

Entanglement and renormalisation of spin systems in the generalised Landau paradigm

Clement Delcamp
IHES



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MOTIVATION

How to efficiently compute the ground states of a spin system?

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In collaboration w/ L. Lootens & F. Verstraete
arXiv:2408.06334

One-dimensional quantum lattice model

- Finite subset $\Lambda \ni i$ of \mathbb{Z} \rightsquigarrow Chain
- Local Hilbert space $\mathcal{H}_{\{i\}} = \mathbb{C}^d$ \rightsquigarrow Degrees of freedom / states
 $\hookrightarrow \mathcal{H}_\Lambda = \bigotimes_{i \in \Lambda} \mathcal{H}_{\{i\}}$
- Local algebra $\mathcal{A}_{\{i\}} = \text{Mat}_d(\mathbb{C})$ \rightsquigarrow Operators
 $\hookrightarrow \mathcal{A}_\Lambda = \bigotimes_{i \in \Lambda} \mathcal{A}_{\{i\}}$
- Self-adjoint matrix $h(I) \in \mathcal{A}_I$, $I \subset \Lambda$ \rightsquigarrow Interaction
- Local Hamiltonian $H_\Lambda = \sum_{I \subset \Lambda} h(I) \in \mathcal{A}_\Lambda$ \rightsquigarrow Dynamics

...[Nachtergaele, Sims '05] [Hastings, Koma '05]

Discrete real set of eigenvalues of $H_\Lambda = H_\Lambda^\dagger$

$$E_\Lambda^{0,1} \leq E_\Lambda^{0,2} \leq \dots \leq E_\Lambda^{0,k} < E_\Lambda^1 \leq \dots$$

If

- The ground state of H_Λ is quasi-degenerate, i.e.

$$\lim_{|\Lambda| \rightarrow \infty} \max_{i,j} \{ |E_\Lambda^{0,i} - E_\Lambda^{0,j}| \} = 0$$

- Distance between $\{E_\Lambda^{0,1}, \dots, E_\Lambda^{0,k}\}$ and E_Λ^1 is larger than a positive constant independent of the volume $|\Lambda|$

Then, the spin chain is said to be (uniformly) **gapped**

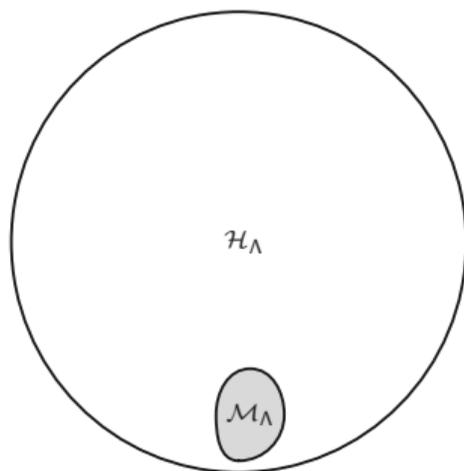
What are the ground states of a gapped spin chain?

Density Matrix Renormalisation Group (DMRG)

[White '92]

↳ Variational algorithm within the subspace of Matrix Product States

[Hastings '04]



Low-energy states of a gapped spin chain are **weakly entangled**

$$\arg \min_{\psi \in \mathcal{H}_\Lambda} \frac{\langle \psi, H_\Lambda \psi \rangle}{\langle \psi, \psi \rangle} \rightsquigarrow \arg \min_{\psi \in \mathcal{M}_\Lambda} \frac{\langle \psi, H_\Lambda \psi \rangle}{\langle \psi, \psi \rangle}$$

Matrix Product State (MPS)

$$\psi = \sum_{i_2, \dots, i_{|\Lambda|}}^{\chi} \psi_{i_1, i_2} \otimes \psi_{i_2, i_3} \otimes \dots \otimes \psi_{i_{|\Lambda|}, i_{|\Lambda|+1}}, \quad \{\psi_{i,j} \in \mathbb{C}^d\}_{i,j=1}^{\chi}$$

↳ Approximates well states in \mathcal{M}_Λ for fixed χ

Ordinary symmetry w/ finite group G

- Unitary operators $\{U(g)\}_{g \in G}$ s.t. $[U(g), H] = 0, \forall g \in G$
- Composition: $U(g_1) \circ U(g_2) = U(g_1 g_2), \forall g_1, g_2 \in G$

In the presence of a symmetry G , we distinguish two situations for state ψ in ground state subspace \mathcal{H}^0 :

$$U(g)(\psi) = e^{i\theta} \cdot \psi, \quad \forall g \in G$$

(symmetric state)

$$\exists g \in G, \quad U(g)(\psi) = \psi' \in \mathcal{H}^0 \quad \text{and} \quad \langle \psi, \psi' \rangle = 0$$

(non-symmetric state)



Spontaneous symmetry breaking

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Spontaneous symmetry breaking

[Verstraete et al. '05] [Chen, Gu, Wen '10]...

