Double Lomb-Scargle Periodogram: Super-Resolution

Jeff Scargle

Astroparticle Symposium 2025

Variability of astrophysical sources

accross wavelengths,

a tribute to Berrie Giebels

THE USA X-RAY TIMING EXPERIMENT

P. S. Ray[1], K.S. Wood[1], G. Fritz[1], P. Hertz[1], M. Kowalski[1], W.N. Johnson[1], M.N. Lovellette[1], M.T. Wolff[1], D. Yentis[1], R. M. Bandyopadhyay[2], E.D. Bloom[3], B. Giebels[3], G.Godfrey[3], K. Reilly[3,4], P. Saz Parkinson[3,4], G. Shabad[3,4], P. Michelson[4], M. Roberts[4], D.A. Leahy[5], L. Cominsky[6], J. Scargle[7], J. Beall[8], D. Chakrabarty[9], Y. Kim[10]

USA AND *RXTE* OBSERVATIONS OF A VARIABLE LOW-FREQUENCY QPO IN XTE J1118+480

K. S. Wood[1], P. S. Ray[2], R. M. Bandyopadhyay[3], M. T. Wolff,
G. Fritz, P. Hertz[4], M. P. Kowalski, M. N. Lovellette, D. Yentis
E. O. Hulburt Center for Space Research, Naval Research Laboratory, Washington, DC 20375
E. Bloom, B. Giebels, G. Godfrey, K. Reilly, P. Saz Parkinson, G. Shabad
Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309
AND
J. Scargle
NASA/Ames Research Center, Moffett Field, CA 94035

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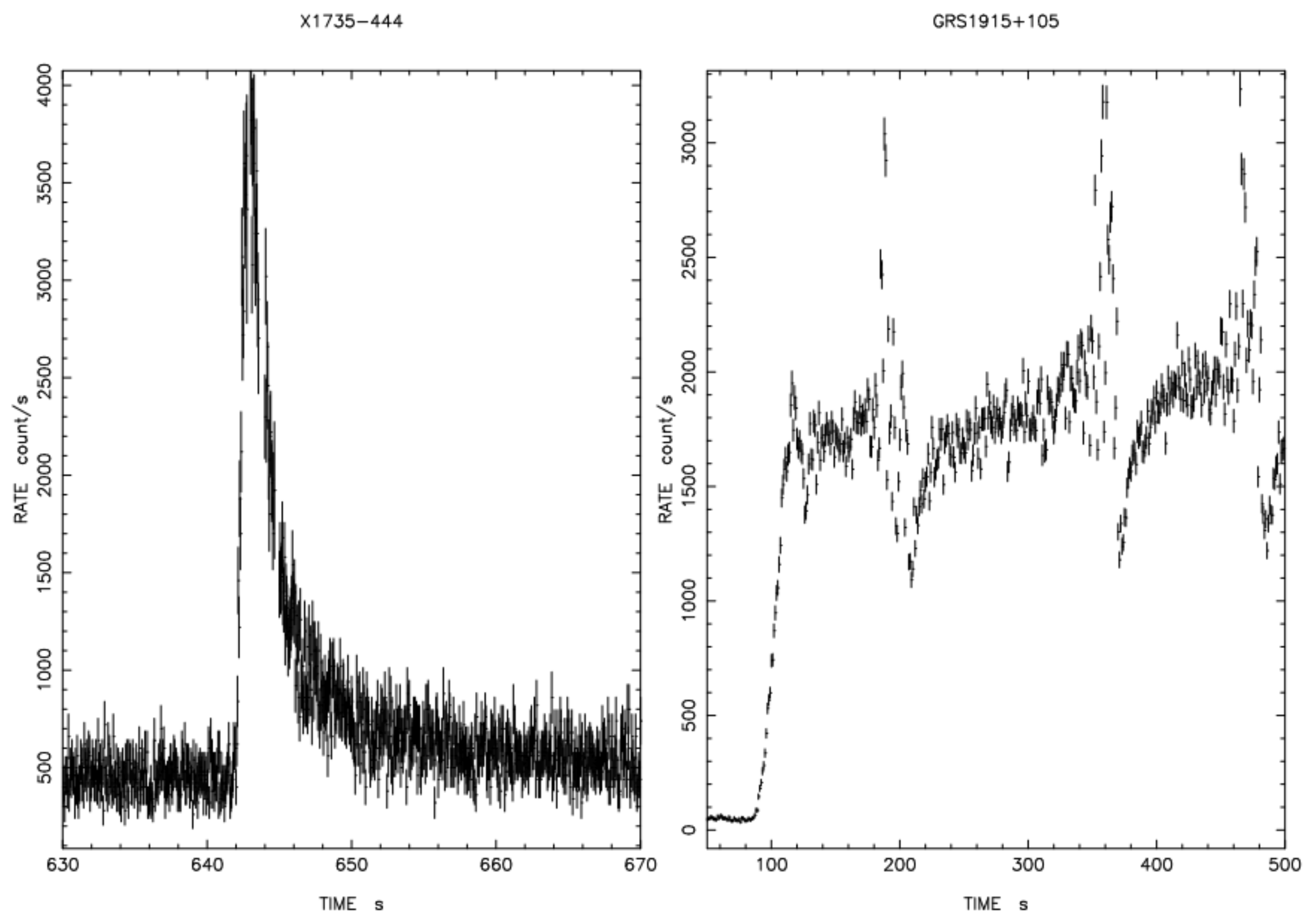
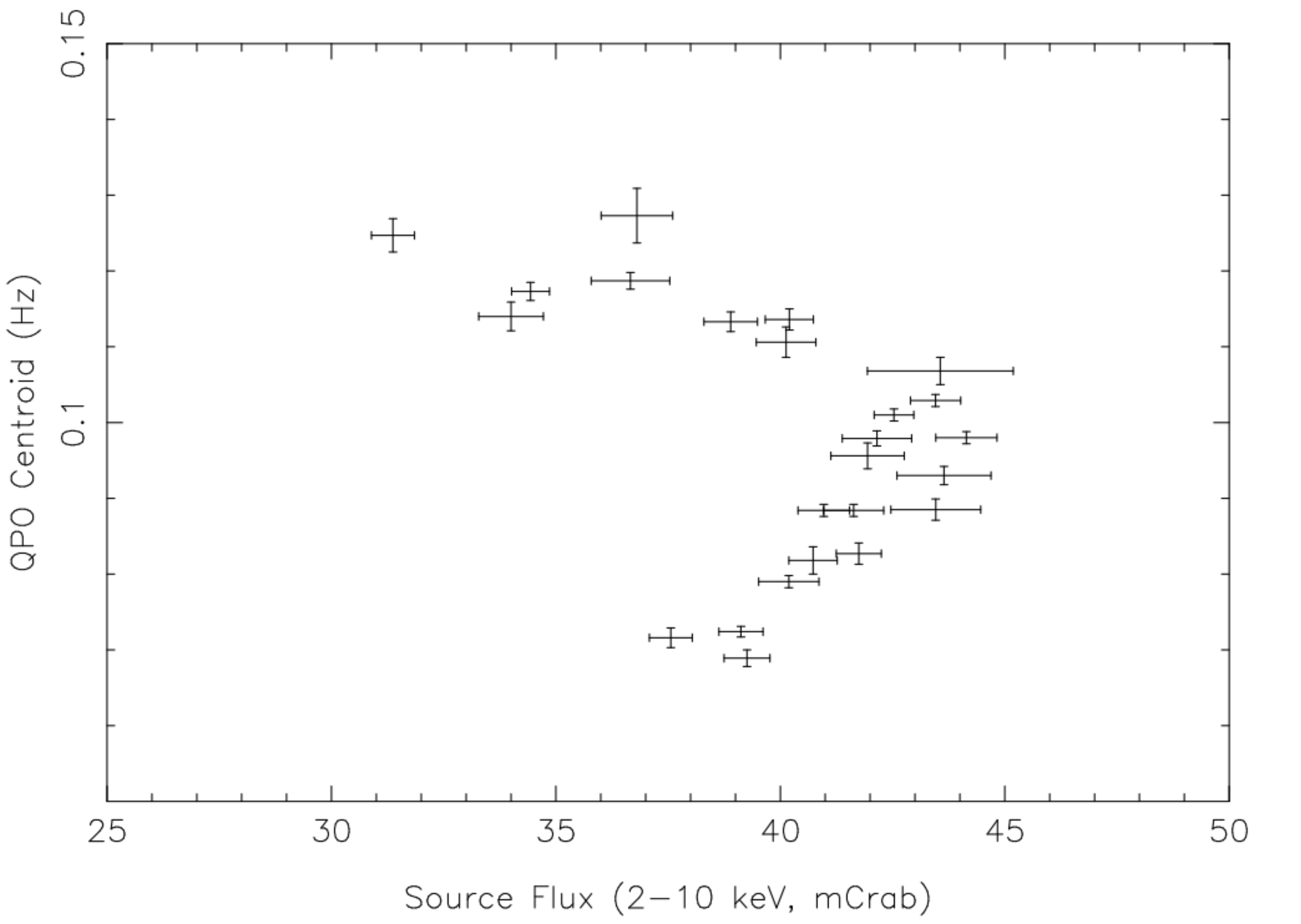


FIGURE 1. Left: A burst from X1735-444. Right: An observation of GRS1915+105



PERIODICITY In Astronomical Systems:
Rotation … Oscillation … Revolution … Waves

The Hunt for Periodicity In Time Series Data:
The PERIODOGRAM (Schuster, 1898) — power spectrum estimate to “Find hidden periodocities”

Problems:

NOISE: does not get smaller as N increases
produces MANY confusing peaks in the periodogram
colored noise background thwarts significance analysis

TRIALS FACTORS — subjective, difficult to estimate

SPECTRAL LEAKAGE — systematic corruption of power spectra

UNEVEN SAMPLING

$$CC = \sum w_n cos^2(\omega t_n)$$
$$SS = \sum w_n sin^2(\omega t_n)$$
$$XC = \sum w_n x_n cos(\omega t_n)$$
$$XS = \sum w_n x_n sin(\omega t_n)$$
$$CS = \sum w_n cos(\omega t_n)\ sin(\omega t_n)$$

$$y_n = a\ cos(\omega t_n) + b\ sin(\omega_1 t_n)$$

$$\log L = -\frac{1}{2}\sum_{n=1}^{N}(\frac{x_n - y_n}{\sigma_n})^2\ ,$$

$$R = \ a^2\ CC + b^2\ SS + 2(ab\ CS - a\ XC - b\ XS)$$

The Lomb Trick:
Trade CS term for phase term

$$CC = \sum w_n cos^2(\omega t_n - \theta)$$
$$SS = \sum w_n sin^2(\omega t_n - \theta)$$
$$XC = \sum w_n x_n cos(\omega t_n - \theta)$$
$$XS = \sum w_n x_n sin(\omega t_n - \theta)$$
$$CS = \sum w_n cos(\omega t_n - \theta) sin(\omega t_n - \theta) = 0$$

Least Squares ≈ Maximum Likelihood

Schuster-like Maximum Likelihood Periodogram

```
P = XC^2 / CC + XS^2 / SS

a = ( SS*XC - CS*XS ) / (CC*SS - CS^2)
b = ( CC*XS - CS*XC ) / (CC*SS - CS^2)
```

$$P_{\text{Schuster}}(\omega) = |\sum x_n e^{-i\omega t_n}\ |^2$$
$$= (\sum x_n cos\ \omega t_n)^2 + (\sum x_n sin\ \omega t_n)^2$$

```
P(Lomb-Scargle) = XC^2 / CC + XS^2 / SS

a = XC/CC
b = XS/SS
```

Also derived from a Fourier Transform (next).

