Analytic structures in $b \to s\ell\ell$

Simon Mutke

Helmholtz-Institut für Strahlen- und Kernphysik,
Bethe Center for Theoretical Physics, Universität Bonn

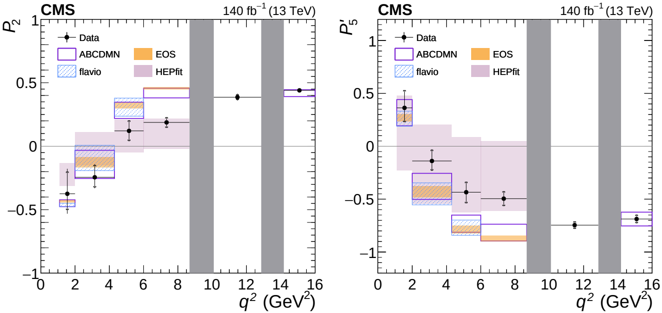
October 23, 2025

Taming Hadronic Uncertainties
in and beyond the Standard Model

Orsay

in collaboration with Martin Hoferichter and Bastian Kubis

- **Flavor anomalies**: sizable deviations from SM predictions for decay rates and angular observables (e.g. P_2, P_5') in $b \to s\mu\mu$ decays (e.g. $B^0 \to K^{*0}\mu^+\mu^-$)



Credit: CMS 2024

$\hookrightarrow$ BSM effects?

$\hookrightarrow$ Unaccounted **non-local effects** in form factors mimicking BSM effects?

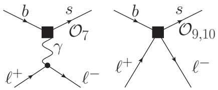
Motivation: non-local form factors in $B \to K^{(*)}\ell\ell$

- Hadronic matrix element for $B \to K^{(*)}\ell\ell$ in Weak Effective Theory

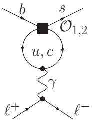
$$\mathcal{A}(B \to K^{(*)}\ell\ell) \sim \mathcal{N}\left[\left(C_9 L_V^\mu + C_{10} L_A^\mu\right)\mathcal{F}_\mu - \frac{L_V^\mu}{q^2}\left(C_7 \mathcal{F}_{T,\mu} + \mathcal{H}_\mu\right)\right]$$

conventions from Gubernari, van Dyk, Virto 2021

- **Local form factors** (FFs) $\mathcal{F}_\mu$ and $\mathcal{F}_{T,\mu}$
  - $\hookrightarrow$ known with good precision
  - $\hookrightarrow$ calculated with Lattice QCD, Light-Cone Sum Rules, . . .



- **Non-local FFs** $\mathcal{H}_\mu$
  - $\hookrightarrow$ still with large uncertainties
  - $\hookrightarrow$ calculated with Operator Product Expansion, QCD factorization, . . .
  - $\hookrightarrow$ extrapolated **analytically** from space-like to semileptonic region
  - $\hookrightarrow$ need good understanding of the **analytic structure**



Analytic structure of hadronic form factors

- Fundamental principles: **analyticity** (causality) and **unitarity** (probability conservation)
- Start with **analyticity**: amplitudes are analytic in all **kinematic invariants**
  - **Meson masses** $p^2 = M_B^2$, $(p-q)^2 = M_{K^{(*)}}^2$
    $\hookrightarrow$ only defined on-shell
  - **Photon virtuality** q^2
    $\hookrightarrow$ can define analytic continuation for arbitrary q^2 in the complex plane
- **Singularities in q^2**
  - **Poles**: (infinitely) narrow bound states
    $\hookrightarrow q^2 = \{M_{J/\psi}^2, M_{\psi(2S)}^2\}$
  - **Thresholds**: branch points of $\gamma^* \to \{\pi^+\pi^-, D\bar{D}, \ldots\}$ cuts
    $\hookrightarrow q^2 = \{4M_\pi^2, 4M_D^2, \ldots\}$
  - **Anomalous thresholds**: anomalous branch points
    $\hookrightarrow$ kinematic singularities, e.g., of the triangle diagram
    $\hookrightarrow$ position depends on all the masses