(Bosonic) G -symmetric gapped phases are labelled by pairs $(A \subseteq G, [\psi] \in H^2(A, \mathbb{C}^\times))$

“Gapped phases with spontaneously broken symmetry are easier to simulate”

[Lootens, CD, Verstraete et al. '21 '22 '23 '24]

Any gapped phase of $H \mapsto$ Spontaneously symmetry broken phase of H^{dual} with dual symmetry
s.t. $D \circ H = H^{\text{dual}} \circ D$ w/ D a unitary Matrix Product Operator

(H^{dual} is obtained via the ψ -twisted gauging of A -subsymmetry)

Caveats: (i) H^{dual} may not be a spin system \rightsquigarrow Generalised DMRG algorithm
(ii) Dual symmetry may be **non-invertible**

Result: Optimal DMRG simulation of any symmetric gapped phase

GAPPED PHASES

“Gapped phases are believed to be described at long distances by topological quantum field theories”

[Atiyah '88]

An n -dimensional TQFT is a symmetric monoidal functor $\mathcal{Z} : \text{Bord}_n \rightarrow \text{Vec}$ from a flavour of the category of n -cobordisms to the category of complex vector spaces

$$\begin{array}{ll}
 \mathcal{Z} : \text{Bord}_n & \rightarrow \text{Vec} \\
 : \Sigma & \mapsto \mathcal{Z}(\Sigma) \in \text{Vec} \\
 : (\Sigma \rightarrow \Sigma') & \mapsto (\mathcal{Z}(\Sigma) \rightarrow \mathcal{Z}(\Sigma')) \\
 : (\emptyset \rightarrow \Sigma) & \mapsto (\mathbb{C} \rightarrow \mathcal{Z}(\Sigma)) \\
 : (\emptyset \rightarrow \emptyset) & \mapsto (\mathbb{C} \rightarrow \mathbb{C})
 \end{array}
 \quad \text{s.t.} \quad
 \begin{array}{l}
 \mathcal{Z}(\emptyset) \cong \mathbb{C} \\
 \mathcal{Z}(\Sigma \rightarrow \Sigma' \cup_{\Sigma'} \Sigma' \rightarrow \Sigma'') = \mathcal{Z}(\Sigma \rightarrow \Sigma') \circ \mathcal{Z}(\Sigma' \rightarrow \Sigma'') \\
 \mathcal{Z}(\Sigma \sqcup \Sigma') \cong \mathcal{Z}(\Sigma) \otimes_{\mathbb{C}} \mathcal{Z}(\Sigma') \\
 \dots
 \end{array}$$

Bosonic G -symmetric $(1+1)d$ gapped phases are described by G -equivariant unitary 2d TQFTs
+ principal G -bundles

[Moore, Segal '06] [Turaev '10]

Classification in terms of G -crossed Frobenius $*$ -algebras

[Kapustin, Turzillo, You '16] [Lauda, Pfeiffer '06]

State-sum construction from semisimple G -equivariant algebras

[Ostrik '03]

Morita equivalence classes of indecomposable semisimple G -equivariant algebras
are classified by pairs $(A \subseteq G, [\psi] \in H^2(A, \mathbb{C}^\times))$

(Same data label topological boundary conditions of 3d topological G -gauge theory)

[Reshetikhin, Turaev '91]

3d TQFT from a **modular tensor category**

[Turaev, Viro '92] [Barrett, Westbury '93]

3d state sum TQFT from a **spherical fusion category** \mathcal{C} so that the Drinfel'd center $\mathcal{Z}(\mathcal{C})$ is a modular tensor category

Fully-extended TQFT s.t. $\mathcal{Z}(\mathbb{S}^1) \simeq \mathcal{Z}(\mathcal{C})$ & $\dim_{\mathbb{C}} \mathcal{Z}(\mathbb{T}^2) = |\text{Irr}(\mathcal{Z}(\mathcal{C}))|$

