TSC exercises

1 **a** $$\hat{H} = \sum_{\underline{k}} (\varepsilon_{\underline{k}} - \mu) c^\dagger_{\underline{k}} c_{\underline{k}} + \frac{1}{2} \sum_{\underline{k}} \left(\Delta_{\underline{k}} c^\dagger_{\underline{k}} c^\dagger_{-\underline{k}} + \Delta^*_{\underline{k}} c_{-\underline{k}} c_{\underline{k}} \right)$$

$$\Psi_{\underline{k}} \equiv \begin{pmatrix} c_{\underline{k}} \\ c^\dagger_{-\underline{k}} \end{pmatrix} \Rightarrow \hat{H} = \frac{1}{2} \sum_{\underline{k}} \Psi^\dagger_{\underline{k}} H^{BdG}_{\underline{k}} \Psi_{\underline{k}}$$

$$H^{BdG}_{\underline{k}} \equiv \begin{pmatrix} P_{\underline{k}} & Q_{\underline{k}} \\ R_{\underline{k}} & S_{\underline{k}} \end{pmatrix} \Rightarrow \hat{H} = \frac{1}{2} \sum_{\underline{k}} \begin{pmatrix} c^\dagger_{\underline{k}} & c_{-\underline{k}} \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} c_{\underline{k}} \\ c^\dagger_{-\underline{k}} \end{pmatrix} + \text{const} =$$

$$= \frac{1}{2} \sum_{\underline{k}} \left(P_{\underline{k}} c^\dagger_{\underline{k}} c_{\underline{k}} + Q_{\underline{k}} c^\dagger_{\underline{k}} c^\dagger_{-\underline{k}} + R_{\underline{k}} c_{-\underline{k}} c_{\underline{k}} + S_{\underline{k}} c_{-\underline{k}} c^\dagger_{-\underline{k}} \right) + \text{cst}$$

To match -terms, we use $\sum_{\underline{k}} f_{\underline{k}} = \sum_{-\underline{k}} f_{-\underline{k}}$, and $\{c_{\underline{k}}, c^\dagger_{\underline{k}}\} = 1$

$$\sum_{\underline{k}} (\varepsilon_{\underline{k}} - \mu) c^\dagger_{\underline{k}} c_{\underline{k}} = \frac{1}{2} \sum_{\underline{k}} \left[(\varepsilon_{\underline{k}} - \mu) c^\dagger_{\underline{k}} c_{\underline{k}} + (\varepsilon_{-\underline{k}} - \mu) c^\dagger_{-\underline{k}} c_{-\underline{k}} \right] =$$

$$= \frac{1}{2} \sum_{\underline{k}} \left[(\varepsilon_{\underline{k}} - \mu) c^\dagger_{\underline{k}} c_{\underline{k}} + (\varepsilon_{-\underline{k}} - \mu)(1 - c_{-\underline{k}} c^\dagger_{-\underline{k}}) \right] =$$

$$= \frac{1}{2} \sum_{\underline{k}} \left[(\varepsilon_{\underline{k}} - \mu) c^\dagger_{\underline{k}} c_{\underline{k}} - (\varepsilon_{-\underline{k}} - \mu) c_{-\underline{k}} c^\dagger_{-\underline{k}} \right] + \frac{1}{2} \sum_{\underline{k}} (\varepsilon_{\underline{k}} - \mu)$$

$$\Rightarrow Q_{\underline{k}} = \Delta_{\underline{k}} \,;\, R_{\underline{k}} = \Delta^*_{\underline{k}} \,;\, P_{\underline{k}} = \varepsilon_{\underline{k}} - \mu \,;\, S_{\underline{k}} = -(\varepsilon_{-\underline{k}} - \mu) \,;\, \text{const} = \frac{1}{2} \sum_{\underline{k}} (\varepsilon_{\underline{k}} - \mu)$$

$$\Rightarrow \boxed{H^{BdG}_{\underline{k}} = \left(\begin{array}{c|c} \varepsilon_{\underline{k}} - \mu & \Delta_{\underline{k}} \\ \hline \Delta^*_{\underline{k}} & -(\varepsilon_{-\underline{k}} - \mu) \end{array} \right)}$$

