Probing the *B* meson DA

with $B \to \ell\ell\ell'\nu$

Aoife Bharucha, CPT Marseille

Based on "Probing the structure of the B meson with $B \to \ell\ell\ell'\nu$",
by AB, Bharti Kindra and Namit Mahajan, arXiv:2102.03193 [hep-ph]

and "On Soft Contributions to the $B \to \gamma^*$ Form Factors",
by AB, Danny van Dyk and Eduardo Velasquez Alvarez, arXiv:2511.XXXXX [hep-ph]

Taming hadronic uncertainties in and beyond the Standard Model
IJCLab, Oct 22 - 24, 2025

What do we know about the B meson's structure ?

Let's start with λ_B!

- Leading moment λ_B of the B-meson distribution amplitude ($\phi_B^+(k,\mu)$), $\lambda_B^{-1}(\mu) = \int_0^\infty \frac{dk}{k}\phi_B^+(k,\mu)$, not well known
- Theory uncertainty on λ_B large, from 200 MeV obtained using non-leptonic decays [Beneke++'03,09] to 460 ± 110 MeV obtained from QCD sum rules [Braun++'03].

$B^+ \to \ell^+\nu\gamma$ probes λ_B for $E_\gamma \gg \Lambda_{\mathrm{QCD}}$ [Beneke'00,Grozin'96,Bosch'03]

Most stringent limits from BaBar collaboration (2009)

$$\mathcal{BR}(B^+ \to \ell^+\nu\gamma) < 15.6 \times 10^{-6} @90\%\,\mathrm{C.L.} \Rightarrow \lambda_B > 300\,\mathrm{MeV}$$

and from Belle (2015):

$$\mathcal{BR}(B^+ \to e^+/\mu^+\nu\gamma) < 4.3 \times 10^{-6}/3.4 \times 10^{-6} @90\%\,\mathrm{C.L.} \Rightarrow \lambda_B > 238\,\mathrm{MeV}$$

Updating this result is a priority at Belle II, and projections look very promising [Gelb++'18,Kou++'18].

What about $B \to \ell\ell\ell'\nu$?

- Measurement of $B \to \gamma\ell\nu$ @LHCb challenging[1]
- However, if we replace the γ by $\gamma^* \to \ell^+\ell^-$, analysis feasible
- LHCb limit for $m_{\mathrm{light}}(\mu^+\mu^-) < 980$ MeV (2018),

$$\mathcal{BR}(B^+ \to \mu^+\mu^-\mu^+\nu) < 1.6 \times 10^{-8}.$$

- Theory status in 2019: Vector meson dominance approach [Danilina++'18,'19], $\mathcal{BR} \sim 3 \times 10^{-7}$, no uncertainties.

Important to provide predictions for $B \to \ell\ell\ell'\nu$ to obtain complementary constraints on λ_B . Important cross-check for the $B \to \gamma\ell\nu$ results from Belle II.

[1] May be possible if photon converted to e^+e^- in VELO [Borsato++'24]

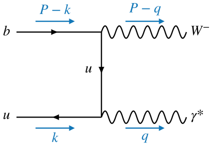
Theory status since 2019

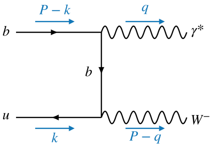
Flurry of papers in 2021

- Three papers working in the QCD factorization framework [Bharucha+'21], [Beneke++'21], [Wang++'21]
- Two papers discussing the dispersive treatment, vector meson dominance [Ivanov++'21,22]
- Slightly later, paper on form factor basis for dispersive treatment [Kürten++'22]

In this talk I will focus on QCD factorization and focus on work in progress

Amplitude for $B^+(p_B) \to \ell^+(q_1)\ell^-(q_2)\ell'^+(p_1)\nu(p_2)$





(a): photon emission from u quark, (b): photon emission from b quark

$$iA = \frac{G_F V_{ub}}{\sqrt{2}} \frac{ie^2}{q^2} (\bar{u}_\ell \gamma^\mu v_\ell)[(\bar{u}_\nu \gamma^\rho P_L v_{\ell'}) T_{\mu\rho} - i(\bar{u}_\nu \gamma_\mu P_L v_{\ell'}) f_B]$$