Dispersion relations

- Next up: **unitarity** of the S-matrix implies **unitarity relation**

  (set $t = q^2$)

  $$\text{disc}\,\mathcal{M}_{if}(t) \equiv \lim_{\varepsilon\to 0}\Big[\mathcal{M}_{if}(t+\mathrm{i}\varepsilon) - \mathcal{M}_{if}(t-\mathrm{i}\varepsilon)\Big] = \mathrm{i}\sum_n \mathcal{M}^*_{fn}\,\mathcal{M}_{in}$$

  $\hookrightarrow$ summing over intermediate states $n \in \{\pi^+\pi^-, D\bar{D}, \ldots\}$

- Amplitudes are analytic with **branch cuts** along real axis

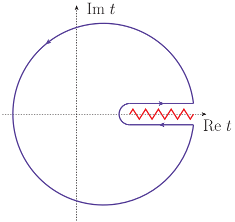
  $\hookrightarrow$ starting at thresholds $t_{\text{thr}} = \{4M_\pi^2, 4M_D^2, \ldots\}$

- Know **discontinuity** along cuts from **unitarity relation**

- Reconstruct from discontinuity via **dispersion relation**:

  $$\mathcal{M}_{if}(t) = \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}t'\,\frac{\mathcal{M}_{if}(t')}{t'-t} = \frac{1}{2\pi\mathrm{i}}\int_{t_{\text{thr}}}^{\infty}\mathrm{d}t'\,\frac{\text{disc}\,\mathcal{M}_{if}(t')}{t'-t}$$

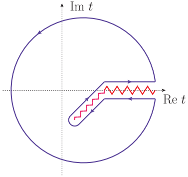
  $\hookrightarrow$ using Cauchy's theorem



Dispersion relations with anomalous thresholds

- Need to modify dispersion relation in presence of additional singularities!

  ↪ **anomalous threshold** leading to additional cuts



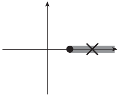
$$\mathcal{M}_{if}(t) = \frac{1}{2\pi i}\int_{t_{\text{thr}}}^{\infty} \mathrm{d}t' \, \frac{\text{disc}\,\mathcal{M}_{if}(t')}{t'-t} + \frac{1}{2\pi i}\int_0^1 \mathrm{d}x \, \frac{\partial t_x}{\partial x} \frac{\text{disc}_{\text{an}}\,\mathcal{M}_{if}(t_x)}{t_x - t}$$

  ↪ with integration contour $t_x = x\, t_{\text{thr}} + (1-x)\, t_{\text{anom}}$

- Three cases: (all three can occur SM, Hoferichter, Kubis 2024)

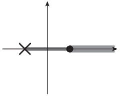
  (1) t_{anom} on normal cut

  ↪ analytic continuation of normal discontinuity



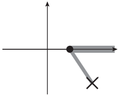
  (2) t_{anom} on negative real axis

  ↪ integration deformed along real axis



  (3) t_{anom} in complex plane

  ↪ integration deformed into complex plane



Anomalous thresholds: where do they come from?

- **Landau equations**: singularities of general loop integral

$$\int \prod_{j=1}^{L} \frac{\mathrm{d}^4 \ell_j}{(2\pi)^4} \prod_{i=1}^{n} \frac{\mathrm{i}}{k_i^2 - m_i^2 + i\varepsilon} \quad \text{singular when} \quad \begin{cases} \alpha_i (k_i^2 - m_i^2) = 0 & \text{for all } i = 1, \ldots, n \\ \sum_{i=1}^{n} \alpha_i k_i \cdot \frac{\partial k_i}{\partial \ell_j} = 0 & j = 1, \ldots, L \end{cases}$$

$\hookrightarrow$ "Leading singularity" $\Leftrightarrow$ all Feynman parameters $\alpha_i \neq 0$

$\hookrightarrow$ "Subleading singularity" $\Leftrightarrow$ some Feynman parameters $\alpha_i = 0$

Anomalous thresholds: where do they come from?

