Theoretical concepts of 2D superconductivity and topology (2h)

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These are lecture notes about spinless superconductivity in two space dimensions (2D). We first introduce Majorana fermions as an alternative representation of a Bogoliubov-de Gennes mean-field superconductor. Such "textbook superconductors" are classified by the 10-fold way periodic table. We consider in particular the D-class in 2D and study the $p_x + ip_y$ superconductor on the lattice, compute its Chern number $\nu \in \mathbb{Z}$, discuss its edge states and the states bound to a superconducting vortex. We then show that a vortex behaves as a bulk anyon and that there are 16 different possible anyon theories depending on ν mod 16. This brings us to the idea that "real-life superconductors" display topological order (à la Wen).

Contents

I. MAJORANA FERMIONS AND BDG FORMALISM

A Majorana fermion (MF) is a "real", i.e. not complex, fermion. It is its own antiparticle and therefore it carries no charge. However it can have a mass, meaning that it does not need to be at zero energy. It can be seen as the real/imaginary part of a Dirac (complex) fermion. It carries only half of the degrees of freedom of a complex fermion. Actually two Majorana fermions are equivalent to a single complex fermion and correspond to a 2-dimensional Hilbert space (a single fermionic mode that can be empty or occupied). For a nice and thorough discussion of the three types of fermions (Dirac, Majorana and Weyl [12]) in 3+1 space-time dimension see [1]. Below we follow the notations and presentation of Kitaev [2, 3].

Let us call a and $a^\dagger$ the annihilation and creation operator for a complex fermionic mode of energy ϵ. They satisfy $\{a, a^\dagger\} = 1$ and $a^2 = 0$ and correspond to a 2-dimensional Fock space with basis vectors $\{|0\rangle, |1\rangle\}$ such that

$$a^\dagger a|n\rangle = n|n\rangle \text{ with } n = 0, 1 \text{ and } a|0\rangle = 0, a^\dagger|0\rangle = |1\rangle, a|1\rangle = |0\rangle, a^\dagger|1\rangle = 0$$

The Hamiltonian is:

$$H = \epsilon(a^\dagger a - \frac{1}{2})$$

The empty state $|0\rangle$ is at energy $-\epsilon/2$ and the occupied state $|1\rangle$ at energy $\epsilon/2$.

We define two Majorana fermion operators:

$$c_1 = a + a^\dagger = 2 \text{ Re } a \text{ and } c_2 = \frac{a - a^\dagger}{i} = 2 \text{ Im } a \text{ such that } a = \frac{c_1 + ic_2}{2} \text{ and } a^\dagger = \frac{c_1 - ic_2}{2}$$

The MF operators are self-adjoint ("real"), they square to 1 (not 0) and they anti-commute with each other (they are fermions):

$$c_{1,2}^\dagger = c_{1,2}, \quad c_{1,2}^2 = 1, \quad \{c_1, c_2\} = 0.$$

The Hamiltonian is

$$H = \frac{i}{2}\epsilon c_1 c_2 = \frac{i}{4}\epsilon(c_1 c_2 - c_2 c_1) = \frac{i}{4}(c_1\ c_2)\begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{i}{4}\sum_{j,l} A_{j,l} c_j c_l \in i\mathbf{R},$$

where $A_{j,l}$ is a real skew-symmetric matrix and ϵ is a coupling between MF. In particular, let us emphasize (in preparation of the discussion to come about Majorana zero modes, MZM) that MF need not correspond to a zero energy mode (here $\epsilon > 0$) and that they are fermions (not non-Abelian anyons).

Majorana fermions are ideal to describe complex fermions that are only conserved modulo 2, i.e. that only conserve fermionic number parity P, i.e. that have a global $\mathbb{Z}_2$ symmetry. This is precisely the case of mean-field superconductivity in the Bogoliubov-de Gennes (BdG) formalism. This should be contrasted with complex fermions that are number conserved Q, i.e. that have a global $U(1)$ symmetry as in a metal or an insulator (typically in a tight-binding model). As a cartoon model, let us imagine having the following Hamiltonian with $n = 2$ complex fermions:

$$H = \epsilon a_1^\dagger a_1 + \epsilon a_2^\dagger a_2 + t a_1^\dagger a_2 + t a_2^\dagger a_1 + \Delta a_1^\dagger a_2^\dagger + \Delta^* a_2 a_1$$