b $$H^{BdG}_{\underline{k}} \equiv \vec{D}_{\underline{k}} \cdot \vec{\sigma} = D_{\underline{k},1} \sigma_x + D_{\underline{k},2} \sigma_y + D_{\underline{k},3} \sigma_z = \begin{pmatrix} D_{\underline{k},3} & D_{\underline{k},1} - i D_{\underline{k},2} \\ D_{\underline{k},1} + i D_{\underline{k},2} & -D_{\underline{k},3} \end{pmatrix}$$

$$\Rightarrow D_{\underline{k},1} = \mathrm{Re}\,\Delta_{\underline{k}} \,;\, D_{\underline{k},2} = -\mathrm{Im}\,\Delta_{\underline{k}} \,;$$

while $D_{\underline{k},3} = \varepsilon_{\underline{k}} - \mu$ if $\varepsilon_{-\underline{k}} = \varepsilon_{\underline{k}}$ is even.

NOTE: IF $\varepsilon_{-\underline{k}} \neq \varepsilon_{\underline{k}}$, we separate $\varepsilon_{\underline{k}} = \dfrac{\varepsilon_{\underline{k}} + \varepsilon_{-\underline{k}}}{2} + \dfrac{\varepsilon_{\underline{k}} - \varepsilon_{-\underline{k}}}{2} \equiv \varepsilon^{even}_{\underline{k}} + \varepsilon^{odd}_{\underline{k}}$.
Then, $D_{\underline{k},3} = \varepsilon^{even}_{\underline{k}} - \mu$, while $H^{BdG}_{\underline{k}} = \vec{D}_{\underline{k}} \cdot \vec{\sigma} + \varepsilon^{odd}_{\underline{k}} \cdot \mathbb{1}$. The last term cannot cause a band crossing and preserves wavefunctions, hence we can simply discard $\varepsilon^{odd}_{\underline{k}}$.

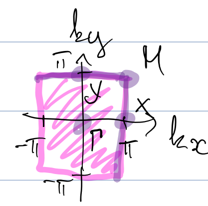
$\vec{D}_k = (\mathrm{Re}\Delta_k, -\mathrm{Im}\Delta_k, \varepsilon_k - \mu)$

BdG bands: $E_{k\pm} = \pm|\vec{D}_k| = \pm\sqrt{(\varepsilon_k-\mu)^2+(\mathrm{Re}\Delta_k)^2+(\mathrm{Im}\Delta_k)^2} = \boxed{\pm\sqrt{(\varepsilon_k-\mu)^2+|\Delta_k|^2}}$

C GAP CLOSES WHEN $\exists k_0 \; E_{k_0+}=0$

$E_{k_0+}\equiv 0 \Rightarrow \varepsilon_{k_0}-\mu \overset{①}{\equiv} 0 \wedge |\Delta_{k_0}| \overset{②}{\equiv} 0$

OUR MODELS: $\varepsilon_k \equiv -t[\cos(k_x)+\cos(k_y)]$; BZ:



① $\Rightarrow -t[\cos k_{0x}+\cos k_{0y}]\equiv\mu$

1A $|\mu|>2t \Rightarrow$ NO SOLUTION FOR k_0

1B $\mu=2t \Rightarrow \cos k_{0x}\equiv -1 \wedge \cos k_{0y}\equiv -1 \Rightarrow k_0=(\pi,\pi)=M$

1C $\mu=-2t \Rightarrow$ - - - $+1$ - - - - $+1 \Rightarrow k_0=(0,0)=\Gamma$

1D $|\mu|<2t \Rightarrow \cos k_{0x}+\cos k_{0y}\equiv -\frac{\mu}{t} \Rightarrow k_0\in$ Fermi surface (FS), $\Gamma, M\notin FS$

$\Delta^{1\pm}_k \equiv \sin k_x \pm i \sin k_y$ ("$p_x \pm i p_y$")

$\Delta^{2\pm}_k \equiv \cos k_x - \cos k_y \pm i \sin k_x\cdot\sin k_y$ ("$d_{x^2-y^2}\pm i d_{xy}$")

② $\Rightarrow |\Delta_{k_0}|\equiv 0 \Rightarrow \mathrm{Re}\Delta_{k_0}\equiv 0 \wedge \mathrm{Im}\Delta_{k_0}\equiv 0$