...[Etingof, Nikshych, Ostrik '05]

Rigid abelian semisimple Vec -enriched monoidal category
w/ finitely many isomorphism classes of simple objects and $\text{End}_{\mathcal{C}}(\mathbb{1}) \cong \mathbb{C}$

E.g. $\mathcal{C} = \text{Mod}(\text{Hopf algebra})$

$\hookrightarrow \text{Mod}(\mathbb{C}^G) \simeq \text{Vec}_G$: Fusion category of finite-dimensional G -graded vector spaces

$\hookrightarrow \text{Mod}(\mathbb{C}[G]) \simeq \text{Rep}(G)$: Fusion category of finite-dimensional representations of G

...[Mueger '03] [Etingof, Nikshych, Ostrik '05] [Drinfel'd et al. '10]

The Drinfel'd center of a fusion category \mathcal{C} is the braided fusion category with objects
($Z \in \mathcal{C}, R_{-,Z} : - \otimes Z \xrightarrow{\sim} - \otimes Z$) s.t. $R_{-,Z}$ fulfill hexagon axioms

E.g. $\mathcal{Z}(\text{Vec}_G) \simeq \text{Mod}(\mathcal{D}(G)) \simeq \text{Mod}(\overline{\Lambda G}) \simeq \text{Mod}(G//G) \simeq \text{Vec}_G^G \simeq \mathcal{Z}(\text{Rep}(G))$

TOPOLOGICAL BOUNDARY CONDITIONS

[Freed, Teleman '21]

Reshetikhin–Turaev theory admits a nonzero boundary theory if and only if it is a Turaev–Viro–Barrett–Westbury theory

[Kitaev, Kong '12] [Fuchs, Schweigert, Valentino '12]

Topological boundary theories are labelled by finite semisimple \mathcal{C} -module categories \mathcal{M} (e.g. $\mathcal{M} = \mathcal{C}$)

Topological lines on the boundary \mathcal{M} form the fusion category $\mathcal{C}_{\mathcal{M}}^{\star} := \text{Func}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ of \mathcal{C} -module endofunctors of \mathcal{M}

[Ostrik '03]

E.g. Every indecomposable Vec_G -module category is of the form $\text{Mod}_{\text{Vec}_G}(\mathbb{C}[A]^{\psi})$ w/ $A \subseteq G$ and $[\psi] \in H^2(A, \mathbb{C}^{\times})$

Fusion categories \mathcal{C} and \mathcal{D} are Morita equivalent if there exists a \mathcal{C} -module category \mathcal{M} s.t. $\mathcal{C}_{\mathcal{M}}^* \simeq \mathcal{D}$

[Müger 03] [Etingof, Gelaki, Nikshych, Ostrik '16]

Invariants of Morita equivalence:

$$\mathcal{Z}(\mathcal{C}) \simeq (\mathcal{C} \boxtimes \mathcal{C}_{\mathcal{M}}^*)_{\mathcal{M}}^* \simeq \mathcal{Z}(\mathcal{C}_{\mathcal{M}}^*)$$

$$\begin{aligned} \text{Mod}(\mathcal{C}) &\xrightarrow{\sim} \text{Mod}((\mathcal{C}_{\mathcal{M}}^*)^{\text{op}}) \\ \mathcal{N} &\mapsto \text{Func}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \end{aligned}$$

E.g. $(\text{Vec}_G)_{\text{Vec}}^* \simeq \text{Rep}(G)$

$\text{Mod}(\text{Hopf algebra})_{\text{Vec}}^* \simeq \text{Mod}(\text{Hopf algebra}^*)$

Let \mathcal{M} be a \mathcal{C} -module category

Let Σ_Υ be a cell decomposition of a 2d oriented surface Σ

Let $\Sigma_\Upsilon^{\mathcal{M}}$ be $\Sigma_\Upsilon \times [0, 1]$ s.t.