with $\{a_j, a_l^\dagger\} = \delta_{j,l}$ and $\{a_j, a_l\} = 0$. It does not commute with the total particle number $Q = a_1^\dagger a_1 + a_2^\dagger a_2$ (with eigenvalues 0, 1 or 2) but commutes with the fermionic parity $P = (1 - 2a_1^\dagger a_1)(1 - 2a_2^\dagger a_2)$ (with eigenvalues ± 1 even/odd). Using the anti-commutation and up to a constant that can be absorbed in a shift of the zero of energy, the Hamiltonian can be rewritten as

$$\begin{aligned} H &= \frac{1}{2}\Big(\epsilon a_1^\dagger a_1 - \epsilon a_1 a_1^\dagger + \epsilon a_2^\dagger a_2 - \epsilon a_2 a_2^\dagger + t a_1^\dagger a_2 - t a_2 a_1^\dagger + t a_2^\dagger a_1 - t a_1 a_2^\dagger + \Delta a_1^\dagger a_2^\dagger - \Delta a_2^\dagger a_1^\dagger + \Delta^* a_2 a_1 - \Delta^* a_1 a_2\Big) \\ &= (a_1^\dagger\ a_2^\dagger\,|\,a_1\ a_2)\frac{1}{2}\left(\begin{array}{cc|cc} t & \epsilon & 0 & \Delta \\ \epsilon & t & -\Delta & 0 \\ \hline 0 & -\Delta^* & -\epsilon & -t \\ \Delta^* & 0 & -t & -\epsilon \end{array}\right)\left(\begin{array}{c} a_1 \\ a_2 \\ \hline a_1^\dagger \\ a_2^\dagger \end{array}\right) = \psi^\dagger\,\frac{1}{2}H_{BdG}\,\psi, \end{aligned} \tag{1}$$

where ψ is a Nambu spinor with artificial doubling of the degrees of freedom and H_{BdG} is a BdG Hamiltonian, the equivalent of a single-particle Hamiltonian of size $2n \times 2n$. By construction, a BdG Hamiltonian has PHS. The charge conjugaison operator (or particle-hole) is anti-unitary and reads $C = \tau_x \otimes \mathbb{1}_2 K$ where $\tau_x \otimes \mathbb{1}_2$ exchanges particle and hole sectors and K takes the complex conjugaison to its right. Here τ_x is the first Pauli matrix in particle/hole space and we take its tensor product with unity $\mathbb{1}_2$ in order to make a 4×4 matrix. For simplicity, we just note $\tau_x \otimes \mathbb{1}_2$ as τ_x. It reads

$$\tau_x = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \text{ and } \tau_x K H_{BdG} K \tau_x = \tau_x H_{BdG}^* \tau_x = -H_{BdG}$$

is the PHS. Note that C *anti-commutes* with H_{BdG}.

It can be rewritten using Majorana fermions

$$\boxed{c_{2j-1} = a_j + a_j^\dagger \text{ (odd integer) and } c_{2j} = \frac{a_j - a_j^\dagger}{i} \text{ (even integer) with } \{c_j, c_l\} = 2\delta_{j,l}, c_j^2 = 1 \text{ and } c_j^\dagger = c_j}$$

as (with $\Delta = \Delta_R + i\Delta_I$)

$$H = (c_1\ c_2\ c_3\ c_4)\frac{i}{4}\begin{pmatrix} 0 & \epsilon & \Delta_I & t-\Delta_R \\ -\epsilon & 0 & -t-\Delta_R & -\Delta_I \\ -\Delta_I & t+\Delta_R & 0 & \epsilon \\ -t+\Delta_R & \Delta_I & -\epsilon & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \frac{i}{4}\sum_{j,l} A_{j,l}c_jc_l = \psi_M^T\ \frac{i}{4}A\ \psi_M, \tag{2}$$

where iA acts as a Majorana Hamiltonian and is purely imaginary. A is a real skew-symmetric matrix of *even* dimension (here $j = 1, ..., 2n$ with $n = 2$). In the Majorana representation the hopping and pairing play similar role (they appear as "Majorana hopping"). The number and parity operators are

$$Q = \sum_{j=1}^{n}\frac{1}{2}(1 + ic_{2j-1}c_{2j}) \text{ and } P = \prod_{j=1}^{n}(-ic_{2j-1}c_{2j}) = i^n c_1c_2....c_{2n}$$

