Correlation between multiple scattering angle and ionization energy loss for fast electrons. Prospects for experimental observation

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Contents

Introduction. I

Molière [1] and Landau [2] theories describe correspondingly multiple Coulomb scattering and ionization energy loss by fast charged particles in matter, but leave the question whether there is a correlation between those processes unanswered.

For a high-energy charged particle, the dominant contribution to correlation between its energy loss and deflection angle arises when the fast particle hits an atomic electron, imparting it energy and transverse momentum simultaneously.

The correlation must be the strongest when the incident particle is an electron or positron, because having the same mass as the knocked-off atomic electrons, it is kinematically allowed to transfer them a substantial fraction of its energy in a single collision.

Introduction. II

In a thick target, the angle-energy correlation generally disappears due to successive interactions with many atoms.
But in multiple Coulomb scattering of pointlike particles there is a significant contribution from single scatterings, purporting the correlation.

The correlation is expected to be stronger for low-Z target materials, because the relative contribution of incoherent scattering for them is higher.

Vanishing of the correlation for normal multiple scattering. I

After N successive identical random collisions, with individual χ_k^2 and $\Delta\epsilon_k$ correlated for the same k, the correlation coefficient between the aggregate energy loss $\epsilon = \sum_{k=1}^{N} \Delta\epsilon_k$ and the square of the deflection angle $\vec{\theta} = \sum_{k=1}^{N} \vec{\chi}_k$ is defined as

$$C = \frac{\left\langle (\theta^2 - \langle\theta^2\rangle)(\epsilon - \langle\epsilon\rangle) \right\rangle}{\sqrt{\left\langle (\theta^2 - \langle\theta^2\rangle)^2 \right\rangle}\sqrt{\left\langle (\epsilon - \langle\epsilon\rangle)^2 \right\rangle}}. \quad (1)$$

If all the single-scattering mean values are finite (multiple scattering is normal), evaluation of (1) gives

$$C = \frac{\left\langle \chi^2 \Delta\epsilon \right\rangle - \left\langle \chi^2 \right\rangle \left\langle \Delta\epsilon \right\rangle}{\sqrt{\langle\chi^4\rangle + (N-2)\langle\chi^2\rangle^2}\sqrt{\left\langle (\Delta\epsilon - \langle\Delta\epsilon\rangle)^2 \right\rangle}} \underset{N \gg 1}{\sim} \frac{1}{\sqrt{N}} \to 0, \quad (2)$$

implying that in the large-N limit (multiple scattering) the correlation must vanish.

Vanishing of the correlation for normal multiple scattering. II

The joint distribution function then tends to a product of a Gaussian and a delta function

$$f(\vec{\theta}, \epsilon, l) \underset{N\to\infty}{\to} \frac{1}{\pi\langle\theta^2\rangle} e^{-\theta^2/\langle\theta^2\rangle} \delta\left(\epsilon - \langle\epsilon\rangle\right).$$

Here the mean values are in proportion $\langle\theta^2\rangle/\langle\epsilon\rangle = \langle\chi^2\rangle/\langle\Delta\epsilon\rangle$, but no correlation arises between θ and ϵ. In the presented derivation it was essential that $\langle\epsilon\rangle \sim N \gg \sqrt{\left\langle(\epsilon - \langle\epsilon\rangle)^2\right\rangle} \sim \sqrt{N}$, whereas for θ the same does not hold, because $\langle\vec{\theta}\rangle = N\langle\vec{\chi}\rangle = 0$ by axial symmetry.

But for multiple Coulomb scattering the diffusion is not normal ($\langle\chi^2\rangle = \infty$).

