6

MICROSCOPIC DERIVATION OF THE TDGL EQUATION

6.1 Preliminaries

We have seen above how the phenomenological approach based on the GL functional allows one to describe fluctuation Cooper pairs (Bose particles) near the superconducting transition and to accounting for their contribution to different thermodynamical and transport characteristics of the system. Now we will consider of the microscopic description of fluctuation phenomena in superconductors. The development of the microscopic approach is necessary for the following reasons:

- This description allows microscopic determination of the values of the phenomenological parameters of the GL theory.
- This method is more powerful than the phenomenological GL approach and allows treatment of fluctuation effects quantitatively even far from the transition point and for magnetic fields strong as H_{c2}, taking into account the contributions of dynamical and short wavelength fluctuations.
- The electron energy relaxation times in metals are relatively large ($\tau_\varepsilon \gg \hbar/T$) which causes the electron low-frequency dynamics to be sensitive to the nearness to the superconducting transition. That is why the temperature dependence of fluctuation corrections can be determined generally speaking not only by the Cooper pair motion but also by changes in the single-electron properties.
- There are some fluctuation phenomena in which the direct Cooper pair contribution is considerably suppressed, or is even absent altogether. Among them we can mention the nuclear magnetic relaxation rate, tunnel conductivity, c-axis transport in strongly anisotropic layered metals, thermoelectric power and heat conductivity where the fluctuation pairing manifests itself by means of the indirect influence on the properties of the single-particle states of electron system.

In Part I the integration over the electron degrees of freedom in process of writing down the GL functional was performed implicitly (it was included in the GL coefficients). In its turn the averaging over the superconducting order parameter has been accomplished explicitly, by means of a functional integration over all possible configurations of the corresponding bosonic fields. In this description we have dealt with the fluctuation Cooper pair related effects only and the method of the functional integration turned out to be simple and effective for their description. In order to take into account more delicate consequences

of the fluctuation pairing in the system of interacting electrons we will develop in the following sections the diagrammatic method of Matsubara temperature Green's functions since it is more adequate for the description of the properties of a Fermi system of interacting electrons.

6.2 The Cooper channel of electron–electron interaction: the fluctuation propagator

Let us start the microscopic description of fluctuation phenomena in a superconductor from the electron Hamiltonian. We will choose it in the simple BCS form:[33]

$$\mathcal{H} = \sum_{\mathbf{p},\sigma} \xi(\mathbf{p}) \widetilde{\psi}^{+}_{\mathbf{p},\sigma} \widetilde{\psi}_{\mathbf{p},\sigma} - g \sum_{\mathbf{p},\mathbf{p}',\mathbf{q},\sigma,\sigma'} \widetilde{\psi}^{+}_{\mathbf{p}+\mathbf{q},\sigma} \widetilde{\psi}^{+}_{-\mathbf{p},-\sigma} \widetilde{\psi}_{-\mathbf{p}',-\sigma'} \widetilde{\psi}_{\mathbf{p}'+\mathbf{q},\sigma'}. \tag{6.1}$$

The momentum conservation law side by side with singlet pairing is already taken into account in the interaction term. Here $\xi(\mathbf{p}) = E(\mathbf{p}) - E_F$ is the quasiparticle energy measured from the Fermi level; $-g$ is the negative constant of electron–electron attraction which is supposed to be momentum independent and different from zero in a narrow domain of momentum space in the vicinity of the Fermi surface where

$$p_F - \frac{\omega_D}{v_F} < |\mathbf{p}|, |\mathbf{p}'| < p_F + \frac{\omega_D}{v_F}.$$

$\widetilde{\psi}^{+}_{\mathbf{p},\sigma}$ and $\widetilde{\psi}_{\mathbf{p},\sigma}$ are the creation and annihilation field operators in the Heisenberg representation, so the first term is just the kinetic energy of the noninteracting Fermi gas. The interaction term is chosen in the traditional form characteristic of the electron-phonon mechanism of superconductivity.[34]

For the description of the properties of an interacting electron system with the Hamiltonian (6.1) we will use the formalism of the Matsubara temperature diagrammatic technique. The state of a noninteracting quasiparticle is described by its Green function

$$G(\mathbf{p}, \varepsilon_n) = \frac{1}{i\varepsilon_n - \xi(\mathbf{p})}, \tag{6.2}$$

where $\varepsilon_n = (2n+1)\pi T$ is a fermion Matsubara frequency.

The effective electron–electron attraction leads to a reconstruction of the ground state of the electron system which formally manifests itself in the appearance at the critical temperature of a pole in the two particle Green function

$$\mathcal{L}(p, p', q) = \langle T_\tau [\widetilde{\psi}_{p+q,\sigma} \widetilde{\psi}_{-p,-\sigma} \widetilde{\psi}^{+}_{p'+q,\sigma'} \widetilde{\psi}^{+}_{-p',-\sigma'}] \rangle, \tag{6.3}$$

where T_τ is the time ordering operator and $4D$ vector notations are used [228]. The two particle Green function can be expressed in terms of the vertex part

[33] We suppose that reader is familiar with the BCS formulation of the theory of superconductivity (see e.g. [228]).

[34] Fluctuations in the framework of more realistic Eliashberg [229] model of superconductivity were studied by Narozhny [230]. He demonstrated that the strong coupling does not change drastically the results of the weak coupling approximation.

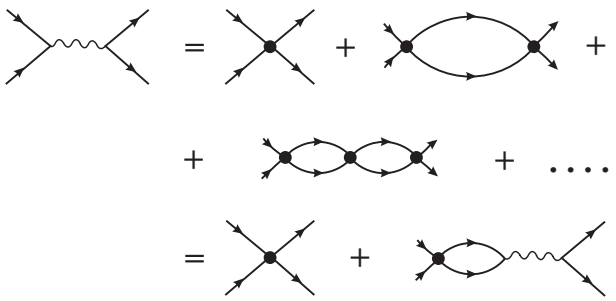

FIG. 6.1. The Dyson equation for the fluctuation propagator (wavy line) in the ladder approximation. Solid lines represent one-electron Green's functions, bold points correspond to the model electron–electron interaction.

[228]. In the case under consideration, it is the vertex part of the electron–electron interaction in the Cooper channel $L(\mathbf{q}, \Omega_k)$ which will be called the fluctuation propagator below. The Dyson equation for $L(\mathbf{q}, \Omega_k)$, accounting for the e–e attraction in the ladder approximation, is represented graphically in Fig. 6.1. It can be written down analytically as

$$L^{-1}(\mathbf{q}, \Omega_k) = -g^{-1} + \Pi(\mathbf{q}, \Omega_k), \tag{6.4}$$

where the polarization operator $\Pi(\mathbf{q}, \Omega_k)$ is defined as a loop of two single-particle Green's functions in the particle–particle channel:

$$\Pi(\mathbf{q}, \Omega_k) = T \sum_{\varepsilon_n} \int \frac{d^3\mathbf{p}}{(2\pi)^3} G(\mathbf{p} + \mathbf{q}, \varepsilon_{n+k}) G(-\mathbf{p}, \varepsilon_{-n}). \tag{6.5}$$

Let us emphasize, that the two quantities introduced above, $\mathcal{L}(p, p', q)$ and $L(q)$, are closely connected with each other. The former being integrated over momenta p and p' becomes an average of the product of the two order parameters:

$$\int dp dp' \mathcal{L}(p, p', q) = \frac{1}{g^2} \left\langle \Delta_q \Delta_q^* \right\rangle, \tag{6.6}$$

where Δ_q is the superconducting gap proportional to the condensate wave function Ψ. The value (6.6) may be written down by means of the polarization operator introduced above as

$$\int dp dp' \mathcal{L}(p, p', q) = -\frac{\Pi}{1 - g\Pi} = \frac{\Pi}{g} L. \tag{6.7}$$

Comparing this equation with Eq. (6.4) for the fluctuation propagator we see that the corresponding expressions are very similar. After analytic continuation to the real frequencies the fluctuation propagator $L(q, i\Omega)$ coincides (up to a constant) with the quantity defined by Eq. (6.6).

One can calculate the propagator (6.4) using the one-electron Green functions of the normal metal (6.2). For sake of convenience of future calculations let us define the correlator of two one-electron Green functions

$$\mathcal{P}(\mathbf{q},\varepsilon_1,\varepsilon_2) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} G(\mathbf{p}+\mathbf{q},\varepsilon_1)G(-\mathbf{p},\varepsilon_2)$$
$$= 2\pi\nu\theta(-\varepsilon_1\varepsilon_2)\left\langle \frac{1}{|\varepsilon_1-\varepsilon_2| + i\Delta\xi(\mathbf{q},\mathbf{p})|_{\epsilon(\mathbf{p})=E_F}} \right\rangle_{F.S.}, \quad (6.8)$$

where $\theta(x)$ is Heavyside step function, ν is the one-electron density of states, $<>_{\text{F.S.}} = \int \frac{d\Omega_{\mathbf{p}}}{4\pi}$ means the averaging over the Fermi surface,

$$\Delta\xi(\mathbf{q},\mathbf{p})|_{\epsilon(\mathbf{p})=E_F} = [\xi(\mathbf{q}+\mathbf{p}) - \xi(-\mathbf{p})]|_{\epsilon(\mathbf{p})=E_F} \approx (\mathbf{v_p q})_{\xi(\mathbf{p})=0}. \quad (6.9)$$

The last approximation is valid not too far from the Fermi surface, i.e. when $(\mathbf{v_p q})_{\xi(\mathbf{p})=0} \ll E_F$.

It is impossible to carry out the angular averaging in (6.8) for an arbitrary anisotropic spectrum. Nevertheless, in the following calculations of fluctuation effects in the vicinity of critical temperature only small momenta $\mathbf{v_p q} \ll T$ will be involved in the integrations, so we can restrict our consideration here to this region, where one can expand the integrand in powers of $\mathbf{v_p q}$. Indeed, the presence of $\theta(-\varepsilon_1\varepsilon_2)$ leaves the difference of the two fermionic frequencies in (6.8) to be of the order of the temperature which allows this expansion. The first term in $\mathbf{v_p q}$ will evidently be averaged out, so with quadratic accuracy one can find:

$$\mathcal{P}(\mathbf{q},\varepsilon_1,\varepsilon_2) = 2\pi\nu\frac{\theta(-\varepsilon_1\varepsilon_2)}{|\varepsilon_1-\varepsilon_2|}\left(1 - \frac{\langle(\mathbf{v_p q})^2\rangle_{F.S.}}{|\varepsilon_1-\varepsilon_2|^2}\right). \quad (6.10)$$

Now one can calculate the polarization operator $\Pi(\mathbf{q},\Omega_k)$ which in the discussed case of a clean superconductor is just the correlator (6.8) summed over the electronic Matsubara frequencies ε_n:

$$\Pi(\mathbf{q},\Omega_k) = T\sum_{\varepsilon_n}\mathcal{P}(\mathbf{q},\varepsilon_{n+k},\varepsilon_{-n}) = \nu\left[\sum_{n\geq 0}\frac{1}{n+1/2+\frac{|\Omega_k|}{4\pi T}}\right.$$
$$\left. - \frac{\langle(\mathbf{v_p q})^2\rangle_{F.S.}}{(4\pi T)^2}\sum_{n=0}^{\infty}\frac{1}{\left(n+1/2+\frac{|\Omega_k|}{4\pi T}\right)^3}\right]. \quad (6.11)$$

The calculation of the sums in (6.11) can be carried out in terms of logarithmic derivatives of the Γ function $\psi^{(n)}(x)$ already familiar to us (see Appendix B).

It worth mentioning that the first sum is well-known in the BCS theory, one can recognize in it the so-called "Cooper logarithm"; its logarithmic divergence at the upper limit ($\psi(x \gg 1) \approx \ln x$) is cut-off by the Debye energy ($N_{\max} = \omega_D/2\pi T$) and one gets:

$$\frac{1}{\nu}\Pi(\mathbf{q}, \Omega_k) = \psi\left(\frac{1}{2} + \frac{|\Omega_k|}{4\pi T} + \frac{\omega_D}{2\pi T}\right) - \psi\left(\frac{1}{2} + \frac{|\Omega_k|}{4\pi T}\right) + \frac{\langle(\mathbf{v_p q})^2\rangle_{F.S.}}{2(4\pi T)^2}\psi''\left(\frac{1}{2} + \frac{|\Omega_k|}{4\pi T}\right). \tag{6.12}$$

The critical temperature in the BCS theory is determined as the temperature $T_{\rm c}$ at which the pole of $L(0, 0, T_{\rm c})$ occurs

$$L^{-1}(\mathbf{q} = 0, \Omega_k = 0, T_{\rm c}) = -g^{-1} + \Pi(0, 0, T_{\rm c}) = 0.$$

Taking into account that $\psi\left(\frac{1}{2}\right) = -C - 2\ln 2$, where $C = 0.577...$ is the Euler constant, one can express the critical temperature in the famous BCS form:

$$T_{\rm c} = \frac{2\gamma_E}{\pi}\omega_D \exp\left(-\frac{1}{\nu g}\right). \tag{6.13}$$

Here $\gamma_E = e^{C_{\rm Euler}} = 1.78...$

Introducing the reduced temperature $\epsilon = \ln(T/T_{\rm c})$, one can write the propagator as

$$L^{-1}(\mathbf{q}, \Omega_k) = -\nu\left[\epsilon + \psi\left(\frac{1}{2} + \frac{|\Omega_k|}{4\pi T}\right) - \psi\left(\frac{1}{2}\right) - \frac{\langle(\mathbf{v_p q})^2\rangle_{F.S.}}{2(4\pi T)^2}\psi''\left(\frac{1}{2} + \frac{|\Omega_k|}{4\pi T}\right)\right]. \tag{6.14}$$

We found (6.14) for bosonic imaginary Matsubara frequencies $i\Omega_k = 2\pi i T k$. These frequencies are necessary for the calculation of fluctuation contributions to any thermodynamic characteristics of the system.