For fermions with global $\mathbb{Z}_2$ symmetry only the fermionic parity P is conserved and not the fermion number Q. In such a case, there is no notion of band filling. Or in other words, one is always at half-filling, with negative energy levels occupied and positive energy levels empty (as if the chemical potential was $\mu = 0$). There is an artificial particle-hole symmetry coming from the Nambu trick of doubling the degrees of freedom. There is always an even number of eigenvalues of H_{BdG} (or equivalently of iA) with $\varepsilon_m \to -\varepsilon_m$ symmetry with $m = 1, ..., n$. After diagonalization, the Hamiltonian takes the form:

$$\boxed{H = \psi^\dagger\,\frac{1}{2}H_{BdG}\,\psi = \frac{i}{4}\sum_{j,l=1}^{2n} A_{j,l}c_jc_l \Rightarrow H = \sum_{m=1}^{n}\varepsilon_m(\tilde{a}_m^\dagger\tilde{a}_m - \frac{1}{2}) = \sum_{m=1}^{n}\frac{i\varepsilon_m}{2}b'_m b''_m}$$

It can be written either in terms of complex fermions $\tilde{a}_m$, $\tilde{a}_m^\dagger$ (usually called Bogoliubov quasiparticle annihilation and creation operators) or in terms of real fermions $b'_m = \tilde{a}_m + \tilde{a}_m^\dagger$, $b''_m = (\tilde{a}_m - \tilde{a}_m^\dagger)/i$.

Majorana fermions are ubiquitous in superconductors. The description in terms of Majorana fermions is compltely equivalent to that in terms of Bogoliubov quasiparticles (BQP) in the BdG formalism. This is explained in detail in [4], see also [5]. The "Majoranas" that are eagerly searched for are not Majorana fermions but are Majorana zero modes that we discuss below.

Remark: in k-space, one needs to pay attention to some signs because the reality condition in direct (real) space $c_j^\dagger = c_j$ becomes $c_k^\dagger = c_{-k}$ in reciprocal space.

$$\{a_k, a_{k'}^\dagger\} = \delta_{k,k'} \text{ and } \{a_k, a_{k'}\} = 0$$

$$c_{1,k} = a_k + a_{-k}^\dagger, \quad c_{2,k} = \frac{a_k - a_{-k}^\dagger}{i}, \quad a_k = \frac{c_{1,k} + ic_{2,k}}{2}, \quad a_k^\dagger = \frac{c_{1,k}^\dagger - ic_{2,k}^\dagger}{2} = \frac{c_{1,-k} - ic_{2,-k}}{2}$$

The Majorana operators $c_{a,k}$ with $a = 1, 2$ are such that $\boxed{c_{a,k}^\dagger = c_{a,-k}}$ and

$$\{c_{a,k}, c_{b,k'}^\dagger\} = \{c_{a,k}, c_{b,-k'}\} = 2\delta_{a,b}\delta_{k,-k'}.$$

Note that $(c_{a,k})^2 = 0$ and not 1.

Remark: BdG formalism allows one to easily solve a quadratic fermionic (second-quantized) Hamiltonian with number parity conservation. The trick is to define the equivalent of a single-particle Hamiltonian and then to solve the many-body problem by filling these modes as for a non-interacting Fermi gas. The price to pay in order to define a single-particle Hamiltonian is to artificially double the degrees of freedom (i.e. introduce a redundancy) by including electrons and holes on the same footing. The single-particle-like Hamiltonian is called H_{BdG}. It is of even dimension and its eigenvalues come in pairs $(\varepsilon_m, -\varepsilon_m)$.

