(Topological) Superconductivity with BdG

OUTLINE:

1. Bogoliubov-de Gennes (BdG) theory
2. Kitaev chain model & Majorana's
3. Engineering a TSC
4. Gapped TSC classes in $d=1,2,3$
5. Point and Line defects in TSC
6. Materials & signatures

→ LECTURE NOTES

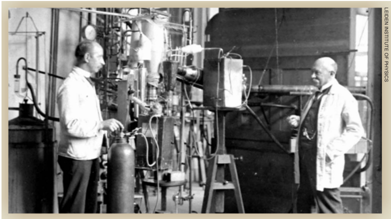
What is missing?

- Microscopic mechanism for pairing (like BCS)
- Phenomenology
- The order parameter(s) (beyond mean-field)
- Exotic states

LATER
THIS
WEEK

Superconductivity

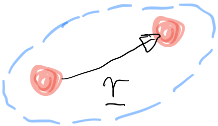
$YBa_2Cu_3O_{7+x}$ $T_c = 93K$



K. ONNES,
LEIDEN $\boxed{1911.}$

Theory $\boxed{1957.}$

Electrons form Cooper pairs
due to effective attraction



Superconductivity

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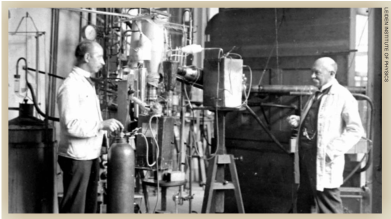

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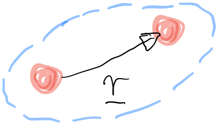


$\langle c^\dagger_{k\sigma} c^\dagger_{-k\sigma'}\rangle_{SC} \neq 0$, $\begin{cases} k = \text{electron momentum} \\ \sigma = \text{electron spin} \end{cases}$

Relevant e-e interaction : $\sum_{\substack{k,k' \\ \sigma,\sigma',\lambda,\lambda'}} V^{\sigma\sigma'\lambda\lambda'}_{k,k'} \underbrace{c^\dagger_{k'\sigma} c^\dagger_{-k'\sigma'}} \underbrace{c_{-k\lambda} c_{k\lambda'}}$

Mean-field

Pairing function: $\Delta^{\sigma\sigma'}_k \equiv \sum_{k',\lambda,\lambda'} V^{\lambda\lambda'\sigma\sigma'}_{k',k} \langle c_{-k'\lambda} c_{k'\lambda'}\rangle$

$$\boxed{\hat{H}_{pair} = \sum_{k\sigma\sigma'} \left(\Delta^{\sigma\sigma'}_k c^\dagger_{k\sigma} c^\dagger_{-k\sigma'} + h.c. \right)}$$

BdG or not? Single-site example

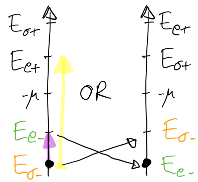
- 1-site, spinful: $\hat{H} = \varepsilon\left(c^\dagger_\uparrow c_\uparrow - c^\dagger_\downarrow c_\downarrow\right) + \left(\lambda c^\dagger_\uparrow c_\downarrow + h.c.\right) - \mu\left(n_\uparrow + n_\downarrow\right) + \left(\Delta c^\dagger_\uparrow c^\dagger_\downarrow + h.c.\right)$

1. Fock space:

$$
\begin{array}{ccc}
 & & \begin{array}{cccc} |0\rangle & |\uparrow\rangle & |\downarrow\rangle & |\uparrow\downarrow\rangle \end{array} \\
e & \begin{array}{c} |0\rangle \\ |\uparrow\rangle \\ |\downarrow\rangle \\ |\uparrow\downarrow\rangle \end{array} & \left(\begin{array}{c|cc|c} \boxed{0} & & & \boxed{\Delta^*} \\ \hline & \boxed{\varepsilon-\mu} & \boxed{\lambda} & \\ & \boxed{\lambda^*} & \boxed{-\varepsilon-\mu} & \\ \hline \boxed{\Delta} & & & \boxed{-2\mu} \end{array}\right) \Big\} 2^{|S_1|} \\
 & \sigma &
\end{array}
$$

$$E_{\sigma\pm} = \pm\sqrt{\varepsilon^2+|\lambda|^2} - \mu$$

$$E_{e\pm} = \pm\sqrt{\mu^2+|\Delta|^2} - \mu$$



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① Fock space:

$$\begin{array}{c|cccc} & |0\rangle & |\uparrow\rangle & |\downarrow\rangle & |\uparrow\downarrow\rangle \\ \hline |0\rangle & \boxed{0} & & & \boxed{\Delta^*} \\ |\uparrow\rangle & & \varepsilon-\mu & \lambda & \\ |\downarrow\rangle & & \lambda^* & -\varepsilon-\mu & \\ |\uparrow\downarrow\rangle & \boxed{\Delta} & & & -2\mu \end{array} \Bigg\} 2^{|\mathcal{S}_1|}$$

$$E_{\sigma\pm} = \pm\sqrt{\varepsilon^2+|\lambda|^2} - \mu$$

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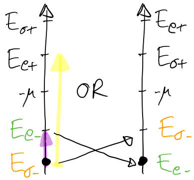
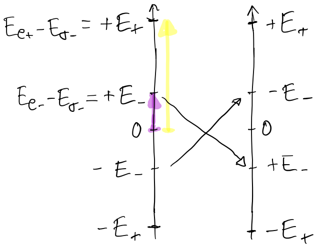


② BdG: $\hat{H} = \frac{1}{2}\hat{\Psi}^\dagger H_{BdG}\hat{\Psi} + \text{const}$

$$\boxed{\hat{\Psi} \equiv \begin{pmatrix} c_\uparrow \\ c_\downarrow \\ \hdashline c_\uparrow^\dagger \\ c_\downarrow^\dagger \end{pmatrix}} \Rightarrow$$

$$H_{BdG} = \begin{array}{c|cc|cc} & c_\uparrow & c_\downarrow & c_\uparrow^\dagger & c_\downarrow^\dagger \\ \hline c_\uparrow^\dagger & \varepsilon-\mu & \lambda & 0 & \Delta \\ c_\downarrow^\dagger & \lambda^* & -\varepsilon-\mu & -\Delta & 0 \\ \hline c_\uparrow & 0 & -\Delta^* & -\varepsilon+\mu & -\lambda^* \\ c_\downarrow & \Delta^* & 0 & -\lambda & \varepsilon+\mu \end{array} \Bigg\} 2\cdot|\mathcal{S}_1|$$

$$E^{BdG}_{\pm, a=\pm} = \pm\left(\sqrt{\varepsilon^2+|\lambda|^2} + a\cdot\sqrt{\mu^2+|\Delta|^2}\right) \equiv \pm E_a$$



BdG solution: Single-site example

2 BdG: $\hat{H} = \frac{1}{2}\hat{\Psi}^\dagger H_{BdG}\hat{\Psi}$, where $\hat{\Psi}^T = (c_\uparrow, c_\downarrow, c_\uparrow^\dagger, c_\downarrow^\dagger)$ is fermion spinor $\{\hat{\Psi}_i, \hat{\Psi}_j^\dagger\} = \delta_{ij}$

$H_{BdG}\,\chi_{a\pm} = \pm E_a \chi_{a\pm}$; $(\chi \in \mathbb{C}^4) \Rightarrow \hat{H} = \frac{1}{2}\sum_{a=\pm} \pm E_a \hat{\gamma}_{a\pm}^\dagger \hat{\gamma}_{a\pm}$, where

$\Downarrow$

Bogoliubon (fermion): $\boxed{\hat{\gamma}_{a\pm}^\dagger} = \chi_{a\pm}\cdot\hat{\Psi}^\dagger = \chi_{a\pm}^1 \boxed{c_\uparrow^\dagger} + \chi_{a\pm}^2 \boxed{c_\downarrow^\dagger} + \chi_{a\pm}^3 \boxed{c_\uparrow} + \chi_{a\pm}^4 \boxed{c_\downarrow}$; $\{\hat{\gamma}_A, \hat{\gamma}_B^\dagger\} = \delta_{AB}$

electron „u"　　hole „v"

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electron „u" hole „v"

Particle-Hole (PH) constraint (doubling): $(\hat{\Psi}_{1/2})^\dagger = \hat{\Psi}_{3/4}$, with $\tau_x = \left(\begin{array}{cc|cc} & & 1 & 0 \\ & & 0 & 1 \\ \hline 1 & 0 & & \\ 0 & 1 & & \end{array}\right)$, $\Psi_i^\dagger = \tau_x^{ij}\Psi_j$

$$\boxed{\hat{H} = \frac{1}{2}\hat{\Psi}^\dagger H_{BdG}\hat{\Psi} = \frac{1}{2}(\tau_x\hat{\Psi})H_{BdG}(\tau_x\hat{\Psi}^\dagger) = \frac{1}{2}\hat{\Psi}^\dagger(-\tau_x H_{BdG}^T\tau_x)\hat{\Psi}}$$