$\Rightarrow$ FOR $\Delta^{1\pm}$: $\sin k_{0x}\equiv 0\wedge \sin k_{0y}\equiv 0 \Rightarrow \left.\begin{matrix} k_{0x}=0,\pi \\ k_{0y}=0,\pi \end{matrix}\right\} \Rightarrow k_0\in\{\Gamma,X,Y,M\}$

FOR $\Delta^{2\pm}$: $\cos k_{0x}\equiv\cos k_{0y}\wedge(\sin k_{0x}\equiv 0 \vee \sin k_{0y}\equiv 0)\Rightarrow k_0\in\{\Gamma,M\}$

Combine ① & ②: $\Delta^{1\pm}$: $\mu\equiv 2t$ ($k_0=M$)

$\mu\equiv -2t$ ($k_0=\Gamma$)

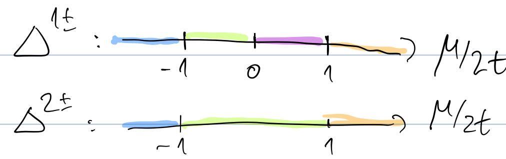
$\left.\begin{matrix} k_0=X \Rightarrow \cos\pi+\cos 0 \equiv -\frac{\mu}{t}\Rightarrow \mu\equiv 0 \\ k_0=Y\Rightarrow \cos 0+\cos\pi=-\frac{\mu}{t}\Rightarrow\mu\equiv 0 \end{matrix}\right\}$ ($k_0=X,Y$ simultaneously)

$\Delta^{2\pm}$: $\mu\equiv 2t$ ($k_0=M$)

$\mu\equiv -2t$ ($k_0=\Gamma$)

NOTE: $\Delta^{1\pm}_k = -\Delta^{1\pm}_{-k}$ is odd, allowing a gap closing @TRIM inside the band.

GAPPED PHASES:



d) $$Ch_1(H^{BdG}) = \int_{BZ} \hat{\vec{D}}_k \cdot (\partial_{k_x}\hat{\vec{D}}_k \times \partial_{k_y}\hat{\vec{D}}_k) = \int_{BZ} \frac{1}{|\vec{D}_k|^3}\vec{D}_k\cdot(\partial_{k_x}\vec{D}_k \times \partial_{k_y}\vec{D}_k)$$
$$= \sum_n p^{(n)}$$

$$p^{(n)} = \mathrm{sgn}\left[\vec{D}_k\cdot(\partial_{k_x}\vec{D}_k \times \partial_{k_y}\vec{D}_k)\right]\Big|_{k=k^{(n)}} \quad ; \quad \vec{D}_{k^{(n)}} = (a,0,0),\ a\in[0,+\infty)$$

$$\vec{D}_k = (\mathrm{Re}\Delta_k, -\mathrm{Im}\Delta_k, \varepsilon_k-\mu)$$
$n = 1,2,\dots$ solution counter

→ Solve $\Delta^{1\pm}_k$ case:

$$\Rightarrow \vec{D}^{1\pm}_k = (\sin k_x, \mp\sin k_y, -t(\cos k_x+\cos k_y)-\mu)$$
$$\partial_{k_x}\vec{D}^{1\pm}_k = (\cos k_x,\ 0,\ +t\sin k_x)$$
$$\partial_{k_y}\vec{D}^{1\pm}_k = (0,\ \mp\cos k_y,\ +t\sin k_y)$$

$\vec{D}_{k^{(n)}} = (|a|,0,0) \sim \hat{x} \Rightarrow$
$$\vec{D}^{1\pm}_k\cdot\{\partial_{k_x}D^{1\pm}\times\partial_{k_y}D^{1\pm}\}\Big|_{k=k^{(n)}} =$$
$$= D_{k^{(n)},x}\left(\partial_{k_x}D_y^{1\pm}\,\partial_{k_y}D_z^{1\pm} - \partial_{k_x}D_z^{1\pm}\,\partial_{k_y}D_y^{1\pm}\right)\Big|_{k=k^{(n)}}$$