II. 10-FOLD WAY OF NON-INTERACTING GAPPED FERMIONS: "TEXTBOOK" SUPERCONDUCTORS

There is a 10-fold classification of non-interacting gapped fermions [6, 7], see Fig. 1. It uses ideas of topology to classify band insulators and superconductors/superfluids in the same framework by attributing topological invariants to single-particle bands. It relies on having quadratic fermionic Hamiltonians (either non-interacting or mean-field like) with gapped ground state and either conserving fermion number Q (i.e. $U(1)$ global symmetry) or conserving only the fermion parity P (i.e. $\mathbb{Z}_2$ global symmetry). These many-body problems can be solved by first finding the energy spectrum of a single-particle Hamiltonian: either a Bloch Hamiltonian $H(\mathbf{k})$ for number-conserved fermions or a BdG Hamiltonian $H_{BdG}(\mathbf{k})$ (or equivalently a Majorana Hamiltonian) when only the parity is conserved. The table is organized in 10 symmetry classes (the so-called Altland-Zirnbauer classes) depending on external symmetries such as time-reversal (TRS), particle-hole (PHS) and chiral or sublattice symmetry (SLS). The two first symmetries T and P are anti-unitary and can either be absent or present in two forms as $T^2 = \pm 1$ and $P^2 = \pm 1$. T commutes with H whereas P anticommutes. The last external symmetry is unitary, is called chirality (or sub-lattice symmetry) S and anti-commutes with $H(\mathbf{k})$. Here chirality refers to a sub-lattice-like symmetry (as in graphene with its honeycomb lattice made of 2 sublattices) and has nothing to do with TR breaking and with clockwise or counterclockwise motion of edge states. It is typically present in bipartite tight-binding models. The classification also depends on space dimension.

Here we will concentrate on the D-class in 2D, see Fig. 1. This class describes BdG superconductors that break TRS and do not conserve $SU(2)$ spin invariance. It has TRS=0 and PHS=+1. According to this table, 2D spinless superconductors that break TRS are classified by an integer $\mathbb{Z}$.

TABLE 2. Summary of the *main result of this paper*: listed are again the ten symmetry classes of single particle Hamiltonians (from TABLE 1) classified in terms of the presence or absence of time-reversal symmetry (TRS) and particle-hole symmetry (PHS), as well as sublattice (or "chiral") symmetry (SLS) [17, 18, 19]. The last three columns list all possible topologically non-trivial quantum ground states as a function of symmetry class and spatial dimension d. The symbols $\mathbf{Z}$ and $\mathbf{Z}_2$ indicate that the space of quantum ground states is partitioned into different topological sectors labeled by an integer ($\mathbf{Z}$), or a $\mathbf{Z}_2$ quantity (two sectors only), respectively.

System	Cartan nomenclature	TRS	PHS	SLS	$d=1$	$d=2$	$d=3$
standard (Wigner-Dyson)	A (unitary)	0	0	0	-	$\mathbf{Z}$	-
	AI (orthogonal)	+1	0	0	-	-	-
	AII (symplectic)	−1	0	0	-	$\mathbf{Z}_2$	$\mathbf{Z}_2$
chiral (sublattice)	AIII (chiral unit.)	0	0	1	$\mathbf{Z}$	-	$\mathbf{Z}$
	BDI (chiral orthog.)	+1	+1	1	$\mathbf{Z}$	-	-
	CII (chiral sympl.)	−1	−1	1	$\mathbf{Z}$	-	$\mathbf{Z}_2$
BdG	D	0	+1	0	$\mathbf{Z}_2$	$\mathbf{Z}$	-
	C	0	−1	0	-	$\mathbf{Z}$	-
	DIII	−1	+1	1	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}$
	CI	+1	−1	1	-	-	$\mathbf{Z}$

FIG. 1: From Schnyder et al. [6].

We remark that the BdG description of a superconductor concentrates on the fermions and that the electromagnetic field barely appears (at most it is treated as a background static gauge field). The latter has no dynamics and plays no role in this classification. This is what we call a "textbook" superconductor following Xiao-Gang Wen.

III. 2D $p_x + ip_y$ SUPERCONDUCTOR

This section is inspired from [8] and [9]. Beware that in the following c and $c^\dagger$ now refer to complex fermionic operators, unlike in the previous section.

A. Bulk Chern number

We now consider spinless complex fermions on a square lattice. Sites are labelled $(m, n) \in \mathbb{Z}^2$. The Hamiltonian describes complex fermions hopping t and with nearesgt-neighbor (p-wave) pairing Δ:

$$H = \sum_{m,n} -t(c^\dagger_{m+1,n}c_{m,n} + c^\dagger_{m,n+1}c_{m,n} + \text{ h.c. }) - (\mu - 4t)c^\dagger_{m,n}c_{m,n} + \Delta(c^\dagger_{m+1,n}c^\dagger_{m,n} + ic^\dagger_{m,n+1}c^\dagger_{m,n}) + \text{ h.c.}$$

with

$$\{c_{m,n}, c^\dagger_{m',n'}\} = \delta_{m,m'}\delta_{n,n'} \text{ and } \{c_{m,n}, c_{m',n'}\} = 0.$$

Note the i factor (in red) in the pairing term along the y direction. This breaks TRS and defines a chirality (replacing with $-i$ would give the opposite chirality). There is also a chemical potential μ and the zero of energy has been chosen so that the band bottom is at zero energy (this explains the $-4t$ next to μ). In the absence of pairing ($\Delta = 0$), $\mu < 0$ describes an empty band while $\mu > 8t$ corresponds to a completely filled band. Both cases correspond to a band insulator.