- 1st term describes emission of γ^* from B meson, 2nd term from the lepton
- At L.O. 2nd term can trivially be written in terms of f_B, 1st term more complicated

On imposing EM current conservation and applying the equation of motion in limit of massless leptons obtain Beneke basis:

$$T_{\mu\rho} = iF_A(g_{\mu\rho}(v \cdot q) - v_\mu q_\rho) + F_V \epsilon_{\rho\mu\lambda\sigma} v^\lambda q^\sigma - iF_\parallel v_\mu q_\nu + if_B g_{\mu\rho},$$

where $p_B = m_B v$ and $q = q_1 + q_2$

Bases for form factors à la KZKvD [Kürten++'22]

Split matrix element so $T_{\rm FSR}$ restores gauge invariance:

$$\left\langle \ell^- \bar{\nu}_\ell \gamma^* \middle| J_{\rm L}^\nu(0) J_{{\rm H}\nu}(0) \middle| B^- \right\rangle = eQ_{B^-} \varepsilon^*_\mu \left[\underbrace{T_{\rm H}^{\mu\nu}(k,q)}_{\text{hadronic tensor}} + \overbrace{T_{\rm FSR}^{\mu\nu}(p_\ell, p_\nu)}^{\text{FSR tensor}} \right] \underbrace{L_\nu}_{\langle \ell^- \bar{\nu} | J_L^\nu | 0\rangle}$$

$$\underbrace{T_{\rm H}^{\mu\nu}}_{T_{\rm hom.} + T_{\rm inhom.}} = \int dx\; e^{iq\cdot x} \left\langle 0 \middle| T\{J_{\rm em}^\mu(x) J_{\rm H}^\nu(0)\} \middle| B^-(q+k) \right\rangle$$

$T_{\rm inhom.}$ defined so $T_{\rm inhom.} + T_{\rm FSR}$ and $T_{\rm hom.}$ satisfy Ward identity independently and $T_{\rm hom.}$ divergence-free

$$T_{\rm hom.}^{\mu\nu}(k,q) = \frac{1}{m_B}\left[(k\cdot q) g^{\mu\nu} - k^\mu q^\nu\right] F_1\left(k^2, q^2\right) + \frac{1}{m_B}\left[\frac{q^2}{k^2} k^\mu k^\nu - \frac{k\cdot q}{k^2} q^\mu k^\nu + q^\mu q^\nu - q^2 g^{\mu\nu}\right] F_2\left(k^2, q^2\right)$$
$$+ \frac{1}{m_B}\left[\frac{k\cdot q}{k^2} q^\mu k^\nu - \frac{q^2}{k^2} k^\mu k^\nu\right] F_3\left(k^2, q^2\right) + \frac{\mathrm{i}}{m_B} \epsilon^{\mu\nu\rho\sigma} k_\rho q_\sigma F_4\left(k^2, q^2\right)$$

Related to conventional basis:

$$F_V = F_4\,,\; F_{A_\perp} = \frac{(k\cdot q) F_1(k^2,q^2) - q^2 F_2(k^2,q^2)}{(k+q)\cdot q}\,,\; F_{A_\parallel} = q^2\left(\frac{F_1(k^2,q^2) + F_2(k^2,q^2)}{(k+q)\cdot q} + \frac{F_2(k^2,q^2) - F_3(k^2,q^2)}{k^2}\right)$$

For $q^2 \to 0$, $F_{A_\perp} = F_1(k^2, q^2)$ and $F_{A_\parallel} = 0$. F_P not defined in limit $m_\ell \to 0$ and would be related to F_3

Form factors at leading order

in $1/m_b$ and α_s

- In $B \to \gamma \ell \nu$, factorization holds for $E_\gamma \sim \mathcal{O}(m_b)$, in our case require hard-collinear virtual photon coupling to u quark. Coupling to b power suppressed.
- In light-cone coordinates ($l \equiv (l_+, l_-, l_\perp)$) where $l^\mu = \frac{l_+}{2} n_-^\mu + l_\perp + \frac{l_-}{2} n_+^\mu$. Spectator quark is soft: $k = (k_+, k_-, k_\perp) \sim (\lambda, \lambda, \lambda)$, virtual photon hard collinear: $q = (q_+, q_-, q_\perp) \sim (\lambda, 1, \lambda^{1/2})$
- u quark propagator can be expressed as

$$\frac{\not{q} - \not{k}}{(q-k)^2} = \frac{q_- \not{n}_+/2}{q^2 - q_- k_+} - \left(\frac{k_+ \not{n}_-/2}{q^2 - q_- k_+} + \frac{k_- \not{n}_+/2}{q^2 - q_- k_+} + \frac{\not{k}_\perp}{q^2 - q_- k_+} \right).$$

