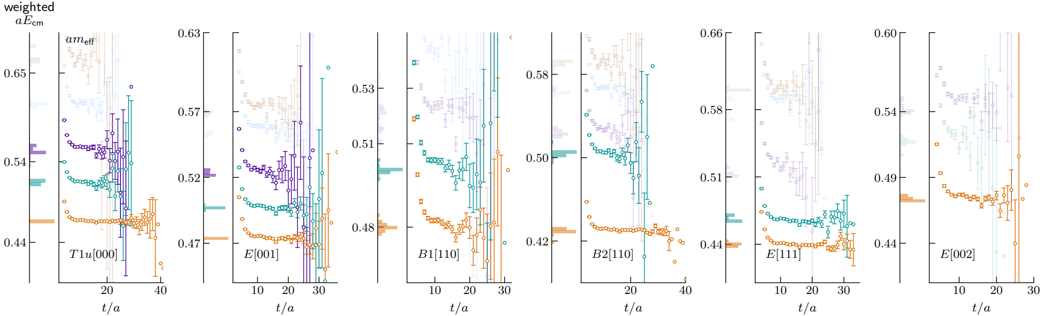


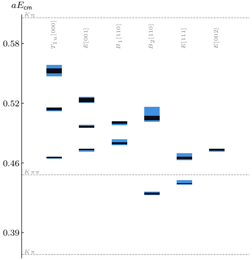




- histogram shows distribution for each energy level
- statistical error is estimated by repeating the data driven method on $\mathrm{N_b} = 2000$ bootstrap samples
- band shows combined *statistical* and *systematic* error

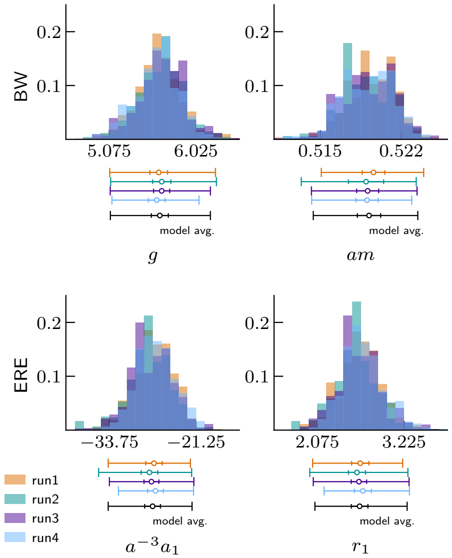
FV SPECTRUM Kπ



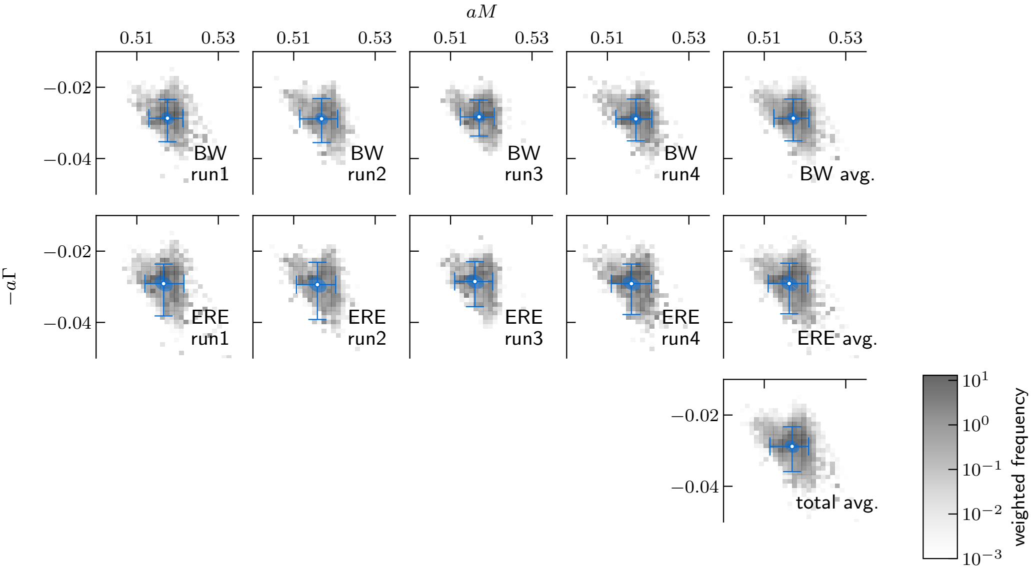


- same exercise repeated for all levels entering Kπ analysis
- 13 energy levels, 6 irreps, 5 momentum frames
- we don't use irreps mixing P-wave and S-wave