- **Landau equations**: singularities of general loop integral

$$\int \prod_{j=1}^{L} \frac{\mathrm{d}^4 \ell_j}{(2\pi)^4} \prod_{i=1}^{n} \frac{\mathrm{i}}{k_i^2 - m_i^2 + i\varepsilon} \quad \text{singular when} \quad \begin{cases} \alpha_i (k_i^2 - m_i^2) = 0 & \text{for all } i = 1, \ldots, n \\ \sum_{i=1}^{n} \alpha_i k_i \cdot \frac{\partial k_i}{\partial \ell_j} = 0 & j = 1, \ldots, L \end{cases}$$

$\hookrightarrow$ "Leading singularity" $\Leftrightarrow$ all Feynman parameters $\alpha_i \neq 0$

$\hookrightarrow$ "Subleading singularity" $\Leftrightarrow$ some Feynman parameters $\alpha_i = 0$

- **Leading singularities** behave as (scalar case)

$$\sim \begin{cases} (t - t_0)^{\frac{4L-n-1}{2}} \log(t - t_0), & \text{if } 4L - n - 1 \text{ is even and nonnegative,} \\ (t - t_0)^{\frac{4L-n-1}{2}}, & \text{else.} \end{cases}$$

Anomalous thresholds: where do they come from?

- **Landau equations**: singularities of general loop integral

$$\int \prod_{j=1}^{L} \frac{\mathrm{d}^4 \ell_j}{(2\pi)^4} \prod_{i=1}^{n} \frac{\mathrm{i}}{k_i^2 - m_i^2 + i\varepsilon} \quad \text{singular when} \quad \begin{cases} \alpha_i (k_i^2 - m_i^2) = 0 & \text{for all } i = 1, \ldots, n \\ \sum_{i=1}^{n} \alpha_i k_i \cdot \frac{\partial k_i}{\partial \ell_j} = 0 & j = 1, \ldots, L \end{cases}$$

$\hookrightarrow$ "Leading singularity" $\Leftrightarrow$ all Feynman parameters $\alpha_i \neq 0$

$\hookrightarrow$ "Subleading singularity" $\Leftrightarrow$ some Feynman parameters $\alpha_i = 0$

- **Leading singularities** behave as (scalar case)

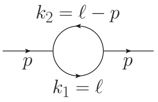
$$\sim \begin{cases} (t - t_0)^{\frac{4L-n-1}{2}} \log(t - t_0), & \text{if } 4L - n - 1 \text{ is even and nonnegative,} \\ (t - t_0)^{\frac{4L-n-1}{2}}, & \text{else.} \end{cases}$$

- **Two-particle threshold**: $L = 1, n = 2 \rightarrow$ behaves as $\sim (t - t_0)^{1/2}$

$$\alpha_i (k_i^2 - m_i^2) = 0, \qquad \alpha_1 k_1 + \alpha_2 k_2 = 0 \qquad \Rightarrow \qquad t = p^2 = (m_1 \pm m_2)^2$$

$\hookrightarrow$ Two-particle threshold at $t_{\text{thr}} = (m_1 + m_2)^2$

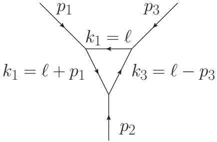
$\hookrightarrow$ Pseudothreshold $(m_1 - m_2)^2 \rightarrow$ left-hand cut on 2nd sheet



Analytic structure of the triangle diagram

- **Triangle diagram**: $L = 1$, $n = 3$

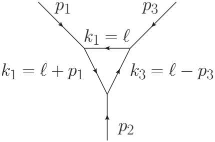
$$\alpha_i(k_i^2 - m_i^2) = 0, \qquad \sum_{i=1}^{3} \alpha_i k_i = 0$$



Analytic structure of the triangle diagram

- **Triangle diagram**: $L = 1$, $n = 3$

$$\alpha_i(k_i^2 - m_i^2) = 0, \qquad \sum_{i=1}^{3} \alpha_i k_i = 0$$



- **Normal thresholds**: subleading singularities

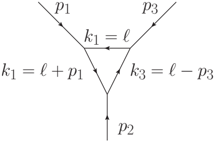
$\hookrightarrow$ e.g. $\alpha_1 = 0 \Rightarrow t \equiv p_2^2 = (m_2 \pm m_3)^2$

$\hookrightarrow$ inherits $\sim (t - t_0)^{1/2}$ behavior

Analytic structure of the triangle diagram

- **Triangle diagram**: $L = 1$, $n = 3$

$$\alpha_i (k_i^2 - m_i^2) = 0, \qquad \sum_{i=1}^{3} \alpha_i k_i = 0$$



- **Normal thresholds**: subleading singularities

  $\hookrightarrow$ e.g. $\alpha_1 = 0 \Rightarrow t \equiv p_2^2 = (m_2 \pm m_3)^2$