- Form factor at leading power ($F_V = F_A$)[2]:

$$F_{B \to \gamma^*}(q^2, p^2) = Q_u f_B \, m_B \int_0^\infty dk_+ \frac{\phi_B^+(k_+)}{q_- \, k_+ - q^2 - i\epsilon}.$$

[2] G. P. Korchemsky et al, hep-ph/9911427, Descotes-Genon and Sachrajda, hep-ph/0209216, Lunghi et al, hep-ph/0210091 and Bosch et al, hep-ph/0301123.

NLL corrections: $F_X = F_X^{\rm NLL} + \xi_X^{\rm NLL} + \Delta F_X^{(u)} + \Delta F_X^{(b)}$ $(X = V/A)$,

The factor $R(E_\gamma, \mu)$ (Beneke Rohrwild 2011) can be adapted to our case:

$$R(q_-, q^2, \mu, \mu_{h1}, \mu_{h2}) = C(q_-, \mu_{h1}) K^{-1}(\mu_{h2}) U(q_-, \mu_{h1}, \mu_{h2}, \mu) J(q_-, q^2, \mu).$$

Form factor @ LO in $1/m_b$, NLL in α_s expansion

$$F_{V/A}^{\rm NLL} = Q_u f_B\, m_B \int_0^\infty \frac{d\omega\, \phi_B^+(\omega, \mu)}{q_-\, \omega - q^2 - i\varepsilon} R(q_-, q^2, \mu, \mu_{h1}, \mu_{h2}),$$

- $C(q_-, \mu_{h1})$ obtained from matching QCD heavy-to-light current to SCET I current at μ_{h1} (Lunghi 2002,Bosch 2003, Bauer 2000).
- $K(\mu_{h2})$ accounts for conversion from static f_B in the SCET current to the standard definition in QCD, f_B
- $J(q_-, q^2, \mu)$ accounts for the hard-collinear radiative corrections, and was calculated for $B \to \gamma \ell \nu$ in (Lunghi 2002, Bosch 2003). For $B \to \ell\ell\ell'\nu$ where $q^2 \neq 0$ we can adopt the expression given in ALERT(Wang 2016).
- RGE factor $U(q_-, \mu_{h1}, \mu_{h2}, \mu)$ resums logarithms to all orders by solving a renormalization group equation (Bosch:2003fc), depends on three scales: the hard scales μ_{h1} and μ_{h2} are taken to be q_- and m_b respectively, while the hard-collinear scale μ is set to $(m_b \Lambda_{\rm QCD})^{1/2}$.

KZKvD basis

$$F_i(q^2, k^2) = \frac{ef_B}{m_B} \int_0^\infty d\omega \underbrace{T_i^{\text{tw2}}(\omega, n_+ q, n_- q)}_{hard-collinearkernel} \cdot \overbrace{\phi_+(\omega)}^{\text{leading 2pt LCDA}} + \mathcal{O} \frac{\Lambda_{\text{had}}}{\{n_+ q, m_b\}}^2 .$$

Hard scattering kernels for F_{1-4}:

$$T_1^{\text{tw2}}(\omega, n_+ q, n_- q) = \frac{C_V m_B^2 Q_u J}{n_+ q(\omega - n_- q)}, \quad T_2^{\text{tw2}}(\omega, n_+ q, n_- q) = 2\frac{C_V m_B(m_B - n_+ q) Q_u J}{(n_+ q)^2(\omega - n_- q)},$$

$$T_3^{\text{tw2}}(\omega, n_+ q, n_- q) = 0 + \mathcal{O}\alpha_s \frac{\Lambda}{m_b}, \quad T_4^{\text{tw2}}(\omega, n_+ q, n_- q) = \frac{C_V m_B^2 Q_u J}{n_+ q(\omega - n_- q)},$$