After Fourier transform, we can rewrite this in BdG form:

$$H = \sum_{\mathbf{p}} \psi^\dagger_{\mathbf{p}} \frac{1}{2} H_{BdG}(\mathbf{p})\, \psi_{\mathbf{p}} \text{ with } H_{BdG}(\mathbf{p}) = \begin{pmatrix} \varepsilon_0(\mathbf{p}) & 2i\Delta(\sin p_x + i\sin p_y) \\ -2i\Delta^*(\sin p_x - i\sin p_y) & -\varepsilon_0(\mathbf{p}) \end{pmatrix},$$

where the band dispersion $\varepsilon_0(\mathbf{p}) = -2t(\cos p_x + \cos p_y) - (\mu - 4t)$ is defined in the first Brillouin zone (BZ) $-\pi < p_x, p_y \leq \pi$. Near the center of the BZ $\mathbf{p} \to 0$

$$H_{BdG}(\mathbf{p}) = \begin{pmatrix} \frac{\mathbf{p}^2}{2m} - \mu & 2i\Delta(p_x + ip_y) \\ -2i\Delta^*(p_x - ip_y) & -\frac{\mathbf{p}^2}{2m} + \mu \end{pmatrix},$$

with the effective band mass $m \equiv \frac{1}{2t}$. This is called a $p_x + ip_y$ superconductor. It exists in two chiralities $p_x \pm ip_y$.

Let $\Delta = |\Delta|e^{i\theta}$ with a uniform phase θ and we perform a gauge transformation $c_{\mathbf{p}} \to e^{i\theta/2}c_{\mathbf{p}}$ so that $c^\dagger_{\mathbf{p}} \to e^{i\theta/2}c^\dagger_{\mathbf{p}}$ that brings the BdG Hamiltonian into the following form:

$$H_{BdG}(\mathbf{p}) = \begin{pmatrix} \varepsilon_0(\mathbf{p}) & 2i|\Delta|(\sin p_x + i\sin p_y) \\ -2i|\Delta|(\sin p_x - i\sin p_y) & -\varepsilon_0(\mathbf{p}) \end{pmatrix} = -2|\Delta|(\sin p_y \tau_x + \sin p_x \tau_y) + \varepsilon_0(\mathbf{p})\tau_z = \vec{h}(\mathbf{p}) \cdot \vec{\tau}$$

where the Pauli matrices τ_x, τ_y, τ_z act in electron-hole (Nambu) space and $\vec{h}(\mathbf{p})$ is an effective momentum-dependent magnetic field. Near the BZ center, it takes the form of a 2+1 massive Dirac Hamiltonian:

$$H_{BdG}(\mathbf{p}) \simeq -2|\Delta|(p_y\tau_x + p_x\tau_y) + (\frac{\mathbf{p}^2}{2m} - \mu)\tau_z = v(-p_y\tau_x - p_x\tau_y) + M(p)v^2\tau_z,$$

where v is a velocity and $M(p)$ a mass (that depends on momentum).

TRS is broken as $H_{BdG}(-\mathbf{p})^* \neq H_{BdG}(\mathbf{p})$. This is due to the i in between $\sin p_x$ and $\sin p_y$. This sets the chirality. Here TR acts as the anti-unitary operator $T = K$, i.e. it is just the complex conjugaison K. It satisfies $T^2 = +1$. PHS must be present in a BdG problem. It manifests as $\tau_x H_{BdG}(-\mathbf{p})^*\tau_x = -H_{BdG}(\mathbf{p})$. The charge conjugaison (or PH operator) is anti-unitary and reads $C = \tau_x K$. It satisfies $C^2 = \tau_x\tau_x^* = +1$. These two properties (TRS=0 and PHS=+1) show that our model is indeed in the D class of the 10-fold way. It should therefore have a $\mathbb{Z}$ topological invariant.