$\Rightarrow \boxed{H_{BdG} = -P H_{BdG} P^{-1}}$, with $\boxed{P \equiv \tau_x K}$ $\Rightarrow H_{BdG}\chi_{a+} = +E_a\chi_{a+} \Rightarrow H_{BdG}(P\chi_{a+}) = -E_a P\chi_{a+}$

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$H_{BdG}\chi_{a\pm} = \pm E_a \chi_{a\pm} \Rightarrow \hat{H} = \frac{1}{2}\sum_{a=\pm} \pm E_a \hat{\gamma}_{a\pm}^\dagger \hat{\gamma}_{a\pm}$, where

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(doubling)

$$\left[\hat{H} = \frac{1}{2}\hat{\Psi}^\dagger H_{BdG}\hat{\Psi} = \frac{1}{2}(\tau_1\hat{\Psi})H_{BdG}(\tau_1\hat{\Psi}^\dagger) = \frac{1}{2}\hat{\Psi}^\dagger(-\tau_1 H_{BdG}^T\tau_1)\hat{\Psi}\right]$$

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① $\chi_{a-} = \tau_x \chi_{a+}^*$ ② $E_{-,a} = -E_{+,a}$ ③ $\boxed{\gamma_{a-}^\dagger = \gamma_{a+}}$ $\left[\chi_{a+}^*\cdot\hat{\Psi} = \chi_{a+}^*\tau_x\hat{\Psi}^\dagger = \chi_{a-}\hat{\Psi}^\dagger\right]$

$$\Rightarrow \boxed{\hat{H} = \frac{1}{2}\sum_a\left(+E_a\hat{\gamma}_{a+}^\dagger\hat{\gamma}_{a+} - E_a\hat{\gamma}_{a-}^\dagger\hat{\gamma}_{a-}\right)} = \boxed{\sum_a |E_a|\hat{\gamma}_{a+}^\dagger\hat{\gamma}_{a+} + \text{const}}$$

BdG structure: Single-site example

- Bogoliubov free fermion structure:

① $|GS\rangle, \hat{\gamma}^+_{a=1}|GS\rangle, \hat{\gamma}^+_{a=2}|GS\rangle, \gamma^+_{a=1}\gamma^+_{a=2}|GS\rangle$: Fock states $2^{|\mathcal{S}_1|}$ from $|\mathcal{S}_1|$ fermionic excitations

② $|GS\rangle$ = vacuum of (+) Bogoliubons : $\boxed{\gamma_{a+}|GS\rangle \equiv 0}$, $\forall a$

Build $|GS\rangle \equiv \prod_a \gamma^+_{a+}|\Psi_0\rangle$, where $|\Psi_0\rangle = |vac\rangle$ or $c^+_\sigma|vac\rangle$, so that $|GS\rangle \neq 0$.

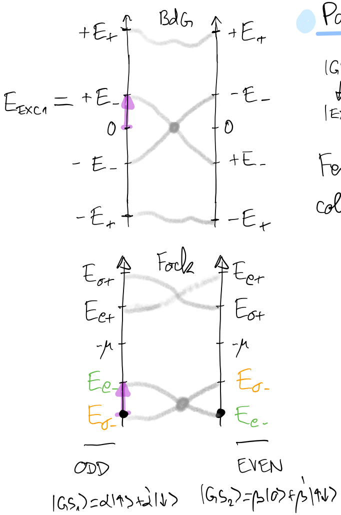
$\Rightarrow$ BCS-like : $\gamma^+_{a+}|\Psi_0\rangle = \sum_\sigma (u_\sigma c^+_\sigma + v_\sigma c_\sigma)|\Psi_0\rangle \sim (v_\sigma + u_\sigma c^+_\sigma c^+_{\bar\sigma})|\tilde{\Psi}_0\rangle$

③ Fermion parity $\hat{H} \sim c^+c + c^+c^+ + cc$ mixes $\begin{cases} \text{only } N = 0, 2, 4 \ldots \\ \text{only } N = 1, 3, 5 \ldots \end{cases}$ (Z_2 charge due to $c^+_\sigma \to -c^+_\sigma$)

$\hat{F} \equiv (-1)^{\hat{N}} = \pm 1$

Parity and topology: Single-site example

Gap-closing adiabatic transitions



Parity switch:

$|GS_1\rangle$

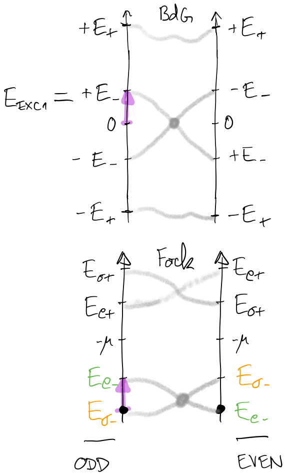
$\downarrow$

$|EXC_1\rangle = \boxed{\gamma^{\dagger}_{a=-}|GS_1\rangle = |GS_2\rangle}$

Fermion excitation collapses → parity switch.

Parity and topology: Single-site example

- Gap-closing adiabatic transitions



$|GS_1\rangle = \alpha|\uparrow\rangle + \alpha'|\downarrow\rangle$ $\quad |GS_2\rangle = \beta|0\rangle + \beta'|\uparrow\downarrow\rangle$

- Parity switch:

$$|GS_1\rangle \downarrow |EXC_1\rangle = \boxed{\gamma^+_{a=-}|GS_1\rangle = |GS_2\rangle}$$

Fermion excitation collapses → parity switch.

- Parity = Topological Invariant

From H_{BdG}? „Sign change" of E_a?

ALMOST: $\boxed{F_{GS1}\cdot F_{GS2} = (\mathrm{sgn}\,\mathrm{Pf}X_1)\cdot(\mathrm{sgn}\,\mathrm{Pf}X_2)}$

Majorana basis $\quad \lambda_{1\sigma} = c_\sigma + c_\sigma^+$

$\lambda_{2\sigma} = i(c_\sigma - c_\sigma^+)$

$$\hat{H} = \frac{1}{2}\hat{\Psi}^+ H_{BdG}\hat{\Psi} = \frac{1}{2}i\hat{\lambda}X\hat{\lambda}$$

$$X^T = -X \Rightarrow X \simeq \begin{pmatrix} 0 & \boxed{E_{a_1}} & & \\ -E_{a_1} & 0 & & \\ & & 0 & \boxed{E_{a_2}} \\ & & -E_{a_2} & 0 \end{pmatrix}$$

$$\mathrm{Pf}X = E_{a=+}\cdot E_{a=-}$$

Interesting in $d>0$ dimensions.

BdG theory in general: setup

Second quantized Hamiltonian: $\hat{H} = \sum_{k,a,b}^{p} (\varepsilon_k^{ab} - \mu\delta_{ab}) c_{ka}^{\dagger} c_{kb} + \frac{1}{2}\sum_{k,a,b}^{p} \left(\Delta_k^{ab} c_{ka}^{\dagger} c_{-kb}^{\dagger} + \Delta_k^{ab*} c_{-kb} c_{ka} \right)$

Key idea: identify fermionic quasiparticles

$a, b = 1 \ldots p$

$\boxed{\text{p degrees of freedom in unit cell}}$

- Nambu spinor:

$$\boxed{\psi_k} \equiv \begin{pmatrix} c_{k1} \\ \vdots \\ c_{kp} \\ \hline c_{-k1}^{\dagger} \\ \vdots \\ c_{-kp}^{\dagger} \end{pmatrix}_{2p} \begin{matrix} \}\text{electrons} \\ \\ \}\text{holes} \end{matrix} \implies \boxed{\psi_k^{\dagger}} = (\underbrace{c_{k1}^{\dagger}, \ldots, c_{kp}^{\dagger}}_{\text{electrons}} \,\vdots\, \underbrace{c_{-k1}, \ldots, c_{-kp}}_{\text{holes}})_{2p}$$

- Indeed fermions:

$$\{\psi_{ka}, \psi_{k'a'}\} = \{\psi_{ka}^{\dagger}, \psi_{k'a'}^{\dagger}\} = 0$$

$$\{\psi_{ka}, \psi_{k'a'}^{\dagger}\} = \delta_{kk'}\delta_{aa'}$$

NOTE: Fourier space

$$\underline{\psi_k} = \frac{1}{\sqrt{N}} \sum_{\alpha} e^{i\underline{k}\underline{\alpha}} \begin{pmatrix} c_{\underline{\alpha}1} \\ \vdots \\ c_{\underline{\alpha}p} \\ \hline c_{\underline{\alpha}1}^{\dagger} \\ \vdots \\ c_{\underline{\alpha}p}^{\dagger} \end{pmatrix}$$

BdG theory: recipe

Goal: Rewrite $\hat{H} \equiv \frac{1}{2}\sum_{k,i,j}^{2P} \psi^{+}_{ki} H^{ij}_{k} \psi_{kj} + \text{const} \leftarrow i,j = 1 \ldots 2p \; (e \,\&\, h)$