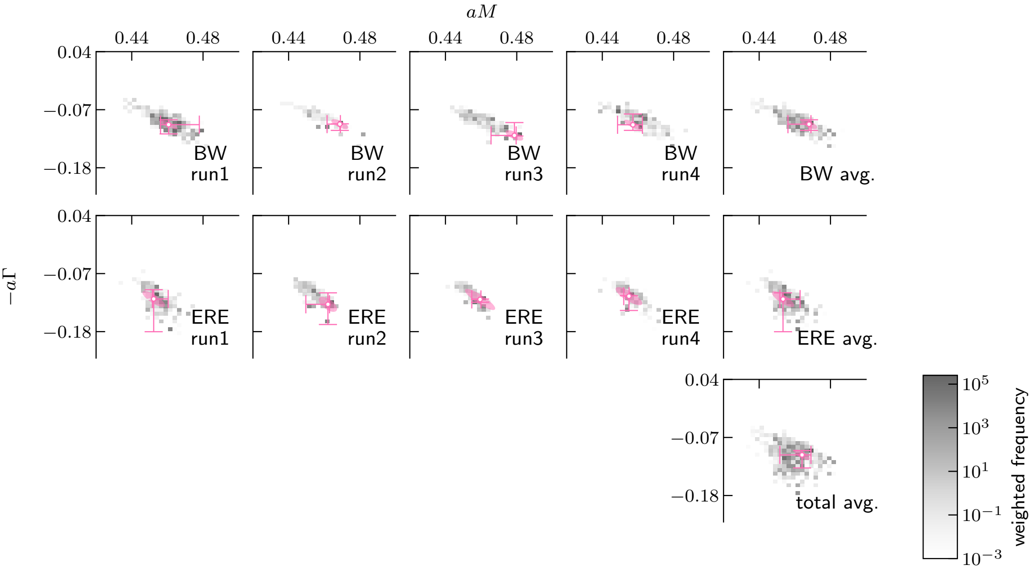
- results for Breit-Wigner with $\alpha^{\text{mod}} = \{m_{K^*}, g_{K^*K\pi}\}$
- results for Effective Range Expansion with $\alpha^{\text{mod}} = \{a_1, r_1\}$
- Four choices of allowed fit ranges, called **runs**



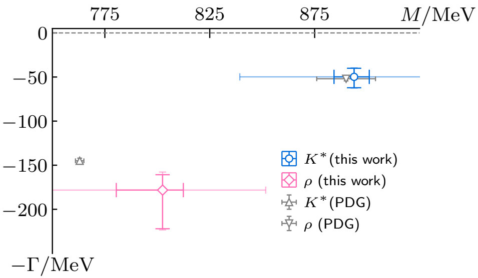
Pole position $K\pi$



Pole position $\pi\pi$



ρ, K^* AT M_π^{phys}: SUMMARY



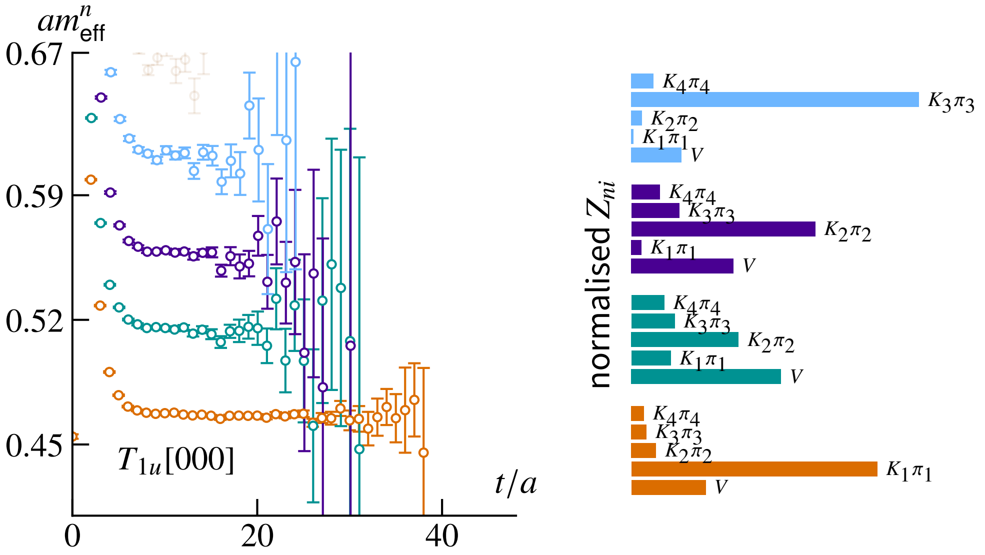
- pole positions of ρ, K^*, including statstical and systematic error
- M_π^{phys}, single lattice spacing