In the vicinity of the transition point one can restrict oneself in summations of the expressions with $L(\mathbf{q}, \Omega_k)$ over Matsubara frequencies to the so-called static approximation, taking into account the term with $\Omega_k = 0$ only, which turns out to be the most singular term in $\epsilon \ll 1$. This approximation physically means that the product of Heisenberg field operators $\widetilde{\psi}_{p,\sigma}\widetilde{\psi}_{-p,-\sigma}$ appears here like a classical field Ψ, which in the phenomenological approach corresponds to the Cooper pair wave function and in the vicinity of critical temperature is proportional to the fluctuation order parameter. Having in mind the GL region of temperatures we restricted ourselves above to the assumption of small momenta $\mathbf{v_p q} \ll T$. In these conditions the static propagator reduces to

$$L(\mathbf{q}, 0) = -\frac{1}{\nu}\frac{1}{\epsilon + \xi^2\mathbf{q}^2}. \tag{6.15}$$

With an accuracy of a numerical factor and the total sign this correlator coincides with the expression (3.5) for $\left\langle|\Psi_{\mathbf{q}}|^2\right\rangle$. By this expression we also have

finally obtained the microscopic value of the coherence length ξ for a clean superconductor with an isotropic D-dimensional Fermi surface which was often mentioned previously (compare with (A.3))

$$\xi^2_{(D)} = \frac{7\zeta(3)\mathbf{v}^2_{\mathbf{F}}}{16D\pi^2T^2}. \tag{6.16}$$

In order to describe the fluctuation contributions to transport phenomena one has to start from the analytic continuation of the propagator (6.14) from the discrete set of $\Omega_k \geq 0$ to the whole upper half-plane of imaginary frequencies. The analytic properties of $\psi^{(n)}(x)$ functions (which have poles at $x = 0, -1, -2, ...$) allow one to obtain the retarded propagator $L^R(\mathbf{q}, -i\Omega)$ by simple substitution $i\Omega_k \to \Omega$. For small $|\Omega| \ll T$ the $\psi-$functions can be expanded in $-i\Omega/4\pi T$ and the propagator acquires the simple pole form:

$$L^R(\mathbf{q}, \Omega) = -\frac{1}{\nu}\frac{1}{-\frac{i\pi}{8T}\Omega + \epsilon + \xi^2\mathbf{q}^2} = \frac{8T}{\pi\nu}\frac{1}{i\Omega - \left(\tau_{\mathrm{GL}}^{-1} + \frac{8T}{\pi}\xi^2\mathbf{q}^2\right)}. \tag{6.17}$$

This expression provides us with the microscopic value of the GL relaxation time $\tau_{\mathrm{GL}} = \frac{\pi}{8(T-T_c)}$, widely used above in the phenomenological theory. Moreover, comparison of the microscopically derived Eq. (6.17) with the phenomenological expressions (3.3), (3.8) and (3.19) shows that $\alpha T_c = \nu$ and $\gamma_{\mathrm{GL}} = \pi\alpha/8 = \pi\nu/8T_c$ (see Appendix A).

In evaluating $L(\mathbf{q}, \Omega_k)$ we neglected the effect of fluctuations on the one-electron Green functions. This is correct when fluctuations are small, i.e. not too near the transition temperature. The exact criterion of this approximation will be discussed in the following.

6.3 Diagrammatic representation of fluctuation corrections

The propagator (3.8) can be considered as the effective inter-electron interaction, therefore using the rules of the diagrammatic technique [228] one can easily write down the series of the perturbation theory diagrams for the thermodynamical potential of the system. These diagrams are presented by the closed loops of increasing complexity; the corresponding first order correction is demonstrated in Fig. 6.2 (a). Drawing the diagrams with fluctuation propagator one has to remember the rule concerning the arrows directions: the diagrams represent the interaction in the particle–particle channel, hence every couple of solid lines entering or exiting the propagator must be directed contrarily. This rule restrains the existence of some types of diagrams. For example, the second-order diagram with the overlapping wavy lines cannot be realized.

The fluctuation corrections to the magnetization, entropy and other first derivatives of the thermodynamical potential can be obtained by direct graphical differentiation of the corresponding diagram (Fig. 6.2a). Indeed, the Green function derivatives in general case can be expressed by means of the Ward

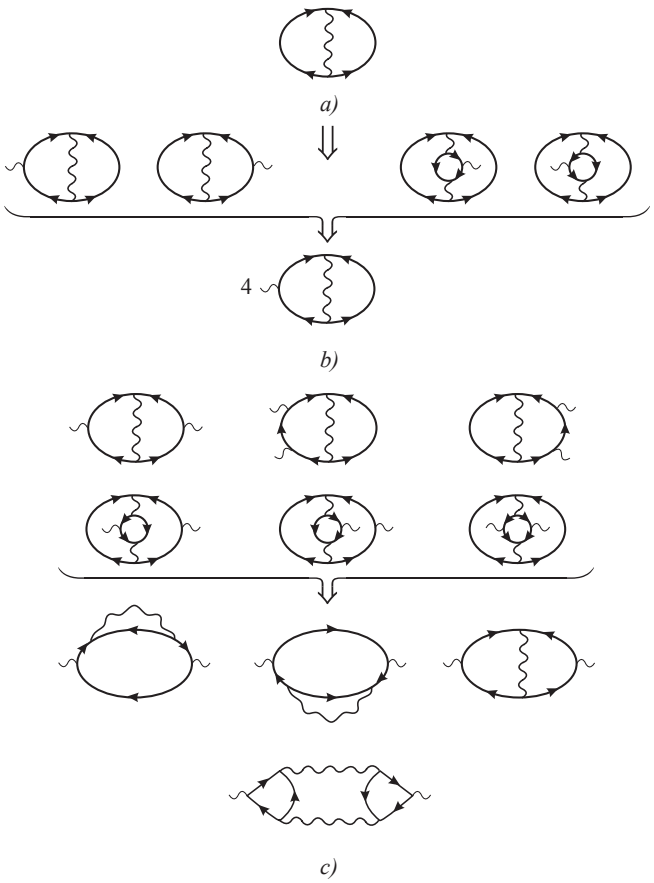

FIG. 6.2. The diagrammatic representation of the first-order fluctuation corrections to: (a) thermodynamical potential; (b) magnetization, entropy; (c) diamagnetic susceptibility, heat capacity. Solid and wavy lines correspond to bare one-electron Green functions and fluctuation propagators respectively. The external vertices correspond to the differentiation procedure.

identities [228] relating them with some convolutions of the Green function's square and corresponding vertex parts. In our case, dealing with the bare Green functions, the situation is trivial. For example,

$$\frac{\partial G^{(0)}\left(\mathbf{p},\varepsilon_n\right)}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \frac{1}{i\varepsilon_n - \xi\left(\mathbf{p}\right)} = G^{(0)}\left(\mathbf{p},\varepsilon_n\right) \mathbf{v} G^{(0)}\left(\mathbf{p},\varepsilon_n\right). \tag{6.18}$$

Hence, graphically the differentiation means the interruption of the Green function by some vertex (in the case of the differentiation over momentum this vertex is the velocity operator). One has to remember, that the fluctuation propagator

itself contains the loops of the bare Green functions, so looking for the complete set of the fluctuation diagrams in the certain order of perturbation theory it is necessary to extract explicitly these loops from fluctuation propagator and to differentiate them too. Such a procedure is demonstrated in Fig. 6.2(b). The first two diagrams are the result of the direct differentiation of the first-order diagram for the free energy. The third and fourth diagrams appear as the derivatives of the propagator, but as is seen, topologically they are equivalent to the first two.

In such a way one can draw the diagrams corresponding to the first-order fluctuation corrections for diamagnetic susceptibility and heat capacity. They are presented in Fig. 6.2(c). The first three follow directly from Fig. 6.2(b), as the next order derivatives of the first two diagrams. The differentiation of the hidden in fluctuation propagator electron loop results in the appearance of the nontrivial last diagram in Fig. 6.2(c). Interrupting of the wavy line in the diagram results in the increase of the number of propagators in it and hence increases its order in the perturbation series. One can see that the diagram equivalent to the last one in Fig. 6.2(c) can be obtained as the second derivative of the second diagram in Fig. 6.2(a).

Collecting carefully the equivalent diagrams appearing in the process of differentiation one can define the corresponding combinatorial factor.

6.4 Superconductor with impurities

6.4.1 *Accounting for electron scattering by impurities*

In order to study fluctuations in real systems like superconducting alloys one has to perform an impurity average in the graphical equation for the fluctuation propagator (see Fig. 6.1). This procedure can be done in the framework of the Abrikosov–Gor'kov approach [228], which we shortly recall below.

The one-particle Green function can be presented in the form of the expansion over products of the exact eigenfunctions of the Hamiltonian $\widehat{H}$:

$$G(\mathbf{r},\mathbf{r}',\varepsilon_n)=\sum_i \frac{\psi_i(\mathbf{r})\,\psi_i^*(\mathbf{r}')}{i\varepsilon_n-E_i}, \tag{6.19}$$

where $\psi_i(\mathbf{r})$ are wave-functions corresponding to the exact electron states of this Hamiltonian. In order to accounting for the effect of impurities with potential $U(\mathbf{r})$ let us start from the corresponding equation for the electron Green function:

$$\left(E-U(\mathbf{r})-\widehat{H}\right)G_E(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'). \tag{6.20}$$

If we solve this equation using the perturbation theory for the impurity potential and average the solution, then the average product of two Green functions, can be presented as series, each term of which is associated with a graph drawn according the rules of a diagrammatic technique (see Fig. 6.3). In this technique solid lines correspond to bare Green functions and dashed lines to random potential correlators. We assume that the impurity system random potential $U(r)$

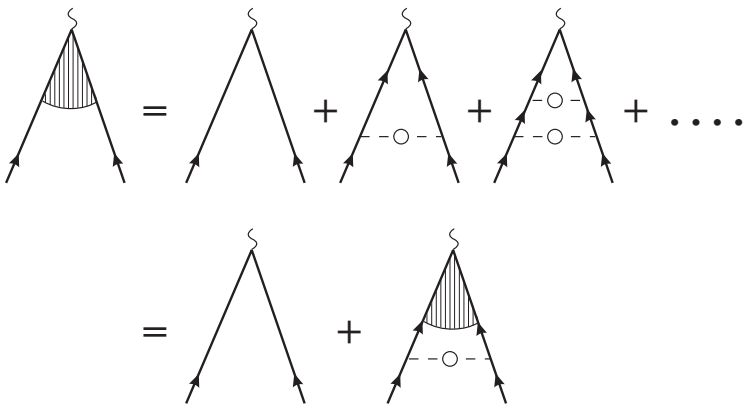

FIG. 6.3. The equation for the vertex part $\lambda(q, \omega_1, \omega_2)$ in the ladder approximation. Solid lines correspond to bare one-electron Green functions and dashed lines to the impurity random potential correlators.

is distributed according to the Gauss law. In this case the high rank correlators can be represented as the products of pair correlators. The latter for impurities with short-range potential can be assumed as δ-functions:

$$\langle U(r)\rangle = 0,\ \langle U(r)U(r')\rangle = \left\langle U^2\right\rangle \delta(r-r'), \tag{6.21}$$

where the angle brackets denote averaging over the impurity configuration. Equation (6.21) corresponds to the Born approximation for the electron interaction with short range impurities, and $\left\langle U^2\right\rangle = n_i \left(\int V(\mathbf{r})d\mathbf{r}\right)^2$ where n_i is the impurity concentration and $V(\mathbf{r})$ is the potential of the single impurity.

In conductors (far enough from the metal–insulator transition) the mean free path is much greater than the electron wavelength $l \gg \lambda = 2\pi/p_F$ (which in practice means the mean free path up to tens of interatomic distances). In the case of the electron spectra with dimensionality $D > 1$ the angular integration in momentum space reduces considerably the contribution of the diagrams with intersecting impurity lines that allows us to omit them up to the leading approximation in $(p_F l)^{-1}$ [228]. In this approximation the one-electron Green function keeps the same form as the bare one (6.2) with the only substitution

$$\varepsilon_n \ \Rightarrow\ \widetilde{\varepsilon}_n = \varepsilon_n + \frac{1}{2\tau}\mathrm{sign}(\varepsilon_n), \tag{6.22}$$

where $1/\tau = 2\pi\nu\left\langle U^2\right\rangle$ is the frequency of elastic collisions.