- RECIPE for BdG matrix (first quantized)

$$(H_{BdG} =) \quad H_k = \begin{array}{c c} & \begin{array}{cc} e & h \end{array} \\ \begin{array}{c} e \\ h \end{array} & \left(\begin{array}{c|c} \xi_k & \Delta_k \\ \hline \Delta_k^{+} & -\xi_{-k}^{T} \end{array} \right)_{2p \times 2p} \end{array} \; ; \quad \begin{array}{ll} [\xi_k]_{p\times p} = \varepsilon_k^{ab} - \mu \,, & a,b = 1 \ldots p \\ [\Delta_k]_{p\times p} = \frac{1}{2}\left(\Delta_k^{ab} - \Delta_{-k}^{ba}\right), & a,b = 1 \ldots p \end{array}$$

- Fermion constraint: $\boxed{\Delta_k \equiv -\Delta_{-k}^{T}}$

$$\left[\text{because} \quad \sum_{k,ab}^{p} \boxed{\Delta_k^{ab}}\, c^{+}_{ka} c^{+}_{-kb} = \sum_{k,a,b}^{p} -\Delta_k^{ab}\, c^{+}_{-kb} c^{+}_{ka} \;\underset{a \leftrightarrow b}{\overset{k \to -k}{=\!=}}\; \sum_{k,a,b}^{p} \boxed{\left(-\Delta_{-k}^{ba}\right)} c^{+}_{ka} c^{+}_{-kb} \right]$$

BdG theory: solution

Bogoliubons: $\hat{H} = \frac{1}{2}\sum_k \Psi_k^\dagger H_k \Psi_k = \frac{1}{2}\sum_k \sum_{i=1}^{2p} E_k^{(i)} \gamma_k^{(i)\dagger} \gamma_k^{(i)}$, $i = 1 \dots 2p$

Bogoliubov fermion

- $2p$ free Bogoliubov bands $E_k^{(i)}$, $i = 1 \dots 2p$

BdG theory: solution

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Bogoliubov fermion

- 2p free Bogoliubov bands $E_k^{(i)}$, $i=1\dots 2p$

- RECIPE: $H_k \chi_k^{(i)} = E_k^{(i)} \chi_k^{(i)}$; $\chi_k^{(i)} = \begin{pmatrix} u_{k1}^{(i)} \\ \vdots \\ u_{kp}^{(i)} \\ \hline v_{k1}^{(i)} \\ \vdots \\ v_{kp}^{(i)} \end{pmatrix}$

2p complex numbers

Bogoliubon: $\boxed{\gamma_k^{(i)\dagger}} \equiv \sum_{a=1}^{2p} \chi_{ka}^{(i)} \Psi_{ka}^\dagger = \sum_{a=1}^{p} \left(u_{ka}^{(i)} c_{ka}^\dagger + v_{ka}^{(i)} c_{-ka} \right)$

BdG theory: PHS constraint

Bogoliubons are NOT independent !

- $$\Psi_k \equiv \begin{array}{c} e\left\{ \vphantom{\begin{matrix} a \\ b \\ c \end{matrix}} \right. \\ h\left\{ \vphantom{\begin{matrix} a \\ b \\ c \end{matrix}} \right. \end{array} \begin{pmatrix} \boxed{c_{k1}} \\ \vdots \\ c_{kp} \\ \hline \boxed{c^\dagger_{-k1}} \\ \vdots \\ c^\dagger_{-kp} \end{pmatrix} \qquad \Psi^\dagger_{-k,1} = \Psi_{k,p+1}$$

DEFINE: $\tau_x = \begin{array}{cc} & \begin{matrix} e & h \end{matrix} \\ \begin{matrix} e \\ h \end{matrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array} \equiv \sigma_x \otimes \mathbb{1}_p = \left(\begin{array}{c|c} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \end{array} \right)_{2p \times 2p}$

$\Rightarrow$ Particle-Hole Symmetry (constraint): $\boxed{\Psi^\dagger_k = (\tau_x \Psi_{-k})^T}$ (second quantized)

$\Rightarrow \tau_x H^T_{-k} \tau_x = \boxed{\tau_x H^*_{-k} \tau_x = -H_k}$ (first quantized)

$P \equiv \tau_x K$

$\boxed{P H_{-k} P^{-1} = -H_k}$ (first quantized)

REMINDER:

$$H_k = \begin{array}{cc} & \begin{matrix} e & h \end{matrix} \\ \begin{matrix} e \\ h \end{matrix} & \left(\begin{array}{c|c} \xi_k & \Delta_k \\ \hline \Delta^\dagger_k & -\xi^T_{-k} \end{array} \right)_{2p \times 2p} \end{array}$$

BdG theory: Interpretation

Bogoliubov doubling = PHS:

1st quant.: ① $(k, E_k^{(i)}) \Rightarrow (-k, -E_k^{(i)})$ $_{i=1\ldots 2p}$ ② $\chi_{-k}^{(i_-)} = \tau_x \chi_k^{(i_+)*}$ $_{i_\pm = 1\ldots p}$ $\Rightarrow$ 2nd q. ③ $\boxed{\gamma_{-k}^{(i_-)\dagger} = \gamma_k^{(i_+)}}$

- Division into $E^{(i_+)} > 0$ and $E^{(i_-)} < 0$ at each k ; $i_+, i_- = 1 \ldots p$

BdG theory: Interpretation

Bogoliubov doubling = PHS:

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- Division into $E^{(i_+)} > 0$ and $E^{(i_-)} < 0$ at each k ; $i_+, i_- = 1\ldots p$

$$\Rightarrow \hat{H} = \frac{1}{2}\sum_k \sum_i^{2p} E_k^{(i)} \gamma_k^{(i)\dagger} \gamma_k^{(i)} = \sum_k \boxed{\sum_{i_+}^{p} |E_k^{(i_+)}| \gamma_k^{(i_+)\dagger} \gamma_k^{(i_+)}} + \text{const}$$

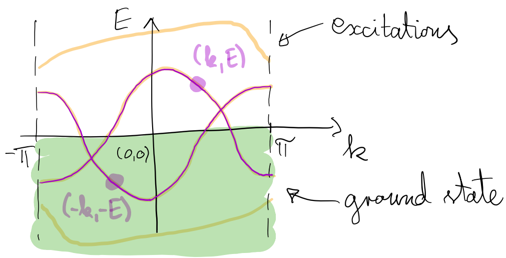
- GS is vacuum of positive Bogoliubons: $\gamma_k^{(i_+)} |GS\rangle \equiv 0, \ \forall k, \ i_+ = 1\ldots p$

- Bogoliubov excitations: $\gamma_k^{(i_+)\dagger} |GS\rangle, \ldots \quad E_k^{(i_+)} > 0$

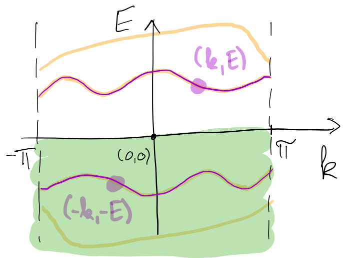
$$\boxed{\frac{1}{2}\sum_k \sum_{i_-}^{p} E_k^{(i_-)} \gamma_k^{(i_-)\dagger} \gamma_k^{(i_-)} \underset{=\!=}{\overset{k \to -k}{=}} \frac{1}{2}\sum_k \sum_{i_-}^{p} \overbrace{E_{-k}^{(i_-)}}^{E_k^{(i_+)}} \left(-\gamma_k^{(i_+)\dagger} \gamma_k^{(i_+)} + 1\right)}$$

BdG theory: Implications

- BdG bands



- HERE: GAPPED (T)SC



- Band topology

Kitaev chain, PBC

- "Spinless" electron model $\quad \alpha = N - 1 - 2 - 3 \;\cdots\; N - 1$

Real space: $\hat{H} \equiv -\frac{1}{2}\sum_\alpha \left(t\, c^\dagger_\alpha c_{\alpha+1} + h.c.\right) - \mu \sum_\alpha c^\dagger_\alpha c_\alpha + \frac{1}{2}\sum_\alpha \left(\bar{\Delta}\, c^\dagger_\alpha c^\dagger_{\alpha+1} + h.c.\right)$

Momentum: $\hat{H} = \sum_k \left(-t\cos k - \mu\right) c^\dagger_k c_k + \frac{1}{2}\sum_k \left(\bar{\Delta} e^{-ik} c^\dagger_k c^\dagger_{-k} + \bar{\Delta}^* e^{+ik} c_{-k} c_k\right)$

- BdG (RECIPE:) (assume $t>0$, $\bar{\Delta} \in \mathbb{R}$)