NLP corrections

$$\frac{\not{q}-\not{k}}{(q-k)^2} = \frac{q_- \not{n}_+/2}{q^2 - q_- k_+} - \left(\overbrace{\frac{k_+ \not{n}_-/2}{q^2 - q_- k_+}}^{\Delta F_X^{(u)}} + \frac{k_- \not{n}_+/2}{q^2 - q_- k_+} + \overbrace{\frac{\not{k}_\perp}{q^2 - q_- k_+}}^{F_\parallel} \right).$$

$F_\parallel = F_\parallel^{\mathrm{NLP}} + \xi_\parallel^{\mathrm{NLL}}$, Computed within Wandzura-Wilczek approx., for hard collinear q^2 [Beneke++'21]:

$$F_\parallel^{\mathrm{NLP}} = -4 f_B m_B Q_u \frac{q_+}{q_-^2} \int_0^\infty dk_+ \frac{\phi_B^-(k_+)}{q_-\, k_+ - q^2 - i\epsilon}.$$

$F_X = F_X^{\mathrm{NLL}} + \xi_X^{\mathrm{NLL}} + \Delta F_X^{(u)} + \Delta F_X^{(b)}$ $(X = V/A)$

Leading power suppressed term in the u-quark propagator[3], we find

$$\Delta F_V^{(u)} = -\Delta F_A^{(u)} = Q_u\, f_B\, m_B \int_0^\infty dk_+ \phi_B^+ \frac{1}{q_-^2} \left(\frac{q^2}{q_-\, k_+ - q^2} + 1 \right) = \frac{Q_u}{q_-^2} \left(f_B\, m_B + q^2\, F_{B\to\gamma^*} \right)$$

Power-suppressed emission from the b-quark leg also yields symmetry breaking correction

$$\Delta F_V^{(b)} = -\Delta F_A^{(b)} = \frac{Q_b f_B m_B}{q^2 - 2 m_b\, v.q}.$$

Note that in our analysis we adopt : $\phi_B^+(k,\mu) = \frac{k}{\lambda_B^2(\mu)} e^{-k/\lambda_B(\mu)}$

[3] symmetry breaking corrections at $\mathcal{O}(1/m_b, 1/E_\gamma)$ in QCD+soft corrections at twist-3 to 6 (Beneke et al. 2018,Wang 2016,18)

Higher twist corrections in KZKvD basis

Calculate 2-particle higher twist corrections (see also [Wang++'21])

$$T_{\mathrm{H}}^{\mu\nu} = -\frac{e_u f_B m_B}{2\pi^2}\int_0^\infty d\omega \int d^4x \frac{e^{i(q-\omega v)\cdot x}}{x^4} x_\rho \left[-iv^\rho g^{\mu\nu} + \epsilon^{\mu\nu\rho\sigma} v_\sigma + i\left(g^{\mu\rho}v^\nu + v^\mu g^{\rho\nu}\right)\right]$$

$$\left[\phi_+(\omega) + x^2 g_+^{WW}(\omega) - \frac{x^2}{2(v\cdot x)^2}\phi_-^{t3}(\omega) - \frac{x^2}{4}\int_0^\infty d\overline{\omega}\int_0^1 du\, e^{-i\overline{\omega}u(v\cdot x)}\left[\psi_4 - \tilde{\psi}_4\right]^{t4}(\omega,\overline{\omega})\right.$$

$$\left. + \frac{x^2}{2}\int_0^\omega d\eta \int_0^\eta d\rho\, \phi_-^{t3}(\rho)\right]$$

Form factors for $g_+^{WW}(\omega)$:

$$F_1^{g_+^{WW}} = -F_2^{g_+^{WW}} = F_4^{g_+^{WW}} = -\frac{f_B m_B Q_u}{(v\cdot q)^2}\int_0^\infty ds\, \frac{g_+^{WW\prime}(s/2(v\cdot q))}{s-q^2}$$

Form factors for $\phi_-^{t3}(\omega)$:

$$F_1^{\phi_-^{t3}} = -F_2^{\phi_-^{t3}} = F_4^{\phi_-^{t3}} = -\frac{f_B m_B Q_u}{2(v\cdot q)^2}\int_0^\infty ds\, \frac{1}{s-q^2}\int_0^{s/2(v\cdot q)} d\rho\, \phi_-^{t3}(\rho)$$