Another effect of the coherent scattering on the same impurities by both electrons forming a Cooper pair is the renormalization of the vertex part $\lambda(\mathbf{q}, \varepsilon_1, \varepsilon_2)$ in the particle–particle channel. Let us demonstrate the details of its calculation. The renormalized vertex $\lambda(\mathbf{q}, \varepsilon_1, \varepsilon_2)$ is determined by a graphical equation of the ladder type (see Fig. 6.3). Here after the averaging over the impurity configurations the value $\left\langle U^2\right\rangle = 1/2\pi\nu\tau$ is associated with the dashed line.

The corresponding integral equation is

$$\lambda(\mathbf{q},\varepsilon_1,\varepsilon_2)=1+\frac{1}{2\pi\nu\tau}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\lambda(\mathbf{q},\varepsilon_1,\varepsilon_2)G(\mathbf{p}+\mathbf{q},\widetilde{\varepsilon}_1)G(-\mathbf{p},\widetilde{\varepsilon}_2), \tag{6.23}$$

which, in the momentum representation, is reduced to the algebraic one

$$\lambda^{-1}(\mathbf{q},\varepsilon_1,\varepsilon_2)=1-\frac{1}{2\pi\nu\tau}\mathcal{P}(\mathbf{q},\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2), \tag{6.24}$$

The correlator $\mathcal{P}(\mathbf{q},\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2)$ was defined by Eq. (6.8) but now it depends on the arguments $\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2$.

Now one has to perform a formal averaging of the general expression (6.8) over the Fermi surface ($\langle\ ...\rangle_{\mathrm{F.S.}}$). Restricting consideration to small momenta

$$\Delta\xi(\mathbf{q},\mathbf{p})|_{|\mathbf{p}|=p_F}\ll|\widetilde{\varepsilon}_1-\widetilde{\varepsilon}_2|, \tag{6.25}$$

the calculation of $\lambda(\mathbf{q},\omega_1,\omega_2)$ for the practically important case of an arbitrary spectrum can be done analogously to (6.10). Indeed, expanding the denominator of (6.8) one can find

$$\lambda(\mathbf{q},\varepsilon_1,\varepsilon_2)=\frac{|\widetilde{\varepsilon}_1-\widetilde{\varepsilon}_2|}{|\varepsilon_1-\varepsilon_2|+\frac{\langle(\Delta\xi(\mathbf{q},\mathbf{p})|_{|\mathbf{p}|=p_F})^2\rangle_{\mathrm{F.S.}}}{\tau|\widetilde{\varepsilon}_1-\widetilde{\varepsilon}_2|^2}\theta(-\varepsilon_1\varepsilon_2)}. \tag{6.26}$$

The assumed restriction on momenta is not too severe and is almost always satisfied in calculations of fluctuation effects at temperatures near T_{c}. In this region of temperatures the effective propagator momenta are determined by $|\mathbf{q}|_{\mathrm{eff}}\sim[\xi^{\mathrm{GL}}(T)]^{-1}=\xi^{-1}\sqrt{\epsilon}\ll\xi^{-1}$, while the Green function $\mathbf{q}$-dependence becomes important for much larger momenta $q\sim\min\{l_T^{-1},l^{-1}\}$,[35] which is equivalent to the limit of the condition (6.25).

The average in (6.26) can be calculated for some particular types of spectra. For example, in the cases of $2D$ and $3D$ isotropic spectra it is expressed in terms of the diffusion coefficient $\mathcal{D}_{(D)}$:

$$\langle(\Delta\xi(\mathbf{q},\mathbf{p})|_{|\vec{p}|=p_F})^2\rangle_{\mathrm{F.S.}(D)}=\tau^{-1}\mathcal{D}_{(D)}q^2=\frac{v_F^2q^2}{D}. \tag{6.27}$$

6.4.2 *Propagator for superconductor with impurities*

In section 6.1, in the process of the microscopic derivation of the TDGL equation, the fluctuation propagator was introduced. This object is of first importance for the microscopic fluctuation theory and it has to be generalized for the case of an impure metal with an anisotropic electron spectrum. This is easy to do using the averaging procedure presented in the previous section. Formally, it is enough

[35] $l_T=\sqrt{\mathcal{D}_{(D)}/T}$ is so-called "temperature length". In clean superconductor it coincides with the accuracy of the numerical factor with ξ.

to use in Eq. (6.4) the polarization operator $\Pi(\mathbf{q},\Omega_k)$ averaged over impurity positions, which can be expressed in terms of $\mathcal{P}(\mathbf{q},\widetilde{\varepsilon}_{n+k},\widetilde{\varepsilon}_{-n})$ introduced above:

$$\Pi(\mathbf{q},\Omega_k) = T\sum_{\varepsilon_n}\lambda(q,\varepsilon_{n+k},\varepsilon_{-n})\mathcal{P}(\mathbf{q},\widetilde{\varepsilon}_{n+k},\widetilde{\varepsilon}_{-n})$$
$$= T\sum_{\varepsilon_n}\frac{1}{\mathcal{P}^{-1}(\mathbf{q},\widetilde{\varepsilon}_{n+k},\widetilde{\varepsilon}_{-n}) - 1/\left(2\pi\nu\tau\right)}. \tag{6.28}$$

For relatively small $\mathbf{q}$ ($\Delta\xi(\mathbf{q},\mathbf{p})|_{|E(\mathbf{p})|=E_F} \ll |\widetilde{\varepsilon}_{n+k}-\widetilde{\varepsilon}_{-n}| \sim \max\{T,\tau^{-1}\}$) and $\Omega \ll T$ one can find an expression for the fluctuation propagator, which can be useful in studies of fluctuation effects near T_c ($\epsilon \ll 1$) for the dirty and intermediate but not very clean case ($T\tau \ll 1/\sqrt{\epsilon}$). Substituting in Eq. (6.28) Eq. (6.10) for $\mathcal{P}(\mathbf{q},\widetilde{\varepsilon}_{n+k},\widetilde{\varepsilon}_{-n})$ and expanding in powers of $\langle(\mathbf{v_p q})^2\rangle_{\text{F.S.}}/|\widetilde{\varepsilon}_{n+k}-\widetilde{\varepsilon}_{-n}|^2$ one can find

$$\Pi(\mathbf{q},\Omega_k) = 2\pi\nu T\sum_{\varepsilon_n}\frac{\theta(\varepsilon_{n+k}\varepsilon_n)\left(1-\frac{\langle(\mathbf{v_p q})^2\rangle_{F.S.}}{|\widetilde{\varepsilon}_{n+k}-\widetilde{\varepsilon}_{-n}|^2}\right)}{|\widetilde{\varepsilon}_{n+k}-\widetilde{\varepsilon}_{-n}|\left[1-\frac{1}{\tau}\frac{1}{|\widetilde{\varepsilon}_{n+k}-\widetilde{\varepsilon}_{-n}|}+\frac{\langle(\mathbf{v_p q})^2\rangle_{F.S.}}{\tau|\widetilde{\varepsilon}_{n+k}-\widetilde{\varepsilon}_{-n}|^3}\right]}$$
$$= 4\pi\nu T\left[\sum_{n=0}\frac{1}{2\varepsilon_n+\Omega_k}-\sum_{n=0}\frac{\langle(\mathbf{v_p q})^2\rangle_{F.S.}}{\left(2\varepsilon_n+\Omega_k+\frac{1}{\tau}\right)\left(2\varepsilon_n+\Omega_k\right)^2}\right]. \tag{6.29}$$

Being in the vicinity of the transition temperature we can omit the Ω_k - dependence of the term proportional to small $\langle(\mathbf{v_p q})^2\rangle_{\text{F.S.}}$. Performing the remaining summation over Matsubara frequencies we find the result very similar Eq. (6.12) for the polarization operator in clean case:

$$\frac{1}{\nu}\Pi(\mathbf{q},\Omega_k) = \ln\frac{\omega_D}{2\pi T}-\psi\left(\frac{1}{2}+\frac{|\Omega_k|}{4\pi T}\right)+\tau^2\langle(\mathbf{v_p q})^2\rangle_{\text{F.S.}}$$
$$\times\left[\psi(\frac{1}{2}+\frac{1}{4\pi T\tau})-\psi(\frac{1}{2})-\frac{1}{4\pi T\tau}\psi'(\frac{1}{2})\right]. \tag{6.30}$$

Now one can write the fluctuation propagator $L^R(q,\Omega)$ in the vicinity of the transition and for the small arguments essential for our needs, $\Omega \ll T$, $\xi(T\tau)q \ll 1$ in the form completely coinciding with Eq. (6.17):

$$L^R(\mathbf{q},\Omega) = -\frac{1}{\nu}\frac{1}{\epsilon - i\frac{\pi\Omega}{8T}+\xi^2_{(D)}(T\tau)\mathbf{q}^2} \tag{6.31}$$
$$= -\frac{1}{\nu}\frac{1}{\epsilon+\frac{\pi}{8T}\left(-i\Omega+\widehat{\mathcal{D}}\mathbf{q}^2\right)}. \tag{6.32}$$

The only difference with respect to the clean case is in the appearance of a natural dependence of the effective coherence length on the elastic relaxation time. In the isotropic D-dimensional case it can be written as

$$\xi^2_{(D)}(T\tau) = \frac{\pi}{8T}\widehat{\mathcal{D}} = (4m\alpha T)^{-1} = \eta_{(D)}$$

$$= -\frac{\tau^2 v_F^2}{D}\left[\psi\left(\frac{1}{2}+\frac{1}{4\pi T\tau}\right)-\psi\left(\frac{1}{2}\right)-\frac{1}{4\pi T\tau}\psi'\left(\frac{1}{2}\right)\right] \quad (6.33)$$

(we introduced here the parameter $\eta_{(D)}$ frequently used in the microscopic theory).[36] Let us stress that the phenomenological coefficient $\gamma_{\rm GL}$ turns out to be equal to the same value $\pi\nu/8T$ as in clean case, and hence does not depend on the impurity concentration.

We have introduced here the generalization of the notion of diffusion coefficient $\widehat{\mathcal{D}}$:

$$\widehat{\mathcal{D}} = \begin{cases} \tau v_F^2/D, \, T\tau \ll 1, \\ v_F^2/2DT, \, T\tau \gg 1, \end{cases} \quad (6.34)$$

which in the clean case is formed by means of the substitution $\tau \to 1/2T$. In the case of a complex spectrum $\widehat{\mathcal{D}}$ can become operator (see the following section).

One has to remember that Eq. (6.32) was derived under the assumption of small momenta $\Delta\xi(\mathbf{q},\mathbf{p})|_{|E(\mathbf{p})|=E_F} \ll |\widetilde{\varepsilon}_{n+k}-\widetilde{\varepsilon}_{-n}| \sim \max\{T,\tau^{-1}\}$, so the range of its applicability is restricted to the region of long wavelength fluctuations ($\epsilon = \ln(T/T_{\rm c}) \ll 1$), where the integrands of diagrammatic expressions have singularities at small momenta of the Cooper pair center of mass.

Another useful formula for the polarization operator can be carried over from Eq. (6.29) for the dirty case ($T\tau \ll 1$). In this approximation the summation can be performed without the expansion over small q and general expression for $\Pi(\mathbf{q},\Omega_k)$ valid for $\max\left\{\Omega_k, \mathcal{D}q^2\right\} \ll \omega_D$ reads as

$$\Pi(\mathbf{q},\Omega_k) = 4\pi\nu T\sum_{\varepsilon_n>0}\frac{1}{2\varepsilon_n+|\Omega_k|+\tau\langle(\mathbf{v_p q})^2\rangle_{F.S.}}$$
$$= \nu\left[\ln\frac{\omega_D}{2\pi T}-\psi\left(\frac{1}{2}+\frac{|\Omega_k|+\mathcal{D}q^2}{4\pi T}\right)\right]. \quad (6.35)$$

This formula will be useful for study of fluctuation phenomena far from transition temperature and other cases.

Finally let us express the Gi parameter for the important $2D$ case in terms of the microscopic parameter $\eta_{(2)}$. In accordance with (2.87) and the definition (6.33):

$$Gi_{(2)}(T\tau) = \frac{7\zeta(3)}{16\pi^2}\,\frac{1}{mT_{\rm c}\eta_{(2)}(T\tau)}. \quad (6.36)$$

One can see that this general definition in the limiting cases of clean and dirty metals results in the same values $Gi_{(2c)}$ and $Gi_{(2d)}$ as were reported in Table 2.1.