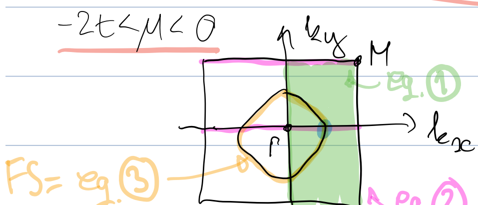
$$\Rightarrow p^{(n)}_{1\pm} = \mathrm{sgn}\{\sin k_x^{(n)}(\pm t\sin k_x^{(n)}\cos k_y^{(n)})\} = \pm\,\mathrm{sgn}\{\cos k_y^{(n)}\}$$
NOTE: $t>0$ by definition

NEED $k^{(n)}$: $\vec{D}^{1\pm}_{k^{(n)}} = (|a|,0,0) \Rightarrow \sin k_x^{(n)} = |a| \wedge \sin k_y^{(n)} = 0 \wedge -t(\cos k_x^{(n)}+\cos k_y^{(n)})-\mu = 0$ ① ② ③

- IN GAPPED PHASES $|\mu|>2t$, there is NO solution to ③ equation.
So no $k^{(n)}$ exists $\Rightarrow$ $\boxed{\mathrm{Chern}\left[H^{BdG}_{1\pm}(|\mu|>2t)\right] = 0}$

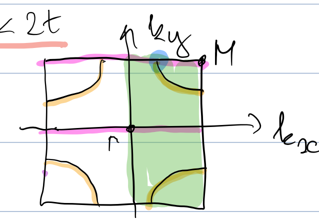
GRAPHICAL SOLUTION FOR $k^{(n)}$

$-2t<\mu<0$



SOLUTIONS
$k^{(1)} = (|k_F|, 0)$

$0<\mu<2t$



SOLUTIONS
$k^{(1)} = (|k_F|, \pi)$

- **Evaluate $p^{(1)}(\Delta^{1\pm}_k)$**, for $-2t<\mu<0$; $k^{(1)} = (|k_F|, 0)$

$$p^{(1)}_{1\pm} = \pm\,\mathrm{sgn}\{\cos k_y^{(1)}\} = \pm 1.$$

$$\boxed{\mathrm{Chern}\left(H^{BdG}_k(\Delta^{1\pm}_k)\right) = \pm 1 \text{ for } -2t<\mu<0.}$$

- Evaluate $p^{(1)}_{1\pm}$ for $0<\mu<2t$, $\underline{k}^{(1)}=(|k_F|,\pi)$.

$$p^{(1)}_{1\pm} = \pm\,\mathrm{sgn}\{\cos k_y^{(1)}\} = \mp 1$$

$$\boxed{\mathrm{Chern}\left(H^{BdG}_{k}(\Delta^{1\pm}_{k})\right) = \mp 1 \text{ for } 0<\mu<2t.}$$

$\boxed{\text{Solve the } \Delta^{2\pm}_{k} \text{ case}}$: NOTATION: $\cos k_x \equiv cx$; $\sin k_x \equiv sx$; $\cos k_y = cy$; $\sin k_y = sy$

$$\vec{D}^{2\pm}_{k} = (cx-cy,\ \mp sx\,sy,\ -t(cx+cy)-\mu)$$

$$\partial_{k_x}\vec{D}^{2\pm}_{k} = (-sx,\ \mp cx\,sy,\ +t\,sx)$$

$$\partial_{k_y}\vec{D}^{2\pm}_{k} = (sy,\ \mp sx\,cy,\ +t\,sy)$$

$\vec{D}_{k^{(m)}} \equiv (|a|,0,0) \sim \hat{x} \Rightarrow$

$$\vec{D}^{2\pm}_{k}\cdot\{\partial_{k_x}\vec{D}^{2\pm}\times\partial_{k_y}\vec{D}^{2\pm}\}\big|_{k=k^{(m)}} =$$

$$= D^{2\pm}_{k^{(m)},x}\left(\partial_{k_x}D^{2\pm}_{y}\,\partial_{k_y}D^{2\pm}_{z} - \partial_{k_x}D^{2\pm}_{z}\,\partial_{k_y}D^{2\pm}_{y}\right)\big|_{k=k^{(m)}}$$