The Lomb-Scargle Periodogram

* Any time series data — measurements, events (photons), etc.
* Any sampling in time — random, systematic, gaps, jitter …
* Easy application of statistical weights
* Window function
    * different from sinc**2 if sampling not even
    * Does not correct for the sampling; deconvolve the window in post
    * Application of (multi-)tapers tames the spectral leakage
* Derivable from a Fourier transform for arbitrarily sampled data
    * LSP = | FT | **2
    * Complex FT enables cross-correlation functions, cross-spectra, co-spectra
    * Invertible, with high frequencies (large effective Nyquist frequency)

Fourier Transform of Unevenly Spaced Time Series

time tags

weights

amplitudes

frequencies
(arbitrary)

$$FT_X(\omega) = F_0 \left\{ \frac{\sum_n w_n x_n \cos \omega(\ t_n - \tau(\omega)\ )}{[\ \sum_m w_m \cos^2 \omega(\ t_m - \tau(\omega)\ )\ ]^{1/2}} + i\, \frac{\sum_n w_n x_n \sin \omega(\ t_n - \tau(\omega)\ )}{[\ \sum_m w_m \sin^2 \omega(\ t_m - \tau(\omega)\ )]^{1/2}} \right\}$$

where

$$\tau(\omega) = \frac{1}{2\omega} \arctan[\ (\sum w_n \sin 2\omega t_n)\ /\ (\sum w_m \cos 2\omega t_m)\ ]$$

where all sums are over $n = 1, 2, ..., N$.

Fourier Transform of Unevenly Spaced Time Series

time tags

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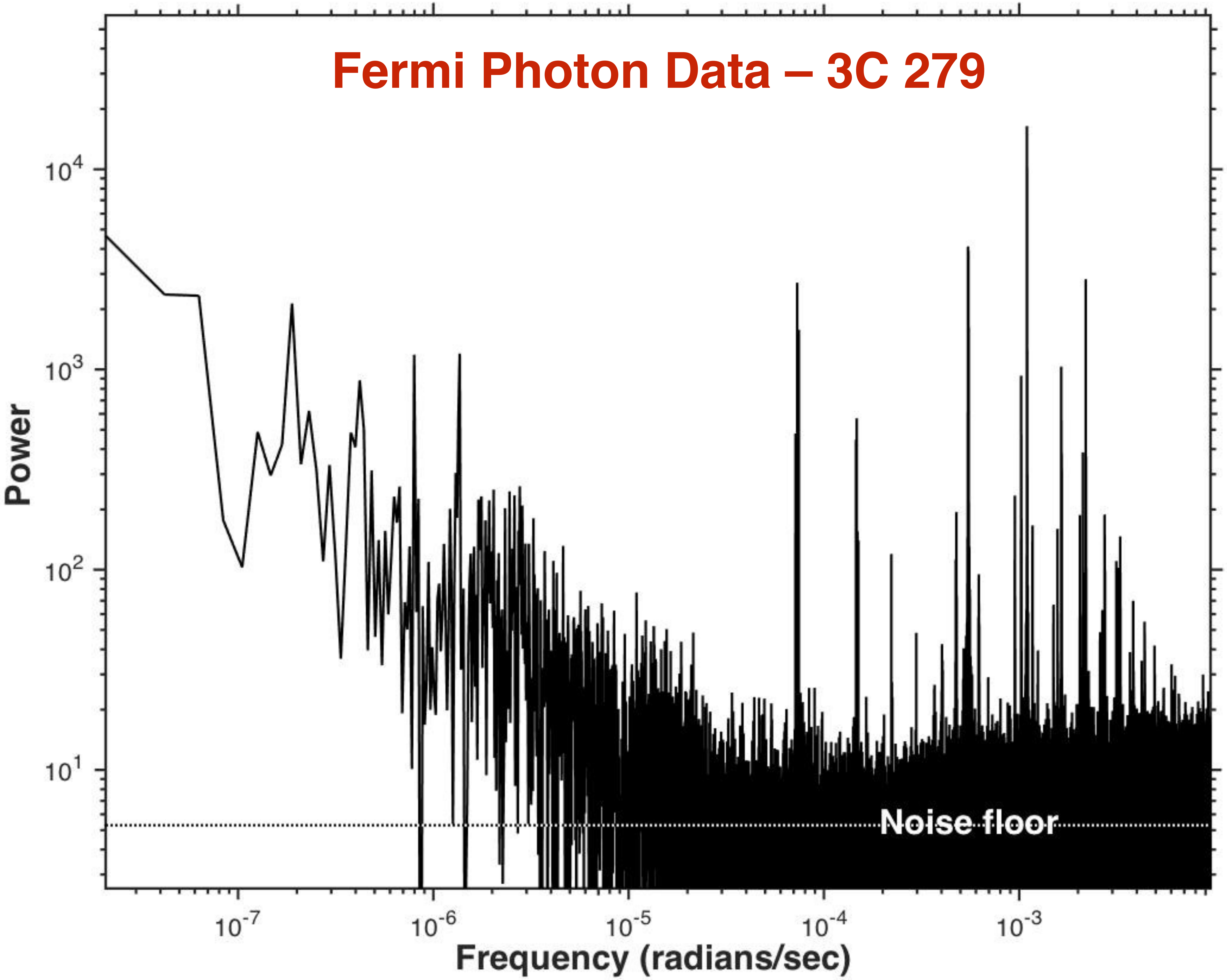
frequencies
(arbitrary)

$$FT_X(\omega) = F_0 \left\{ \frac{\sum_n w_n x_n \cos \omega(\, t_n - \tau(\omega)\,)}{[\,\sum_m w_m \cos^2 \omega(\, t_m - \tau(\omega)\,)\,]^{1/2}} + i\, \frac{\sum_n w_n x_n \sin \omega(\, t_n - \tau(\omega)\,)}{[\,\sum_m w_m \sin^2 \omega(\, t_m - \tau(\omega)\,)]^{1/2}} \right\}$$

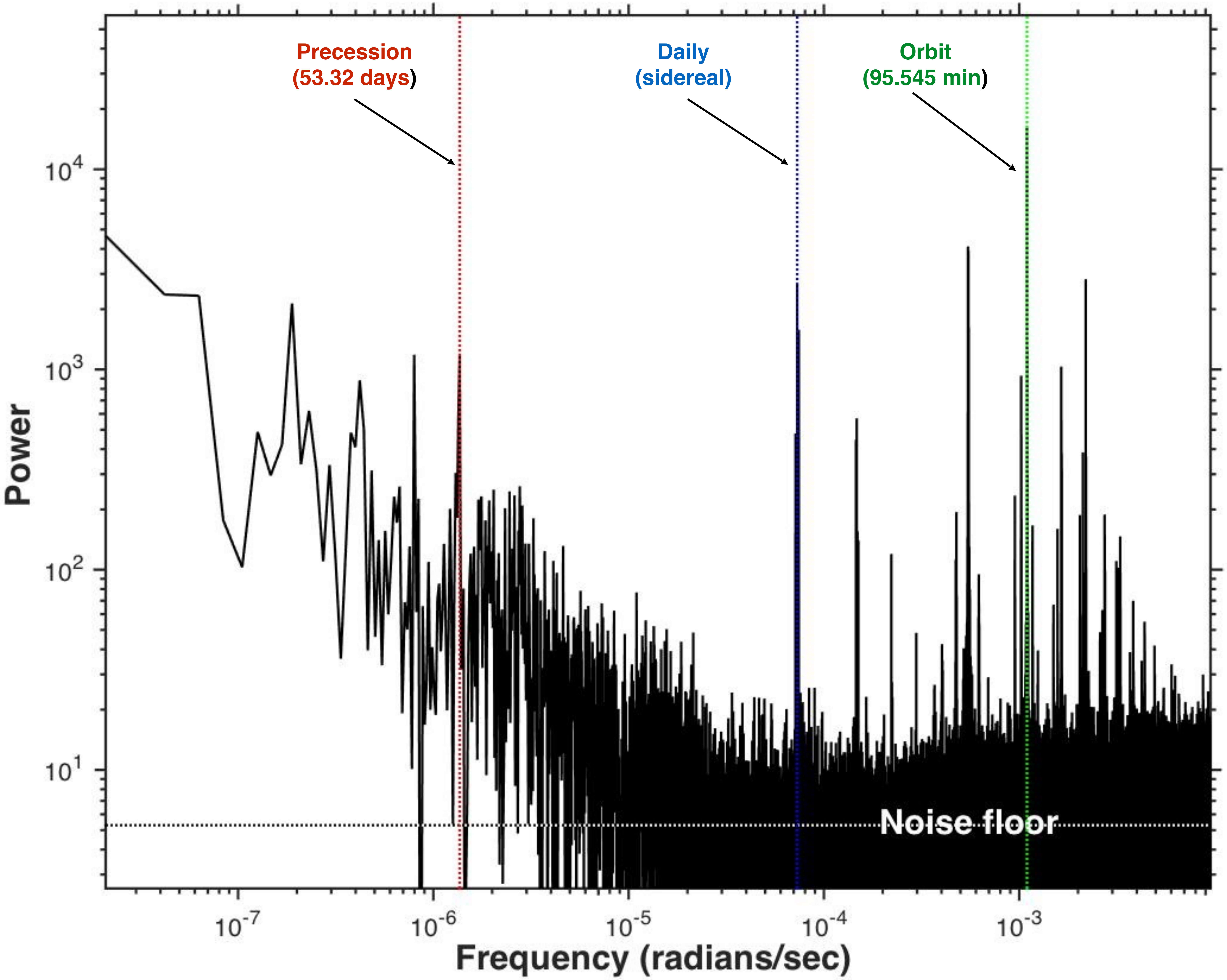
where

$$\tau(\omega) = \frac{1}{2\omega} \arctan[\,(\sum w_n \sin 2\omega t_n) \,/\, (\sum w_m \cos 2\omega t_m)\,]$$