The dispersion relation of BQP is made of two bands that are generically separated by a gap:

$$\varepsilon_\pm(\mathbf{p}) = \pm|\vec{h}(\mathbf{p})| = \pm\sqrt{4|\Delta|^2(\sin^2 p_x + \sin^2 p_y) + \varepsilon_0(\mathbf{p})^2}.$$

For this type of pairing there is a full gap.

Do these bands carry a topological invariant? One needs to consider the mapping $\vec{n}(\mathbf{p}) = \vec{h}(\mathbf{p})/|\vec{h}(\mathbf{p})|$ from the BZ $= T^2$ to the Bloch sphere S^2 (corresponding to a 2-level system for each wavevector $\mathbf{p}$):

$$\mathbf{p} \in BZ =]-\pi, \pi]^2 \to \vec{n}(\mathbf{p}) = (-2|\Delta|\sin p_y, -2|\Delta|\sin p_x, \varepsilon_0(\mathbf{p}))/\varepsilon_+(\mathbf{p}) \in S^2$$

when $\varepsilon_+(\mathbf{p}) \neq 0$ for all $\mathbf{p}$ which means that the system is gapped. These maps $T^2 \to S^2$ [13] are classified by the second homotopy group $\Pi_2(S^2) = \mathbb{Z}$ that defines a wrapping number (here equivalent to the Chern number). For a two band system the Chern number (for the lower band) can be computed as

$$\nu = \frac{1}{2}\sum_D \chi_D \text{ sign } (M_D),$$

where the sum is over Dirac points D (defined as the $\mathbf{p}$ points in BZ such that $h_x(\mathbf{p}_D) = h_y(\mathbf{p}_D) = 0$), χ_D is the chirality (or winding number of the Dirac point D) and $M_D = h_z(p_D)$ is its mass (or gap). There are four Dirac points $\mathbf{p}_D = (0,0),(\pi,0),(0,\pi),(\pi,\pi)$. These are high-symmetry points of the BZ known as the 4 time-reversal invariant momenta (TRIM): Γ, X, Y and M. The chiralities are $\chi_\Gamma = -1$, $\chi_X = +1$, $\chi_Y = +1$ and $\chi_M = -1$, as near Γ, $H_{BdG}(\mathbf{q}) \sim -q_y\tau_x - q_x\tau_y$; near X, $H_{BdG}(\mathbf{q}) \sim -q_y\tau_x + q_x\tau_y$; near Y, $H_{BdG}(\mathbf{q}) \sim q_y\tau_x - q_x\tau_y$; and near M, $H_{BdG}(\mathbf{q}) \sim q_y\tau_x + q_x\tau_y$. And the masses are $M_\Gamma = -\mu$, $M_X = M_Y = 4t - \mu$ and $M_M = 8t - \mu$. So that the Chern number $\nu = 0$ when $\mu < 0$ or $\mu > 8t$, $\nu = 1$ when $0 < \mu < 4t$ and $\nu = -1$ when $4t < \mu < 8t$. At the phase transitions, the gap closes at one (when $\mu = 0$ or $8t$) or two (when $\mu = 4t$) Dirac points.

In summary, when $\mu < 0$ or $\mu > 8t$, this is strong pairing and the system is a trivial superconductor with $\nu = 0$. When $0 < \mu < 4t$ or $4t < \mu < 8t$, this is weak pairing and the system is a topological chiral superconductor with $\nu = \pm 1$.

B. Gapless chiral-Majorana edge states

The $p_x + ip_y$ SC is very similar to the integer quantum Hall effect at $\nu = 1$ except that it only has half the degrees of freedom (real versus complex fermion). When $\nu = \pm 1$, we therefore except a chiral gapless edge state similar to a QH edge channel, except that it is "real" rather than complex and therefore charge neutral. There is no quantum Hall effect but there is a quantized thermal Hall effect: heat transport is quantized [3]. The measure is the chiral central charge $c_- = \nu/2 = \pm 1/2$. A chiral-Majorana mode is also known as a Weyl-Majorana mode (it only exists in 1+1 spacetime dimension).

Imagine a disk of $p_x + ip_y$ SC (with $0 < \mu \ll 4t$) with perimeter L, surrounded by the vacuum modelled as $\mu < 0$ (i.e. $\nu = 0$), see Fig. 2(a). The fact that the Chern number is an integer and can only change in a discrete manner via a gap closing, means that near the edge the gap must close, corresponding to the gapless edge mode. This is an example of bulk-edge correspondence. If the bulk was characterized by a larger $\nu \in \mathbb{Z}$, there would be more gapless chiral-Majorana edge states. The number of edge channels is actually equal to the bulk Chern number ν.