[36] Let us recall that its square determines the product of the GL parameter α and the Cooper pair mass entering in the GL functional. In the clean case we supposed the latter equal to two free electron masses and defined α in accordance with (A.12). As we just have seen in the case of the impure superconductor ξ depends on impurity concentration and this dependence, in principle, can be attributed both to α or m. For our further purposes it is convenient to leave α in the same form (A.12) as in the case of a clean superconductor. The Cooper pair mass in this case becomes dependent on the electron mean free pass what physically can be attributed to the diffusion motion of the electrons forming the pair.

6.5 Layered superconductor

The main object for applications of the microscopic theory of superconducting fluctuations in this book will be a layered superconductor. We have already studied above the manifestation of fluctuations in some of its properties in the framework of the phenomenological Lawrence–Doniach model and were convinced in its richness. The quasi-$2D$ fluctuation Cooper pair spectrum allows us to obtain in a unique way both $3D$ and $2D$ results, to study crossovers between them, to elucidate the role of anisotropy, to get directly results for thin superconducting films, to study the out-of-plane transport, to apply corresponding results to the tunneling and Josephson systems etc. Finally, the class of HTS most investigated during the last two decades consists mainly of layered systems and it will be potential target of our consideration.

Intending to operate in the framework of the microscopic theory we begin our discussion from the properties of the quasiparticle normal state energy spectrum and the choice of the appropriate model. While models of layered superconductors with several conducting layers per unit cell and with either intralayer or interlayer pairing have been considered [231], it has been shown [232] that all of these models give rise to a Josephson pair potential that is periodic in k_z, the wave-vector component parallel to the c-axis, with period $2\pi/s$ where s is the c-axis repeat distance. While such models differ in their superconducting densities of states, they all give rise to qualitatively similar fluctuation propagators, which differ only in the precise definitions of the parameters and in the precise form of the Josephson coupling potential. Ignoring the rather unimportant differences between such models in the Gaussian fluctuation regime above $T_{\rm c}$, we therefore consider the simplest model of a layered superconductor, in which there is one layer per unit cell, with intralayer pairing.

Some remarks regarding the normal–state quasiparticle momentum relaxation time are necessary. In the "old" layered superconductors the materials were generally assumed to be in the dirty limit (like $TaS_2(\text{pyridine})_{1/2}$). In the HTS cuprates, however, both single crystals and epitaxial thin films are nominally in the "intermediate" regime, with $l/\xi_{xy} \approx 2-5$. In addition, the situation in the cuprates is complicated by the presence of phonons for $T \simeq T_{\rm c} \simeq 100\,K$, the nearly localized magnetic moments on the Cu^{2+} sites, and by other unspecified inelastic processes. We will suppose the impurities to be located in conducting layers and the electron scattering to be elastic.

Below we will mainly restrict our consideration by the local limit of the fluctuation pairs motion. This means that we consider the case of not too clean superconductors, keeping the impurity concentration n_i and reduced temperature such that the resulting electron mean-free path satisfies the requirement $l < \xi_{xy}(T) = \xi_{xy}/\sqrt{\varepsilon}$ and the impurity vertex can be taken in the local form (6.26) with $\langle(\Delta\xi(\mathbf{q},\mathbf{p})|)^2\rangle_{\rm F.S.}$ determined by (6.38). What concerns the phase-breaking time τ_φ it will be supposed much larger than τ.

The important example is already familiar case of quasi-$2D$ electron motion in a layered metal:

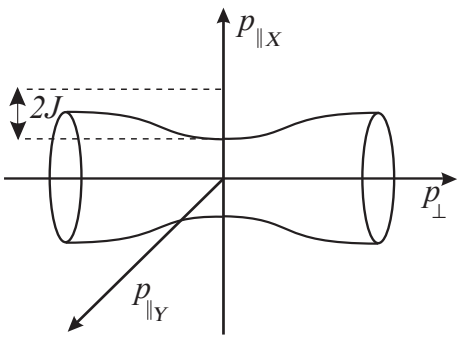

FIG. 6.4. The Fermi surface in the form of a corrugated cylinder.

$$\xi(\mathbf{p}) = E(\mathbf{p}_\|) + J\cos(p_z s) - E_F, \tag{6.37}$$

where $E(\mathbf{p}_\|) = \mathbf{p}_\|^2/(2m)$, $\mathbf{p} \equiv (\mathbf{p}_\|, p_z)$, $\mathbf{p}_\| \equiv (p_x, p_y)$, J is the effective nearest-neighbor interlayer hopping energy for quasiparticles. We note that J characterizes the width of the band in the c-axis direction taken in the strong-coupling approximation and can be identified with the effective energy of electron tunneling between planes (see Eqs. (2.45), (2.45) and footnote therein). The Fermi surface, defined by the condition $\xi(\mathbf{p}) = 0$, is a corrugated cylinder (see Fig. 6.4). In this case the average (6.27) is written in a more sophisticated form:

$$\langle(\Delta\xi(\mathbf{q},\mathbf{p})|_{|\vec{p}|=p_F})^2\rangle_{F.S.} = \frac{1}{2}\left(v_F^2\mathbf{q}^2 + 4J^2\sin^2(q_z s/2)\right) = \tau^{-1}\widehat{\mathcal{D}}q^2, \tag{6.38}$$

where we have introduced the definition of the generalized diffusion coefficient operator $\widehat{\mathcal{D}}$ in order to deal with an arbitrary anisotropic spectrum. The generalization of (6.32) for the case of a layered superconductor with a quasi $2D$ electronic spectrum is evident:

$$L^R(q,\Omega) = -\frac{1}{\nu}\frac{1}{\epsilon - i\frac{\pi\Omega}{8T} + \eta_{(2)}\mathbf{q}_\|^2 + r\sin^2(q_z s/2)}. \tag{6.39}$$

Let us stress, that the anisotropy parameter $r = \frac{J^2}{T} = \frac{4\xi_z^2(0)}{s^2}$ more than once used in the phenomenological description of fluctuations (see Eq. (2.45)) appears here in the natural way. The coefficient $\eta_{(2)}$ in its turn is nothing else as the in-plane coherence length ξ_{xy} square (see Eq. (6.33)).

7

MICROSCOPIC DERIVATION OF THE GL FUNCTIONAL

7.1 GL functional of a conventional superconductor

In this chapter we will demonstrate how the GL functional itself can be carried out from the microscopic theory of superconductivity. For this aim we will use the method of functional integration alternative to the diagrammatic technique approach.

Let us start again from the BCS Hamiltonian (6.1) and write it in the form:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}, \tag{7.1}$$

where

$$\mathcal{H}_0 = \widetilde{\psi}^{+}(\mathbf{r}) \left(-\frac{\nabla^2}{2m} + U(\mathbf{r}) - \mu \right) \widetilde{\psi}(\mathbf{r}), \tag{7.2}$$

$$\mathcal{H}_{\text{int}} = -g \int \widetilde{P}^{+}(\mathbf{r}) \widetilde{P}(\mathbf{r}) \, d\mathbf{r}, \tag{7.3}$$

and

$$\widetilde{P}(\mathbf{r}) = \widetilde{\psi}_{\downarrow}(\mathbf{r}) \widetilde{\psi}_{\uparrow}(\mathbf{r}) \tag{7.4}$$

is the operator of the Cooper pair annihilation in the Heisenberg representation. The potential $U(\mathbf{r})$ describes, as above, the interaction of electrons with impurities.

Our task now consists of the calculation of the partition function

$$Z = \operatorname{tr} \exp\left(-\int_0^{\beta} \mathcal{H} d\tau \right) = \operatorname{tr} \exp\left(-\frac{\mathcal{H}}{T} \right), \tag{7.5}$$

but in contrast to Eq. (2.1) of chapter 1, with the microscopic Hamiltonian (7.1) instead of the phenomenological functional (2.4).

In the ideal gas $g = 0$, and $\mathcal{H}_0$ is just the quadratic form of the Heisenberg electron field operators $\widetilde{\psi}(\mathbf{r})$ and $\widetilde{\psi}^{+}(\mathbf{r})$. One can diagonalize it and carry out the trace operation in (2.1) to calculate the partition function. In order to calculate the partition function of the interacting electron gas with Hamiltonian (7.1) one has in advance to present the operator $\mathcal{H}_{\text{int}}$ as the quadratic form over the field operators.

Let us separate the exponent with $\mathcal{H}_{\text{int}}$ in the expression for the partition function in the form of multiplier. The problem is nontrivial because the operators $\mathcal{H}_0$ and $\mathcal{H}_{\text{int}}$ do not commute. Nevertheless, introducing the operator

of the imaginary time ordering T_τ and the interaction representation for the Hamiltonian $\mathcal{H}$ [228] one can present the partition function in the form:

$$Z = \mathrm{tr}\left\langle \exp\left(-\int_0^\beta \mathcal{H}_0 d\tau\right) T_\tau \exp\left(-\int_0^\beta \mathcal{H}_{\mathrm{int}}\left(\tau\right) d\tau\right)\right\rangle. \tag{7.6}$$

Now let us present the $\mathcal{H}_{\mathrm{int}}$ in the form of the quadratic form over the field operators by means of the Hubbard–Stratonovich transformation. For this purpose let us write

$$\exp\left(-\int_0^\beta \mathcal{H}_{\mathrm{int}}\left(\tau\right) d\tau\right) = \prod_x \exp\left(g\widetilde{P}^+(x)\widetilde{P}(x)d^4x\right), \tag{7.7}$$

where $x = (\mathbf{r}, \tau)$. Each multiplier in the product (7.7) can be presented in the form of the integral[37]

$$e^{g\widetilde{P}^+(x)\widetilde{P}(x)d^4x}, = \int d^2\Delta(x) \exp\left[\left(-\frac{|\Delta(x)|^2}{g} - \Delta^*(x)\widetilde{P}\left(x\right) - \Delta(x)\widetilde{P^+}\left(x\right)\right) d^4x\right]. \tag{7.8}$$

Here $d^2\Delta(x) = d\left(\mathrm{Im}\,\Delta(x)\right) d\left(\mathrm{Re}\,\Delta(x)\right)$. The product of all these multipliers is the functional integral over $\mathfrak{D}\Delta(\mathbf{r},\tau)\mathfrak{D}\Delta^*(\mathbf{r},\tau)$. Finally [233],

$$Z = tr\left\langle T_\tau \int \mathfrak{D}\Delta(\mathbf{r},\tau)\mathfrak{D}\Delta^*(\mathbf{r},\tau) \exp\left(\int_0^\beta -\frac{|\Delta(\mathbf{r},\tau)|^2}{g} - \Delta^*(\mathbf{r},\tau)\widetilde{\psi}_\downarrow\left(\mathbf{r}\right)\widetilde{\psi}_\uparrow\left(\mathbf{r}\right) - \Delta(\mathbf{r},\tau)\widetilde{\psi}^+_\downarrow\left(\mathbf{r}\right)\widetilde{\psi}^+_\uparrow\left(\mathbf{r}\right) - \mathcal{H}_0 d\tau\right)\right\rangle. \tag{7.9}$$

Now, contained in the exponent form is quadratic over the operators $\widetilde{\psi}$ and the problem is reduced to the description of one electron motion in the fluctuation field $\Delta(\mathbf{r},\tau)$ being an arbitrary function of the coordinate $\mathbf{r}$ and imaginary time τ.

Let us introduce the normal $G_\Delta\left(\mathbf{r},\tau,\mathbf{r}',\tau'\right) = -\left\langle T_\tau \widetilde{\psi}_\downarrow\left(\mathbf{r},\tau\right)\widetilde{\psi}^+_\downarrow\left(\mathbf{r}',\tau'\right)\right\rangle$ and anomalous $F_\Delta\left(\mathbf{r},\tau,\mathbf{r}',\tau'\right) = < T_\tau \widetilde{\psi}_\downarrow\left(\mathbf{r},\tau\right)\widetilde{\psi}_\uparrow\left(\mathbf{r}',\tau'\right) >$ electron Green functions of superconducting state which can be written in the matrix Gor'kov–Nambu representation

$$\widehat{G}_\Delta = \begin{pmatrix} G_\Delta\left(\mathbf{r},\mathbf{r}',\tau,\tau'\right) & F_\Delta\left(\mathbf{r},\mathbf{r}',\tau,\tau'\right) \\ F^+_\Delta\left(\mathbf{r},\mathbf{r}',\tau,\tau'\right) & -G^+_\Delta\left(\mathbf{r}',\mathbf{r},\tau',\tau\right) \end{pmatrix}. \tag{7.10}$$

They satisfy the Gor'kov equation written in the form:

[37] We omit here the unimportant coefficient.

$$\left[\tau_z \frac{\partial}{\partial \tau} + \tau_0 \widehat{\xi} + i\tau_y \operatorname{Re}\Delta(\mathbf{r},\tau) + i\tau_x \operatorname{Im}\Delta(\mathbf{r},\tau)\right] \widehat{G}_\Delta = \delta(\mathbf{r}-\mathbf{r}')\,\delta(\tau-\tau'). \tag{7.11}$$

Here $\tau_i (i = x, y, z)$ are the Pauli matrices, τ_0 is the unit matrix, $\widehat{\xi} = -\frac{\nabla^2}{2m} + U(\mathbf{r}) - \mu$, the magnetic field is supposed to be equal zero.