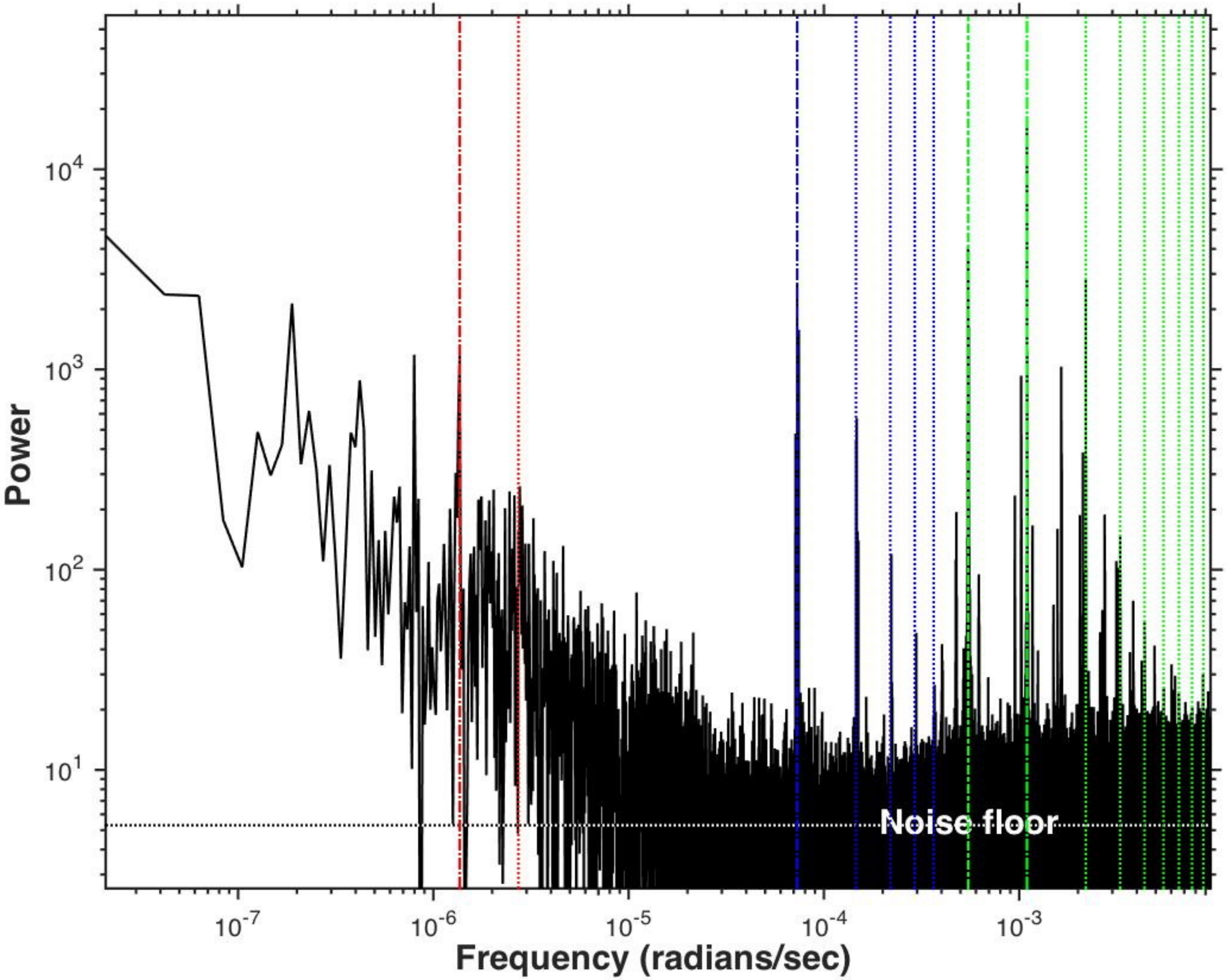
Treat event (photon) data as $\delta(t - t_n)$ ... that is, $x_n = 1$



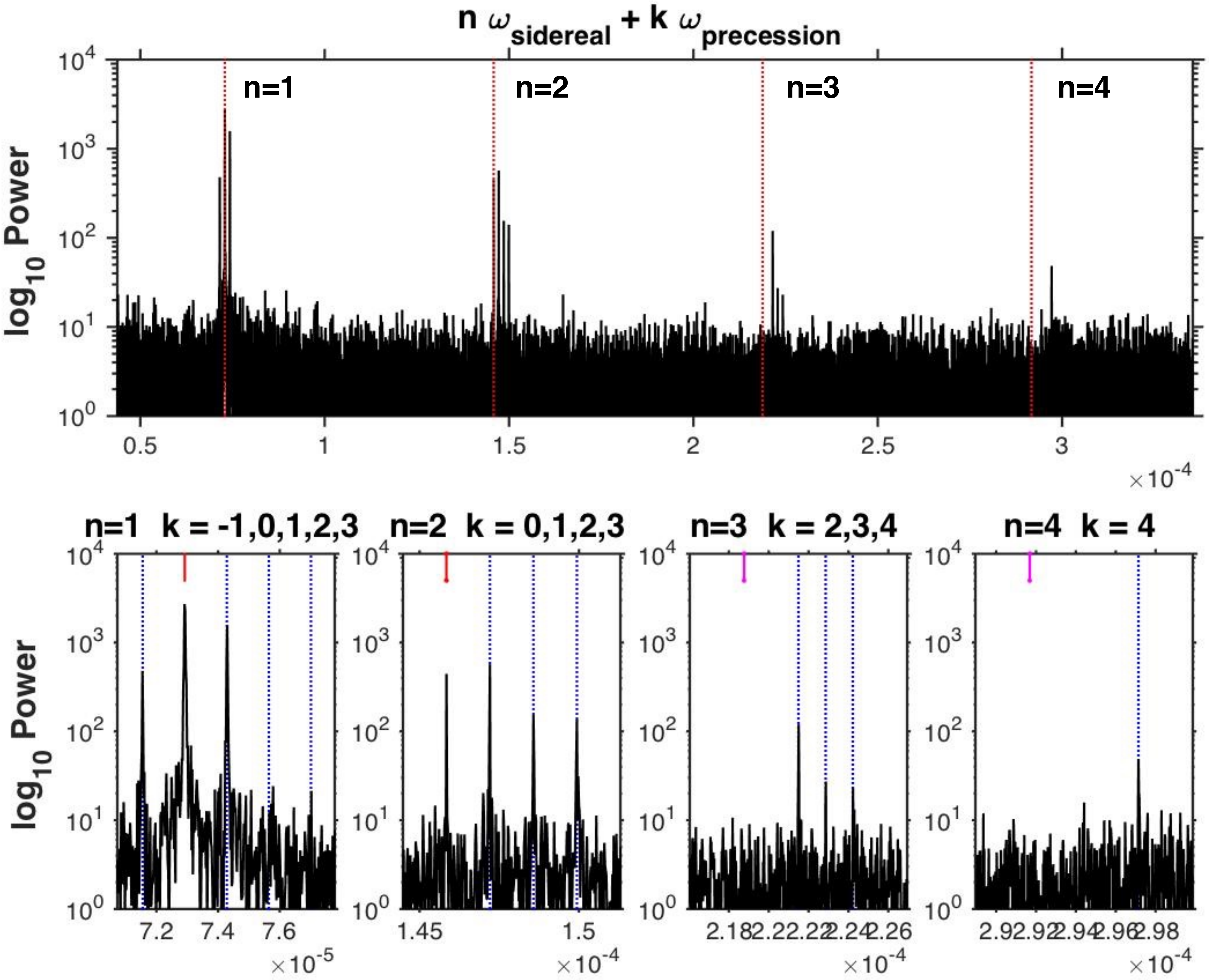
Fermi Photon Data – 3C 279
Power
10^4
10^3
10^2
10^1
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
Frequency (radians/sec)
Noise floor



Precession
(53.32 days)
Daily
(sidereal)
Orbit
(95.545 min)
Noise floor
Power
10^4
10^3
10^2
10^1
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
Frequency (radians/sec)



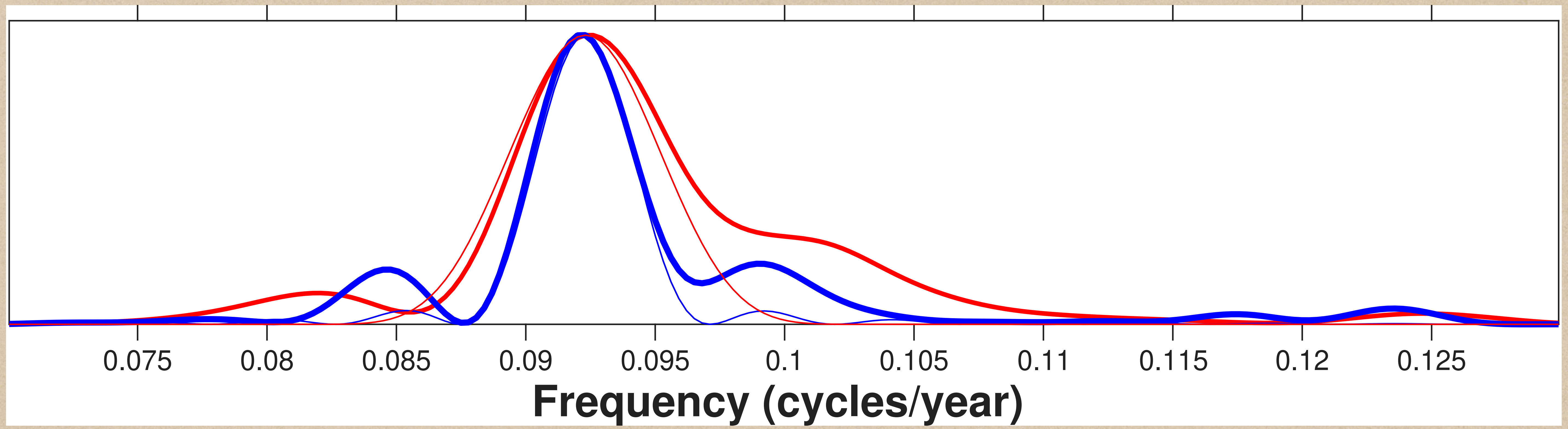
Power
Frequency (radians/sec)
Noise floor
10^{4}
10^{3}
10^{2}
10^{1}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}



n ω sidereal + k ω precession
n=1
n=2
n=3
n=4
log10 Power
×10^-4
n=1 k = -1,0,1,2,3
n=2 k = 0,1,2,3
n=3 k = 2,3,4
n=4 k = 4
×10^-5
×10^-4

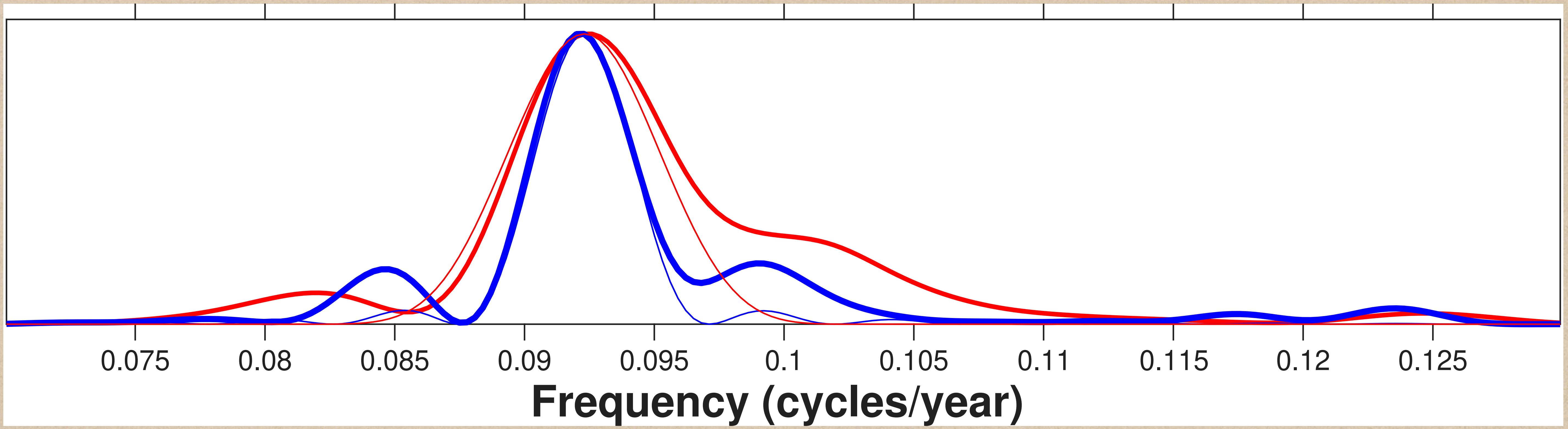
The Great QPO Hunt

LSP of Sunspot Number Time Series



- Frequency Uncertainty ≈ width of peak
- Best Resolution ≈ width of peak

LSP of Sunspot Number Time Series



- Frequency (Ur … ty = width of peak
- Best Res … width of peak

Wrong!