$\Sigma_\Upsilon \times \{0\}$ is a topological boundary labelled by \mathcal{M} &
 $\Sigma_\Upsilon \times \{1\}$ a gluing boundary

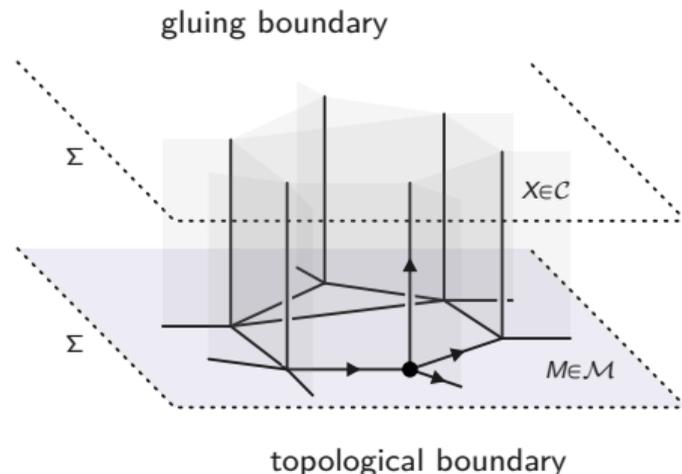
By def., $\mathcal{L}(\Sigma_\Upsilon^{\mathcal{M}}) \in \mathcal{L}(\Sigma_\Upsilon) \cong \mathcal{L}(\Sigma_{\Upsilon'}) \subset \mathcal{H}(\Sigma_\Upsilon)$

$$\text{w/ } \mathcal{H}(\Sigma_\Upsilon) = \bigoplus_{X \in \text{Irr}(\mathcal{C})} \bigotimes_{e \in E(\text{int})} \text{Hom}_{\mathcal{C}}(\mathbb{1}, \bigotimes_{p \supset e} X(e))$$

$$\mathcal{L}(\Sigma_\Upsilon \times [0, 1]) : \mathcal{H}(\Sigma_\Upsilon) \rightarrow \mathcal{H}(\Sigma_\Upsilon)$$

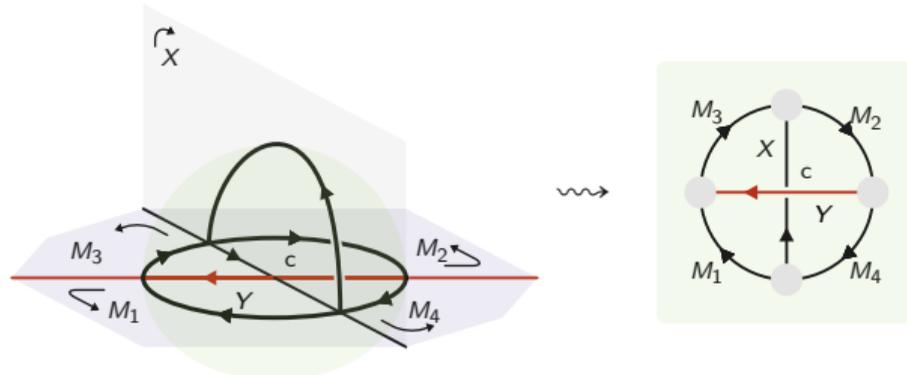
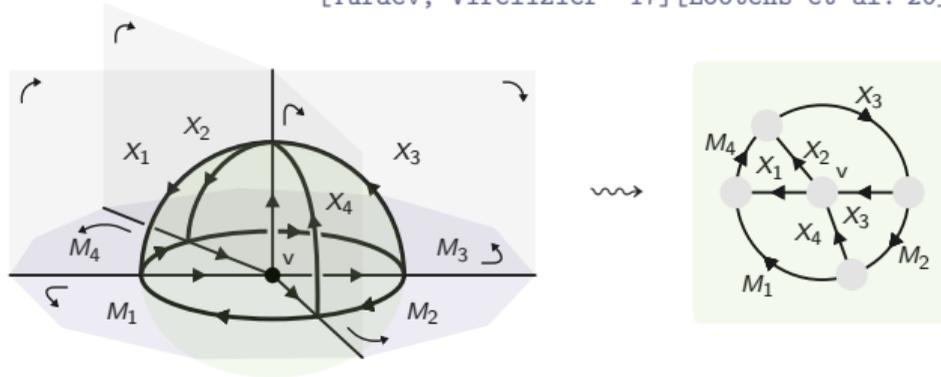
$$\mathcal{L}(\Sigma_\Upsilon) = \text{im } \mathcal{L}(\Sigma_\Upsilon \times [0, 1])$$

Basis vectors in $\mathcal{L}(\Sigma_\Upsilon)$ are obtained by inserting appropriate topological lines in $\mathcal{C}_{\mathcal{M}}^*$



EVALUATION MAPS

[Turaev, Virelizier '17] [Lootens et al.'20] [CD, Ishtiaque '24]...