Transport equation. I

The correlated process of small-angle scattering of a fast electron on atoms in an amorphous substance, and of the associated ionization energy loss, is governed by a transport equation

$$\begin{aligned}\frac{\partial}{\partial l} f(\vec{\theta}, \epsilon, l) \quad &= \quad -n_a \int d\sigma_{el}(\chi) \left[f(\vec{\theta}, \epsilon, l) - f(\vec{\theta} - \vec{\chi}, \epsilon, l) \right] \\ &\quad -n_a \iint d\sigma_{in}(\chi, \Delta\epsilon) \left[f(\vec{\theta}, \epsilon, l) - f(\vec{\theta} - \vec{\chi}, \epsilon - \Delta\epsilon, l) \right], \qquad (3)\end{aligned}$$

which must be subjected to initial condition $f(\vec{\theta}, \epsilon, 0) = \delta(\vec{\theta})\delta(\epsilon)$.
The anomalous character of multiple inelastic Coulomb scattering enters through the Rutherford asymptotics of the elastic scattering

$$n_a l d\sigma_{el}(\chi) \underset{\chi \gg \chi_a}{\simeq} n_a l d\sigma_{Ruth}(\chi) = 2Z^2 \bar{\chi}_c^2 \frac{d\chi}{\chi^3}, \qquad (4)$$

and inelastic scattering differential cross-sections:

$$n_a l d\sigma_{in}(\chi, \Delta\epsilon) \underset{\chi \gg \chi_a}{\simeq} 2Z \bar{\chi}_c^2 \delta\left(\Delta\epsilon - \frac{p^2}{2m}\chi^2 \right) \frac{d\chi}{\chi^3} d\Delta\epsilon. \qquad (5)$$

Transport equation. II

Here $\frac{d\sigma_{Ruth}}{d\chi} = \frac{8\pi Z^2 e^4}{p^2 v^2 \chi^3}$, with Ze the atom nucleus charge, and v the particle velocity, p its momentum.
The factor [3]

$$\bar{\chi}_c^2 = \frac{4\pi e^4 n_a l}{p^2 v^2} \tag{6}$$

incorporates all the properties of the target except Z.

Correlated solution in the multiple scattering regime. I

The resulting distribution function depends on the following dimensionless variables:

$$\Theta = \theta / Z\bar{\chi}_c, \qquad \text{(reduced deflection angle)} \tag{7}$$

$$\lambda(Z, I, \epsilon) = \frac{2m}{p^2 Z\bar{\chi}_c^2}\epsilon - \lambda_0, \qquad \lambda_0 = \ln \frac{p^2\gamma^2 Z\bar{\chi}_c^2}{I_\delta^2} - \gamma_E + \gamma^{-2}, \tag{8}$$

with I – the mean ionization potential,

$$y_0(I, Z) = 4Z\left(Z\bar{\chi}_c^2 / \chi_{at}'^2\right)^{1+1/Z} e^{\gamma_E/Z} \gg 1 \qquad \text{(thickness parameter)}, \tag{9}$$

$$\chi_{at}' = \chi_a'^{\frac{Z}{Z+1}} \chi_{in}'^{\frac{1}{Z+1}} \qquad \text{(screening angle)}. \tag{10}$$

Correlated solution in the multiple scattering regime. II

The solution of the transport equation (3) is given by a double integral

$$\frac{p^2}{2m} Z^3 \bar{\chi}_c^{-4} f(Z, l, \theta, \epsilon) = F(Z, y_0, \Theta, \lambda)$$
$$= \frac{1}{4\pi} \int_0^{y_{\max}} dy J_0(\sqrt{y}\Theta) e^{\Omega_{el}(y_0, y)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} du e^{\lambda u + \Omega_{in}(y/4Z, u)}, \quad (11)$$

with J_0 the Bessel function,

$$\Omega_{el}(y_0, y) = -\frac{y}{4} \ln \frac{y_0}{y} \qquad \text{(the function appearing in the Molière theory),} \quad (12)$$

$$\Omega_{in}(Y, u) = (u + Y) [\ln u + \mathrm{Ein}\,(Y/u)] + u \left(1 - e^{-Y/u}\right). \quad (13)$$