Generalizations of The Lomb-Scargle Periodogram

* Non-sinusoidal periodic signals
* Damped sinusoids (amplitude modulation)
* Additive Constant
* Multivariate
* Tapered, Stacked, etc.
* Basis functions other than sinusoids
* Corresponding Fourier Transform
* Multi-frequency

  Considered by many authors:

  Barning (1963); Bretthorst; Lomb; Baluev;

  Seilmayer, Wondrak and Garcia

  What's new in this talk? Explicit Algorithms …

Criticisms of The Lomb-Scargle Periodogram

* Does not include weights
* Window function complicated
* Spectral Leakage
* Nyquist Frequency not Defined
* Restricted model: not good for non-white noise
* Restricted model: single-frequency sinusoid

Criticisms of The Lomb-Scargle Periodogram

* Does not include weights. Yes it does! (Gilliland and Baliunas, Scargle)
* Window function complicated This was not promised; deconvolve it yourself.
* Spectral Leakage LSP can reduce aliasing and other leakage.
* Nyquist Frequency not Defined. Define your own! (vanderPlas)
* Restricted model: not good for non-white noise. Try it, you'll like it!
* Restricted model: single-frequency sinusoid. The rest of this talk …

$$\log L = -\frac{1}{2}\sum_{n=1}^{N}(\frac{x_n - y_n}{\sigma_n})^2 \ ,$$

$$\begin{aligned} y_n = & a_1\ cos(\omega_1 t_n - \theta) + b_1\ sin(\omega_1 t_n - \theta) \\ & + a_2\ cos(\omega_2 t_n - \theta) + b_2\ sin(\omega_2 t_n - \theta)\ , \end{aligned}$$

Least-Squares

$$\begin{aligned} R = a_1^2\ CC11 + b_1^2\ SS11 + a_2^2\ CC22 + b_2^2\ SS22 \\ - 2(a_1 XC1 + b_1 XS1 + a_2 XC2 + b_2 XS2) \\ + 2(a_1 b_1 CS11 + a_1 a_2 CC12 + a_1 b_2 CS12 \\ + b_1 a_2 SC12 + b_1 b_2 SS12 + a_2 b_2 CS22\ ) \end{aligned}$$

$$\begin{aligned} CC11 &= \sum w_n cos^2(\Omega_1) \\ SS11 &= \sum w_n sin^2(\Omega_1) \\ CC22 &= \sum w_n cos^2(\Omega_2) \\ SS22 &= \sum w_n sin^2(\Omega_2) \\ XC1 &= \sum w_n x_n cos(\Omega_1) \\ XS1 &= \sum w_n x_n sin(\Omega_1) \\ XC2 &= \sum w_n x_n cos(\Omega_2) \\ XS2 &= \sum w_n x_n sin(\Omega_2) \\ CC12 &= \sum w_n cos(\Omega_1) cos(\Omega_2) \\ CS12 &= \sum w_n cos(\Omega_1) sin(\Omega_2) \\ SC12 &= \sum w_n sin(\Omega_1) cos(\Omega_2) \\ SS12 &= \sum w_n sin(\Omega_1) sin(\Omega_2) \\ CS11 &= \sum w_n cos(\Omega_1) sin(\Omega_1) \\ CS22 &= \sum w_n cos(\Omega_2) sin(\Omega_2) \end{aligned}$$

Bayesian Marginalized Posterior

$$P(\omega_1, \omega_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L(a_1, b_1, a_2, b_2) da_1 db_1 da_2 db_2\ .$$

$$\int_{-\infty}^{\infty} e^{-(Ax^2+Bx+C)}\ dx = \sqrt{\frac{\pi}{A}} e^{(B^2/4A)-C}\ .$$

$$\begin{aligned} RC &= \text{coeffs}(R, a1) \\ A &= RC(3) \\ B &= RC(2) \\ C &= RC(1) \\ R &= -(B^2/(4A)) + C \end{aligned}$$

$$\log L = -\frac{1}{2}\sum_{n=1}^{N}(\frac{x_n - y_n}{\sigma_n})^2 \ ,$$

$$\begin{aligned} y_n =& a_1\ cos(\omega_1 t_n - \theta) + b_1\ sin(\omega_1 t_n - \theta) \\ &+ a_2\ cos(\omega_2 t_n - \theta) + b_2\ sin(\omega_2 t_n - \theta)\ , \end{aligned}$$

Least-Squares

$$\begin{aligned} R = & a_1^2\ CC11 + b_1^2\ SS11 + a_2^2\ CC22 + b_2^2\ SS22 \\ & - 2(a_1 XC1 + b_1 XS1 + a_2 XC2 + b_2 XS2) \\ & + 2(a_1 b_1 CS11 + a_1 a_2 CC12 + a_1 b_2 CS12 \\ & + b_1 a_2 SC12 + b_1 b_2 SS12 + a_2 b_2 CS22\ ) \end{aligned}$$

$$\begin{aligned} CC11 &= \sum w_n cos^2(\Omega_1) \\ SS11 &= \sum w_n sin^2(\Omega_1) \\ CC22 &= \sum w_n cos^2(\Omega_2) \\ SS22 &= \sum w_n sin^2(\Omega_2) \\ XC1 &= \sum w_n x_n cos(\Omega_1) \\ XS1 &= \sum w_n x_n sin(\Omega_1) \\ XC2 &= \sum w_n x_n cos(\Omega_2) \\ XS2 &= \sum w_n x_n sin(\Omega_2) \\ CC12 &= \sum w_n cos(\Omega_1) cos(\Omega_2) \\ CS12 &= \sum w_n cos(\Omega_1) sin(\Omega_2) \\ SC12 &= \sum w_n sin(\Omega_1) cos(\Omega_2) \\ SS12 &= \sum w_n sin(\Omega_1) sin(\Omega_2) \\ CS11 &= \sum w_n cos(\Omega_1) sin(\Omega_1) \\ CS22 &= \sum w_n cos(\Omega_2) sin(\Omega_2) \end{aligned}$$

Bayesian Marginalized Posterior

$$P(\omega_1, \omega_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L(a_1, b_1, a_2, b_2) da_1 db_1 da_2 db_2\ .$$

$$\int_{-\infty}^{\infty} e^{-(Ax^2+Bx+C)}\ dx = \sqrt{\frac{\pi}{A}} e^{(B^2/4A)-C}\ .$$

$$\begin{aligned} RC &= \text{coeffs}(R, a1) \\ A &= RC(3) \\ B &= RC(2) \\ C &= RC(1) \\ R &= -(B^2/(4A)) + C \end{aligned}$$

Other orthogonal bases: just plug in!

$$\begin{aligned} y_n = & a_1\ cos((\omega_1+\omega_2)t_n)\ cos((\omega_1-\omega_2)t_n) \\ & + b_1\ sin((\omega_1+\omega_2)t_n)\ cos((\omega_1-\omega_2)t_n) \\ & + a_2\ cos((\omega_1+\omega_2)t_n)\ sin((\omega_1-\omega_2)t_n)/(\omega_2-\omega_1) \\ & + b_2\ sin((\omega_1+\omega_2)t_n)\ sin((\omega_1-\omega_2)t_n)/(\omega_2-\omega_1) \end{aligned}$$

Baluev

```
 denominator = (- CC12^2*SS12^2 + SS11*SS22*CC12^2 + 2*CC12*CS12*SC12*SS12 ...
                - CS12^2*SC12^2 + CC22*SS11*CS12^2 + CC11*SS22*SC12^2 ...
                + CC11*CC22*SS12^2 - CC11*CC22*SS11*SS22)

a1_numerator = (CC22*SS12^2*XXC1 - CC12*SS12^2*XXC2 - CS12*SC12^2*XXS2 ...
              + SC12^2*SS22*XXC1 + CC12*SC12*SS12*XXS2 ...
              + CC22*CS12*SS11*XXS2 - CC22*CS12*SS12*XXS1 ...
              - CC12*SC12*SS22*XXS1 + CS12*SC12*SS12*XXC2 ...
              + CC12*SS11*SS22*XXC2 - CC22*SS11*SS22*XXC1)

b1_numerator = (CC22*CS12^2*XXS1 - CS12^2*SC12*XXC2 - CC12^2*SS12*XXS2 ...
              + CC12^2*SS22*XXS1 + CC12*CS12*SC12*XXS2 + CC11*CC22*SS12*XXS2 ...
              - CC11*CC22*SS22*XXS1 + CC12*CS12*SS12*XXC2 ...
              - CC22*CS12*SS12*XXC1 + CC11*SC12*SS22*XXC2 ...
              - CC12*SC12*SS22*XXC1)

a2_numerator = (CC11*SS12^2*XXC2 - CS12^2*SC12*XXS1 - CC12*SS12^2*XXC1 ...
              + CS12^2*SS11*XXC2 - CC12*CS12*SS11*XXS2 ...
              + CC12*CS12*SS12*XXS1 - CC11*SC12*SS12*XXS2 ...
              + CC11*SC12*SS22*XXS1 + CS12*SC12*SS12*XXC1 ...
              - CC11*SS11*SS22*XXC2 + CC12*SS11*SS22*XXC1)

b2_numerator = (CC11*SC12^2*XXS2 - CS12*SC12^2*XXC1 + CC12^2*SS11*XXS2 ...
              - CC12^2*SS12*XXS1 + CC12*CS12*SC12*XXS1 - CC11*CC22*SS11*XXS2 ...
              + CC11*CC22*SS12*XXS1 - CC12*CS12*SS11*XXC2 ...
              - CC11*SC12*SS12*XXC2 + CC12*SC12*SS12*XXC1 ...
              + CC22*CS12*SS11*XXC1)

               a1 = a1_numerator / denominator
               b1 = b1_numerator / denominator
               a2 = a2_numerator / denominator
               b2 = b2_numerator / denominator

Double Periodogram  = CC11*a1^2 + 2*CC12*a1*a2 + 2*CS12*a1*b2 - 2*XXC1*a1 + CC22*a2^2 ...
                  + 2*SC12*a2*b1 - 2*XXC2*a2 + SS11*b1^2 ...
                  + 2*SS12*b1*b2 - 2*XXS1*b1 + SS22*b2^2 - 2*XXS2*b2;
```