$\langle \mathcal{Z}(\Sigma_\Gamma^{\mathcal{M}}), \psi \rangle$, for any $\psi \in \mathcal{H}(\Sigma_\Gamma)$, is the partition function of a 2d theory that can be coupled to topological lines in $\mathcal{C}_{\mathcal{M}}^*$, i.e. a $\mathcal{C}_{\mathcal{M}}^*$ -symmetric theory

(ψ encodes the local dynamics)

$\langle \mathcal{Z}(\Sigma_\Gamma^{\mathcal{M}}), \psi \rangle$ and $\langle \mathcal{Z}(\Sigma_\Gamma^{\mathcal{N}}), \psi \rangle$, for any two $\mathcal{M}, \mathcal{N} \in \text{Mod}(\mathcal{C})$, are the partition functions of two dual 2d theories

FROBENIUS ALGEBRA

[Thorngren, Wang '19] [Komargodski, Ohmori, Roumpedakis, Seifnashri '20]

\mathcal{C} -symmetric unitary 2d TQFTs are classified by (finite semisimple) \mathcal{C} -module categories

[Etingof, Gelaki, Nikshych, Ostrik '16]

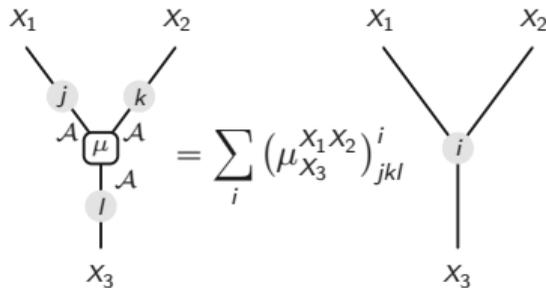
Every finite semisimple \mathcal{C} -module category \mathcal{P} is of the form $\text{Mod}_{\mathcal{C}}(A)$ for an algebra object A in \mathcal{C}

[Fuchs, Runkel, Schweigert '02] [Fuchs, Stigner '09] [Kong, Zheng '19]

A can be endowed the structure of a Δ -separable symmetric Frobenius algebra $(A, \mu, \eta, \Delta, \epsilon)$ in \mathcal{C}

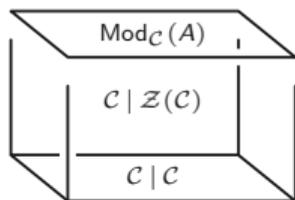
$$\begin{aligned} \mu : A \otimes A &\rightarrow A, \quad \eta : \mathbb{C} \rightarrow A \\ \Delta : A &\rightarrow A \otimes A, \quad \epsilon : A \rightarrow \mathbb{C} \end{aligned}$$

$$\begin{aligned} \text{s.t. } (\mu \otimes \text{id}) \circ (\text{id} \otimes \Delta) &= (\text{id} \otimes \mu) \circ (\Delta \otimes \text{id}) = \Delta \circ m \\ \& \quad m \circ \Delta = \text{id} \quad \& \quad \text{symmetric} \end{aligned}$$

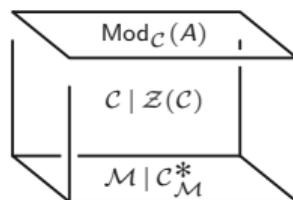


MAPPING GAPPED PHASES

[Lootens, CD, Verstraete et al. '21 '22 '23 '24]



\mathcal{C} -symmetric gapped phase
labelled by \mathcal{C} -module category $\text{Mod}_{\mathcal{C}}(A)$



$\mathcal{C}_{\mathcal{M}}^*$ -symmetric gapped phase
labelled by $\mathcal{C}_{\mathcal{M}}^*$ -module category $\text{Func}_{\mathcal{C}}(\mathcal{M}, \text{Mod}_{\mathcal{C}}(A))$

Spontaneously symmetry broken phase is obtained by choosing $\mathcal{M} = \text{Mod}_{\mathcal{C}}(A)$

Duality operator D is associated with a simple object in $\text{Func}_{\mathcal{C}}(\mathcal{C}, \text{Mod}_{\mathcal{C}}(A))$

CONCLUSION

“Every symmetric gapped phase of matter can be studied from the viewpoint of a symmetric gapped phase where the symmetry is spontaneously broken”