$\Rightarrow$ faint error: naive esimation accounting for use of single lattice spacing

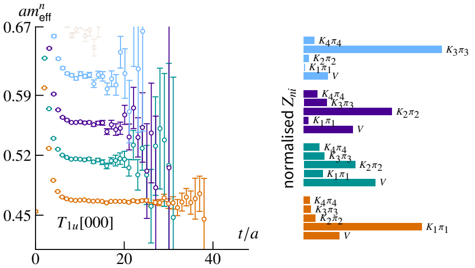
- Agreement with PDG at the level of $< 1\sigma$ for all but Γ_ρ, where agreement is at the level of 1.6σ

Multi-Hadron Decays on the Lattice

GEVP from 5 interpolators



- On the lattice, all states with the same quantum numbers mix
- GEVP (same as for energies) provides eigenvectors v_i
- These eigenvectors define an **optimized linear combination** of interpolators
- That linear combination $\Rightarrow$ our lattice K^* state
- We can then insert this K^* state into 3-point functions with currents



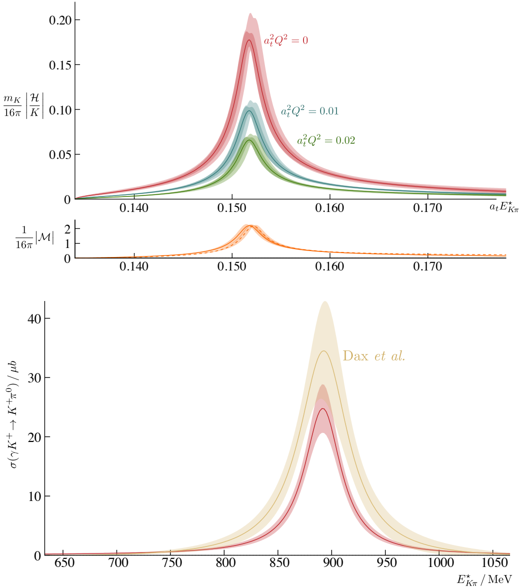
The optimized K^* state: $|K^*\rangle = \sum_i v_i |O_i\rangle$

From lattice K^* to form factors

- Build the lattice K^* state using GEVP eigenvectors
- Compute 3pt functions with this state and a current J^μ
- This gives a **finite-volume form factor** F_L [Briceño, Dudek, Leskovec; PRD 21]
- Finite-volume F_L is related to the physical form factor F via the **Lellouch–Lüscher factor** (known, computable)
- After this step: treat just like non-resonant channels (e.g. $B \to K\ell^+\ell^-$)
- Method already applied to $\pi\gamma \to \pi\pi$ and $K\gamma \to K\pi$ transition amplitudes [Briceño et al., PRD 2016] [Alexandrou et al., PRD 2018] [Radhakrishnan et al., PRD 2022]

  as well as recently to $B \to \rho\ell\nu$ [Leskovec et al., PRL 2025]

$K\gamma \to K\pi$

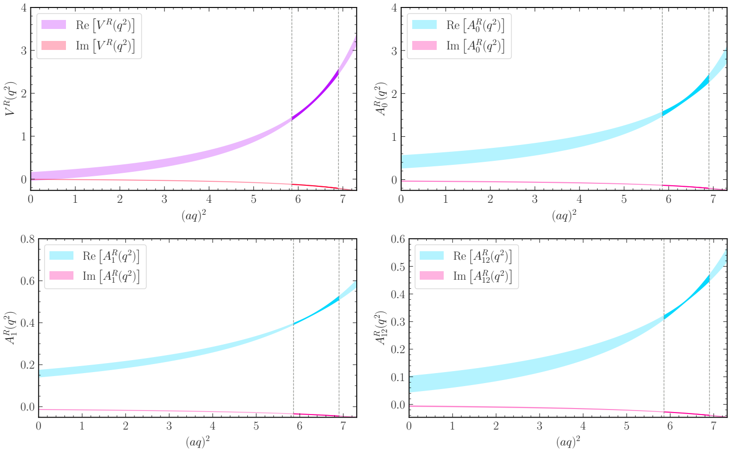


- $K\gamma \to K\pi$ transition amplitude

  [Radhakrishnan et al.; PRD 22]

- multiple photon virtualities aQ^2
- extrapolation to M_π^{phys} simplistic, but in reasonable agreement with PDG