Calculating the trace over electron field operators $\widetilde{\psi}_\downarrow, \widetilde{\psi}^+_\uparrow$ one finds the corresponding contribution to the partition function

$$Z_\Delta = e^{\mathcal{S}_0} \int \mathfrak{D}\Delta(\mathbf{r},\tau)\mathfrak{D}\Delta^*(\mathbf{r},\tau) \exp\left\{-\mathcal{S}\left[\Delta(\mathbf{r},\tau)\right]\right\} \tag{7.12}$$

with

$$\mathcal{S}\left[\Delta(\mathbf{r},\tau)\right] = -\int d\mathbf{r} \int_0^\beta d\tau \left[tr\left\{\ln \widehat{G}^{-1}_\Delta\right\} - \frac{|\Delta(\mathbf{r},\tau)|^2}{g}\right] = \mathcal{S}_0 + \mathcal{S}_\Delta. \tag{7.13}$$

The trace here is supposed to be carried out over the Nambu–Gor'kov subscripts. The independent on the order parameter part of action

$$\mathcal{S}_0 = -\int d\mathbf{r} \int_0^\beta d\tau \left[\operatorname{tr}\left\{\ln \widehat{G}^{-1}_{\Delta=0}\right\}\right] \tag{7.14}$$

determines the ideal gas partition function and is introduced in order to provides the natural normalization $Z_\Delta(\Delta = 0) = 1$, what corresponds $\mathcal{S}_\Delta(\Delta = 0) = 0$.

In the case when $\Delta(\mathbf{r},\tau) = \text{const}$ one can calculate the matrix $\widehat{G}^{-1}$ by means of the Fourier transform of Eq. (7.11):

$$\widehat{G}_\Delta(\mathbf{p},\varepsilon_n) = -\frac{i\tau_z\varepsilon_n + \tau_0\xi(\mathbf{p}) - i\tau_y \operatorname{Re}\Delta - i\tau_x \operatorname{Im}\Delta}{\Delta^2 + \varepsilon_n^2 + \xi^2(\mathbf{p})}. \tag{7.15}$$

Here ε_n are the fermionic Matsubara frequencies and $\xi(\mathbf{p})$ are the eigenvalues of the $\widehat{\xi}-$operator. In the absence of impurities $\xi(\mathbf{p}) = p^2/2m - \mu$. Recalling that the free energy is related to the action as $F = T\mathcal{S}$, and substituting these values in Eq. (7.12), one can find

$$-\frac{T\mathcal{S}_\Delta}{V} = T\sum_{\varepsilon_n} \nu \int d\xi \ln\left(1 + \frac{\Delta^2}{\varepsilon_n^2 + \xi^2}\right) - \frac{|\Delta|^2}{g}. \tag{7.16}$$

The mean field approximation as usual corresponds to the calculation of the functional integral in Eq. (7.12) by the steepest descend method. One can see that the saddle point condition ($\partial\mathcal{S}/\partial\Delta = 0$) coincides with the BCS self-consistency equation

$$1 = gT\nu \sum_{\varepsilon_n} \int d\xi \frac{1}{\varepsilon_n^2 + \xi^2 + \Delta^2}. \tag{7.17}$$

Let us notice that Eq. (7.16) describes not only the mean field approximation but also the order parameter fluctuations independent on the imaginary time

and space variables. Expanding its right hand side in the vicinity of the critical temperature in the series over the order parameter powers one can reproduce the famous expression for the free energy in the Landau form

$$\frac{F}{V} = A\Delta^2 + \frac{B}{2}\Delta^4 \tag{7.18}$$

with the coefficients defined by means of the microscopic characteristics of superconductor:

$$A = \nu\left(\frac{1}{g} - \ln\frac{\omega_D}{2\pi T}\right) = \nu \ln\frac{T}{T_{\mathrm{c}}}, \tag{7.19}$$

$$B = \frac{\nu T}{2}\sum_{\varepsilon_n}\int\frac{d\xi}{\left(\varepsilon_n^2+\xi^2\right)^2} = \frac{7\zeta(3)}{8\pi^2 T^2}\nu. \tag{7.20}$$

It worth mentioning that these coefficients, and therefore the critical temperature T_{c}, do not depend on the concentration of nonmagnetic impurities which do not break the time reversal symmetry. This statement remains correct until the interaction (7.3)–(7.4) has the point character, and corresponding order parameter is coordinate independent. The situation turns out to be certainly different for the order parameter with p- or d-wave symmetry. In these cases even nonmagnetic impurities suppress the superconducting state [234]. Below we will derive the GL functional for these cases.

When the order parameter depends on $\mathbf{r}$ and τ the fluctuation part of action $\mathcal{S}_\Delta$ can be presented in the form of the series over $\Delta(\mathbf{r},\tau)$. It takes the form:

$$\begin{aligned}-\mathcal{S}_\Delta = &\int \Delta^*(\mathbf{r}_1,\tau_1)\,\widehat{L}^{-1}(\mathbf{r}_1,\tau_1,\mathbf{r}_2,\tau_2)\,\Delta(\mathbf{r}_2,\tau_2)\,d\mathbf{r}_1 d\tau_1 d\mathbf{r}_2 d\tau_2\\ &-\frac{1}{2}\int \Delta^*(\mathbf{r}_1,\tau_1)\,\Delta^*(\mathbf{r}_2,\tau_2)\,\widehat{B}(\mathbf{r}_1,\tau_1,\mathbf{r}_2,\tau_2,\mathbf{r}_3,\tau_3,\mathbf{r}_4,\tau_4)\\ &\times\Delta(\mathbf{r}_3,\tau_3)\,\Delta(\mathbf{r}_4,\tau_4)\,d\mathbf{r}_1 d\tau_1 d\mathbf{r}_2 d\tau_2 d\mathbf{r}_3 d\tau_3 d\mathbf{r}_4 d\tau_4\end{aligned} \tag{7.21}$$

In Eq. (12.19), operator $\widehat{L}$ is the fluctuation propagator: $\hat{L}_\omega = \left[-g^{-1}+\hat{\Pi}_\omega\right]^{-1}$; polarization operator $\hat{\Pi}_\omega$ in the coordinate representation has the form

$$\Pi_\omega(\mathbf{r},\mathbf{r}') = T\sum_\varepsilon \Pi_\omega(\mathbf{r},\mathbf{r}';\varepsilon) = T\sum_\varepsilon G_\varepsilon(\mathbf{r},\mathbf{r}')G_{\omega-\varepsilon}(\mathbf{r},\mathbf{r}'), \tag{7.22}$$

where G_ε is the one-electron Green function of a normal metal at Matsubara frequency. Nonlinear operator $\widehat{B}$ in Eq. (12.19) corresponds to the product of the four such Green functions.

The condition for saddle point definition $\delta\mathcal{S}_\Delta/\delta\Delta(\mathbf{r},t) = 0$ in the case of a gapless superconductor results in the TDGL equation [104].

Close to T_{c} the coordinate dependence of the fluctuating order parameter is smooth while the dependence of $\Delta(\mathbf{r},\tau)$ on τ can be omitted at all. As the result

the function B remains constant while L^{-1} can be expanded over the powers of q^2 (see (6.32)). In result the free energy (7.16) is presented in the form of the GL series (A.6) with the coefficients A and B determined by Eqs. (7.19)–(7.20). The q^2 coefficient C is related to the square of coherence length by Eq. (A.9) (see also Eq. (6.33)).

Let us mention that for most problems $\Delta(\mathbf{r}, t)$ is the smooth self-averaging function, hence only the coefficient C has to be averaged over impurities configuration. After such averaging the expression for propagator L in the frequency-momentum representation is determined by Eqs. (6.4) and (6.30). Nevertheless for some problems of the mesoscopic character (for instance, see the problem of the optimal fluctuation in Part IV), where the disorder fluctuations are important themselves, it is necessary first to integrate over the order parameter fluctuations and only subsequently to perform the impurity average.

In order to get the match between the developed microscopic and phenomenological theory results (7.18) and (2.4) the correspondence between the phenomenological order parameter Ψ and the microscopic fluctuation field intensity Δ has to be established. Here, it is necessary to make the following comment. The choice of the coefficient of this proportionality is the delicate procedure where some arbitrariness takes place. The most common are two following choices. In the first one the identity $\Delta \equiv \Psi$ is postulated. In this case $a = A, b = B, 1/4m = C$. Such a choice is convenient since in this case the value of Δ is equal to the value of one-particle spectrum gap, following from the microscopic theory. Yet such a choice becomes embarrassing in the case of an impure superconductor, where the Cooper pair mass $2m$ ceases to be universal anymore and starts to depend on the impurity concentration. The second choice assumes the Cooper pair mass to be fixed and equal to two free electron masses. In this case Δ, a and b are determined by Eqs. (A.10) and (7.1).

Due to the short range nature of the interaction in the BCS model we succeeded, applying Hubbard–Stratonovich transformation, to separate it and in this way to reduce the problem to the accounting for only long wave length fluctuations of Δ. In some problems (such as those discussed in [92, 95], where the inter-electron interaction is considered in strongly disordered superconductors) the long range character of the Coulomb interaction, or, in other words, the electric field fluctuations, is of the first importance. Accounting for such interaction means the appearance in Hamiltonian (7.1) of the additional term

$$\mathcal{H}_Q = \frac{\rho(\mathbf{r})\,\rho(\mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|}, \tag{7.23}$$

with $\rho(\mathbf{r}) = e\,|\Psi|^2$. After the Hubbard–Stratonovich transformation in action appears the additional term

$$-\mathcal{S}_e = -\frac{(\nabla\varphi)^2}{8\pi} + \varphi\rho, \tag{7.24}$$

where $\varphi(\mathbf{r})$ is the electric field potential. Strong Coulomb interaction results in the smallness of the Debye radius ($|r_D|^2 = e^2\nu$) with respect to the Cooper pair size (coherence length ξ), while the plasma frequency $\omega_{\mathrm{p}}^2 = 4\pi e^2 n/m$ is large in comparison with Δ. This is the reason why in the most part of the problems instead of functional integration over the fields $\varphi(\mathbf{r})$ one can restrict himself to accounting for only neutrality constrain $\rho = 0$.

7.2 GL functional in the case of a nontrivial order parameter symmetry[38]

Let us derive the GL functional for the general case of the anisotropic order parameter. The inter-electron interaction can be written in the most general form as

$$V_{ee}(\mathbf{p}, \mathbf{p}_1, \mathbf{q}) = V_{\mathbf{p},\mathbf{p}_1,\mathbf{q}} \widetilde{\psi}^{+}_{\mathbf{p}} \widetilde{\psi}^{+}_{-\mathbf{p}+\mathbf{q}} \widetilde{\psi}_{\mathbf{p}_1} \widetilde{\psi}_{-\mathbf{p}_1+\mathbf{q}}. \tag{7.25}$$

We will be interested in fluctuations with the wave vectors q much smaller than p_F and it is natural to suppose that both electron momenta $\mathbf{p}, \mathbf{p}_1$ belong to the vicinity of the Fermi surface. In this case the matrix elements $V_{\mathbf{p},\mathbf{p}_1,\mathbf{q}}$ weakly depend on q while their dependence on $\mathbf{p}, \mathbf{p}_1$ is reduced to the function of their directions only. For example, in the case of a $2D$ isotropic metal $V_{\mathbf{p},\mathbf{p}_1,\mathbf{q}}$ depends only on the angle, $\varphi = \varphi - \varphi_1$, between the vectors $\mathbf{p}$ and $\mathbf{p}_1$. Let us expand this function in the Fourier series:

$$V(\varphi) = -\sum_l g_l \left(\cos l\varphi \cos l\varphi_1 + \sin l\varphi \sin l\varphi_1\right). \tag{7.26}$$

Each positive g_l in principle can generate the pairing with corresponding order parameter symmetry, but that one conformable to the largest g_l is realized. For example, in the case when the coefficient g_0 is the largest one the s-pairing takes place. If the g_2 is larger than all other g_l the d-wave pairing is realized in the system.