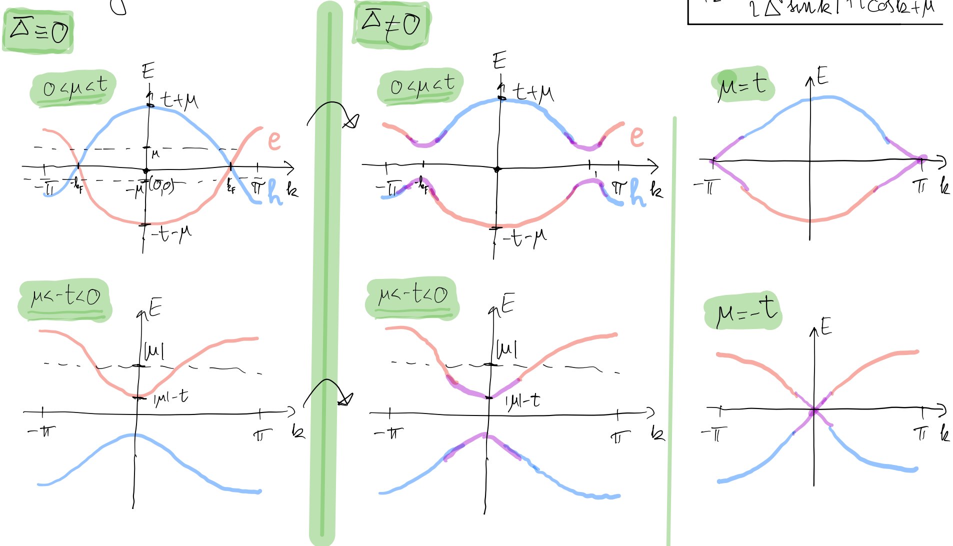
$\Psi_k \equiv \begin{pmatrix} c_k \\ c^\dagger_{-k} \end{pmatrix}$; $\xi_k = -t\cos k - \mu$; $\Delta_k \equiv \frac{1}{2}\left(\bar{\Delta} e^{-ik} - \bar{\Delta} e^{+ik}\right) = -i\bar{\Delta}\sin k$

$$\Rightarrow H_k = \begin{array}{c|c} \xi_k & -i\bar{\Delta}\sin k \\ \hline i\bar{\Delta}^*\sin k & -\xi_k \end{array} \Rightarrow \boxed{E^\pm_k} = \pm\sqrt{\xi_k^2 + |\Delta_k|^2} = \pm\sqrt{\left(t\cos k + \mu\right)^2 + |\bar{\Delta}|^2 \sin^2 k}$$

Kitaev chain, BdG bands

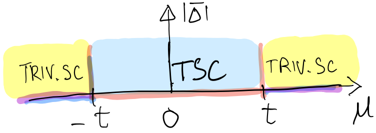
- Mixing electrons and holes

$$H_k = \left(\begin{array}{c|c} -t\cos k - \mu & -i\Delta \sin k \\ \hline i\Delta^* \sin k & +t\cos k + \mu \end{array}\right)$$



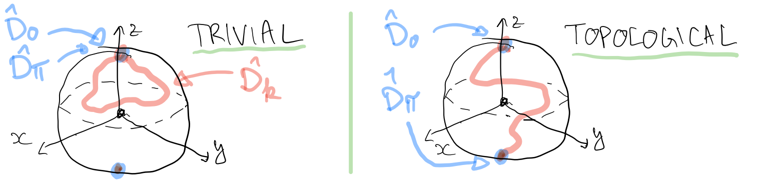
Kitaev chain, topology

- Phase diagram



$$H_k = \begin{pmatrix} -t\cos k - \mu & -i\bar\Delta \sin k \\ i\bar\Delta^* \sin k & +t\cos k + \mu \end{pmatrix}$$

- Bulk topology: $H_k = \vec D_k \cdot \vec\tau$; At $K = 0, \pi$, $\Delta_K = 0 \Rightarrow \boxed{\hat D_K = \mathrm{sgn}(\xi_K)\cdot \hat z}$



$\mathbb{Z}_2$ topological invariant: $\quad W = \mathrm{sgn}(\xi_0)\cdot\mathrm{sgn}(\xi_\pi) = \begin{cases} -1, & |\mu| < t \\ +1, & |\mu| > t \end{cases}$

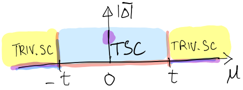
(for two bands)

Kitaev chain, open

- Edge states

$\alpha = $ 1 — 2 — 3 · · · N-1 — N

$$\hat{H} \equiv -\frac{1}{2}\sum_{\alpha}^{N-1}\left(t\, c^\dagger_\alpha c_{\alpha+1} + h.c.\right) - \mu \sum_{\alpha}^{N} c^\dagger_\alpha c_\alpha + \frac{1}{2}\sum_{\alpha}^{N-1}\left(\tilde{\Delta}\, c^\dagger_\alpha c^\dagger_{\alpha+1} + h.c.\right)$$



● : $\mu \equiv 0\,,\; t \equiv \tilde{\Delta} > 0$

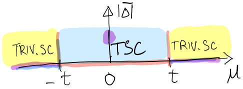
$$\boxed{\hat{H}_T = \frac{t}{2}\sum_{\alpha=1}^{N-1}\left(-c^\dagger_\alpha c_{\alpha+1} - c^\dagger_{\alpha+1} c_\alpha + c^\dagger_\alpha c^\dagger_{\alpha+1} + c_{\alpha+1} c_\alpha\right)}$$

Kitaev chain, open

- Edge states

$$\alpha = 1 - 2 - 3 \;\cdots\; N-1 - N$$

$$\hat H \equiv -\frac{1}{2}\sum_{\alpha}^{N-1}\left(t\, c^\dagger_\alpha c_{\alpha+1} + h.c.\right) - \mu \sum_{\alpha}^{N} c^\dagger_\alpha c_\alpha + \frac{1}{2}\sum_{\alpha}^{N-1}\left(\tilde\Delta\, c^\dagger_\alpha c^\dagger_{\alpha+1} + h.c.\right)$$



●: $\mu \equiv 0\,,\; t \equiv \tilde\Delta > 0$

$$\boxed{\hat H_T = \frac{t}{2}\sum_{\alpha=1}^{N-1}\left(-c^\dagger_\alpha c_{\alpha+1} - c^\dagger_{\alpha+1} c_\alpha + c^\dagger_\alpha c^\dagger_{\alpha+1} + c_{\alpha+1} c_\alpha\right)}$$

- Majorana fermions

$$\left.\begin{aligned} \lambda_{\alpha R} &\equiv c_\alpha + c^\dagger_\alpha \\ \lambda_{\alpha I} &\equiv i\left(c_\alpha - c^\dagger_\alpha\right) \end{aligned}\right\} \quad \boxed{\begin{aligned} \lambda^\dagger_{\alpha R} &= \lambda_{\alpha R} \\ \lambda^\dagger_{\alpha I} &= \lambda_{\alpha I} \end{aligned}}$$

„Real" fermion, not „complex"

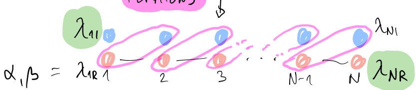
$$\begin{aligned} c^\dagger_\alpha &= \frac{1}{2}\left(\lambda_{\alpha R} + i\lambda_{\alpha I}\right) \\ c_\alpha &= \frac{1}{2}\left(\lambda_{\alpha R} - i\lambda_{\alpha I}\right) \end{aligned}$$

$$\hat H_T = \frac{t}{2}\sum_{\alpha=1}^{N-1} i\lambda_{\alpha R}\lambda_{\alpha+1, I} = -t\sum_{\beta=1}^{N-1}\left(d^\dagger_\beta d_\beta - \frac{1}{2}\right),$$

λ_{1I}, λ_{NR} do not appear

BULK COMPLEX FERMIONS

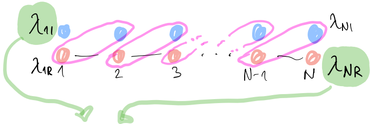
$d^\dagger_\beta$



$\lambda_{1I}\,;\,\lambda_{NR}$

EDGE MAJORANAS

Majorana Bound States (MBS)



(our chain)

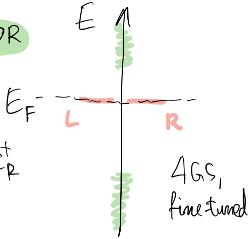
$d_0^+ \equiv \frac{1}{2}(\lambda_{NR} + i\lambda_{11}) = \frac{1}{2}(c_N - c_1 + c_N^+ + c_1^+)$: COMPLEX NON-LOCAL FERMION

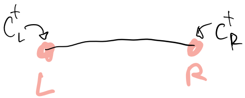
Consequences

① Ground State (GS) degeneracy : $\{|GS\rangle, d_0^+|GS\rangle\}$

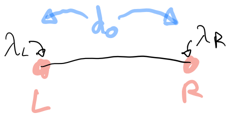
② Non-locality → non-Abelian anyon

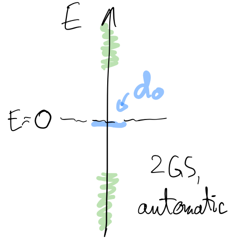
SPINLESS INSULATOR (SSH)





TOPOLOGICAL SC (KITAEV CHAIN)