  $\hookrightarrow$ inherits $\sim (t - t_0)^{1/2}$ behavior

- **Anomalous threshold**: all $\alpha_i \neq 0 \rightarrow$ behaves as $\sim \log(t - t_0)$

  $\hookrightarrow$ $t_\pm \equiv p_1^2 \frac{m_1^2 + m_3^2}{2m_1^2} + p_3^2 \frac{m_1^2 + m_2^2}{2m_1^2} - \frac{p_1^2 p_3^2}{2m_1^2} - \frac{(m_1^2 - m_2^2)(m_1^2 - m_3^2)}{2m_1^2} \pm \frac{1}{2m_1^2} \sqrt{\lambda(p_1^2, m_1^2, m_2^2)\lambda(p_3^2, m_1^2, m_3^2)}$

  $\hookrightarrow$ can be complex-valued

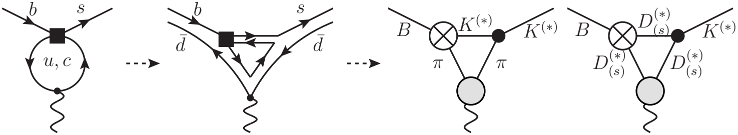
  $\hookrightarrow$ t_- always on 2$^{\text{nd}}$ sheet

  $\hookrightarrow$ t_+ can move to 1$^{\text{st}}$ sheet, if external masses big enough (B decays!),

$$m_3 p_1^2 + m_2 p_3^2 - (m_2 + m_3)(m_1^2 + m_2 m_3) > 0$$

Triangle loops in non-local $B \to K^{(*)}\gamma^*$ form factors

- **Triangle loop** contributions to non-local FFs:



- Start with ***u*-quark loop** and **$\pi\pi$ intermediate states**:
  - CKM-suppressed $\sim \lambda^4$ compared to ***c*-quark loop** $\sim \lambda^2$
  - Input (form factors, branching ratios, polarization fractions ...) well known
  - Sizable energy gap to next state $\pi\omega$

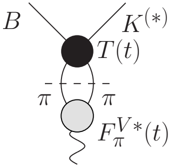
    $\hookrightarrow$ cf. various $D_{(s)}^{(*)}\bar{D}_{(s)}^{(*)}$ for hadronization of charm loop within close proximity
- Build **dispersive framework**

Form factor dispersion relation

- **Unitarity relation** for $B \to K^{(*)}\gamma^*$ FF with intermediate $\pi\pi$

$$\mathrm{disc}\,\Pi(t) = 2i\,t\,\sigma_\pi(t)^3\,T(t)\,F_\pi^{V*}(t)$$

$\hookrightarrow$ pion vector FF $F_\pi^V(t)$, $B \to K^{(*)}\pi\pi$ P-wave amplitude $T(t)$



- FF **dispersion relation**

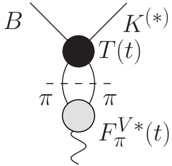
$$\Pi(t) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}t'\,\frac{t'\,\sigma_\pi(t')^3\,T(t')\,F_\pi^{V*}(t')}{t'-t}$$

Form factor dispersion relation

- **Unitarity relation** for $B \to K^{(*)}\gamma^*$ FF with intermediate $\pi\pi$

$$\mathrm{disc}\,\Pi(t) = 2i\, t\, \sigma_\pi(t)^3\, T(t)\, F_\pi^{V*}(t)$$

$\hookrightarrow$ pion vector FF $F_\pi^V(t)$, $B \to K^{(*)}\pi\pi$ P-wave amplitude $T(t)$



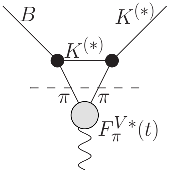
- FF **dispersion relation**

$$\Pi(t) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}t'\, \frac{t'\, \sigma_\pi(t')^3\, T(t')\, F_\pi^{V*}(t')}{t' - t}$$

- Left-hand cut from crossed-channel $K^{(*)}$-exchange in $T(t)$

$\hookrightarrow$ leads to **triangle topology**

$\hookrightarrow$ add **anomalous cut**, $\Pi(t) = \Pi^{\mathrm{norm}}(t) + \Pi^{\mathrm{anom}}(t)$