Form factors for $\left[\psi_4 - \tilde{\psi}_4\right]^{t4}(\omega,\overline{\omega})$:

$$F_1^{[\psi_4-\tilde{\psi}_4]^{t4}} = -F_2^{[\psi_4-\tilde{\psi}_4]^{t4}} = F_4^{[\psi_4-\tilde{\psi}_4]^{t4}} = \frac{f_B m_B Q_u}{4(v\cdot q)^2}\int_0^\infty \frac{ds}{s-q^2}\int_0^\infty \frac{d\bar{s}}{s+\bar{s}-q^2}\left[\psi_4 - \tilde{\psi}_4\right]^{t4}$$

Can all be written in dispersive form

LCSR Set-up [Braun++'03,Beneke++'18,Wang'16,'18]

$$F_i(k^2,q^2) = \frac{f_\rho \mathcal{F}_i^{B\to\rho}(k^2)}{m_\rho^2 - q^2} + \frac{1}{\pi}\int_{s_0}^{\infty} ds \, \frac{\overbrace{\mathcal{I}m\left\{F_i(k^2,s)\right\}}^{\mathcal{I}m\left\{F_i^{\text{QCDF}}(k^2,s)\right\}\text{ for } s>s0}}{s-q^2}$$

Isolate ρ contribution and perform Borel transform:

$$f_\rho \mathcal{F}_i^{B\to\rho}(k^2) = \frac{1}{\pi}\int_0^{s_0} ds\, e^{-(s-m_\rho^2)/M^2} \mathcal{I}m\left\{F_i^{\text{QCDF}}(k^2,s)\right\}$$

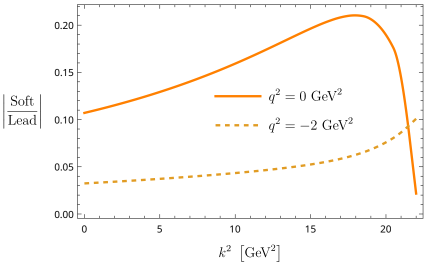
This gives the QCDF+soft contribution to the form factor

$$F_i(k^2,q^2) = \int_0^\infty \frac{ds}{\pi}\frac{\mathcal{I}m\left\{F_i^{\text{QCDF}}(k^2,s)\right\}}{s-q^2} + \int_0^{s_0}\frac{ds}{\pi}\mathcal{I}m\left\{F_i^{\text{QCDF}}(k^2,s)\right\}\left[\frac{e^{-(s-m_\rho^2)/M^2}}{m_\rho^2-q^2} - \frac{1}{s-q^2}\right]$$

In KZKvD basis this gives

$$F_i(n_+q,n_-q) = \frac{ef_B}{m_B(n_+q)}\int_0^\infty ds\, T_i^{\text{tw2}}\left(\frac{s}{n_+q}, n_+q, n_-q\right)\phi_+\left(\frac{s}{n_+q}\right)$$
$$+ \frac{ef_B}{m_B(n_+q)}\int_0^{s_0} ds\, T_i^{\text{tw2}}\left(\frac{s}{n_+q}, n_+q, n_-q\right)\phi_+\left(\frac{s}{n_+q}\right)$$
$$\left[\frac{s-(n_+q)(n_-q)}{m_\rho^2-(n_+q)(n_-q)}e^{-(s-m_\rho^2)/M^2} - 1\right]$$

Numerical Results



- soft corrections **under control** over all phase space, $< 5\%$
- dispersion relation allows **extrapolation** from negative q^2 to physical positive q^2

where $q^2 = (n_+ q)(n_- q)$ and $k^2 = m_B(m_B - n_+ q)$

Soft@NLL: $F_X = F_X^{\rm NLL} + \xi_X^{\rm NLL} + \Delta F_X^{(u)} + \Delta F_X^{(b)}$ $(X = V/A)$, $F_\parallel = F_\parallel^{\rm NLP} + \xi_\parallel^{\rm NLL}$

Extend $\xi_{\rm soft}^{\rm Eucl.}$ to $q^2 > 0$ via once subtracted dispersion relation (following Guadagnoli et al. 2017, Kozachuk et al. 2017),