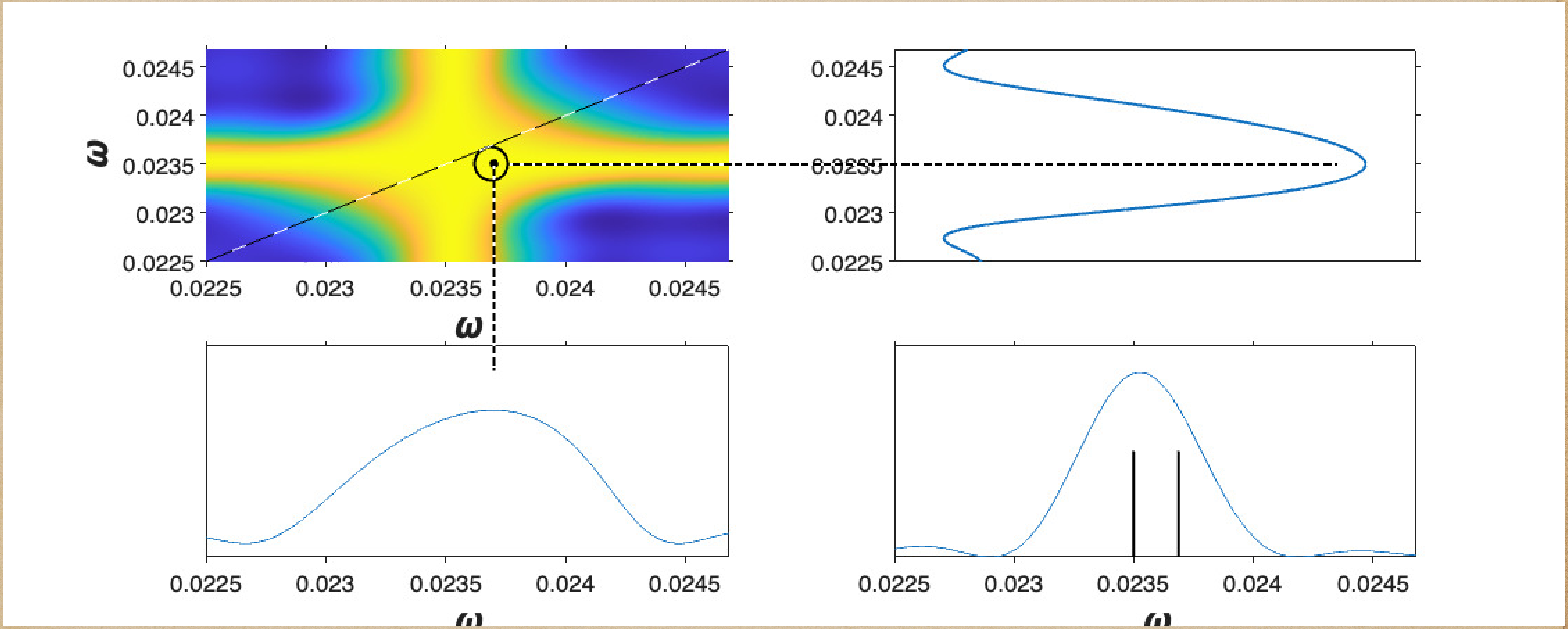
```
Bayes Numerator = ...
            - 2*CC12^2*SS12*XXS1*XXS2 + SS22*CC12^2*XXS1^2
            + SS11*CC12^2*XXS2^2 + 2*CC12*CS11*CS22*XXS1*XXS2
            - 2*CC12*CS11*SC12*XXS2^2 + 2*CC12*CS11*SS12*XXC2*XXS2
            - 2*SS22*CC12*CS11*XXC2*XXS1 - 2*CC12*CS12*CS22*XXS1^2
            + 2*CC12*CS12*SC12*XXS1*XXS2 + 2*CC12*CS12*SS12*XXC2*XXS1
            - 2*SS11*CC12*CS12*XXC2*XXS2 + 2*CC12*CS22*SS12*XXC1*XXS1
            - 2*SS11*CC12*CS22*XXC1*XXS2 + 2*CC12*SC12*SS12*XXC1*XXS2
            - 2*SS22*CC12*SC12*XXC1*XXS1 - 2*CC12*SS12^2*XXC1*XXC2
            + 2*SS11*SS22*CC12*XXC1*XXC2 - 2*CS11^2*CS22*XXC2*XXS2
            + SS22*CS11^2*XXC2^2 + CC22*CS11^2*XXS2^2
            + 2*CS11*CS12*CS22*XXC2*XXS1 + 2*CS11*CS12*SC12*XXC2*XXS2
            - 2*CS11*CS12*SS12*XXC2^2 - 2*CC22*CS11*CS12*XXS1*XXS2
            - 2*CS11*CS22^2*XXC1*XXS1 + 2*CS11*CS22*SC12*XXC1*XXS2
            + 2*CS11*CS22*SS12*XXC1*XXC2 - 2*SS22*CS11*SC12*XXC1*XXC2
            - 2*CC22*CS11*SS12*XXC1*XXS2 + 2*CC22*SS22*CS11*XXC1*XXS1
            - 2*CS12^2*SC12*XXC2*XXS1 + SS11*CS12^2*XXC2^2
            + CC22*CS12^2*XXS1^2 + 2*CS12*CS22*SC12*XXC1*XXS1
            - 2*SS11*CS12*CS22*XXC1*XXC2 - 2*CS12*SC12^2*XXC1*XXS2
            + 2*CS12*SC12*SS12*XXC1*XXC2 - 2*CC22*CS12*SS12*XXC1*XXS1
            + 2*CC22*SS11*CS12*XXC1*XXS2 + SS11*CS22^2*XXC1^2
            + CC11*CS22^2*XXS1^2 - 2*CS22*SC12*SS12*XXC1^2
            - 2*CC11*CS22*SC12*XXS1*XXS2 - 2*CC11*CS22*SS12*XXC2*XXS1
            + 2*CC11*SS11*CS22*XXC2*XXS2 + SS22*SC12^2*XXC1^2
            + CC11*SC12^2*XXS2^2 - 2*CC11*SC12*SS12*XXC2*XXS2
            + 2*CC11*SS22*SC12*XXC2*XXS1 + CC22*SS12^2*XXC1^2
            + CC11*SS12^2*XXC2^2 + 2*CC11*CC22*SS12*XXS1*XXS2
            - CC22*SS11*SS22*XXC1^2 - CC11*SS11*SS22*XXC2^2
            - CC11*CC22*SS22*XXS1^2 - CC11*CC22*SS11*XXS2^2
```

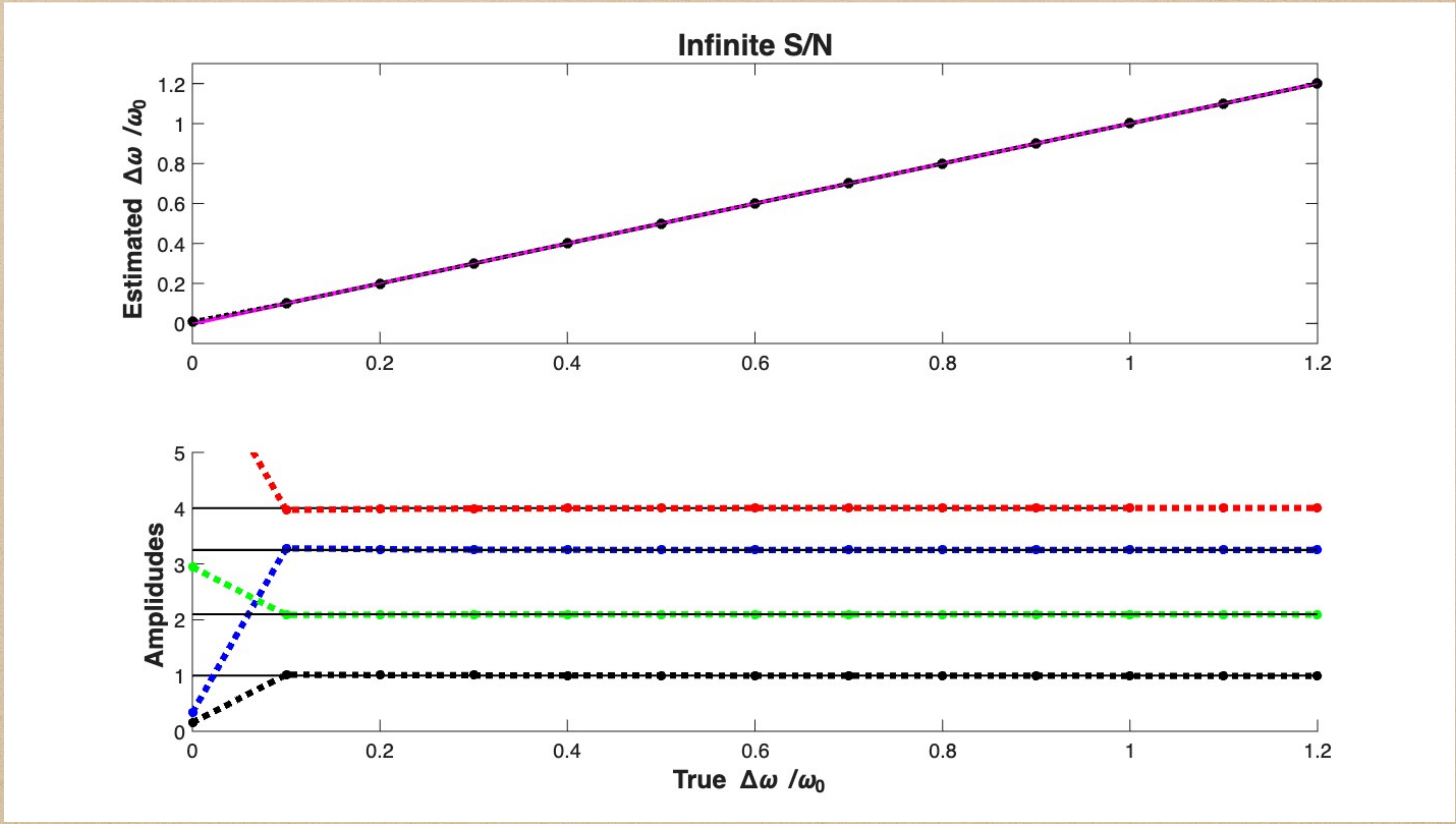
```
Bayes Denominator = ...
          - CC12^2*SS12^2 + SS11*SS22*CC12^2 + 2*CC12*CS11*CS22*SS12
        - 2*SS22*CC12*CS11*SC12 - 2*SS11*CC12*CS12*CS22
         + 2*CC12*CS12*SC12*SS12 - CS11^2*CS22^2 + CC22*SS22*CS11^2
         + 2*CS11*CS12*CS22*SC12 - 2*CC22*CS11*CS12*SS12
         - CS12^2*SC12^2 + CC22*SS11*CS12^2 + CC11*SS11*CS22^2
         - 2*CC11*CS22*SC12*SS12 + CC11*SS22*SC12^2
         + CC11*CC22*SS12^2 - CC11*CC22*SS11*SS22
```