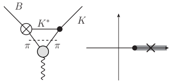


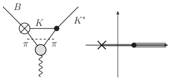
- Simple Born amplitude violates unitarity (Watson's theorem)

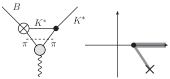
$\hookrightarrow$ need to include **$\pi\pi$-rescattering**

$\hookrightarrow$ unitarize via Muskhelishvili–Omnès representation

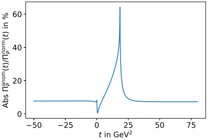
Anomalous fractions for $B^0 \to K^{(*)0} \gamma^*$

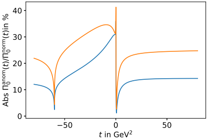


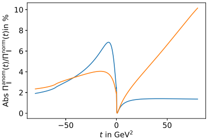


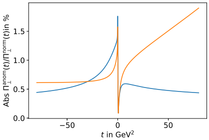


t_+ / GeV^2 = 18.6 (case 1) | -57.8 (case 2) | $0.5 - 4.2i$ (case 3)









SM, Hoferichter, Kubis 2024

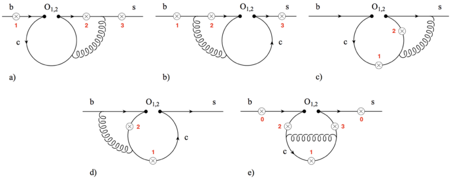
$\lambda = 0$ | $\lambda = \|$ | $\lambda = \perp$

- How important are **anom. contributions**? $\to$ anom. fraction $|\Pi^{\text{anom}}(t)/\Pi^{\text{norm}}(t)|$
- All parameters fixed from **data**!

  $\hookrightarrow$ one ambiguity left due to lack of Dalitz plot data (blue and orange curve)
- **Anomalous contributions** can be $\gtrsim$ **10%** away from thresholds, resonances

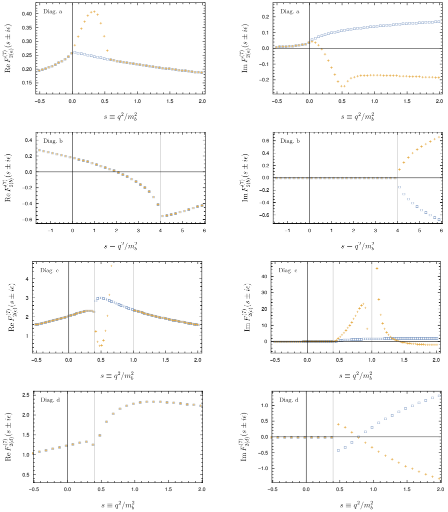
Comparison: analytic structures in $b \to s\ell\ell$ at two-loop

- What about the **quark level**?
- Consider $b \to s\ell\ell$ at two-loop order



Asatrian, Greub, Virto 2019

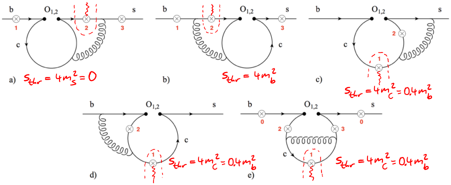
- Leads to interesting analytic structures



Asatrian, Greub, Virto 2019

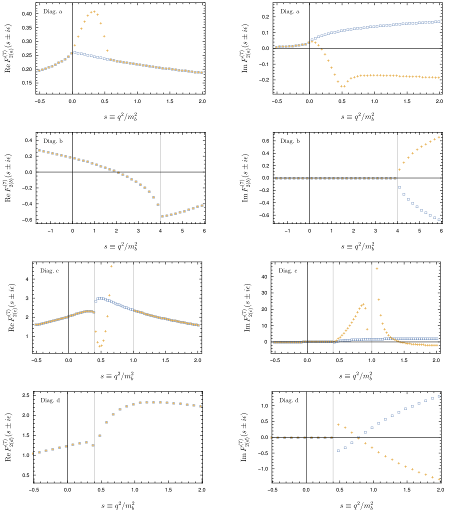
Comparison: analytic structures in $b \to s\ell\ell$ at two-loop

- What about the **quark level**?
- Consider $b \to s\ell\ell$ at two-loop order



Asatrian, Greub, Virto 2019

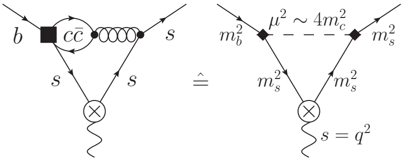
- Leads to interesting analytic structures
  - ↪ usual **two-particle thresholds**
  - ↪ what is going on with the rest?
  - ↪ anomalous thresholds at play?