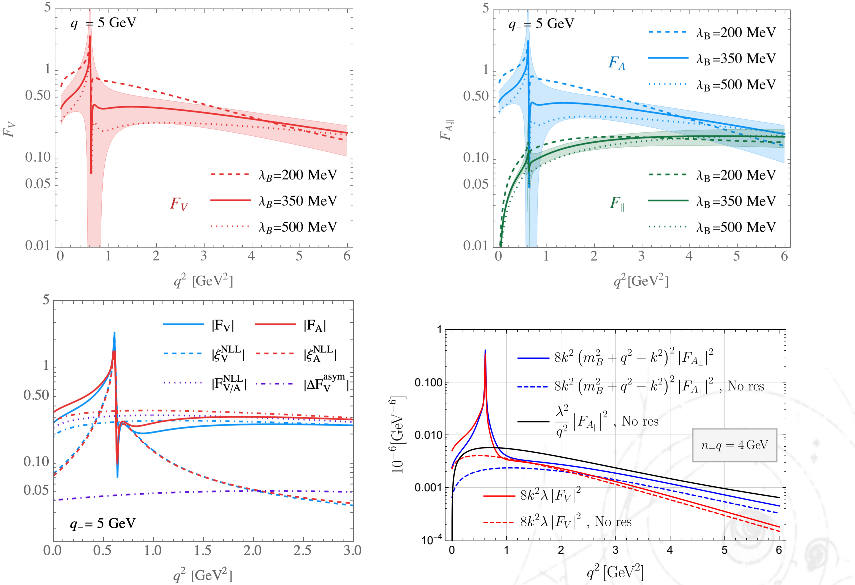
$$\xi_X^{\rm NLL}(q^2) = \xi_{X,\,{\rm soft}}^{\rm Eucl.}(q_0^2) + \sum_R \frac{q^2 - q_0^2}{m_R^2 - q_0^2} \frac{c_R f_R m_R e^{i\delta_R}}{m_R^2 - q^2 - i m_R \Gamma_R} F_X^{B\to R}$$

where $X = V, A$ and $\parallel$, $\xi_{V/A,\,{\rm soft}}^{\rm Eucl.}(q_0^2) = \xi_{\rm soft}^{\rm Eucl.}(q_0^2)$ and $\xi_{\parallel,\,{\rm soft}}^{\rm Eucl.}(0) = 0$ q_0^2 is the subtraction point, $R = \rho, \omega$, $c_\rho = \frac{1}{2}$, $c_\omega = \frac{1}{6}$ and

$$F_V^{B\to R} = \frac{2m_B}{m_B + m_R} V^{B\to R}(p^2), \quad F_A^{B\to R} = -\frac{m_B + m_R}{v.q} A_1^{B\to R}(p^2), \quad F_\parallel^{B\to R} = -\frac{q^2}{v.q}\frac{m_B + m_R}{m_B^2 + p^2}$$

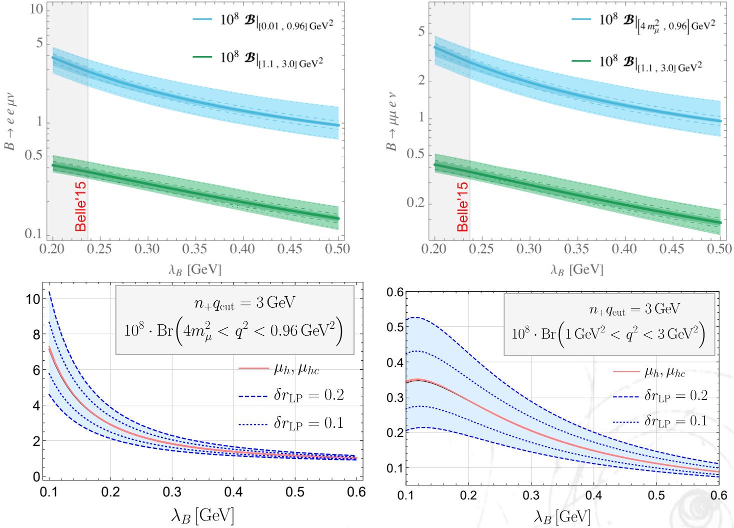
Other options: Breit Wigner [Beneke++'21] and no soft corrections [Wang++'21]