Pairing: Electron spin

- Cooper pair: orbitals and spins

$$\Delta^{\sigma\sigma'}_{\mathbf{k}ab}\, c^{+}_{\mathbf{k}a\sigma} c^{+}_{-\mathbf{k}b\sigma'} |vac\rangle = \Delta^{\sigma\sigma'}_{\mathbf{k}ab}\, \underbrace{|\mathbf{k}a;-\mathbf{k}b\rangle}_{\text{orbital}} \otimes \underbrace{|\sigma\sigma'\rangle}_{\text{spin}} \; ; \; \sigma,\sigma' = \uparrow,\downarrow \text{ for } S_z = \pm\tfrac{1}{2}$$

$a, b = 1 \ldots r$ for orbitals

- Adding two spins $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$

$$\sum_{\sigma,\sigma'}^{\uparrow,\downarrow} \Delta^{\sigma\sigma'}_{\mathbf{k}ab} |\mathbf{k}a;-\mathbf{k}b\rangle \otimes |\sigma\sigma'\rangle = \left(\frac{\Delta^{\uparrow\downarrow} - \Delta^{\downarrow\uparrow}}{2} |\mathbf{k}a;-\mathbf{k}b\rangle\right) \otimes \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \quad : S_z = 0 \} S = 0 \quad \boxed{\text{ODD}} \text{ in } \sigma \leftrightarrow \sigma'$$

$$+ \left(\Delta^{\uparrow\uparrow} |\mathbf{k}a;-\mathbf{k}b\rangle\right) \otimes \left(|\uparrow\uparrow\rangle\right) \quad : S_z = +1$$

$$+ \left(\Delta^{\downarrow\downarrow} |\mathbf{k}a;-\mathbf{k}b\rangle\right) \otimes \left(|\downarrow\downarrow\rangle\right) \quad : S_z = -1$$

$$+ \left(\frac{\Delta^{\uparrow\downarrow} + \Delta^{\downarrow\uparrow}}{2} |\mathbf{k}a;-\mathbf{k}b\rangle\right) \otimes \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \quad : S_z = 0$$

$S_z = +1, -1, 0$: $S = 1$ $\boxed{\text{EVEN}}$ in $\sigma \leftrightarrow \sigma'$

- Fermion constraint:

$$|A_1;A_2\rangle \otimes |s_1 s_2\rangle = -|A_2;A_1\rangle \otimes |s_2 s_1\rangle \Rightarrow \Delta_{\mathbf{k}ab} = -\Delta_{-\mathbf{k}ba}$$

→ EVEN

→ ODD

Pairing: Electron spin

- Cooper pair: orbitals and spins

$$\Delta^{\sigma\sigma'}_{kab}\, c^{\dagger}_{ka\sigma} c^{\dagger}_{-kb\sigma'}|vac\rangle = \Delta^{\sigma\sigma'}_{kab}\, \underbrace{|ka;-kb\rangle}_{\text{orbital}} \otimes \underbrace{|\sigma\sigma'\rangle}_{\text{spin}}$$

; $\sigma,\sigma' = \uparrow,\downarrow$ for $S_z = \pm\frac{1}{2}$

$a,b = 1\ldots r$ for orbitals

- Adding two spins $\frac{1}{2}\otimes\frac{1}{2} = 0 \oplus 1$

$$\sum_{\sigma,\sigma'}^{\uparrow,\downarrow} \Delta^{\sigma\sigma'}_{kab}|ka;-kb\rangle\otimes|\sigma\sigma'\rangle = \left(\frac{\Delta^{\uparrow\downarrow}-\Delta^{\downarrow\uparrow}}{2}|ka;-kb\rangle\right)\otimes\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \quad : S_z=0 \} \ S=0 \quad \boxed{\text{ODD}} \text{ in } \sigma\leftrightarrow\sigma'$$

$$+\left(\Delta^{\uparrow\uparrow}|ka;-kb\rangle\right)\otimes\left(|\uparrow\uparrow\rangle\right) \quad : S_z=+1$$

$$+\left(\Delta^{\downarrow\downarrow}|ka;-kb\rangle\right)\otimes\left(|\downarrow\downarrow\rangle\right) \quad : S_z=-1 \qquad S=1 \quad \boxed{\text{EVEN}} \text{ in } \sigma\leftrightarrow\sigma'$$

$$+\left(\frac{\Delta^{\uparrow\downarrow}+\Delta^{\downarrow\uparrow}}{2}|ka;-kb\rangle\right)\otimes\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \quad : S_z=0$$

- Fermion constraint: $|A_1;A_2\rangle\otimes|s_1 s_2\rangle = -|A_2;A_1\rangle\otimes|s_2 s_1\rangle \Rightarrow \Delta_{kab} = -\Delta_{-kba}$ → EVEN / → ODD

- Kitaev chain: $c_k \equiv c_{k\uparrow}$ (spin polarized) $\Rightarrow$ pairing is triplet, odd in k ($\Delta \sin k$)

(PROBLEM) Materials? MBS  $\downarrow$ $\uparrow$ → gapped out

Pairing: Electron spin

- Cooper pair: orbitals and spins

$$\Delta^{\sigma\sigma'}_{\underline{k}ab}\, c^{\dagger}_{\underline{k}a\sigma} c^{\dagger}_{-\underline{k}b\sigma'}|vac\rangle = \Delta^{\sigma\sigma'}_{\underline{k}ab} \underbrace{|\underline{k}a;-\underline{k}b\rangle}_{\text{orbital}} \otimes \underbrace{|\sigma\sigma'\rangle}_{\text{spin}}\,;\quad \sigma,\sigma'=\uparrow,\downarrow \text{ for } S_z=\pm\tfrac{1}{2}$$

$a,b=1\ldots r$ for orbitals

- Adding two spins $\frac{1}{2}\otimes\frac{1}{2}=0\oplus 1$

$$\sum_{\sigma,\sigma'}^{\uparrow,\downarrow} \Delta^{\sigma\sigma'}_{\underline{k}ab}|\underline{k}a;-\underline{k}b\rangle\otimes|\sigma\sigma'\rangle = \left(\frac{\Delta^{\uparrow\downarrow}-\Delta^{\downarrow\uparrow}}{2}|\underline{k}a;-\underline{k}b\rangle\right)\otimes(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle) \quad : S_z=0 \}\ S=0 \quad \boxed{\text{ODD}} \text{ in } \sigma\leftrightarrow\sigma'$$

$$+\left(\Delta^{\uparrow\uparrow}|\underline{k}a;-\underline{k}b\rangle\right)\otimes(|\uparrow\uparrow\rangle) \quad : S_z=+1$$

$$+\left(\Delta^{\downarrow\downarrow}|\underline{k}a;-\underline{k}b\rangle\right)\otimes(|\downarrow\downarrow\rangle) \quad : S_z=-1 \quad S=1 \quad \boxed{\text{EVEN}} \text{ in } \sigma\leftrightarrow\sigma'$$

$$+\left(\frac{\Delta^{\uparrow\downarrow}+\Delta^{\downarrow\uparrow}}{2}|\underline{k}a;-\underline{k}b\rangle\right)\otimes(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle) \quad : S_z=0$$

- Fermion constraint:

$$|A_1;A_2\rangle\otimes|s_1 s_2\rangle = -|A_2;A_1\rangle\otimes|s_2 s_1\rangle \Rightarrow \Delta_{\underline{k}ab} = -\Delta_{-\underline{k}ba}$$

→ EVEN

→ ODD

- RECIPE: DEFINITION OF ψ, $\vec{d}$

$$[\Delta_{\underline{k}ab}]_{ss'} = \begin{pmatrix} -d_x+id_y & d_z+\phi \\ d_z-\phi & d_x+id_y \end{pmatrix}_{ss'} = \left[\left(\phi^{ab}_{\underline{k}}\,\mathbb{1} + \vec{d}^{\,ab}_{\underline{k}}\cdot\vec{\sigma}\right)i\sigma_y\right]_{ss'};\quad \begin{cases} \phi_{\underline{k}} = \phi^{T}_{-\underline{k}} \\ \vec{d}_{\underline{k}} = -\vec{d}^{\,T}_{-\underline{k}} \end{cases}$$

Pairing: Symmetries

- SC is a spontaneously symmetry-broken phase ⇒ Landau theory

  Normal state $G_N = U(1)\times TR \times SG \xrightarrow{T<T_c} G_{SC} < G_N$. $\Delta^{\sigma\sigma'}_{\mathbf{k}\,ab}$ is order parameter, labeled by IRREPS of G_N.