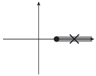
Asatrian, Greub, Virto 2019

Analytic structure of $F^{(7)}_{2,(a)}$

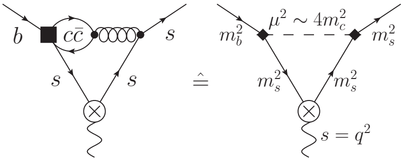
- Example: diagram $F^{(7)}_{2,(a)}$ from Asatrian, Greub, Virto 2019



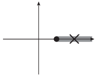
- Which singularities in s do we expect? ($m_c^2 = 0.1\, m_b^2$ and $m_s^2 = 0$)
  - Two-particle $s\bar{s}$ threshold at $s = 4m_s^2 = 0 \rightarrow$ unitarity cut
  - **Anomalous thresholds** $s_- = 0$ and $s_+ = m_b^2 - \mu^2 \sim m_b^2 - 4m_c^2 = 0.6\, m_b^2$
    $\hookrightarrow$ on 2nd sheet, i.e. at $s - i\varepsilon$ on the unitarity cut



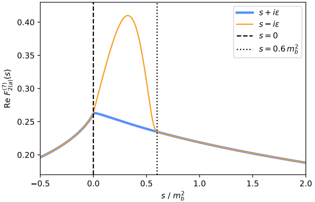
- Example: diagram $F^{(7)}_{2,(a)}$ from Asatrian, Greub, Virto 2019

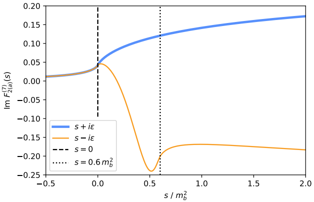


- Which singularities in s do we expect? ($m_c^2 = 0.1\, m_b^2$ and $m_s^2 = 0$)
  - Two-particle $s\bar{s}$ threshold at $s = 4m_s^2 = 0 \to$ unitarity cut
  - **Anomalous thresholds** $s_- = 0$ and $s_+ = m_b^2 - \mu^2 \sim m_b^2 - 4m_c^2 = 0.6\, m_b^2$
    $\hookrightarrow$ on 2nd sheet, i.e. at $s - i\varepsilon$ on the unitarity cut



- Matches the observation (plots from Asatrian, Greub, Virto 2019 with more data points)



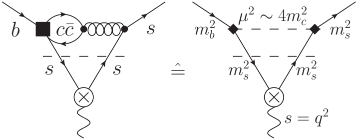


Dispersive representation for $F^{(7)}_{2,(a)}$

- Want **dispersion relation** to confirm suspected **analytic structures**

$$F^{(7)}_{2,(a)}(s) = F^{(7)}_{2,(a)}(s_0) + \frac{s - s_0}{2\pi \mathrm{i}} \int_{s_{\text{thr}}=0}^{\infty} \mathrm{d}s' \, \frac{\text{disc}\, F^{(7)}_{2,(a)}(s')}{(s' - s_0)(s' - s)}$$

- Discontinuity determined by two-particle cut



$\hookrightarrow$ take known discontinuity from scalar **triangle diagram** (with $\mu^2 = 4m_c^2$)

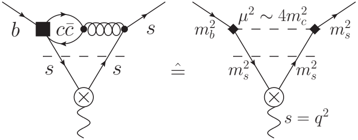
$\hookrightarrow$ adjust prefactors for correct threshold behavior

Dispersive representation for $F^{(7)}_{2,(a)}$

- Want **dispersion relation** to confirm suspected **analytic structures**

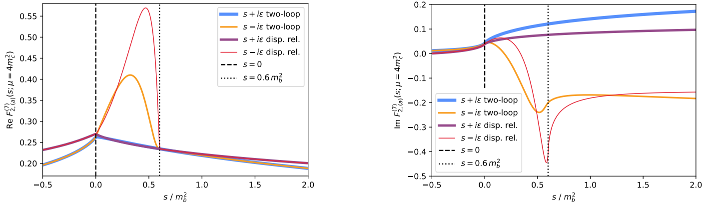
$$F^{(7)}_{2,(a)}(s) = F^{(7)}_{2,(a)}(s_0) + \frac{s - s_0}{2\pi \mathrm{i}} \int_{s_{\mathrm{thr}}=0}^{\infty} \mathrm{d}s' \, \frac{\mathrm{disc}\, F^{(7)}_{2,(a)}(s')}{(s' - s_0)(s' - s)}$$