In the crystalline lattice

$$V_{\mathbf{p},\mathbf{p}_1} = \sum_l g_l \chi_l(\mathbf{p}) \chi_l(\mathbf{p}_1). \tag{7.27}$$

Here the quantum number l does not have anymore the rigorous sense of the angular momentum but just shows how many zeros the function χ_l has. The habitual presentation for d-wave case $\chi_2 = p_x^2 - p_y^2$ means that the function χ_2 has four zeros which take place in directions $p_x = \pm p_y$. We will ignore the exotic case when two interaction constants g_l and g_m are almost equal and will suppose the interaction to be weak. So in the series (7.27) one can leave only the term

[38] In this section we base on the results of [234].

with the largest g_ρ. The corresponding interaction Hamiltonian will be described by the same Eq. (7.3) but the operators (7.4) must be written in the form

$$\widetilde{P}_\rho(\mathbf{r}) = \sum_{\mathbf{p},\mathbf{q}} \chi_\rho(\mathbf{p})\, \widetilde{\psi}_{\mathbf{p}+\mathbf{q}/2}(\mathbf{r})\, \widetilde{\psi}_{\mathbf{p}-\mathbf{q}/2}(\mathbf{r}) \exp(i\mathbf{q}\mathbf{r}). \tag{7.28}$$

The formulae (7.7)–(7.21) will be valid with the simple substitution of Δ by the anisotropic gap $\Delta(\mathbf{p}) = \Delta\chi_\rho(\mathbf{p})$ while the function $\chi_\rho(\mathbf{p})$ is normalized according to

$$\overline{\chi_\rho^2} = \frac{1}{\nu}\int \chi_\rho^2(\mathbf{p})\,\delta(\varepsilon_{\mathbf{p}} - \mu)\, d\mathbf{p} = 1. \tag{7.29}$$

The GL functional maintains its form (A.6) but the coefficients B and C must me substituted by their anisotropic values

$$B_\rho = B\overline{\chi_\rho^4}, \tag{7.30}$$

$$C_\rho = C\overline{\left(\chi_\rho^2(\mathbf{p})\,\mathbf{v}_{\mathbf{p}}^2\right)}/v_F^2. \tag{7.31}$$

Namely by this reason all physical consequences of the GL phenomenology are applicable to the p- and d-superconductors ($\rho = 1, 2$).

The important difference of an anisotropic case in comparison with the isotropic one consists in the role of impurities. This is due to the fact that in an anisotropic superconductor the presence of nonmagnetic impurities changes, with respect to the clean case, not only coefficient C but also A, B and T_{c}. Let us demonstrate this.

Due to the scattering by impurities the electron momentum changes with time in such a way that the correlator

$$\langle \mathbf{p}(t)\,\mathbf{p}(0)\rangle = p_F^2 e^{-t/\tau_p}. \tag{7.32}$$

In the case of p-pairing $\chi_p(\mathbf{p}) \sim \mathbf{p}$ and the same relaxation law can be written for $\langle \chi_p(\mathbf{p},t)\,\chi_p(\mathbf{p},0)\rangle \sim \exp(-t/\tau_p)$. The corresponding average value $\overline{\chi_p} = 0$.[39] One can check that the same law but with $\tau_d = \tau_p/2$ is valid also for d-pairing: $\langle \chi_d(\mathbf{p},t)\,\chi_d(\mathbf{p},0)\rangle \sim \exp(-t/\tau_d)$,and again $\overline{\chi_d} = 0$. These statements are valid for any type of pairing with $\rho \neq 0$. The exception is the case of s-pairing when in the BCS approximation $\overline{\chi_s} = 1$. Nevertheless even in the case of s-pairing but with the anisotropic electron spectrum $\overline{\chi_s}$ can differ from 1. Let us see to what consequences this will lead for the critical temperature. In the general case $\overline{\chi_\rho} \neq 0$ correlator of the parameter $\chi_l(\mathbf{p})$ relaxes to its average value as

$$\langle \chi_\rho(\mathbf{p},t)\,\chi_\rho(\mathbf{p},0)\rangle = e^{-t/\tau_\rho} + \left(\overline{\chi_\rho(\mathbf{p})}\right)^2\left(1 - e^{-t/\tau_\rho}\right). \tag{7.33}$$

Performing the Fourier transform of expression (7.33) one can find

[39] The sign $\overline{(\cdots)}$ means the averaging over the state with the energy close to the Fermi level.

$$\langle \chi_\rho(\mathbf{p},t)\,\chi_\rho(\mathbf{p},0)\rangle_\omega = \frac{1-\left(\overline{\chi_\rho(\mathbf{p})}\right)^2}{i\omega+1/\tau_\rho} + \frac{\left(\overline{\chi_\rho(\mathbf{p})}\right)^2}{i\omega} \tag{7.34}$$

On can calculate this correlator for electron with energy ξ_l in the basis of the electron eigenfunctions of the Hamiltonian with the exact impurity potential

$$\langle \chi_\rho(\mathbf{p},t)\,\chi_\rho(\mathbf{p},0)\rangle = \theta(t)\sum_{m,k}\left|\left(\chi_\rho(\mathbf{p})\right)_{km}\right|^2 \exp(i\omega_{mk}t)\,\delta(\xi_k), \tag{7.35}$$

where $\omega_{mk}=\xi_k-\xi_m$. After the Fourier transform one finds

$$\langle \chi_\rho(\mathbf{p},t)\,\chi_\rho(\mathbf{p},0)\rangle_\omega = \sum_{m,k}\frac{\left|\left(\chi_\rho(\mathbf{p})\right)_{km}\right|^2}{\omega-\omega_{mk}}\delta(\xi_k). \tag{7.36}$$

In the same basis of the electron eigenfunctions one can write the equation for the critical temperature which follows from (7.17):

$$\begin{aligned} 1 &= -g_\rho T_\mathrm{c}\sum_{\varepsilon_n}\sum_{m,k}\left|\left(\chi_\rho(\mathbf{p})\right)_{mk}\right|^2\frac{1}{i\varepsilon_n-\xi_m}\frac{1}{i\varepsilon_n+\xi_k} \\ &= -g_\rho T_\mathrm{c}\sum_{\varepsilon_n}\sum_{m,k}\frac{\left|\left(\chi_\rho(\mathbf{p})\right)_{mk}\right|^2}{2\varepsilon_n-i\omega_{mk}}\left(\frac{1}{\varepsilon_n+i\xi_m}+\frac{1}{\varepsilon_n-i\xi_k}\right). \end{aligned} \tag{7.37}$$

The summation over eigen-states can be performed in the following way. Recalling that $\varepsilon_n\sim T\ll E_F$ one can substitute the last parenthesis as the Lorentzian $2\varepsilon_n\left(\xi_k^2+\varepsilon_n^2\right)^{-1}$, which, in its turn, to take as the delta function $\delta(\xi_k)$ and finally to use Eq. (7.36):

$$\begin{aligned} 1 &= g_\rho\nu T_\mathrm{c}\sum_{\varepsilon_n}\sum_{m,k}\frac{\left|\left(\chi_\rho(\mathbf{p})\right)_{mk}\right|^2}{-2i\varepsilon_n-\omega_{mk}}\delta(\xi_k) = \\ &. = g_\rho\nu T_\mathrm{c}\sum_{\varepsilon_n}\left[\frac{\left(\overline{\chi_\rho(\mathbf{p})}\right)^2}{2|\varepsilon_n|}+\frac{1-\left(\overline{\chi_\rho(\mathbf{p})}\right)^2}{2|\varepsilon_n|+1/\tau_\rho}\right]. \end{aligned} \tag{7.38}$$

For the traditional case of s-pairing in not very anisotropic superconductor $\overline{\chi_\rho(\mathbf{p})}=1$ and Eq. (7.38) is reduced to Eq. (7.17) resulting in the BCS value of the critical temperature T_{c0}.

Here it is necessary to underline that in the case of *s*-pairing in superconductor with the strongly anisotropic electron spectrum $\overline{\chi_\rho(\mathbf{p})}\neq 1$. In result, the presence of even nonmagnetic impurities reduces the critical temperature according to the relation:

$$\frac{1}{\nu g_\rho}=\ln\frac{\omega_D}{2\pi T_\mathrm{c}}-\psi\left(\frac{1}{2}\right)+\left[\left(\overline{\chi_\rho(\mathbf{p})}\right)^2-1\right]\left[\psi\left(\frac{1}{2}+\frac{1}{4\pi T_\mathrm{c}\tau_\rho}\right)-\psi\left(\frac{1}{2}\right)\right].$$

Introducing the critical temperature of the corresponding clean superconductor T_{c0} one can express the shifted critical temperature T_c as [234]:

$$\ln \frac{T_{\rm c}}{T_{c0}} = \left[\left(\overline{\chi_\rho \left(\mathbf{p} \right)} \right)^2 - 1 \right] \left[\psi \left(\frac{1}{2} + \frac{1}{4\pi T_{\rm c} \tau_\rho} \right) - \psi \left(\frac{1}{2} \right) \right]. \tag{7.39}$$

Nevertheless the anisotropy in conventional superconductors usually is small and corresponding renormalization of the critical temperature is weak and ignored (the so-called Anderson theorem) [235, 236].

In the cases of pairing with higher momenta $\overline{\chi}_\rho = 0$ and the corresponding equation for critical temperature takes the form

$$\ln \frac{T_{\rm c}}{T_{c0}} = \psi \left(1/2 \right) - \psi \left(\frac{1}{2} + \frac{1}{4\pi T_{\rm c} \tau_\rho} \right). \tag{7.40}$$

That is the nonmagnetic impurities for p- and d-superconductors produce the same effect on critical temperature as the magnetic impurities do for the conventional s-wave superconductor.

Evidently exists some critical impurities concentration ($\tau_\rho^{(\rm cr)} \sim T_{c0}^{-1}$) when impurities completely destroy the state of superconductivity. Let us find the corresponding $\tau_\rho^{(\rm cr)}$. For this purpose one can expand the function ψ in Eq. (7.40) according to Eq. (B.16) and, putting $T_{\rm c} = 0$, to obtain

$$\ln \frac{1}{4\pi \tau_\rho^{(\rm cr)} T_{c0}} = \psi \left(\frac{1}{2} \right) = - \ln 4\gamma_{\rm E}, \tag{7.41}$$

what means that the critical concentration corresponds to

$$\tau_\rho^{(\rm cr)} = \frac{\gamma_{\rm E}}{\pi T_{c0}}. \tag{7.42}$$

8

MICROSCOPIC THEORY OF FLUCTUATION CONDUCTIVITY

8.1 Qualitative discussion of the different fluctuation contributions

In chapter 3, the direct fluctuation effect on conductivity, related to the charge transfer by means of fluctuation Cooper pairs, was discussed in detail. Nevertheless, in this section we return to its discussion and will demonstrate the corresponding calculations by means of the microscopic theory. This will be done in order to prepare the basis for studies of the AL contribution to a variety of physical values like magnetoconductivity near the upper critical field, conductivity far from transition point and in ultra-clean limit, Hall conductivity, etc.

The microscopic approach allows us also to calculate the above-cited indirect fluctuation effects such as the so-called DOS and MT contributions. We will start now from their qualitative discussion.

The important consequence of the presence of fluctuating Cooper pairs above T_c is the decrease in the one-electron density of states at the Fermi level. Indeed, if some electrons are involved in pairing they cannot simultaneously participate in charge transfer and energy absorbtion as single-particle excitations. Nevertheless, the total number of the electronic states cannot be changed by the Cooper interaction, and only a redistribution of the levels along the energy axis is possible [169,237]. In this sense one can speak of the opening of a fluctuation pseudogap at the Fermi level. The decrease of the one-electron DOS at the Fermi level leads to a reduction of the Drude conductivity. This, indirect, fluctuation correction to the conductivity is called the DOS contribution and it appears side by side with the paraconductivity (or AL contribution). It has the opposite (negative) sign and turns out to be much less singular in $(T-T_c)^{-1}$ in comparison with the AL contribution, so that in the vicinity of T_c it was usually omitted. However, in many cases [52, 108, 238–241], when for some special reasons the main, most singular, corrections are suppressed, the DOS correction becomes of major importance. Such a situation is realized in study of fluctuation quasiparticle current in tunnel structures, of fluctuation c-axis transport in strongly anisotropic HTS, fluctuation corrections to the NMR relaxation rate or thermoelectric power.

The correction to the normal state conductivity above the transition temperature related to the fluctuation DOS renormalization can be evaluated qualitatively. Indeed, the fact that some electrons ($\Delta\mathcal{N}_e$ per unit volume) participate in fluctuation Cooper pairings means that the effective number of carriers taking part in one-electron charge transfer diminishes leading to a decrease of conductivity (we deal here with the longitudinal component):

$$\delta\sigma_{xx}^{DOS} = -\frac{\Delta\mathcal{N}_e e^2\tau}{m_e} = -\frac{2n_s e^2\tau}{m_e}, \tag{8.1}$$

where n_s is the fluctuation Cooper pairs concentration above transition point. The latter was already calculated in chapter 2 and is given by the Eq. (2.151). Let us recall, that the mass m in Eq. (2.151) represents the effective Cooper pair mass and generally speaking should not coincide with the doubled electron mass. Substituting the Eq. (2.151) to the Eq. (8.1) and using the explicit form for the Ginzburg–Landau coefficient α in accordance with Eq. (A.12) one can find that

$$\delta\sigma_{xx}^{DOS} \sim -\frac{e^2}{s}\ln\frac{2}{\sqrt{\epsilon}+\sqrt{\epsilon+r}}, \tag{8.2}$$

what coincides up to the accuracy of numerical coefficient with the microscopic expression (8.28), which will be carried out below in result of much more cumbersome microscopic analysis.