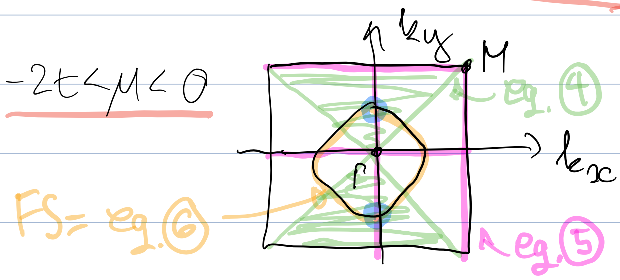
$$p^{(m)}_{2\pm} = \mathrm{sgn}\{(cx-cy)\cdot t(\mp cx\,s^2y \pm s^2x\,cy)\} = \mp\,\mathrm{sgn}\{(cx-cy)(cx\,s^2y - s^2x\cdot cy)\}$$

$\boxed{\text{NEED } k^{(m)}}$: $\vec{D}^{2\pm}_{k^{(m)}} \equiv (|a|,0,0) \Rightarrow \overset{(4)}{cx-cy \equiv |a|} \wedge \overset{(5)}{sx\cdot sy = 0} \wedge \overset{(6)}{-t(\cos k_x^{(m)}+\cos k_y^{(m)})-\mu = 0}$

- Again for $|\mu|>2t$, $\boxed{Ch(\Delta^{2\pm};\ |\mu|>2t) = 0}$

$\boxed{\text{GRAPHICAL SOLUTION FOR } k^{(m)}}$ eg. (4): $\cos k_x \geq \cos k_y$

$-2t<\mu<0$

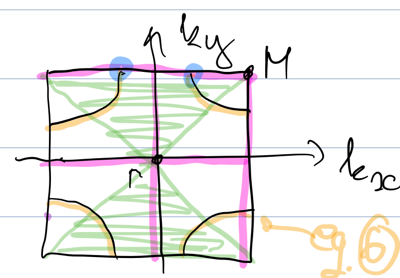


SOLUTIONS

$k^{(1)} = (0, |k_F|)$

$k^{(2)} = (0, -|k_F|)$

$0<\mu<2t$



SOLUTIONS

$k^{(1)} = (|k_F|, \pi)$

$k^{(2)} = (-|k_F|, \pi)$

- $\boxed{\text{Evaluate } P^{(1)}_{2\pm}, P^{(2)}_{2\pm}}$ for $\Delta^{2\pm}_{k_1}$ for $\underline{0<\mu<2t}$; $\underset{\sim}{k}^{(1)} = (|k_F|, \pi)$; $k^{(2)} = (-|k_F|, \pi)$

$$P^{(n)}_{2\pm} = \mp\,\mathrm{sgn}\{(c_x - c_y)(c_x s_y^2 - s_x^2 c_y)\}$$

$$\Rightarrow P^{(1)}_{2\pm} = \mp\,\mathrm{sgn}\{(\cos|k_F|+1)(-\sin^2|k_F|(-1))\} = \mp 1$$

$$\Rightarrow P^{(2)}_{2\pm} = P^{(1)}_{2\pm}(|k_F| \to -|k_F|) = P^{(1)}_{2\pm} = \mp 1$$

$$\boxed{\mathrm{Chern}(\Delta^{2\pm}_{k};\ 0<\mu<2t) = P^{(1)}_{2\pm} + P^{(2)}_{2\pm} = \mp 2}$$

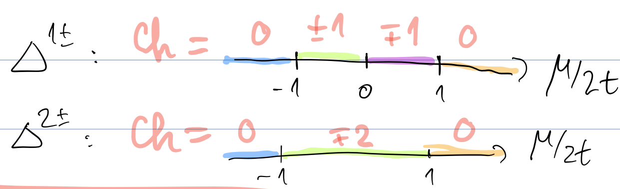
- $\boxed{\text{Evaluate } P^{(1)}_{2\pm}, P^{(2)}_{2\pm}}$ for $\Delta^{2\pm}_{k_1}$ for $\underline{-2t<\mu<0}$; $\underset{\sim}{k}^{(1)} = (0, |k_F|)$; $k^{(2)} = (0, -|k_F|)$

$$\Rightarrow P^{(1)}_{2\pm} = \mp\,\mathrm{sgn}\{(1 - c|k_F|)\cdot s^2|k_F|\} = \mp 1$$

$$\Rightarrow P^{(2)}_{2\pm} = P^{(1)}_{2\pm}(|k_F| \to -|k_F|) = P^{(1)}_{2\pm} = \mp 1$$