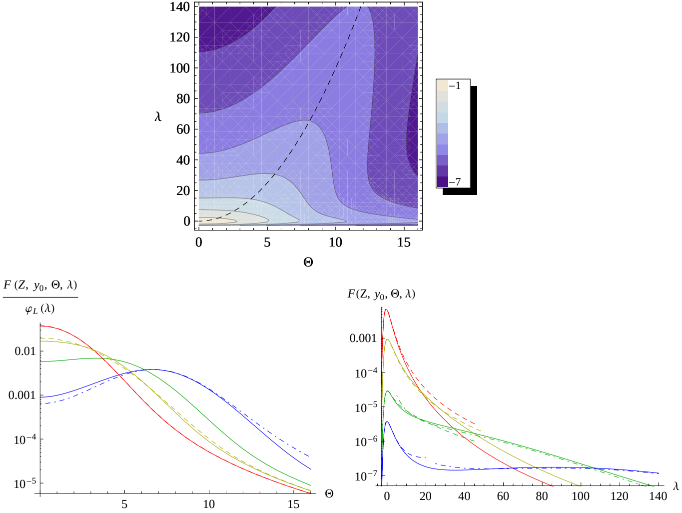


Figure 1: Correlated angle-energy loss distribution of electrons multiple scattered in hydrogen ($Z = 1$) target of reduced thickness $y_0 = 10^4$. a). Contour plot of $\log F$, with F given by Eq. (11). Dashed parabola, $\lambda = Z\Theta^2$. b). Angular distributions at fixed values of the ionization energy loss straggling variable λ. Solid curves, calculation by exact formula (11), at $\lambda = 0$ (red), $\lambda = 7$ (yellow), $\lambda = 20$ (green), $\lambda = 50$ (blue). c). Ionization energy loss distribution at fixed values of the scattering angle. Solid curve, $\Theta = 0$ (red), $\Theta = 4$ (yellow), $\Theta = 8$ (green), $\Theta = 16$ (blue).

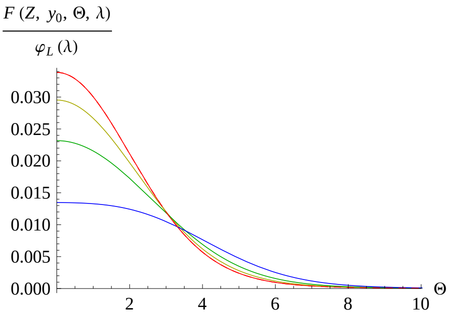


Figure 2: Angular distributions of electrons multiple scattered in a carbon ($Z = 6$) target of reduced thickness $y_0 = 10^4$, at fixed values of the ionization energy loss straggling variable λ: $\lambda = 0$ (red), $\lambda = 7$ (yellow), $\lambda = 20$ (green), $\lambda = 50$ (blue).

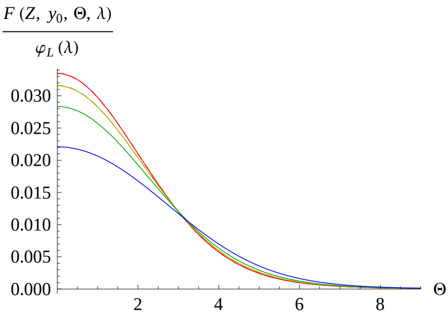
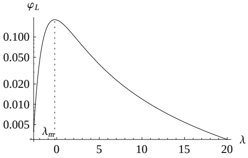


Figure 3: The same as Fig. 2, for silicon ($Z = 14$) target of reduced thickness $y_0 = 10^4$.