- Discontinuity determined by two-particle cut



$\hookrightarrow$ take known discontinuity from scalar **triangle diagram** (with $\mu^2 = 4m_c^2$)

$\hookrightarrow$ adjust prefactors for correct threshold behavior



$\hookrightarrow$ goes in promising direction, but can we do better?

Dispersive representation for $F^{(7)}_{2,(a)}$

- Problem: we fixed $\mu^2 = 4m_c^2$
- Solution: introduce **spectral density function** $\Pi(\mu^2)$ and allow for any $\mu^2 \geq 4m_c^2$

$$F^{(7)}_{2,(a)}(s) = F^{(7)}_{2,(a)}(s_0) + \frac{s - s_0}{2\pi \mathrm{i}} \int_{4m_c^2}^{\infty} \mathrm{d}\mu^2 \, \Pi(\mu^2) \int_0^{\infty} \mathrm{d}s' \, \frac{\text{disc}\, F_{\mu^2}(s')}{(s' - s_0)(s' - s)}$$

$\hookrightarrow$ "smears out" **anomalous threshold** along $s_+ = m_b^2 - \mu^2$ between $[0, 0.6\, m_b^2]$

- Model $\Pi(\mu^2)$ via (3rd order) conformal polynomial in $z(\mu^2)$ and fit to $F^{(7)}_{2,(a)}(s)$

$$z(\mu^2) \equiv z(\mu^2, \mu^2_{\text{thr}} = 4m_c^2, \mu_0^2 = 0) = \frac{\sqrt{\mu^2_{\text{thr}} - \mu^2} - \sqrt{\mu^2_{\text{thr}} - \mu_0^2}}{\sqrt{\mu^2_{\text{thr}} - \mu^2} + \sqrt{\mu^2_{\text{thr}} - \mu_0^2}}$$

Dispersive representation for $F^{(7)}_{2,(a)}$

- Problem: we fixed $\mu^2 = 4m_c^2$
- Solution: introduce **spectral density function** $\Pi(\mu^2)$ and allow for any $\mu^2 \geq 4m_c^2$

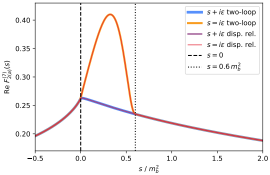
$$F^{(7)}_{2,(a)}(s) = F^{(7)}_{2,(a)}(s_0) + \frac{s - s_0}{2\pi \mathrm{i}} \int_{4m_c^2}^{\infty} \mathrm{d}\mu^2\, \Pi(\mu^2) \int_0^{\infty} \mathrm{d}s' \, \frac{\mathrm{disc}\, F_{\mu^2}(s')}{(s' - s_0)(s' - s)}$$

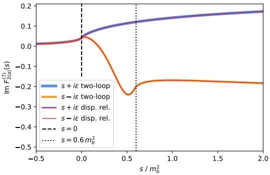
$\hookrightarrow$ "smears out" **anomalous threshold** along $s_+ = m_b^2 - \mu^2$ between $[0, 0.6\, m_b^2]$

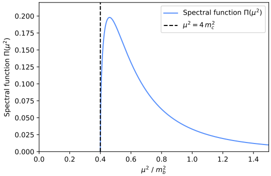
- Model $\Pi(\mu^2)$ via (3rd order) conformal polynomial in $z(\mu^2)$ and fit to $F^{(7)}_{2,(a)}(s)$

$$z(\mu^2) \equiv z(\mu^2, \mu^2_{\mathrm{thr}} = 4m_c^2, \mu_0^2 = 0) = \frac{\sqrt{\mu^2_{\mathrm{thr}} - \mu^2} - \sqrt{\mu^2_{\mathrm{thr}} - \mu_0^2}}{\sqrt{\mu^2_{\mathrm{thr}} - \mu^2} + \sqrt{\mu^2_{\mathrm{thr}} - \mu_0^2}}$$