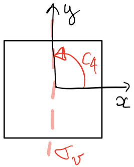
- Inversion symmetry

  if $Inv \in G_N$, then $Inv\,\Delta\,Inv^{-1} = \begin{cases} +\Delta, \text{ EVEN} \Rightarrow \phi \text{ (SINGET)} \\ -\Delta, \text{ ODD} \Rightarrow \vec{d} \text{ (TRIPLET)} \end{cases}$

  $\overbrace{\text{if } Inv \notin G_N}^{\text{SOC}}$, can mix $\phi, \vec{d}$

- Rotation symmetry (EXAMPLE)

  C_{4v} in 2d:



  - ACTION: ON $\mathbf{k}$: $C_4\begin{pmatrix}k_x\\k_y\end{pmatrix} = \begin{pmatrix}k_y\\-k_x\end{pmatrix}$, $\sigma_v\begin{pmatrix}k_x\\k_y\end{pmatrix} = \begin{pmatrix}-k_x\\k_y\end{pmatrix}$
  - IRREPS:
    1-dim.: $A_{1/2}, B_{1/2}$: $\boxed{\text{EVEN}}$
    2-dim.: E : $\boxed{\text{ODD}}$

  IRREP BASIS FUNCTIONS:
  E.g., for tight-binding problem, on Bravais lattice $\{a\hat{x}, a\hat{y}\}$, at nearest neighbor level:

  $$A_1: \; f^{(A_1)}_{\mathbf{k}} = \cos(k_x a) + \cos(k_y a)$$
  $$E: \; \begin{cases} f^{(E,1)}_{\mathbf{k}} = \sin(k_x a) \\ f^{(E,2)}_{\mathbf{k}} = \sin(k_y a) \end{cases}$$

  - SINGLET SC ORDER PARAMETER
    $\phi_{\mathbf{k}} = \eta^{(A_1)} f^{(A_1)}_{\mathbf{k}}$, $\eta^{(A_1)} \in \mathbb{C}$
    "S-wave"
  - TRIPLET SC ORDER PARAMETER
    $\vec{d}_{\mathbf{k}} = \eta^{(E,1)} f^{(E,1)}_{\mathbf{k}} + \eta^{(E,2)} f^{(E,2)}_{\mathbf{k}}$, $\begin{pmatrix}\eta^{(E,1)}\\ \eta^{(E,2)}\end{pmatrix} \in \mathbb{C}^2$
    "P-wave"

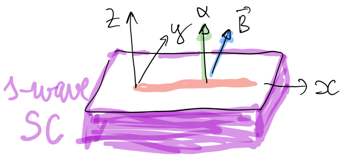
- Chiral SC: In SC such as $\vec{d} = \begin{pmatrix}p_x\\p_y\end{pmatrix}$, system likes $p_x \pm i p_y$.

Realize MBS: Rashba wire

- Solution: Magic of Spin-Orbit Coupling (SOC)

(ONE) RECIPE: Balance of singlet s-wave, SOC, and magnetic field.

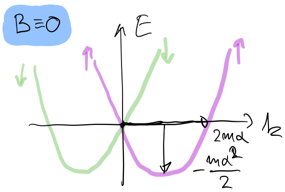
- Rashba SOC wire model:

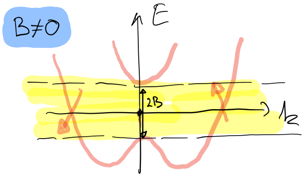


Realize MBS: Rashba wire

- Bulk (PBC) Rashba wire:
$$\Psi_k \equiv \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \\ c^\dagger_{-k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} \Rightarrow \begin{cases} \xi_k = \dfrac{k^2}{2m} - \alpha k \sigma_z + B\sigma_y - \mu \\ \Delta_k = \bar{\Delta}\, i\sigma_2 \quad \text{SINGLET s-wave} \leftarrow (\bar{\Delta} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + h.c.) \end{cases}$$

ξ_k ONLY (not BdG)

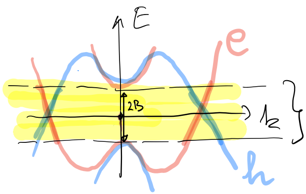




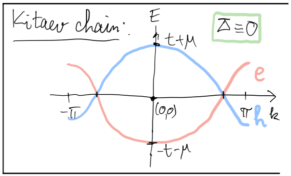
BdG

$\bar{\Delta} \equiv 0$

$(\mu \equiv 0)$



ONE BAND IF $|\mu|, \bar{\Delta} < B$



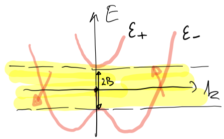
Realize MBS: SOC bands + s-wave pairing

- Triplet pairing introduced by SOC

$$H_k = \left(\begin{array}{c|c} \xi_k & \Delta_k \\ \hline \Delta_k^\dagger & -\xi^T_{-k} \end{array}\right)_{4\times 4} ; \quad \Psi_k = \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \\ c^\dagger_{-k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} \rightarrow \text{Eigenbasis of } \xi_k \text{ („helicity basis")}: \quad \tilde{H}_k = \left(\begin{array}{cc|cc} \varepsilon_{k+} & 0 & \tilde{\Delta}_{++} & \tilde{\Delta}_{+-} \\ 0 & \varepsilon_{k-} & \tilde{\Delta}_{-+} & \tilde{\Delta}_{--} \\ \hline \multicolumn{2}{c|}{\tilde{\Delta}^\dagger} & -\varepsilon_{-k+} & 0 \\ & & 0 & -\varepsilon_{-k-} \end{array}\right) ; \quad \tilde{\Psi}_k = \begin{pmatrix} d_{k,+} \\ d_{k,-} \\ d^\dagger_{-k,+} \\ d^\dagger_{-k,-} \end{pmatrix}$$

INTRABAND $\Delta_{++} = \Delta_{--} = -\frac{i}{2} \cdot \bar{\Delta} \cdot \frac{\alpha k}{\sqrt{(\alpha k)^2 + B^2}}$: ODD in k

INTERBAND $\Delta_{+-} = -\Delta_{-+} = -\frac{i}{2} \cdot \bar{\Delta} \cdot \frac{B}{\sqrt{(\alpha k)^2 + B^2}}$: EVEN in k



Realize MBS: SOC bands + s-wave pairing

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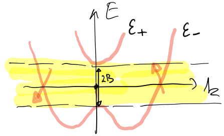
$$H_k = \left(\begin{array}{c|c} \xi_k & \Delta_k \\ \hline \Delta_k^\dagger & -\xi^T_{-k} \end{array}\right)_{4\times 4} ; \quad \Psi_k = \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \\ c^\dagger_{-k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} \rightarrow$$

Eigenbasis of ξ_k („helicity basis"):

$$\tilde{H}_k = \left(\begin{array}{cc|cc} \varepsilon_{k+} & 0 & \tilde{\Delta}_{++} & \tilde{\Delta}_{+-} \\ 0 & \varepsilon_{k-} & \tilde{\Delta}_{-+} & \tilde{\Delta}_{--} \\ \hline \multicolumn{2}{c|}{\tilde{\Delta}^\dagger} & -\varepsilon_{-k+} & 0 \\ & & 0 & -\varepsilon_{-k-} \end{array}\right) ; \quad \tilde{\Psi}_k = \begin{pmatrix} d_{k,+} \\ d_{k,-} \\ d^\dagger_{-k,+} \\ d^\dagger_{-k,-} \end{pmatrix}$$

INTRABAND $\Delta_{++} = \Delta_{--} = -\frac{i}{2}\cdot\bar{\Delta}\cdot\frac{\alpha k}{\sqrt{(\alpha k)^2 + B^2}}$: ODD in k

INTERBAND $\Delta_{+-} = -\Delta_{-+} = -\frac{i}{2}\cdot\bar{\Delta}\cdot\frac{B}{\sqrt{(\alpha k)^2 + B^2}}$: EVEN in k



- Effectively a Kitaev chain in topo SC phase: $\boxed{\sqrt{\mu^2 + \bar{\Delta}^2} < B \Leftrightarrow \text{Topo SC}}$ (full solution)

Controlled by $\mu, B, \bar{\Delta}$ $\Rightarrow$ gating, fields.. control of MBS position

SOC works hard: split bands & allows singlet to be efficient.

GENERAL : Topo-SC from Δ_k or ξ_k : $\Delta_s c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \rightsquigarrow \Delta_s \underbrace{\psi^*_{b\uparrow}(k)\,\psi^*_{b\downarrow}(-k)}_{\Delta_{bb}(k)}\, d^\dagger_{kb}\, d^\dagger_{-kb}$

Classes of gapped SCs

- Strong TSC – from 10-fold table

$$H_{\underline{k}} = \begin{pmatrix} \xi_{\underline{k}} & \Delta_{\underline{k}} \\ \Delta^{+}_{\underline{k}} & -\xi^{T}_{-\underline{k}} \end{pmatrix}_{2p\times 2p} ; \quad \Psi_{\underline{k}} = \begin{pmatrix} c_{\underline{k}1} \\ \vdots \\ c_{\underline{k}p} \\ c^{+}_{-\underline{k}1} \\ \vdots \\ c^{+}_{-\underline{k}p} \end{pmatrix}$$

class	T^2	P^2	S^2	d=0	1	2	3
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$ CS	$\mathbb{Z}$ Ch	
DIII	−1	+1	+1		$\mathbb{Z}_2$ $\widetilde{\text{CS}}$	$\mathbb{Z}_2$ FK	$\mathbb{Z}$ ν_3
C	0	−1	0			$2\mathbb{Z}$	
CI	+1	−1	+1				$2\mathbb{Z}$