Results for the form factors



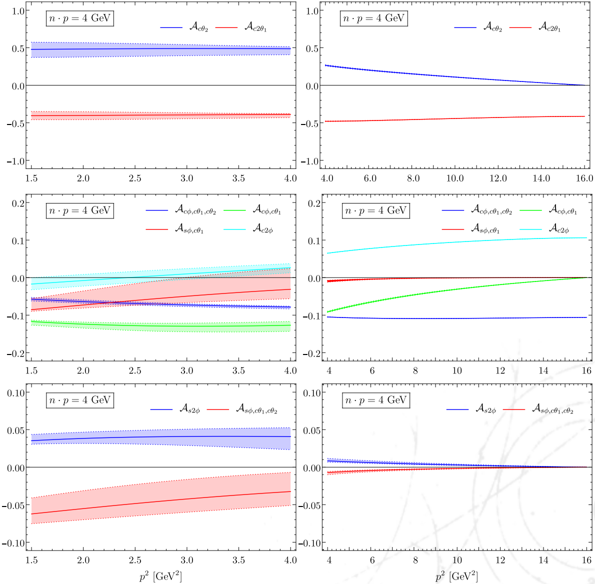
Compare to $\mathcal{R}eF_V$ from [Wang++'22]/BR contr. normalized FFs from [Beneke++'21]:

Results for the Branching Ratio



Small-q^2 bin has similar sensitivity as $B \to \gamma\ell\nu$, but larger uncertainty due to resonance contribution. Higher q^2 bin retains some sensitivity, gradually decreasing.

Asymmetries [Wang++'21]



Thoughts

- Three papers on QCDF treatment for $B \to \ell\ell\ell'\nu$ in 2021. Twist two/LP at NLL ($\propto \lambda_B$) and higher twist/NLP at LO. Studied branching ratios and asymmetries. Different approach to soft corrections.
- Dispersive approach/LCSR require full form factor basis/remove singularities. New basis advocated/recently twist 2 and 3 2-particle form factors calculated.

Problem

- Initially LHCb has measured $B \to 3\mu\nu$), but this has two potential issues:

1. Ambiguity in like-sign muons means different leptons easier to interpret theoretical [4]
2. In ideal kinematic region where largest rate and dependence on λ_B, resonances result in large uncertainties

[4] It turns out that Beneke et al. find for small $q^2_{\rm low}$, cut on $n_- q_{\rm low} > 3$ GeV suffices to resolve this, as cases falling outside this cut (i.e. where true q^2 is $q^2_{\rm high}$, $n_- q$ small) is phase-space suppressed by two powers of $1/m_b$

Thoughts

- Three papers on QCDF treatment for $B \to \ell\ell\ell'\nu$ in 2021. Twist two/LP at NLL ($\propto \lambda_B$) and higher twist/NLP at LO. Studied branching ratios and asymmetries. Different approach to soft corrections.
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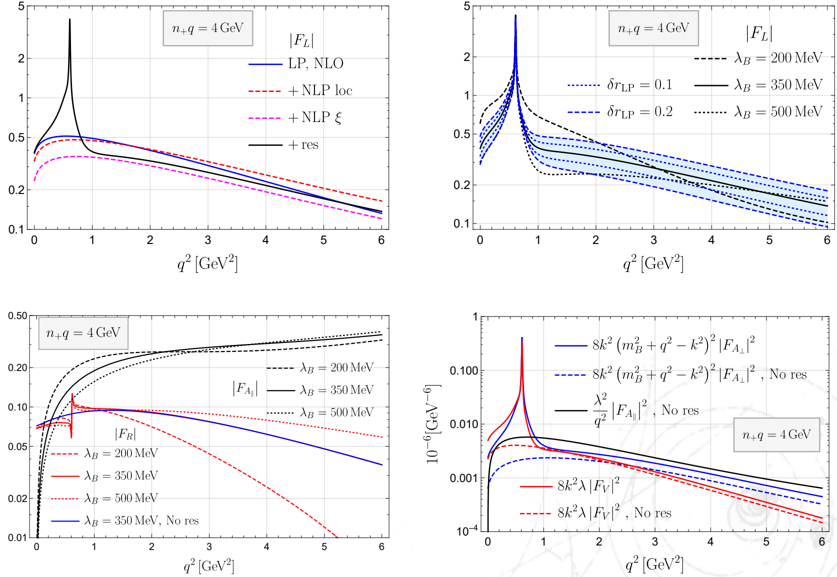
Possible solutions