- Allows one to reproduce $F^{(7)}_{2,(a)}(s)$ from Asatrian, Greub, Virto 2019 very accurately







Dispersive representation for $F^{(7)}_{2,(a)}$: cross-checks

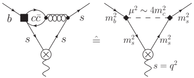
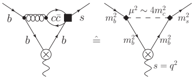
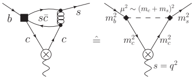
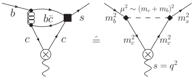
- Comparison between dispersion relation and exact result from Asatrian, Greub, Virto 2019

$s\,/\,m_b^2$	Disp. Rel. for $F^{(7)}_{2(a)}(s)$	Exact $F^{(7)}_{2(a)}(s)$ from AGV 2019
$-3+\mathrm{i}$	$0.1033827 + 0.02230829\mathrm{i}$	$0.103381 + 0.022309\mathrm{i}$
$-1-2\mathrm{i}$	$0.13457343 - 0.06182213\mathrm{i}$	$0.134573 - 0.061823\mathrm{i}$
-4.2	$0.08568058 + 0.00202792\mathrm{i}$	$0.0856783 + 0.00202803\mathrm{i}$
$17+\mathrm{i}\epsilon$	$0.02918665 + 0.25775442\mathrm{i}$	$0.0292152 + 0.257753\mathrm{i}$
$17-\mathrm{i}\epsilon$	$0.02918665 - 0.25884806\mathrm{i}$	$0.0292152 - 0.258846\mathrm{i}$

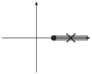
- Accurate up to several digits

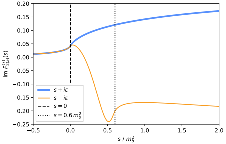
  $\hookrightarrow$ even far away from fit range and in complex plane

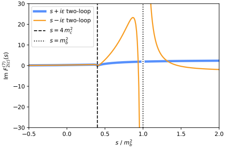
Analytic structure of the two-loop diagrams

(a)	(b)	(c)	(d)
			
$s_{\text{thr}} = 0$	$4m_b^2$	$4m_c^2$	$4m_c^2$
$\mu^2_{\text{thr}} = 4m_c^2$	$4m_c^2$	$(m_c + m_s)^2$	$(m_c + m_b)^2$
$s_+ \sim 0.6\, m_b^2$		$\sim m_b^2$	

- Also considered $(b) - (d)$
  - $\hookrightarrow$ (c) and (d) topologically more complicated
  - $\hookrightarrow$ no triangle topology in (e) $\rightarrow$ analytically simpler
- **Anomalous thresholds** play a role in (a) and (c)!
  - $\hookrightarrow$ still fulfill "unmodified" dispersion relations ("case 1")
  - $\hookrightarrow$ however, requires careful numerical treatment







Summary and Outlook

- **Analytic structure** of non-local FFs richer than previously thought

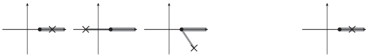
  $\hookrightarrow$ **anomalous thresholds**!

  $\hookrightarrow$ essential for detailed understanding both on **hadronic** and **quark level**

- Anomalous thresholds are **kinematic singularities**

  $\hookrightarrow$ non-trivially depend on all masses involved; position in complex plane different:

hadronic level	**quark level**



  $\hookrightarrow$ not surprising since already normal thresholds differ, e.g., $4m_c^2 \leftrightarrow 4M_D^2$

  $\hookrightarrow$ supports strategy to match at large space-like q^2

Summary and Outlook

- **Analytic structure** of non-local FFs richer than previously thought

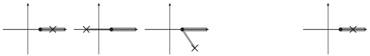
  ↪ **anomalous thresholds**!

  ↪ essential for detailed understanding both on **hadronic** and **quark level**

- Anomalous thresholds are **kinematic singularities**

  ↪ non-trivially depend on all masses involved; position in complex plane different:

  hadronic level   **quark level**

  ↪ not surprising since already normal thresholds differ, e.g., $4m_c^2 \leftrightarrow 4M_D^2$

  ↪ supports strategy to match at large space-like q^2

- **Open questions**:
  - Can we quantify anomalous contributions to charm loop?
  - How to bring hadronic and quark picture together?
  - What are the implications for non-local FF predictions?