SU(2) spin-rotation symmetry (C, CI)

Gapped SCs: d=1, class D

- Kitaev chain class! Has only the PHS constraint : $\tau_x H_k^* \tau_x = -H_{-k}$

class	T^2	P^2	S^2	d=0	1	2	3
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^{CS}$	$\mathbb{Z}^{Ch}$	
DIII	-1	+1	+1		$\mathbb{Z}_2^{\tilde{CS}}$	$\mathbb{Z}_2^{FK}$	$\mathbb{Z}^{\nu_3}$
C	0	-1	0			$2\mathbb{Z}$	
CI	+1	-1	+1				$2\mathbb{Z}$

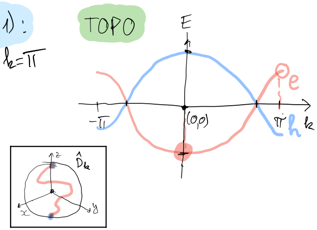
- Topological invariant

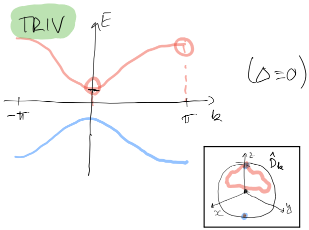
① Take Berry connection : $A_k^{ij} \equiv i \chi_k^{(i)\dagger} \cdot \partial_k \chi_k^{(j)}$, where $H_k \chi_k^{(i)} = E_k^{(i)} \chi_k^{(i)}$

② Chern-Simons integral $CS = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \sum_{i=1}^{p} A_k^{ii} \in \frac{\mathbb{Z}}{2}$ for filled BdG bands, $E_k^{(i)} < 0$

$\Rightarrow$ $\mathbb{Z}_2$: $W \equiv e^{2\pi i \cdot CS} = \text{sgn}\left[\text{Pf}(X_{k=\pi}) \cdot \text{Pf}(X_{k=0}) \right] = \pm 1 \begin{cases} \text{TRIV. SC} \\ \text{TOPO. SC} \end{cases}$

For one electron band $(p=1)$:
Electron filling @ $k=0$ vs. $k=\pi$





$(\Delta = 0)$

Gapped SCs: d=1, class DIII

class	T^2	P^2	S^2	d=0	1	2	3
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^{CS}$	$\mathbb{Z}^{Ch}$	
DIII	−1	+1	+1		$\mathbb{Z}_2^{\widetilde{CS}}$	$\mathbb{Z}_2^{FK}$	$\mathbb{Z}^{\nu_3}$
C	0	−1	0			$2\mathbb{Z}$	
CI	+1	−1	+1				$2\mathbb{Z}$

- Spinful Time-Reversal Symmetry: $\sigma_y H^*_k \sigma_y = H_{-k}$

- Topological invariant $\mathbb{Z}_2 : W_K = \pm 1 \begin{cases} \text{TRIV. SC} \\ \text{TOPO. SC} \end{cases}$

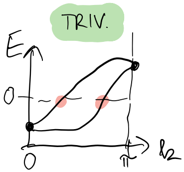
- Key simplification: IF $\Delta \ll E_F$, topology should depend on Fermi surface (FS) and Δ

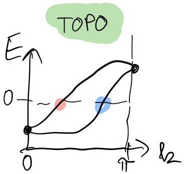
$$\boxed{N_{1d} \equiv \prod_s \mathrm{sgn}\left(\delta_{k_F^{(s)}}\right) = W_K}$$, s labels all Fermi points $0 < k_F^{(s)} < \pi$.

$$\delta_{k_F^{(s)}} \equiv \chi_{k_F^{(s)}}^{(\alpha_s)\dagger} \left(T \Delta_k^\dagger\right) \chi_{k_F^{(s)}}^{(\alpha_s)} \in \mathbb{R} \to \text{„sign of pairing"}$$

$\delta > 0$:

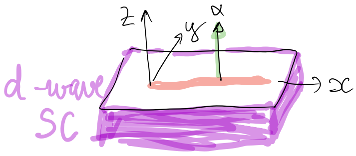
$\delta < 0$:





Rashba wire to realize DIII TSC

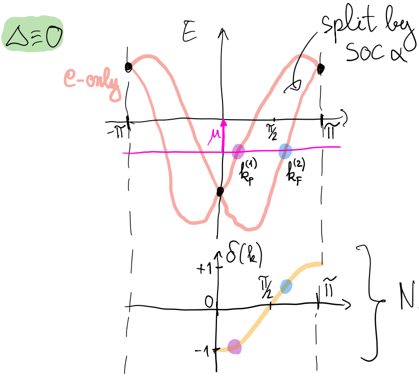
- Rashba wire + d-wave SC, NO B-field



- s-wave : $c^\dagger_{x\uparrow} c^\dagger_{x\downarrow} \to c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}$
- d-wave : $c^\dagger_{x+1,\uparrow} c^\dagger_{x,\downarrow} - c^\dagger_{x+1,\downarrow} c^\dagger_{x,\uparrow} \to \cos k \cdot c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}$

$$\begin{cases} \xi_k = \dfrac{k^2}{2m} - \alpha k \sigma_z - \mu \\ \Delta_k = \tilde{\Delta} \cos k \, (i\sigma_y) \end{cases}$$

SINGLET d-wave



- Precise solution : TSC for $|\mu| < \alpha$.

(Kramers pair of MBS at each end.)

TOPO

$$N_{1d} = \mathrm{sgn}\left[\delta(k_F^{(1)}) \cdot \delta(k_F^{(2)})\right] = -1 \text{ if } |\mu| \ll \alpha$$

Gapped SCs: d=2, class D

- Time-reversal broken. Has only the PHS constraint : $\tau_x H^*_{\mathbf{k}} \tau_x = -H_{-\mathbf{k}}$

class	T^2	P^2	S^2	d=0	1	2	3
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
DIII	-1	+1	+1		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
C	0	-1	0			$2\mathbb{Z}$	
CI	+1	-1	+1				$2\mathbb{Z}$

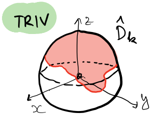
- Topological invariant : The usual Chern number $Ch \in \mathbb{Z}$

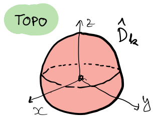
EXAMPLE:

$$H_{\mathbf{k}} = \begin{pmatrix} \xi_{\mathbf{k}} & \Delta(\sin k_x - i \sin k_y) \\ \Delta(\sin k_x + i \sin k_y) & -\xi_{\mathbf{k}} \end{pmatrix} \Rightarrow \vec{D}_{\mathbf{k}} = (D_x, D_y, D_z) = (\Delta \sin k_x, \Delta \sin k_y, \xi_{\mathbf{k}})$$

- Some other forms

For one electron band ($p=1$), $\boxed{H_{\mathbf{k}} = \vec{D}_{\mathbf{k}} \cdot \vec{\tau}}$, and $Ch = \#\text{wraps of } \hat{D}_{\mathbf{k}}$





- TSC in this class is „chiral", and is expected when SC pairing is chiral, e.g., $p_x \pm i p_y$, $d_{x^2-y^2} \pm i d_{xy}$.

} EXERCISE TOMORROW

Gapped SCs: d=2, class DIII

- Spinful Time-Reversal Symmetry: $\sigma_y H^*_k \sigma_y = H_{-k}$

class	T^2	P^2	S^2	d=0	1	2	3
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^{CS}$	$\mathbb{Z}^{Ch}$	
DIII	−1	+1	+1		$\mathbb{Z}_2^{CS}$	$\mathbb{Z}_2^{FK}$	$\mathbb{Z}^{\nu_3}$
C	0	−1	0			$2\mathbb{Z}$	
CI	+1	−1	+1				$2\mathbb{Z}$

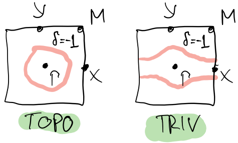
- Topological invariant: $\mathbb{Z}_2$: FK $= \pm 1$ $\begin{cases} \text{TRIV. SC} \\ \text{TOPO. SC} \end{cases}$

- Key simplification: IF $\Delta \ll E_F$:

$$\boxed{N_{2d} \equiv \prod_s \mathrm{sgn}(\delta_s)^{m_s} = FK}$$, s labels all Fermi lines

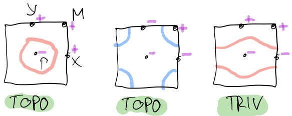
$m_s \equiv \#$ TRIM enclosed by s-th Fermi line

$\delta_s \equiv \chi^{(\alpha_s)\dagger}_{k_F} (T\Delta^\dagger_k) \chi^{(\alpha_s)}_{k_F} \in \mathbb{R} \rightarrow$ "sign of pairing"



- Another simplification: If ξ_k has inversion, and Δ is pure triplet, $p=1$

$$\boxed{N_{2d} = (-1)^{p_0}}$$, where $p_0 = \#$ connected components of Fermi line



Point defects: Majorana Bound State

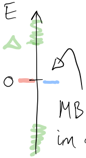
class	T^2	P^2	S^2	d=1	2	3
D	0	+1	0	$\mathbb{Z}_2^{CS_1}$	$\mathbb{Z}_2^{CS_3}$	$\mathbb{Z}_2^{CS_5}$
DIII	-1	+1	+1	$\mathbb{Z}_2^{\widetilde{CS}_1}$	$\mathbb{Z}_2^{\widetilde{CS}_3}$	$\mathbb{Z}_2^{\widetilde{CS}_5}$
C	0	-1	0			
CI	+1	-1	+1			

← Majorana Bound State (MBS)

← Kramers pair of MBS

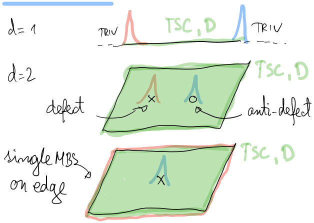
① MBS: class D ($\mathbb{Z}_2$)

ENERGY:



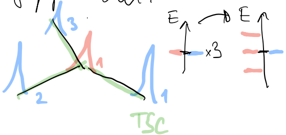
MBS separated in real space

REALSPACE:



TOPOLOGICAL PROTECTION

Locally, odd vs. even # of MBS is distinguished. In TSC, all except one red and one blue can be gapped out.