Double Bayesian Periodogram

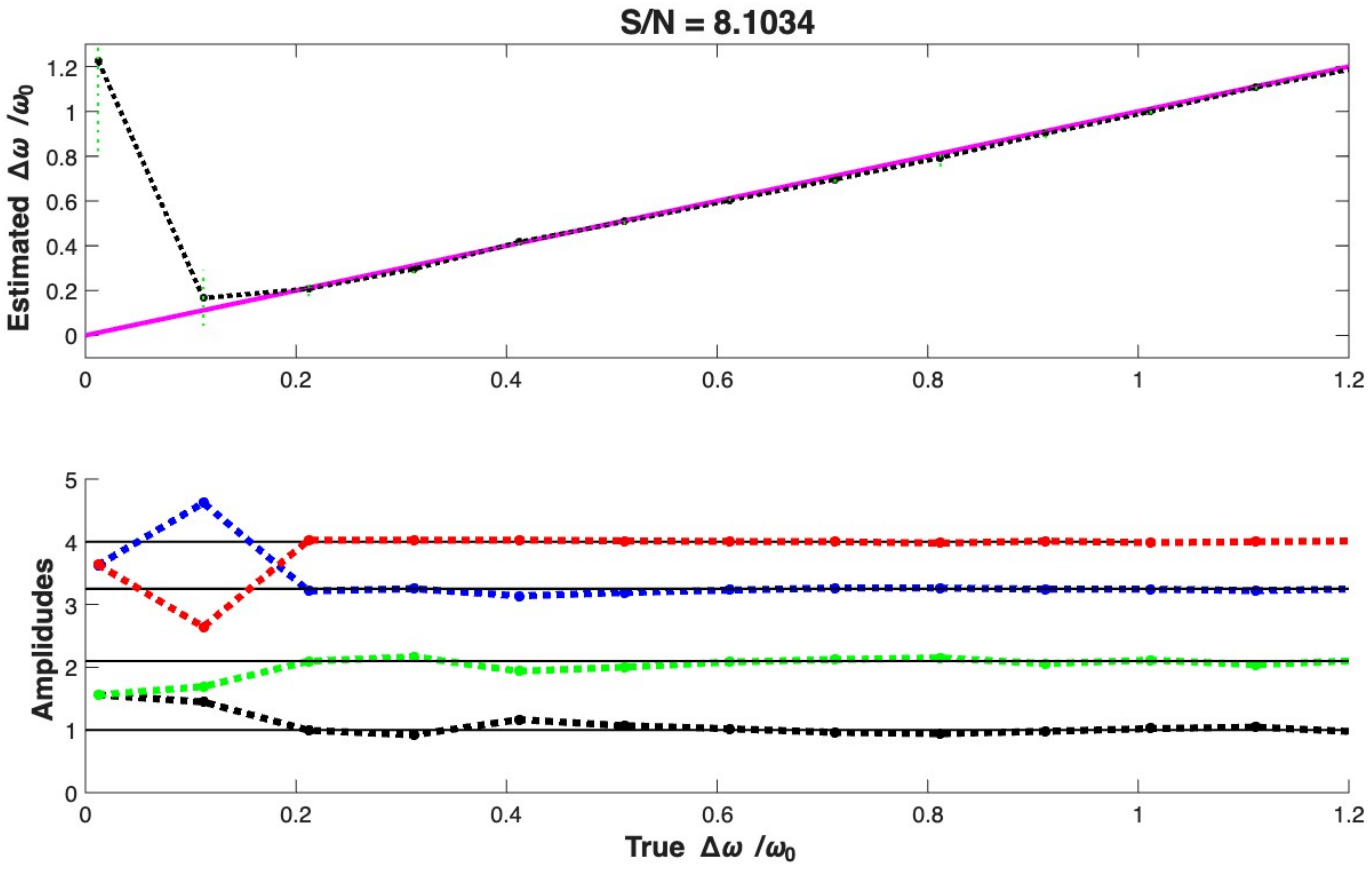
= Bayes Numerator / Bayes Denominator

Two Sinusoids (simulated)

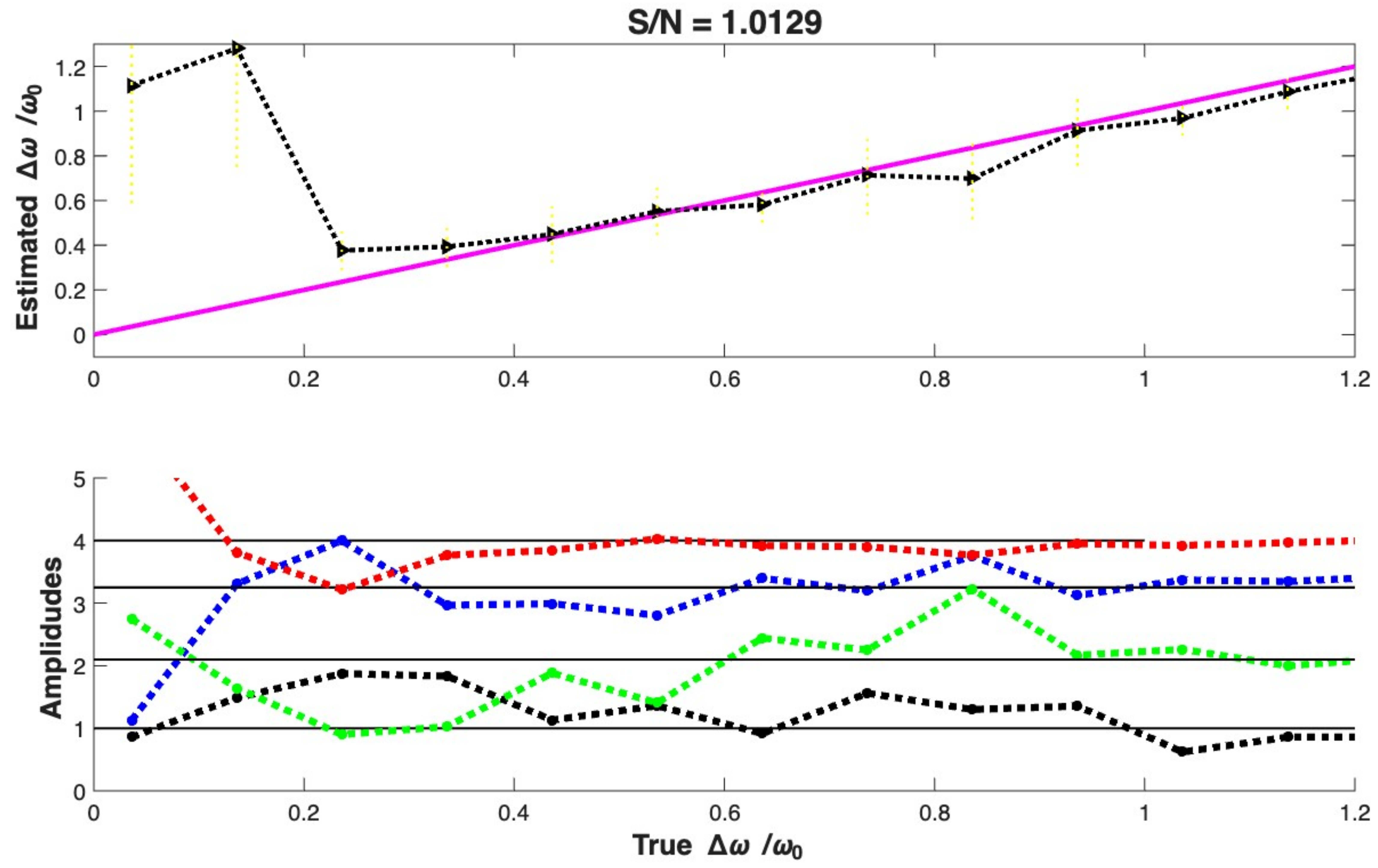


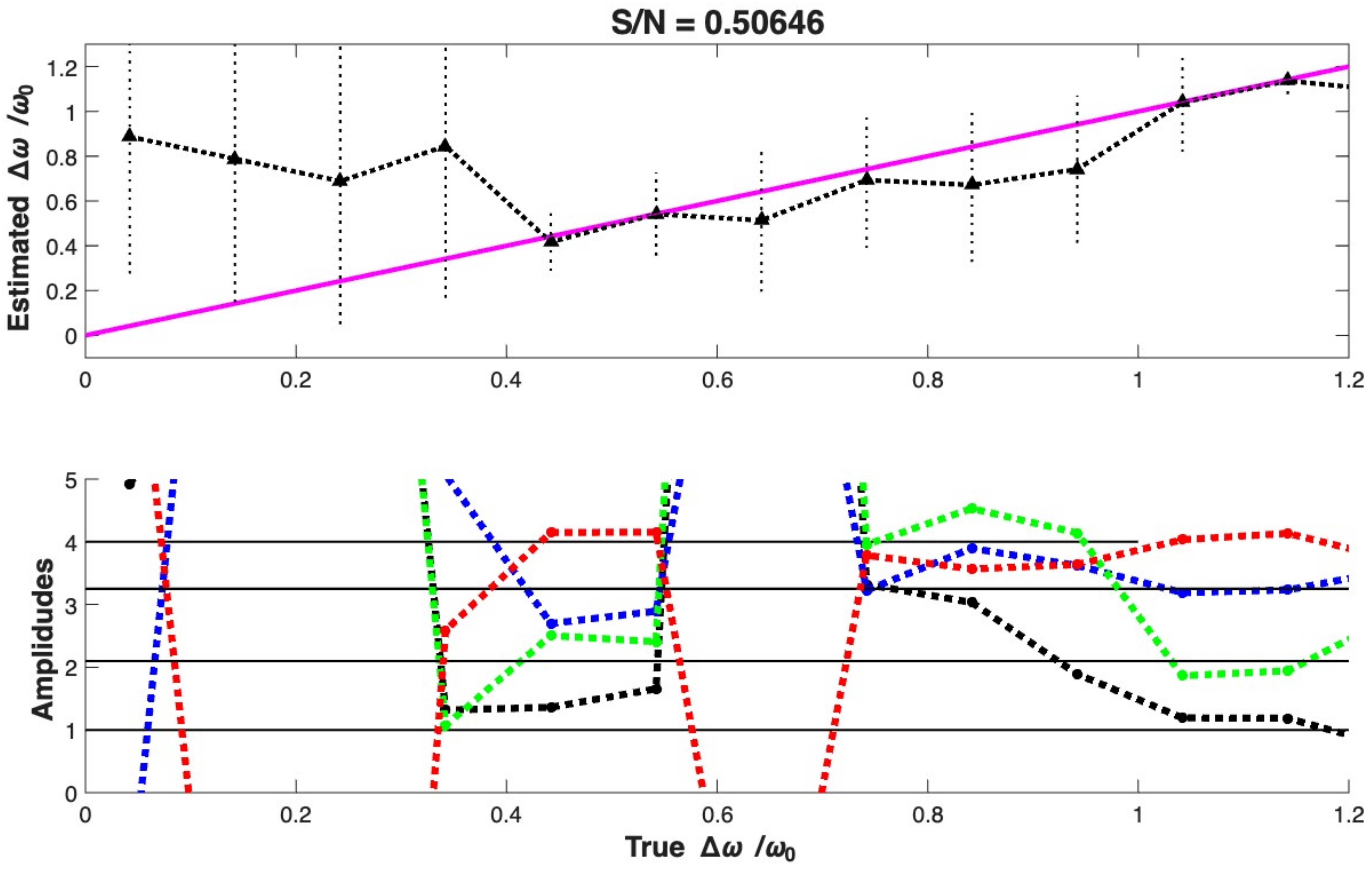


Infinite S/N
Estimated $\Delta\omega\ /\omega_0$
0
0.2
0.4
0.6
0.8
1
1.2
Amplidudes
0
1
2
3
4
5
True $\Delta\omega\ /\omega_0$



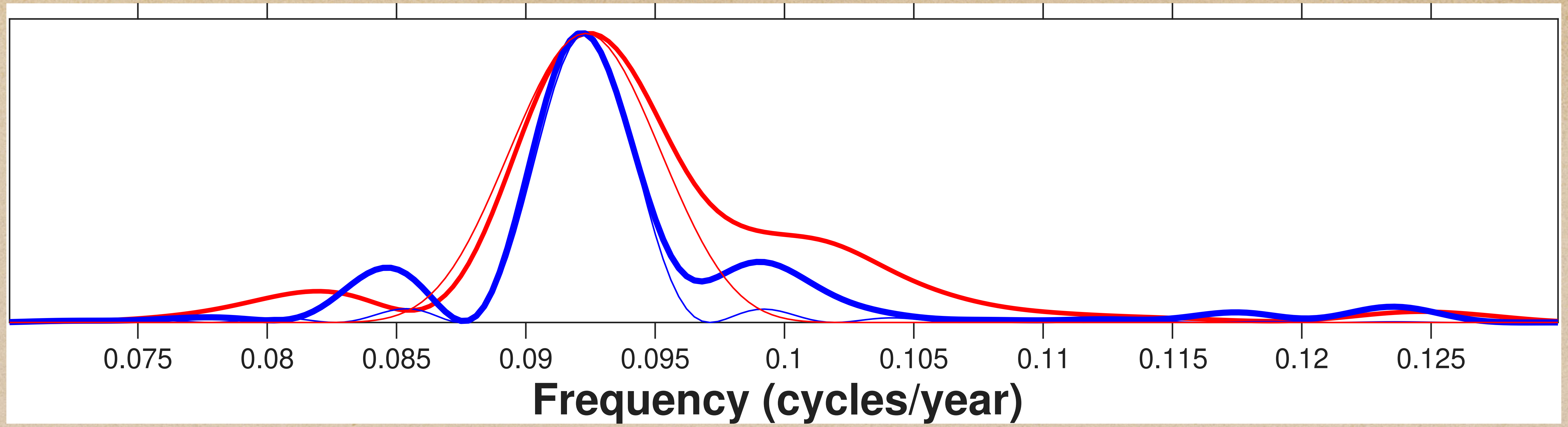
S/N = 8.1034
Estimated Δω /ω0
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
Amplidudes
5
4
3
2
1
0
0
0.2
0.4
0.6
0.8
1
1.2
True Δω /ω0