Consider the chiral edge mode with dispersion $\varepsilon = vp$ where $p \in \mathbb{R}$ (clockwise). The boundary is circular with perimeter L and acts as 1D periodic boundary conditions so that we expect the momentum to be quantized as $p = n2\pi/L$, where $n \in \mathbb{Z}$. Then the mid-gap states would have energy $\varepsilon_n = vn2\pi/L$ including an isolated $\varepsilon_n = 0$ state. But it is not possible for a BdG/Majorana problem in a finite system. Indeed we know that we must have an even number of states with the symmetry $\varepsilon_n \to -\varepsilon_n$, which forbids having an odd number of zero energy eigenvalues. Actually, a more detailed calculation shows that there is a π Berry phase shift so that the correct quantization is actually:

$$\varepsilon_n = (n + 1/2)\frac{2\pi v}{L}$$

Now we have the states at energy $\pm(1/2, 3/2, 5/2, \ldots)v2\pi/L$. This is as if there was anti-periodic boundary conditions along the edge, see Fig. 2(a).

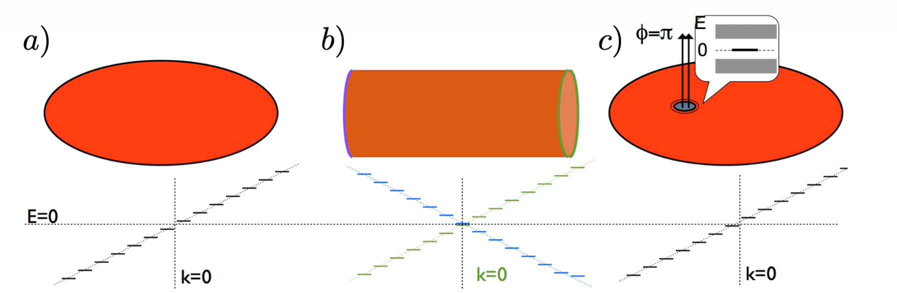

FIG. 2: (a) No MZM. (b) 2 MZMs: 1 per edge. (c) 2 MZMs: 1 on the vortex and one on the outer edge. From Bernevig and Neupert [9].

C. States bound to a SC vortex

Now, we insert a superconducting vortex corresponding to a π flux inside the disk (SC flux is $h/(2e) = \pi$ when $\hbar = 1$ and $e = 1$). Notice that a vortex corresponds to a phase winding of 2π for $\Delta(\mathbf{r}) = |\Delta|e^{i\theta}$ but that it corresponds to a π phase for an electron/hole. It is also known as a π-vortex or as a vison (Ising vortex as in a $\mathbb{Z}_2$ gauge field, the flux can only be 0 or π). Therefore, for an electron/hole the SC vortex acts as an end point for a branch cut across which the phase jumps by π. This shift the boundary condition for the gapless chiral edge mode from $n + 1/2$ to n (again a π phaseshift $\delta pL = \pi$). So that now

$$\varepsilon_n = n\frac{2\pi v}{L}$$

and there is an isolated zero mode at the boundary of the disk. This is not possible, which means that it is accompanied by another zero mode bound to the vortex. Indeed, semiclassically a vortex can be modelled as a hole of finite size with $\mu < 0$. Now the disk geometry has turned into a Corbino (or cylinder) with two boundaries, see Fig. 2(b): an outer one of length L and a much smaller inner one around the vortex of length $L' \ll L$. The inner one also host a chiral gapless edge mode but with opposite chirality (anti-clockwise):

$$\varepsilon_{n'} = -n'\frac{2\pi v}{L'}$$

We therefore have two Majorana zero modes: one $n = 0$ on the outer edge and one $n' = 0$ on the vortex, see Fig. 2(c). This semiclassical expectation is confirmed by a microscopic quantum mechanical calculation.