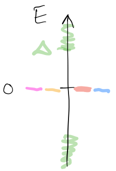
NOTE: Fundamentally, we don't need the blue partner.

Point defects: Majorana Bound State

class	T^2	P^2	S^2	$d=1$	2	3	
D	0	+1	0	$\mathbb{Z}_2^{CS_1}$	$\mathbb{Z}_2^{CS_3}$	$\mathbb{Z}_2^{CS_5}$	← Majorana Bound State (MBS)
DIII	−1	+1	+1	$\mathbb{Z}_2^{\widetilde{CS}_1}$	$\mathbb{Z}_2^{\widetilde{CS}_3}$	$\mathbb{Z}_2^{\widetilde{CS}_5}$	← Kramers pair of MBS
C	0	−1	0				
CI	+1	−1	+1				

② Kramers MBS, class DIII ($\mathbb{Z}_2$)

ENERGY:



REALSPACE:

$\lambda_1 = \sum_a \chi_{1,a} \psi_a$

$\lambda_1' = \sum_a (T\chi_1)_a \psi_a$

λ_1, λ_1' are independent!

$\boxed{\text{Kramers theorem}}$: $T^2 = -\mathbb{1} \Rightarrow \chi_n^\dagger (T\chi_n) = 0$

(Wigner)

TOPOLOGICAL PROTECTION

Topology distinguishes odd vs. even # of K. pairs

If # K.p. = odd, one pair is protected.

Line defects: Chiral Majorana Mode

class	T^2	P^2	S^2	$d=1$	2	3
D	0	+1	0		$\mathbb{Z}^{C_2}$	$\mathbb{Z}^{C_2}$
DIII	−1	+1	+1		$\mathbb{Z}_2^{FK_2}$	$\mathbb{Z}_2^{FK_2}$
C	0	−1	0		$2\mathbb{Z}^{2C_2}$	$2\mathbb{Z}^{2C_2}$
CI	+1	−1	+1			

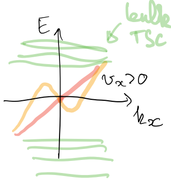
← Chiral Majorana Mode (cMM)

← Helical MM (hMM)

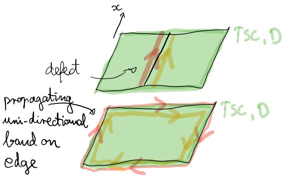
← Chiral Dirac Mode

① Chiral MM, class D ($\mathbb{Z}$)

ENERGY:



REALSPACE:



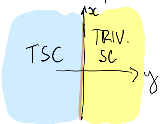
TOPOLOGICAL PROTECTION

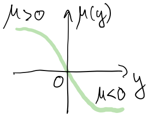
$\mathbb{Z}$ is (# modes $\cdot v_x > 0$) − (# modes $\cdot v_x < 0$).

cMM cannot hybridise with itself or other cMM of same chirality.

An example of Chiral Majorana Mode

- Take spinless (chiral) p-ip SC in 2d





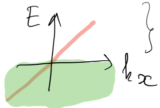
In bulk, $\mu \gtrless 0$ is TSC/TRIV SC.

Here, $\bar{\Delta}(k_x - ik_y) \to \bar{\Delta}(k_x - \partial_y)$.

Neglect $-\frac{(\vec{\nabla})^2}{2m}$, due to slow variation of μ $\Rightarrow$ $H_{BdG} = \begin{pmatrix} -\mu(y) & \bar{\Delta}(k_x - \partial_y) \\ \bar{\Delta}(k_x + \partial_y) & \mu(y) \end{pmatrix}$

Solutions localized around $y=0$:

1st quant: $\boxed{\chi_{k_x,y} \sim e^{+\frac{1}{\bar{\Delta}}\int_0^y dy' \mu(y')} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$, with $\boxed{E = \bar{\Delta} k_x}$



} chiral MM

2nd quant.: $\gamma_{k_x,y} \sim |f(y)| \cdot (c_{k_x,y} + c^+_{-k_x,y})$

NOTE: For $p_x + ip_y$ pairing, $E = -\Delta k_x$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

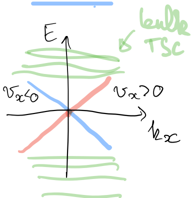
Line defects: Helical Majorana Mode

class	T^2	P^2	S^2	d=1	2	3
D	0	+1	0		$\mathbb{Z}^{C_2}$	$\mathbb{Z}^{C_2}$
DIII	−1	+1	+1		$\mathbb{Z}_2^{FK_2}$	$\mathbb{Z}_2^{FK_2}$
C	0	−1	0		$2\mathbb{Z}^{2C_2}$	$2\mathbb{Z}^{2C_2}$
CI	+1	−1	+1			

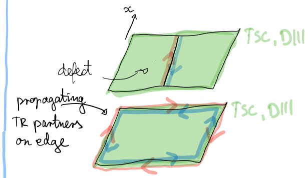
← Chiral Majorana Mode (cMM)
← Helical MM (hMM)
← Chiral Dirac Mode

② Helical MM, class DIII ($\mathbb{Z}_2$)

ENERGY:



REALSPACE:



TOPOLOGICAL PROTECTION

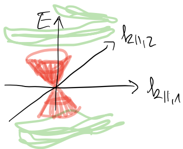
Distinguishes odd vs. even pairs ↑↓. The ↓ of one pair can hybridize with ↑ of another pair, so only 1 pair protected.

Kramers pair: $T\lambda_{k\uparrow}T^{-1} = \lambda_{-k\downarrow}$

Planar defects

class	T^2	P^2	S^2	$d=1$	2	3
D	0	+1	0			
DIII	−1	+1	+1			$\mathbb{Z}$ ν_3
C	0	−1	0			
CI	+1	−1	+1			$2\mathbb{Z}$ $2\nu_3$

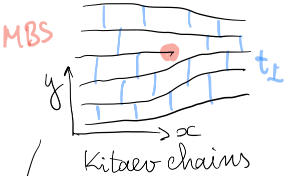
Surface mode of 3d TSC in class D is a „cone" of Majorana's.



MBS from weak topology

- In 2d class D, MBS without chiral SC?

Kitaev wires:



- As $t_\perp$ grows $\longrightarrow$ 2d system with $Ch = 0$.

MBS remains? Weak Topology:

class	T^2	P^2	S^2	d=0	1	2
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2 \leftarrow$	$\mathbb{Z}$
					W	Ch

Weak indices $\vec{G}_W \equiv (W_x, W_y)$.

$\vec{G}_W = (0,1) \Rightarrow$ ① Edge state along y

② MBS on dislocation $\vec{b} = \hat{y}$.

Experimental Signatures

1. Zero-bias conductance peak in STS / tunnel junction
2. Quantized thermal Hall conductivity
3. Thermal conductivity and spin current
4. Anomalous Josephson effect
5. Transport through wire

Where to search for topological SC?

Theorems?

1 Possible (not necessary): (1) odd pairing, (2) gapped; (3) count of FS pockets.
(1) is automatic in spin-split bands; (2) in 2d chiral SC

2 MBS only with broken spin-rotations ⇒ Odd pairing and SOC help.

Candidate systems?

INTRINSIC

- Heavy fermions UTe_2, UPt_3 (TRS breaking)...
- Sr_2RuO_4 disproven p+ip after 20+ years
- Fe-based: $FeSe_{1-x}Te_x$ ... (vortex MBS)
- Doped TIs: $Cu_xBi_2Se_3$ ...
- TMDs, van der Waals (ABC graphene...)

EXTRINSIC

- 1d proximity wires
- 2d TI + s-wave SC
- Defect engineering...

BEYOND:

- Crystalline TSC: doping TCI ($LaNiGa_2$)
- Weak TSC, even with even pairing
- Gapless, chiral-even pairing ...