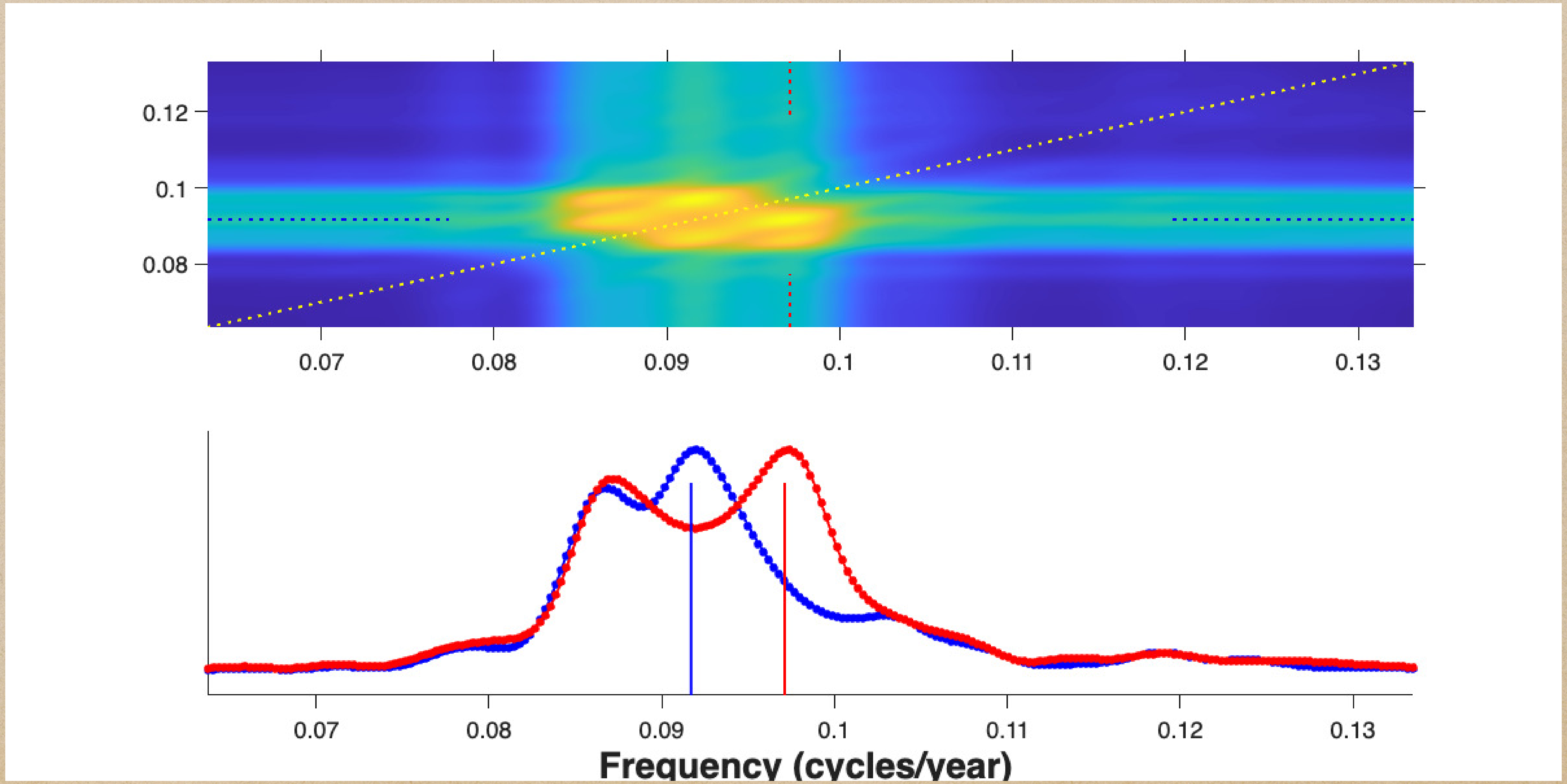

S/N = 1.0129
Estimated Δω /ω0
Amplidudes
True Δω /ω0



S/N = 0.50646
Estimated $\Delta\omega$ $/\omega_0$
Amplidudes
True $\Delta\omega$ $/\omega_0$