The third, purely quantum, fluctuation contribution is generated by the coherent scattering of the electrons forming a Cooper pair on the same elastic impurities. This is the so called anomalous Maki–Thompson (MT) contribution [40,41] which can be treated as the result of Andreev reflection of the electron by fluctuation Cooper pairs. This contribution appears only in transport coefficients and often turns out to be important. Its temperature singularity near T_c is similar to that of the paraconductivity, although being extremely sensitive to electron phase-breaking processes and to the type of orbital symmetry of pairing it can be suppressed. Let us evaluate it.

The physical origin of the MT correction consists in the fact that the Cooper interaction of electrons with nearly opposite momenta changes the mean free path (diffusion coefficient) of electrons. As we have already seen in the previous section the amplitude of this interaction increases drastically when $T \to T_c$:

$$g_{\text{eff}} = \frac{g}{1-\nu g\ln\frac{\omega_D}{2\pi T}} = \frac{1}{\ln\frac{T}{T_c}} \approx \frac{T}{T-T_c} = \frac{1}{\epsilon}.$$

What is the reason for this growth? One can say that the electrons scatter one at another in a resonant way with the virtual Cooper pair formation. Or, it is possible to imagine that the electrons undergo Andreev reflection by fluctuation Cooper pairs, binding in the Cooper pairs themselves. The probability of such induced pair irradiation (let us remember that Cooper pairs are Bose particles) is proportional to their number in the final state, i.e. $n(p)$ (1.5). For small momenta $n(p) \sim 1/\epsilon$.

One can ask why such an interaction does not manifest itself considerably far from the transition point? This is due to the fact that just a small number of electrons with the total momentum $q \lesssim \xi^{-1}(T)$ interacts so intensively. In accordance with the Heisenberg principle the minimal distance between such electrons is of the order of $\sim \xi(T)$. On the other hand, such electrons, in order to interact, have to approach one another approximately up to a distance of

the Fermi length $\lambda_F \sim 1/p_F$. The probability of such event may be estimated in the spirit of the self-intersecting trajectories contribution evaluation in the weak-localization theory [242, 131].

In the process of diffusion motion the distance between two electrons increases with time according to the law: $R(t) \sim (\mathcal{D}t)^{1/2}$. Hence the scattering probability

$$W \sim \int_{t_{\min}}^{t_{\max}} \frac{\lambda_F^{D-1}}{R^D(t)} v_F dt.$$

The lower limit of the integral can be estimated from the condition $R(t_{\min}) \sim \xi(T)$ (only such electrons interact in the resonant way). The upper limit is determined by the phase-breaking time τ_φ since for larger time intervals the phase coherence, necessary for the pair formation, is broken. As the result the relative correction to conductivity due to such processes is equal to the product of the scattering probability on the effective interaction constant: $\delta\sigma^{MT}/\sigma = W g_{\text{eff}}$. In the $2D$ case

$$\delta\sigma^{MT} \sim \frac{e^2}{8\epsilon} \ln \frac{\mathcal{D}\tau_\varphi}{\xi^2(T)}.$$

This result will be confirmed below in the framework of the microscopic consideration.

8.2 The electromagnetic response operator

The most general relation between the current density $\mathbf{j}(\mathbf{r},t)$ and vector-potential $\mathbf{A}(\mathbf{r}', t')$ is given through the so-called electromagnetic response operator $Q_{\alpha\beta}$ [228]:

$$\mathbf{j}_\alpha(\mathbf{r},t) = -\int Q_{\alpha\beta}(\mathbf{r},\mathbf{r}',t,t')\mathbf{A}_\beta(\mathbf{r}',t')\mathbf{d}\mathbf{r}'dt'. \tag{8.3}$$

Assuming space and time homogeneity, one can take the Fourier transform of this relation and compare it with the definition of the conductivity tensor $j_\alpha = \sigma_{\alpha\beta}E_\beta$. This allows us to express the conductivity tensor in terms of the retarded electromagnetic response operator

$$\sigma_{\alpha\beta}(\omega) = -\frac{1}{i\omega}[Q_{\alpha\beta}]^R(\omega)\ . \tag{8.4}$$

The electromagnetic response operator $Q_{\alpha\beta}(\omega_\nu)$, defined on Matsubara frequencies $\omega_\nu = 2\nu\pi T$, can be presented as the correlator of two one-electron Green's functions [228] averaged over impurity positions and accounting for interactions, in our case the particle–particle interactions in the Cooper channel. The appropriate diagrams corresponding to the first order of perturbation theory in the fluctuation amplitude can be drawn according to the rules of the diagrammatic technique (see Fig. 6.2) and they are shown in Fig. 8.1.

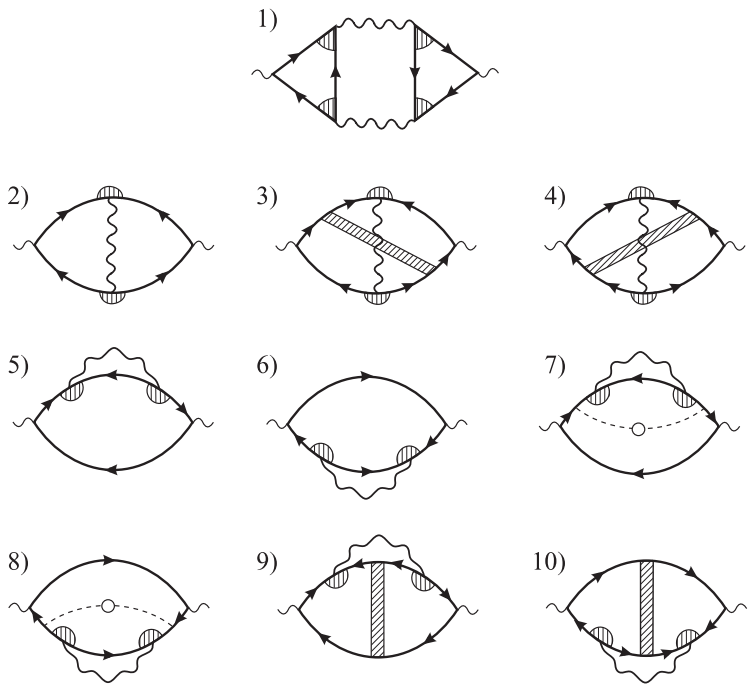

FIG. 8.1. Feynman diagrams for the leading-order contributions to the fluctuation conductivity. Wavy lines are fluctuation propagators, thin solid lines with arrows are impurity-averaged normal state Green's functions, shaded semicircles are renormalized by scattering of electrons by impurities vertex functions, dashed lines with central crosses are additional impurity renormalizations and shaded rectangles are impurity ladders. Diagram 1 represents the AL term; diagrams 2–4 represent the MT type contributions; diagrams 5–10 arise from corrections to the normal state DOS.

With each electromagnetic field component A_α we associate the external vertex $ev_\alpha(p) = e\partial\xi(p)/\partial p_\alpha$. For the longitudinal conductivity tensor elements (parallel to the layers, for which $\alpha = x, y$), the resulting vertex is simply ep_α/m. For the c-axis conductivity, the vertex is given by

$$ev_z(p) = e\frac{\partial\xi(p)}{\partial p_z} = -eJs\sin(p_z s). \tag{8.5}$$

Each solid line in the diagrams represents a one-electron Green's function averaged over impurities (6.2), a wavy line represents a fluctuation propagator $L(\mathbf{q}, \Omega_k)$ (6.32), three-leg vertices were defined by Eq. (6.26). The four-leg impurity vertex, appearing in diagrams 3–4, 9–10 of the Fig. 8.1, is called the Cooperon in the weak localization (WL) theory (see, e.g. [243]). It differs from the above three-leg vertex only by the additional factor $(2\pi\nu\tau)^{-1}$. We do not

renormalize the current vertex since this renormalization only leads to the substitution of the scattering time τ by the transport one $\tau_{\rm tr}$ in the final results (see [228]). We integrate over the internal Cooper pair momentum $\mathbf{q}$ and electron momentum $\mathbf{p}$ and sum over the internal fermionic and bosonic Matsubara frequencies, with momentum and energy conservation at each vertex (fluctuation propagator endpoint) in the analytic expressions for the diagrams presented in Fig. 8.1.

After these necessary introductory remarks and definitions we will consider the microscopic calculation of the different fluctuation contributions.

8.3 Fluctuation conductivity of a layered superconductor in the vicinity of $T_{\rm c}$

Let us consider the microscopic calculation of the fluctuation conductivity in the framework of the microscopic approach on the actual example of a layered superconductor (superconductor with the quasi-$2D$ electron spectrum being in its normal state).

8.3.1 *AL contribution*

We first examine the AL paraconductivity (diagram 1 of Fig. 8.1). Actually this contribution was already studied in section 3.2 in the framework of the TDGL equation but, in order to demonstrate how the method works, we will carry out here the appropriate calculations in the microscopic approach, as was originally done by Aslamazov and Larkin [36].

The AL contribution to the electromagnetic response operator tensor has the form:

$$Q^{AL}_{\alpha\beta}(\omega_\nu) = -4e^2T\sum_{\Omega_k}\int\frac{d^3\mathbf{q}}{(2\pi)^3}B_\alpha(\mathbf{q},\Omega_k,\omega_\nu)L(\mathbf{q},\Omega_k) \times B_\beta(\mathbf{q},\Omega_k,\omega_\nu)L(\mathbf{q},\Omega_k+\omega_\nu), \tag{8.6}$$

where

$$B_\alpha(\mathbf{q},\Omega_k,\omega_\nu) = T\sum_{\varepsilon_n}\lambda(\mathbf{q},\varepsilon_{n+\nu},\Omega_k-\varepsilon_n)\lambda(\mathbf{q},\varepsilon_n,\Omega_k-\varepsilon_n) \times\int\frac{d^3\mathbf{p}}{(2\pi)^3}v_\alpha(\mathbf{p})G(\mathbf{p},\varepsilon_{n+\nu})G(\mathbf{p},\varepsilon_n)G(\mathbf{q}-\mathbf{p},\Omega_k-\varepsilon_n) \tag{8.7}$$

is the block of three Green's functions with the impurity vertex integrated over the electron momentum and summed over the fermionic frequency.

In the vicinity of $T_{\rm c}$, due to the pole structure of the fluctuation propagators in (8.6), the leading contribution to the electromagnetic response operator $Q^{\rm AL(R)}_{\alpha\beta}$ arises from them rather than from the weak frequency dependence of the blocks B_α, so we can neglect the Ω_k- and ω_ν-dependencies of the Green functions blocks and use the expression for $B_\alpha(\mathbf{q},0,0)$ valid for small $\mathbf{q}$ only:

$$B_\alpha(\mathbf{q}) = B_\alpha(\mathbf{q},0,0)$$

$$= -\nu T \sum_{\varepsilon_n} \lambda^2(\mathbf{q}, \varepsilon_n, -\varepsilon_n) \int_{-\infty}^{\infty} \frac{d\xi}{\left(i\widetilde{\varepsilon}_n - \xi\right)^2} \left\langle \frac{v_\alpha(\mathbf{p})}{(i\widetilde{\varepsilon}_n + \xi - \mathbf{vq})} \right\rangle_{FS}. \tag{8.8}$$

Taking into account that only small values of q, defined by the poles of propagator will be involved in the further q-integration, one can perform the angular integration expanding over q the last Green's function. The q-dependence of the impurity vertices can be ignored since it occurs for $\widehat{\mathcal{D}}q^2 \sim \varepsilon_n \sim T$. As a result the first term is averaged out while the second gives:

$$\begin{aligned} B_\alpha(\mathbf{q}) &= -\nu T \left\langle v_\alpha v_\beta q_\beta \right\rangle_{FS} \sum_{\varepsilon_n} \frac{\left|\widetilde{\varepsilon}_n\right|^2}{\left|\varepsilon_n\right|^2} \int_{-\infty}^{\infty} \frac{d\xi}{\left(\xi^2 + \widetilde{\varepsilon}_n^2\right)^2} \\ &= -\nu \left\langle v_\alpha v_\beta q_\beta \right\rangle_{\mathrm{FS}} \pi T \sum_{n=0}^{\infty} \frac{1}{\left(\varepsilon_n + 1/2\tau\right) \varepsilon_n^2}, \end{aligned} \tag{8.9}$$

where $\left\langle v_\alpha v_\beta q_\beta \right\rangle_{FS} = v_F^2 q_\alpha / D$ in the case of D dimensional isotropic spectrum. The last sum already was carried out in evaluation of the (6.28) and finally (8.9) turns out to be proportional to the square of coherence length (6.33). In the case of the isotropic spectrum

$$\begin{aligned} B_\alpha(\mathbf{q}) &= \frac{2\nu\tau^2 v_F^2}{D} \left[\psi(\frac{1}{2} + \frac{1}{4\pi T\tau}) - \psi(\frac{1}{2}) - \frac{1}{4\pi T\tau} \psi'(\frac{1}{2}) \right] q_\alpha \\ &= -2\nu \xi_{(D)}^2 (T\tau) q_\alpha. \end{aligned} \tag{8.10}$$