$B \to (\rho \to \pi\pi)\ell\nu$



- First calculation of a heavy meson decay into a resonant final state

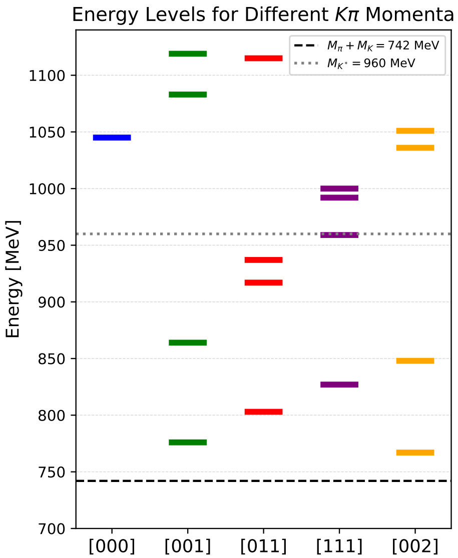
  [Leskovec et al., PRL 25]
- Performed at $M_\pi \approx 320$ MeV, no chiral or continuum extrapolation
- Form factors extracted at high q^2 and extrapolated to low-q^2 region
- Four form factors: V, A_0, A_1, A_{12}

OUTLOOK: $B \to K^*(\to K\pi)\ell^+\ell^-$ DECAYS

- New computing allocation with team from CERN, Edinburgh, Liverpool, BNL
- DWF, $a^{-1} = 2.7$ GeV
- $M_\pi = 232$ MeV, $M_K = 510$ MeV
- 3 DWF heavy quarks $m_c \leqslant m_h \lesssim 0.5 m_b$
- RHQ heavy quark $m_h = m_b$
- we employ the expensive but effective ***distillation*** technique
- three-point functions for 5 values of $\mathbf{p}_{K\pi}$ and 2 values of $\mathbf{p}_B$

$$\langle n; \mathbf{p}_{K\pi} | J_{V/A/T} | B, \mathbf{p}_B \rangle_L$$

- preliminary results very soon!



3PT ANALYSIS

Thanks to the distillation approach, the dataset comprises three-point functions for e.g.

- $B \to K^* \ell^+ \ell^- \quad (b \to s)$
- $B_s \to K^* \ell^+ \ell^- \quad (b \to d)$
- $D \to \rho \ell^+ \ell^- \quad (c \to u)$
- $B \to \rho \ell \nu \quad (b \to u)$
- $K\gamma \to K\pi \quad (s \to s, d \to d)$
- $\pi\gamma \to \pi\pi \quad (d \to d)$

all with true resonant final states.

Outlook & Conclusions

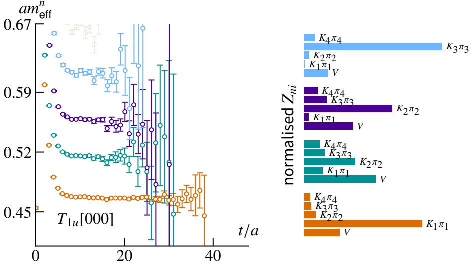
Conclusions for K^*, ρ scattering

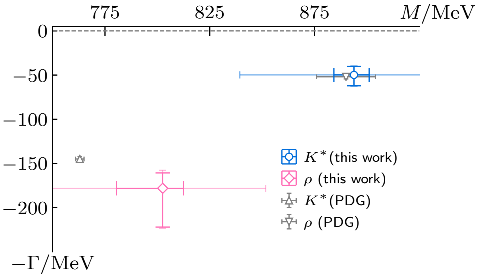
Power-like finite volume effects relate lattice FV spectrum

- discrete
- no branch cut
- hidden resonance pole

to infinite-volume resonance poles

this formalism also relates matrix elements $\langle E_n | J | E_m \rangle$ to weak transition amplitudes etc.



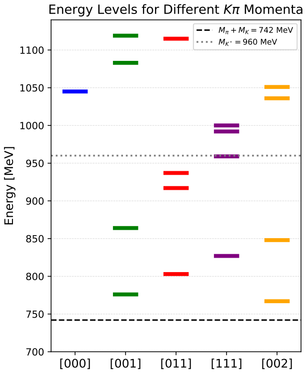


Outlook

- lattice community is making fast progress in multi-hadron weak decays
- Exploratory calculation for $B \to K^* \ell^+ \ell^-$ form factors underway
- systematically improvable, conceptually clear way to control all uncertainties

$\Rightarrow$ potential avenue towards finding new physics in B anomalies

- achieving scrutiny of standard semileptonic decays will need a dedicated collaborative effort



This project has received funding from the European Union's Horizon Europe research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 101106913.