- Measurement of $B \to \ell\ell\ell'\nu$ with $\ell \neq \ell'$ (expect LHCb yield to diminish by a factor 3/4)
- Exclude resonances by choosing bin with $q^2 >$1 GeV^2. (But here there is less dependence on λ_B), or $q^2 = [0, 0.5]$ GeV^2
- Improve understanding of resonance region by experiment/double fit of resonances (including the associated phases) and λ_B? Angular analysis?

Parameters and Uncertainties

Parameter	Value	Ref.
m_B	5.28 GeV	PDG
f_B	192.0 ± 4.3 MeV	PDG
$\lvert V_{ub}\rvert^{\text{excl}}$	$(3.70 \pm 0.16) \times 10^{-3}$	PDG
G_F	1.166×10^{-5} GeV^{-2}	PDG
m_μ	0.105 GeV	PDG
τ_B	$(1.641 \pm 0.008) \times 10^{-12}$ s	PDG
m_ρ	0.775 GeV	PDG
m_e	0.511×10^{-3} GeV	PDG
α_{em}	$1/137$	PDG
λ_B	$[200 - 500]$ MeV	Beneke et al. 2018
s_0	1.5 ± 0.1 GeV^2	Beneke et al. 2018
M^2	1.25 ± 0.25 GeV^2	Beneke et al. 2018

Comparison with arXiv:2102.10060 [hep–ph]



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With $F^{\rm QCD}(E,s) = F_{V/A}^{\rm NLL}$, and $\omega' = s/q_-$ we find:

$$\xi_{\rm soft}^{\rm Eucl.} = Q_u f_B\, m_B\, C(q_-,\mu_{h1}) K^{-1}(\mu_{h2}) U(q_-,\mu_{h1},\mu_{h2},\mu) \int_0^{s_0/q_-} {\rm d}\omega' \left(\frac{e^{-(q_-\,\omega' - m_\rho^2)/M^2}}{m_\rho^2 - q^2 - i\varepsilon} - \frac{1}{q_-\,\omega' - q^2 - i\varepsilon} \right) \phi_B^{+,\rm eff}(\omega',\mu)$$

where $\phi^+_{B,\rm eff}(\omega,\mu) = \phi^+_B(\omega,\mu) + \frac{\alpha_s(\mu) C_F}{4\pi}\, \delta\phi^+_{B,\rm eff}(\omega',\mu)$ (Beneke et al. 2018,Wang 2016):

$$\delta\phi^+_{B,\rm eff} = \int_0^{\omega'} d\omega \left(\frac{2}{\omega-\omega'} \ln \frac{\mu^2}{q_-(\omega'-\omega)} \right)_\oplus \phi^+_B(\omega,\mu) - \omega' \int_0^{\omega'} d\omega \left(\frac{1}{\omega-\omega'} \ln \frac{\omega'-\omega}{\omega'} \right)_\oplus \frac{\phi^+_B(\omega,\mu)}{\omega} + \frac{\omega'}{2} \int_0^{\infty} d\omega \ln^2 \left| \frac{\omega-\omega'}{\omega'} \right| \frac{d}{d\omega} \frac{\phi^+_B(\omega,\mu)}{\omega} - \int_{\omega'}^{\infty} d\omega \left(\ln^2 \frac{\mu^2}{q_-\omega'} - \frac{\pi^2}{2} - 1 \right) \frac{d}{d\omega} \phi^+_B(\omega,\mu) + \omega' \int_{\omega'}^{\infty} d\omega \left(\ln \frac{\mu^2}{q_-\omega'} \ln \frac{\omega-\omega'}{\omega'} - \frac{1}{2} \ln^2 \frac{\mu^2}{q_-(\omega-\omega')} + \frac{1}{2} \ln^2 \frac{\mu^2}{q_-\omega'} + 3 \ln \frac{\omega-\omega'}{\omega'} - \frac{2\pi^2}{3} \right) \frac{d}{d\omega} \frac{\phi^+_B(\omega,\mu)}{\omega}$$