In the other case important for our consideration, the layered superconductor, the Green function block in the vicinity of the transition temperature takes the form

$$B_\alpha(\mathbf{q}) = -2\nu \frac{\eta_{(2)}}{v_F^2} \begin{cases} v_F^2 q_\alpha, \, \alpha = x, y, \\ sJ^2 \sin q_z s, \, \alpha = z. \end{cases} \tag{8.11}$$

Let us return to the study of the general expression (8.6) for $Q_{\alpha\beta}^{\mathrm{AL}}(\omega_\nu)$. The Ω_k-summation in it can be transformed into a contour integral, using the identity [244]

$$T \sum_{\Omega_k} f(\Omega_k) = \frac{1}{4\pi i} \oint_{C_0} dz \coth \frac{z}{2T} f(-iz), \tag{8.12}$$

where $z = i\Omega_k$ is a variable in the plane of complex frequency and the contour C_0 encloses all bosonic Matsubara frequencies over which the summation is carried out (see Fig. 8.2(a)). Applying this transformation to Eq. (8.6) one can write

$$Q_{\alpha\beta}^{AL}(\omega_\nu) = -\frac{e^2}{\pi i} \int \frac{d^3\mathbf{q}}{(2\pi)^3} B_\alpha(\mathbf{q}) B_\beta(\mathbf{q}) \oint_C dz \coth \frac{z}{2T} L(\mathbf{q}, -iz) L(\mathbf{q}, -iz + \omega_\nu), \tag{8.13}$$

where the integration contour C is some continuously deformed contour C_0, which we choose for convenience in further integration. In order to do this let us

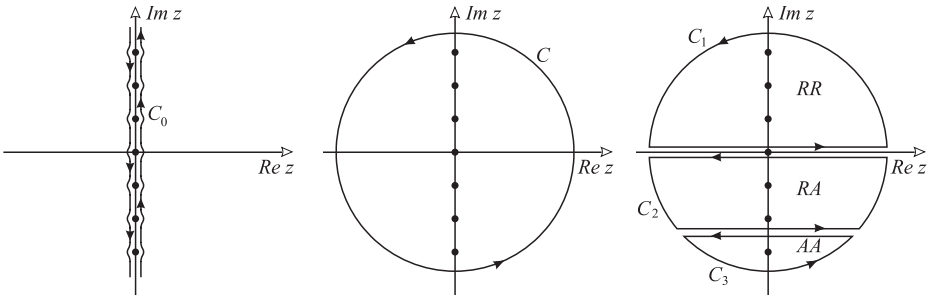

FIG. 8.2. The contour of integration in the plane of complex frequencies.

notice that the integrand function in (8.13) has the breaks of analyticity at the lines $\mathrm{Im}\, z = 0$ and $\mathrm{Im}\, z = -i\omega_\nu$. Indeed, the fluctuation propagator $L(\mathbf{q}, \Omega_k)$ was defined on the bosonic Matsubara frequencies only, while now we have to use it as the function of the continuous variable z. As is known from the properties of Green's functions in the complex plane z, two analytic functions, related to $L(\mathbf{q}, \Omega_k)$, can be introduced. The first one, $L^R(\mathbf{q}, -iz)$ (retarded), is analytic in the upper half-plane ($\mathrm{Im}\, z > 0$), while the second one, $L^A(\mathbf{q}, -iz)$ (advanced), has no singularities in the lower half-plane ($\mathrm{Im}\, z < 0$). In accordance with this observation let us cut the z-plane along the lines $\mathrm{Im}\, z = 0$ and $\mathrm{Im}\, z = -i\omega_\nu$ (see Fig. 8.2(b)) and choose the contour C in (8.13) as the aggregate of three closed contours $C_1 + C_2 + C_3$ (see Fig. 8.2(c)). Each of them already encloses a domain of well-defined analyticity of the integrand function.

Hence the integrand function is analytic along each of these contours and the corresponding integral, according to the Cauchy theorem, is determined by the sum of residues in the poles of $\coth(z/2T)$ (see the formula (8.12)). If the modulus of function $f(-iz)$ on the large circle $|z| = R \to \infty$ tends to zero as $\sim R^{-1-\gamma} (\gamma > 0)$ the value of the integral over this circle in (8.12) tends to zero too. Finally, the contour integral in (8.13) is reduced to the four integrals along the horizontal lines:

$$
\begin{aligned}
I^{AL}_{\alpha\beta}(\mathbf{q}, \omega_\nu) &= \oint_{C_1+C_2+C_3} dz \coth \frac{z}{2T} L(\mathbf{q}, -iz) L(\mathbf{q}, -iz + \omega_\nu) \\
&= \int_{-\infty}^{\infty} dz \coth \frac{z}{2T} L^R(\mathbf{q}, -iz + \omega_\nu) \left[L^R(\mathbf{q}, -iz) - L^A(\mathbf{q}, -iz) \right] \\
&+ \int_{-\infty - i\omega_\nu}^{\infty - i\omega_\nu} dz \coth \frac{z}{2T} L^A(\mathbf{q}, -iz) \left[L^R(\mathbf{q}, -iz + \omega_\nu) - L^A(\mathbf{q}, -iz + \omega_\nu) \right].
\end{aligned}
\tag{8.14}
$$

Now one can shift the variable in the last integral to $z = z' - i\omega_\nu$, take into account that $i\omega_\nu$ is the period of $\coth \frac{z}{2T}$ and get an expression analytic in $i\omega_\nu \to \omega$. Finally:

$$Q_{\alpha\beta}^{AL(R)}(\omega) = -\frac{2e^2}{\pi}\int \frac{d^3\mathbf{q}}{(2\pi)^3} B_\alpha(\mathbf{q}) B_\beta(\mathbf{q}) \times \int_{-\infty}^{\infty} dz \coth\left(\frac{z}{2T}\right) \left[L^R(\mathbf{q}, -iz - i\omega) + L^A(\mathbf{q}, -iz + i\omega)\right] \operatorname{Im} L^R(\mathbf{q}, -iz).$$

Being interested here in the d.c. conductivity one can expand the integrand function in ω. It is possible to show that the zeroth-order term is canceled by the same type contributions from all other diagrams (this cancelation confirms the absence of anomalous diamagnetism above the critical temperature). The remaining integral can be integrated by parts and then carried out taking into account that the contribution most singular in ϵ comes from the region $z \sim \epsilon \ll T$:

$$\sigma_{\alpha\beta}^{AL} = \frac{e^2}{2\pi T}\int \frac{d^3\mathbf{q}}{(2\pi)^3} B_\alpha(\mathbf{q}) B_\beta(\mathbf{q}) \int_{-\infty}^{\infty} \frac{dz}{\sinh^2 \frac{z}{2T}} \left[\operatorname{Im} L^R(\mathbf{q}, -iz)\right]^2. \tag{8.15}$$

Coming back to the case of the layered superconductor one can reproduce the Lawrence–Doniach expression (3.36) for the in-plane paraconductivity. Indeed, separating the imaginary part of Eq. (6.39) and substituting it together with (8.11) to (8.15) one can find:

$$\sigma_{xx}^{AL} = \frac{\pi e^2 \eta_{(2)}^2}{8T}\int \frac{q_x^2 d^2\mathbf{q}}{(2\pi)^2} \int_{-\pi/s}^{\pi/s} \frac{dq_z}{2\pi} \int_{-\infty}^{\infty} \frac{dz}{\left[(\eta_{(2)}\mathbf{q}^2 + \epsilon + r\sin^2\frac{q_z s}{2})^2 + \left(\frac{\pi z}{8T}\right)^2\right]^2} = \frac{\pi e^2 \eta_{(2)}^2}{2s}\int \frac{d^2\mathbf{q}}{(2\pi)^2} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{q^2}{\left[\eta_{(2)}\mathbf{q}^2 + \epsilon + \frac{r}{2}(1-\cos\theta)\right]^3},$$

where the Lawrence–Doniach anisotropy parameter r [50] was already defined by (2.45). The integral over θ can be at first presented as the complete second derivative and then expressed by means of the standard for the theory of layered superconductors integral (C.3). Carrying out the remaining integration over x by parts one can find

$$\sigma_{xx}^{AL} = \frac{e^2}{16s}\int_0^\infty x\frac{d^2}{dx^2}\left[\frac{1}{\sqrt{(\epsilon+x)(\epsilon+x+r)}}\right]dx = \frac{e^2}{16s}\frac{1}{[\epsilon(\epsilon+r)]^{1/2}} \to \frac{e^2}{16s}\begin{cases} 1/\sqrt{\epsilon r}, \ \epsilon \ll r, \\ 1/\epsilon, \ \epsilon \gg r. \end{cases} \tag{8.16}$$

In the same way one can evaluate the AL contribution to the transverse fluctuation conductivity and get the familiar from the phenomenological consideration Eq. (3.37) [238, 245, 246]:

$$\sigma_{zz}^{AL} = \frac{\pi e^2 s r^2}{32}\int \frac{d^2\mathbf{q}}{(2\pi)^2} \frac{1}{\left[(\eta_{(2)}\mathbf{q}^2 + \epsilon)(\eta_{(2)}\mathbf{q}^2 + \epsilon + r)\right]^{3/2}} \tag{8.17}$$

$$= \frac{e^2 s}{32\xi_{xy}^2}\left(\frac{\epsilon + r/2}{[\epsilon(\epsilon + r)]^{1/2}} - 1\right) \to \frac{e^2 s}{64\xi_{xy}^2}\begin{cases}\sqrt{r/\epsilon}, \epsilon \ll r, \\ (r/2\epsilon)^2. \ \epsilon \gg r\end{cases}$$

Note that contrary to the case of in-plane conductivity, the critical exponent for σ_{zz} above the Lawrence–Doniach crossover temperature $T_{\rm LD}$ (for which $\epsilon(T_{\rm LD}) = r$) is 2 instead of 1, so the crossover occurs from the $0D$ to $3D$ regimes. This is related to the tunneling (so from the band structure point of view effectively zero dimensional $0D$) character of electron motion along the c-axis.

8.3.2 *Contributions from fluctuations of the DOS*

In original paper of Aslamazov and Larkin [36] the most singular AL contributions to conductivity, heat capacity and other properties of a superconductor above the critical temperature were considered. The diagrams of the type 5–6 were pictured and correctly evaluated as less singular in ϵ. Nevertheless the specific form of the AL contribution to the transverse conductivity of a layered superconductor, which may be considerably suppressed for small interlayer transparency, suggested to re-examine the contributions from diagrams 5–10 of Fig. 8.1 which are indeed less divergent in ϵ, but turn out to be of lower order in the transmittance and of the opposite sign with respect to the AL one [238,239]. These, so-called DOS, diagrams describe the changes in the normal Drude-type conductivity due to fluctuation renormalization of the normal quasiparticles DOS above the transition temperature (see section 10.1). In the dirty limit, the calculation of contributions to the longitudinal fluctuation conductivity σ_{xx} from such diagrams was discussed in [241,247]. Contrary to the case of the AL contribution, the in-plane and out-of-plane components of the DOS contribution differ only in the square of the ratio of effective Fermi velocities in the parallel and perpendicular directions. This allows us to calculate both components simultaneously. The contribution to the fluctuation conductivity due to diagram 5 (diagram 6 gives an identical contribution) is

$$Q_{\alpha\beta}^{(5)}(\omega_\nu) = 2e^2 T \sum_{\Omega_k} \int \frac{d^3\mathbf{q}}{(2\pi)^3} L(\mathbf{q}, \Omega_k) \sum\nolimits_{\alpha\beta}^{(5)}(\mathbf{q}, \Omega_k), \tag{8.18}$$

where

$$\sum\nolimits_{\alpha\beta}^{(5)}(\mathbf{q}, \Omega_k, \omega_\nu) = T \sum_{\varepsilon_n} \lambda^2(\mathbf{q}, \varepsilon_n, \Omega_k - \varepsilon_n) I_{\alpha\beta}^{(5)}(\mathbf{q}, \varepsilon_n, \Omega_k, \omega_\nu) \tag{8.19}$$

and

$$\begin{aligned} I_{\alpha\beta}^{(5)}(\mathbf{q}, \varepsilon_n, \Omega_k, \omega_\nu) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} v_\alpha(\mathbf{p}) v_\beta(\mathbf{p}) G^2(\mathbf{p}, \varepsilon_n) G(\mathbf{q} - \mathbf{p}, \Omega_k - \varepsilon_n) G(\mathbf{p}, \varepsilon_{n+\nu}) \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{(i\widetilde{\varepsilon}_n - \xi_{\mathbf{p}})^2} \frac{v_\alpha(\mathbf{p}) v_\beta(\mathbf{p})}{(i\widetilde{\varepsilon}_{n+\nu} - \xi_{\mathbf{p}})} \frac{1}{\left(i\widetilde{(\Omega_k - \varepsilon_n)} - \xi_{\mathbf{q}-\mathbf{p}}\right)}. \end{aligned} \tag{8.20}$$