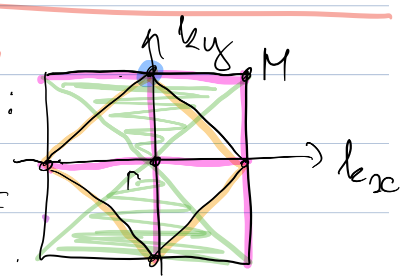
$$\boxed{\mathrm{Chern}(\Delta^{2\pm}_{k};\ -2t<\mu<0) = P^{(1)}_{2\pm} + P^{(2)}_{2\pm} = \mp 2}$$

SUMMARY



Finally, for $\Delta^{2\pm}_k$, also gapped at $\boxed{\mu=0}$:

$\underset{\sim}{k}^{(1)} = (0, \pi) \Leftarrow$



- $\boxed{\text{Evaluate } P^{(1)}_{2\pm}}$ for $\Delta^{2\pm}_{k_1}$ for $\mu = 0$: $\underset{\sim}{k}^{(1)} = (0, \pi)$

$$P^{(1)}_{2\pm} = \pm\,\mathrm{sgn}\{2\cdot 0\} = \mathrm{sgn}\{0\}\ ↯$$

$P^{(1)}_{2\pm} = 0$ means that $\vec{D} \parallel \partial_x \vec{D} \times \partial_y \vec{D}$, it is a critical point of the $\vec{D}_k$ surface, we should choose another ray instead of $\underset{\sim}{\ell} = (|a|, 0, 0)$, e.g., $(0, 0, |a|)$.

The critical point is anticipated also from $\underset{\sim}{k}^{(1)}$ being on a corner of FS, where two solutions $\underset{\sim}{k}^{(1)}, \underset{\sim}{k}^{(2)}$ merged. Exercise for the reader...

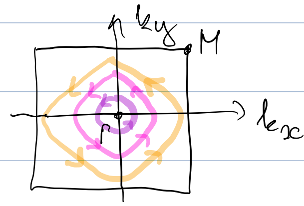
(e) WINDING OF Δ : $W(\mu/t)$

- BY DEFINITION, FOR $|\mu|>2t$ THE $W(\mu)\equiv 0$ AS THERE IS NO FS.
- GAPPED PHASE $\Rightarrow |\Delta_k|\neq 0 \Rightarrow \Delta_k = e^{i\varphi_k}\cdot|\Delta_k|$ and φ_k is well-defined.

$$W(\mu) = \oint_{FS} \frac{dk}{L_{FS}} \cdot \frac{\Delta_k^*}{|\Delta_k|}\left(\nabla_k \frac{\Delta_k}{|\Delta_k|}\right) = \oint_{FS} \frac{dk}{L_{FS}} \cdot e^{-i\varphi_k}\left(\nabla_k e^{i\varphi_k}\right) = \oint_{FS} dk \cdot \nabla_k \varphi_k =$$

$=$ ACCUMULATED PHASE φ on traversing the closed FS.

- FS : for $\boxed{-2t<\mu<0}$



Since $W(\mu)$ is a topological number, we can deform the FS to a circle around Γ. So, FS : $\{ k \equiv k_F(\cos\theta_k, \sin\theta_k);\ k_F \text{ is const} < \pi \}$, we can take $k_F \ll 1$.

$$k_F \ll 1 \Rightarrow \begin{cases} \sin k_x = \sin(k_F\cos\theta_k) \simeq k_F\cos\theta_k \\ \sin k_y = \sin(k_F\sin\theta_k) \simeq k_F\sin\theta_k \\ \cos k_x = \cos(k_F\cos\theta_k) \simeq 1-\frac{k_F^2}{2}\cos^2\theta_k \\ \cos k_y \simeq 1-\frac{k_F^2}{2}\sin^2\theta_k \end{cases}$$

NOTE: θ_k is polar angle of k.