Normalized correlation function

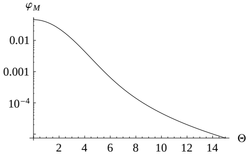
The magnitude of the correlation may be quantified by introducing the normalized correlation function

$$g(Z, y_0, \Theta, \lambda) = \frac{Z+1}{Z} \frac{F(Z, y_0, \Theta, \lambda)}{\varphi_L(\lambda)\varphi_M\left(\frac{4\chi_c^2}{\chi_{at}'^2}, \frac{\theta}{\chi_c}\right)}. \qquad (14)$$



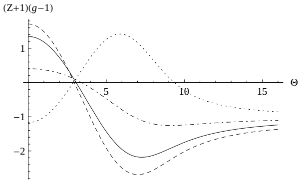
Here in the denominator

$$2\pi \int_0^\infty d\Theta \Theta F(Z, y_0, \Theta, \lambda) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} du e^{\lambda u + \Omega_{in}(0,u)} = \varphi_L(\lambda)$$



is the Landau distribution, and

$$\int_{-\infty}^{\infty} d\lambda F(Z, y_0, \Theta, \lambda) = \frac{Z}{Z+1} \varphi_M\left(\frac{4\chi_c^2}{\chi_{at}'^2}, \frac{\theta}{\chi_c}\right),$$



with $\chi_c^2 = Z(Z+1)\bar{\chi}_c^2$,

$$\varphi_M(\eta_0, \Psi) = \frac{1}{4\pi} \int_0^{\eta_{\max}} d\eta J_0\left(\sqrt{\eta}\Psi\right) e^{-\frac{\eta}{4} \ln \frac{\eta_0}{\eta}}, \qquad (15)$$

is the Molière distribution.

Angular dependence of the mean energy loss

An alternative possibility to express the correlation is to evaluate the conditional (fixed-angle) mean energy loss:

$$\bar{\epsilon}(\theta) = \frac{\int_0^\infty d\epsilon \epsilon f(\theta,\epsilon)}{\int_0^\infty d\epsilon f(\theta,\epsilon)} = \frac{p^2 Z \bar{\chi}_c^2}{2m}\left[\lambda_0 + \bar{\lambda}(\Theta)\right], \quad (16)$$

with

$$\bar{\lambda}(\Theta) = \frac{\int_{-\infty}^{\infty} d\lambda \lambda F(Z, y_0, \Theta, \lambda)}{\int_{-\infty}^{\infty} d\lambda F(Z, y_0, \Theta, \lambda)}, \quad (17)$$

$$\bar{\lambda}(Z,\eta_0,\theta/\chi_c) = \frac{2\int_{\theta/\chi_c}^{\infty} d\Psi \Psi \varphi_M(\eta_0,\Psi)}{\varphi_M(\eta_0,\theta/\chi_c)} - \ln\frac{\eta_0}{4(Z+1)} - \gamma_E, \quad \eta_0 = \frac{4\chi_c^2}{\chi_{at}'^2}. \quad (18)$$

It expresses, in the integral form, solely through the Molière distribution φ_M.

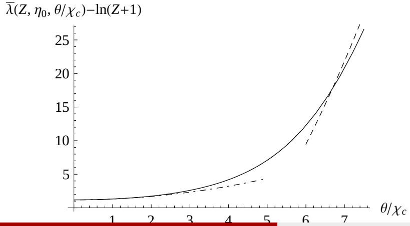


Figure 4: Solid curve, dependence of the mean energy loss defined by Eq. (18) on the scattering angle, for a reduced target thickness $\eta_0 = 10^4$.

Summary

- The amalgamation of Molière and Landau theories reveals a substantial correlation between the deflection angle and ionization energy loss for relativistic electrons in amorphous matter.
  The correlation arises at the single-scattering level, but is not erased by multiple Coulomb scattering, because of the anomalous character of the latter.
- The domain of applicability of the combined theory is just the intersection of the domains of applicability of Molière and Landau theories.
- The experimental verification of the expected correlation must be feasible with silicon targets (the lowest-Z industrial semiconductor), by looking for $\sim Z^{-1} \approx 10\%$ differences between the angular distributions measured at different values of the ionization energy loss.
  Stronger correlation effects ($\sim Z^{-1} \approx 20\%$) are expected for organic semiconductors, used nowadays for flexible electronics [7].

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