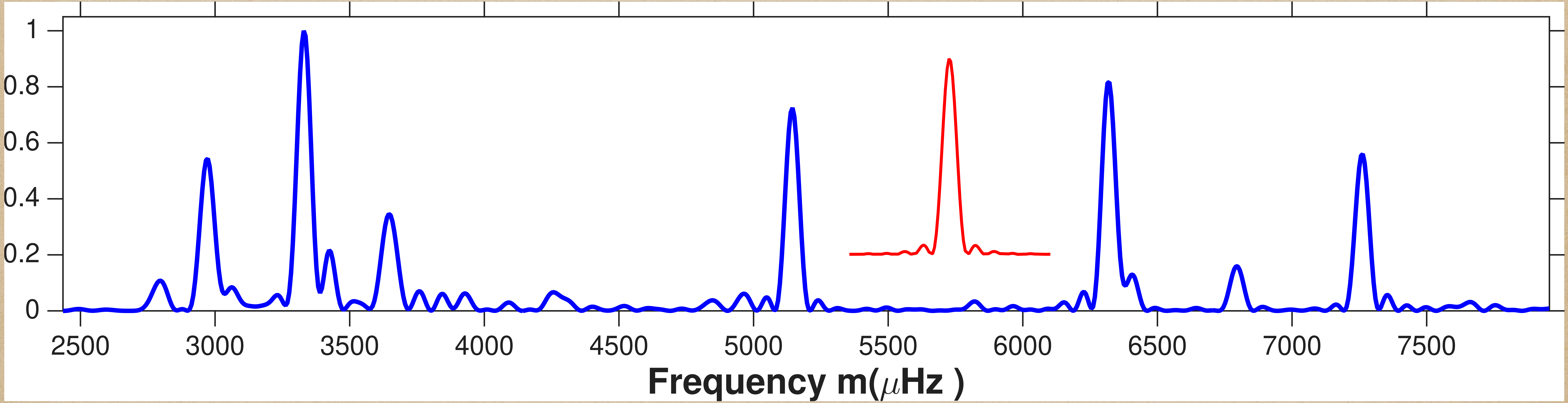
Sunspot Number Time Series

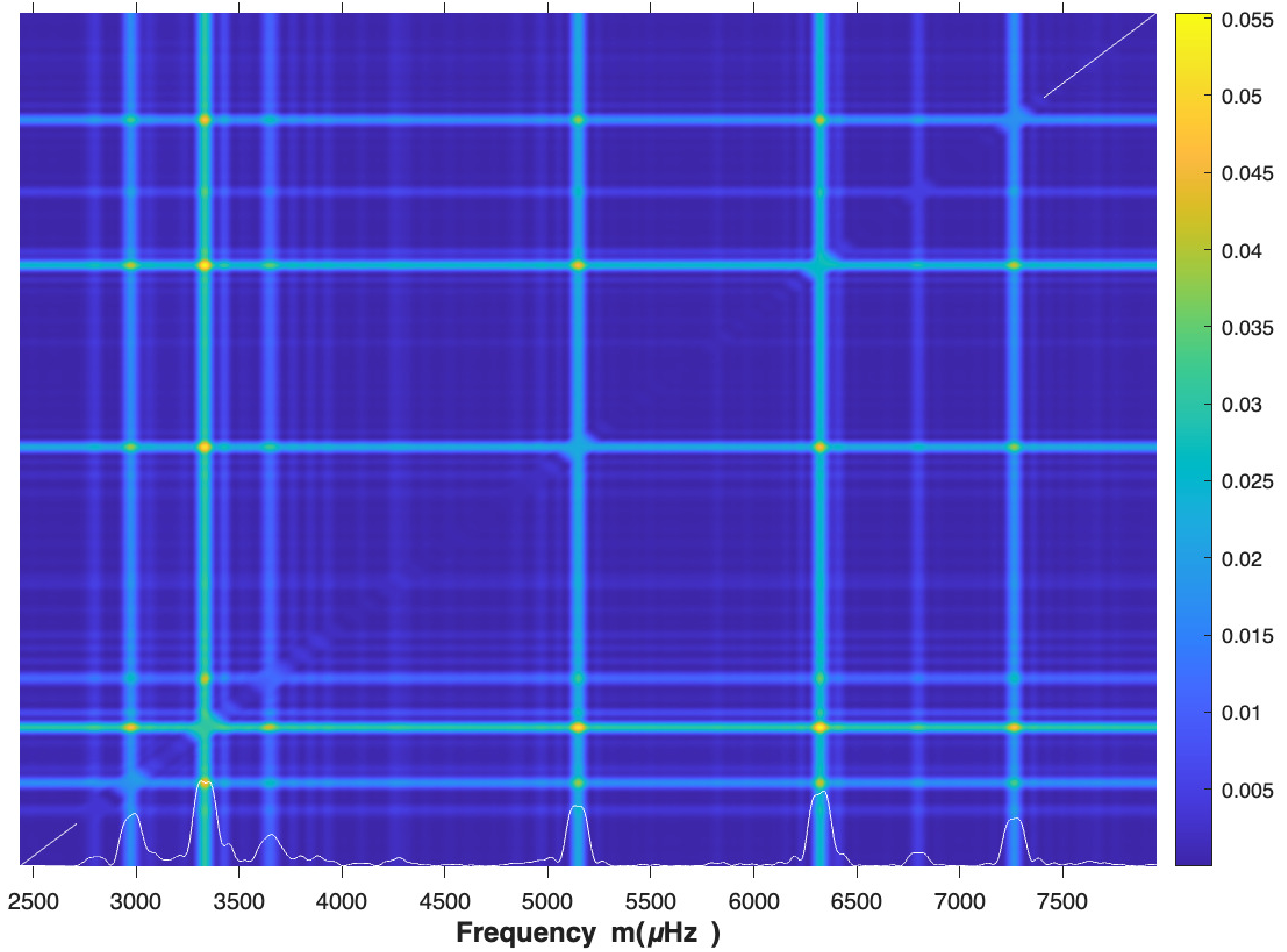




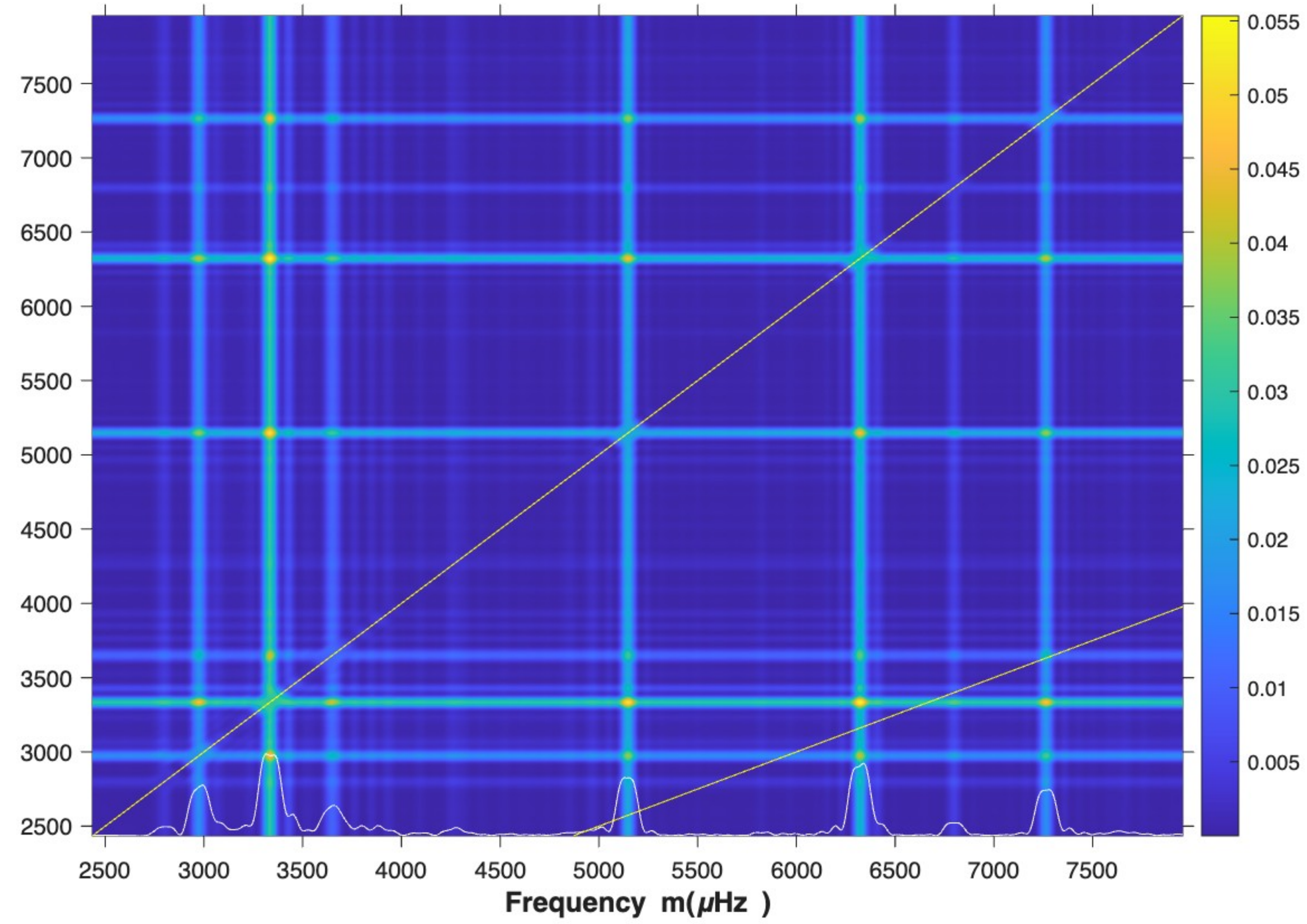
The Pulsating White Dwarf WD J0135+5722

De Geronimo, Uzundag, Rebassa-Mansergas, Brown, Kilic, Corsico, Jewett and Moss, 2025, arXiv:2501.1366



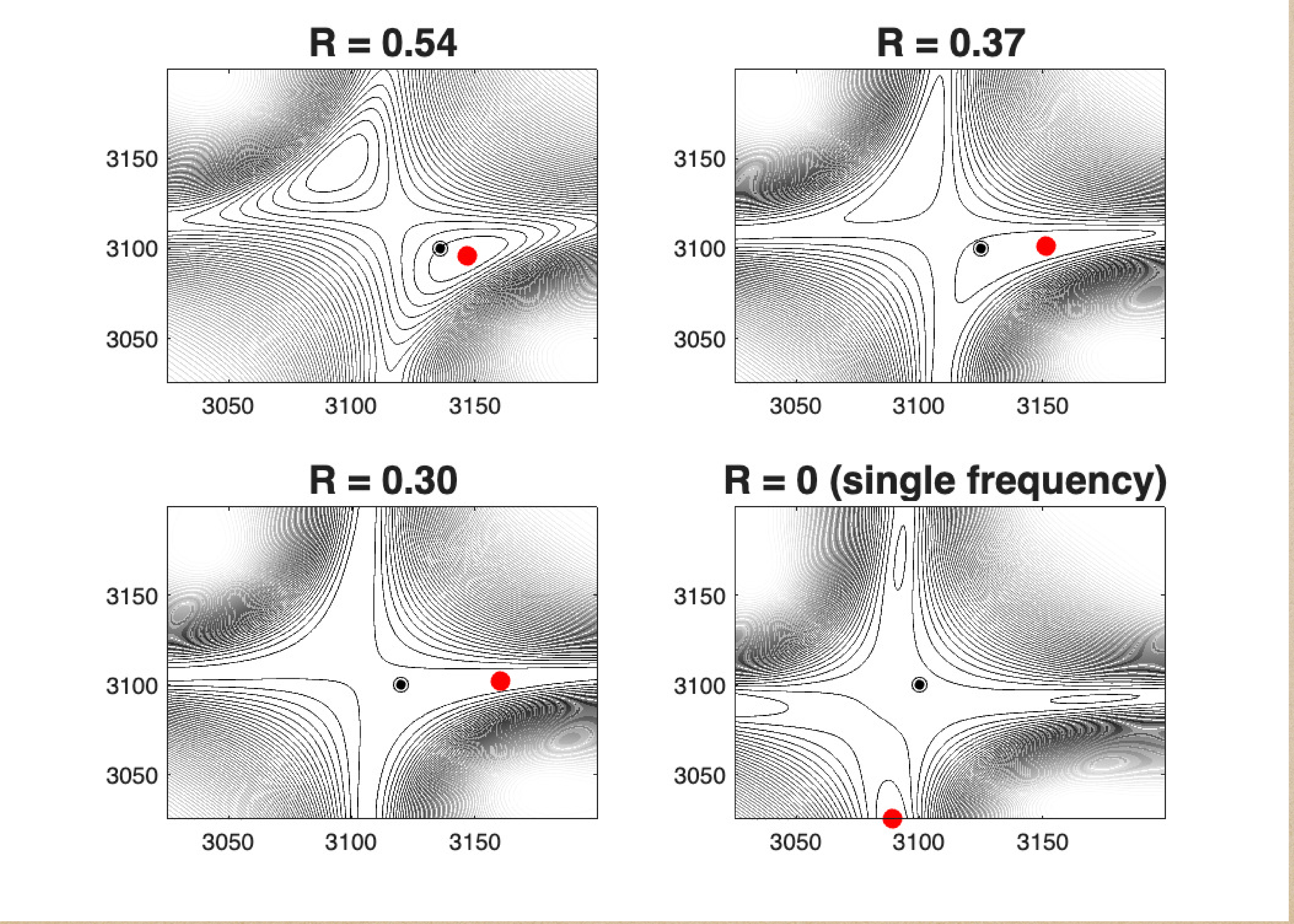


0.055
0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
2500
3000
3500
4000
4500
5000
5500
6000
6500
7000
7500
Frequency m(μHz)

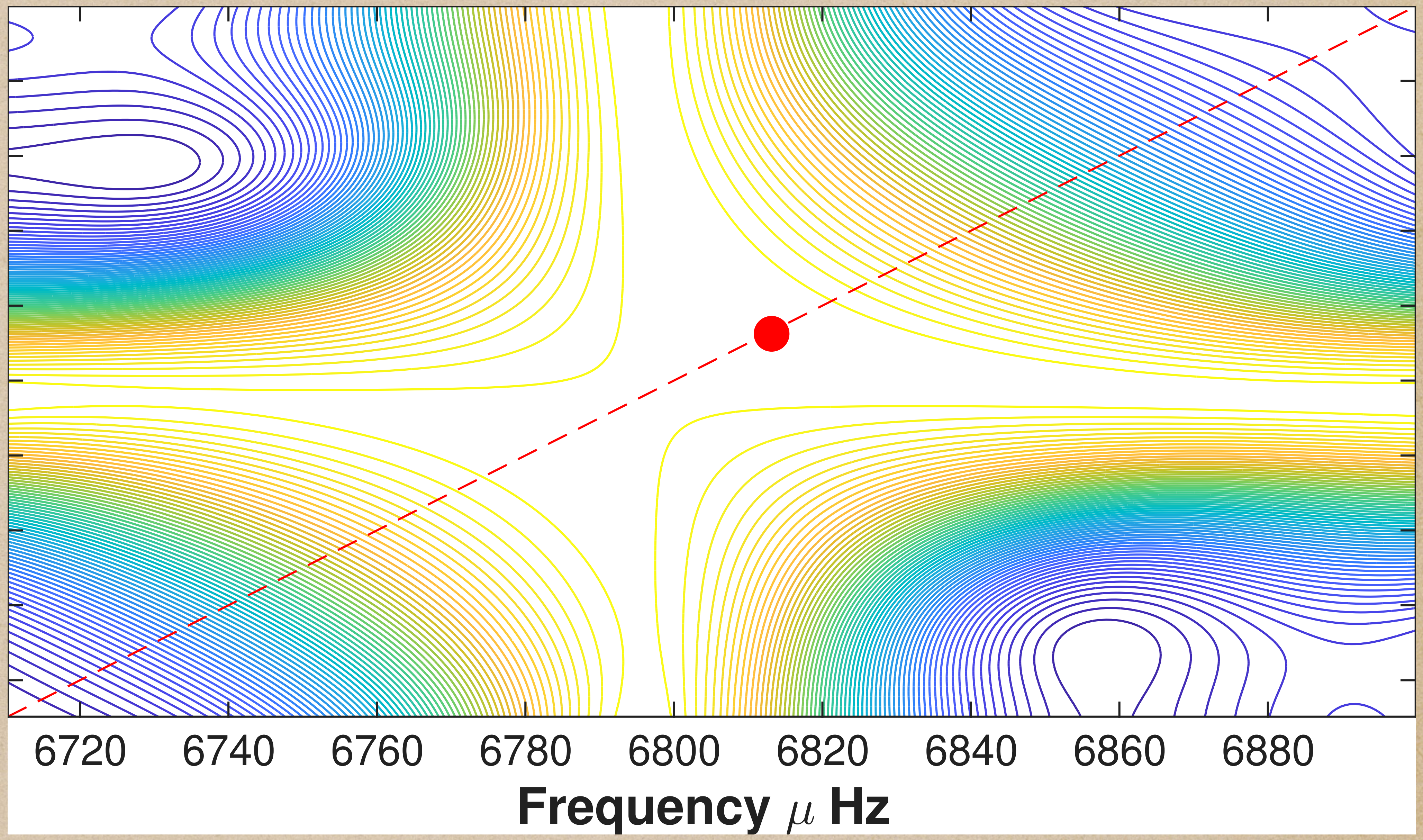


7500
7000
6500
6000
5500
5000
4500
4000
3500
3000
2500
2500 3000 3500 4000 4500 5000 5500 6000 6500 7000 7500
Frequency m(μHz)
0.055
0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005

Evolution of twin-peak resolution



White Dwarf WD J0135+5722
Unresolved Sinusoid



Applications

- Super Resolution
- Harmonics
- …?