$$\Rightarrow \Delta_k^{1\pm} = k_F\, e^{\pm i\theta_k} \Rightarrow \boxed{\varphi_k^{1\pm} = \pm\theta_k}$$

$$\Rightarrow \Delta_k^{2\pm} = \frac{k_F^2}{2}\left(-\cos^2\theta_k+\sin^2\theta_k\right) \pm i\, k_F^2 \sin\theta_k\cdot\cos\theta_k = \frac{k_F^2}{2}\left(-\cos(2\theta_k) \pm i\sin(2\theta_k)\right) =$$
$$= -\frac{k_F^2}{2} e^{\mp i2\theta_k}$$

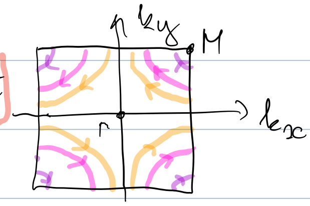
$$\Rightarrow \boxed{\varphi_k^{2\pm} = \mp 2\theta_k}$$

$$\Rightarrow \boxed{W^{1\pm}} \quad \oint_{FS} \frac{dk}{L_{FS}}\cdot\nabla_k\varphi_k = \int_0^{2\pi} \frac{k_F\, d\theta}{2\pi k_F}\, \partial_\theta\varphi(\theta) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\cdot(\pm 1) = \boxed{\pm 1}$$

NOTE: length of FS : $L_{FS} = 2\pi k_F$

$$\Rightarrow \boxed{W^{2\pm} = \mp 2}$$

- FS: for $\boxed{0 < \mu < 2t}$



NOTE: We preserve FS loop orientation, so now the loop is clockwise around M-point.

NOW FS: $\underline{k} = (\pi, \pi) + k_F(\cos\theta_{\underline{k}}, -\sin\theta_{\underline{k}}) \leftarrow$ clockwise around M

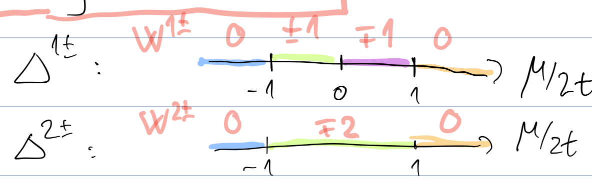
$$k_F \ll 1 \Rightarrow \begin{cases} \sin k_x = \sin(\pi + k_F\cos\theta_{\underline{k}}) \approx -k_F\cos\theta_{\underline{k}} \\ \sin k_y = \sin(\pi - k_F\sin\theta_{\underline{k}}) \approx +k_F\sin\theta_{\underline{k}} \\ \cos k_x = \cos(\pi + k_F\cos\theta_{\underline{k}}) = -\cos(k_F\cos\theta_{\underline{k}}) \simeq -1 + \frac{k_F^2}{2}\cos^2\theta_{\underline{k}} \\ \cos k_y = \cos(\pi - k_F\sin\theta_{\underline{k}}) = -\cos(-k_F\sin\theta_{\underline{k}}) \approx -1 + \frac{k_F^2}{2}\sin^2\theta_{\underline{k}} \end{cases}$$

$$\Rightarrow \Delta^{1\pm}_{\underline{k}} = -k_F(\cos\theta_{\underline{k}} \mp i\sin\theta_{\underline{k}}) = -k_F e^{\mp i\theta_{\underline{k}}}$$

$$\Delta^{2\pm}_{\underline{k}} = \frac{k_F^2}{2}(\cos^2\theta_{\underline{k}} - \sin^2\theta_{\underline{k}} \mp i2\sin\theta_{\underline{k}}\cos\theta_{\underline{k}}) = \frac{k_F^2}{2}e^{\mp i2\theta_{\underline{k}}}$$

$$\boxed{\begin{array}{ll} \Rightarrow \varphi^{1\pm}_{\underline{k}} = \mp\theta_{\underline{k}} & W^{1\pm} = \mp 1 \\ \Rightarrow \varphi^{2\pm}_{\underline{k}} = \mp 2\theta_{\underline{k}} & W^{2\pm} = \mp 2 \end{array}}$$

SUMMARY:



so $\boxed{W = Ch}$

NOTE: The sign of W is tied to choosing counter-clockwise loop orientation.