Actually, Volovik 1999 has shown that there is an even/odd effect for the states bound to a SC vortex. When ν is even:

$$\varepsilon_{n'} = (n' + 1/2)\frac{2\pi v}{L'}.$$

These are known as Caroli-de Gennes-Matricon (CdGM) states. The lowest eigen-states are at finite energy $\pm v\pi/L$ and correspond to a single complex fermionic mode "at energy $v2\pi/L$". And when ν is odd:

$$\varepsilon_{n'} = n'\frac{2\pi v}{L'}.$$

The $\varepsilon_{n'=0} = 0$ is known as a MZM. It corresponds to an unpaired Majorana fermion at zero energy and bound to the vortex. It only corresponds to half a complex fermionic mode. Because of that, the corresponding Hilbert space at zero energy is only well-defined thanks to the presence of another unpaired Majorana fermion. MZMs always appear in pairs in finite systems (the pair is related by a branch cut). In an infinite system, it is possible to have a single MZM by sending the partner to infinity (the branch cut is semi-infinite, starting at the vortex and going to infinity).

D. MZM as a non-Abelian anyon

If we introduce a second well-separated vortex in the disk, the edge mode shifts back to $\varepsilon_n = v(n + 1/2)2\pi/L$ and each vortex hosts a MZM. Therefore, a pair of well-separated vortices describe a complex fermion mode at zero energy $\varepsilon_0 = 0$ that can be empty or occupied. It therefore corresponds to a 2-fold ground state degeneracy (GSD).

If we have $2N$ well-separated vortices, the GSD $= 2^N = \sqrt{2}^{2N}$ so that each vortex or MZM corresponds to a non-integer quantum dimension $d_\sigma = \sqrt{2}$. Among these 2^N degenerate states, 2^{N-1} are of even parity and 2^{N-1} of odd parity [14]. It is usual to notate σ the MZM or the vortex in an odd ν topological SC. The fact that two MZM code for a single complex fermion mode that can be empty or occupied is written as a fusion rule

$$\sigma \times \sigma = 1 + \psi$$

where 1 indicates the empty mode or vacuum (i.e. the condensate of Cooper pairs) and ψ indicates an occupied mode or a fermion (i.e. a Bogoliubov quasiparticle).

If we move adiabatically the vortices while keeping them well-separated, the GSD is not changed and we stay in the degenerate subspace. Exchanging the position of the vortices should therefore correspond to a unitary transformation in this subspace. This is precisely the definition of non-Abelian anyons. Abelian anyons are such that exchanging two, one goes back to the same (non-degenerate) initial state up to a phase that need not be 0 (as for bosons) or π (as for fermions) but can by any phase (actually π times a rational number), hence the name any-on. Here it is not a simple phase but a unitary operator.

In conclusion, vortices in topological superconductors with odd Chern number trap a Majorana zero mode that behaves as a non-Abelian anyon. Remark that a MZM is a MF at zero energy and bound to a topological defect (a vortex or a domain wall). It is only the composite object "MF + vortex + at zero energy" that behaves as a non-Abelian anyon. A MF by itself is just a fermion.

Remark: Because a vortex carries a branch cut, when we exchange two vortices, one of the two must cross the branch cut of the other and acquires a π phase. This means that....

IV. 16-FOLD WAY: "REAL-LIFE" SUPERCONDUCTORS

A "real-life" superconductor is one in which the coupling to the electromagnetic gauge field is considered seriously, in which the gauge field is not a spectator but an actor. Starting from a "textbook" superconductor described as fermions with global $\mathbb{Z}_2$ symmetry, one needs to gauge the fermion parity by introducing a $\mathbb{Z}_2$ gauge field, that is dynamical (it has quantum fluctuations) and local. In practice, it means treating the SC vortices as excitations of the superconductor on the same footing as BQP.

For example, if $\nu = 1$, we have 3 topological sectors or 3 types of topological quasiparticles:
- 1 = vacuum, or Cooper pair condensate
- ψ = BQP (or more properly a MF)
- σ = vortex, MZM

A consistent theory for these three anyon types requires to define fusion and braiding. In mathematics, this is known as a modular tensor category MTC and is equivalent to a topological quantum field theory TQFT (for a pedagogical introduction see [10]). The fusion rules are:

$$1 \times x = x \times 1 = x \text{ (where } x = 1, \psi, \sigma), \psi \times \psi = 1, \sigma \times \sigma = 1 + \psi \text{ and } \sigma \times \psi = \psi \times \sigma = \sigma.$$

$\psi \times \psi = 1$ means that ψ is its own antiparticle, this is the MF (or BQP). $\sigma \times \sigma = 1 + \psi$ means that σ is a non-Abelian anyon: collectively 2 σ's share a 2-dimensional Hilbert space. They fractionnalize an internal space. The corresponding quantum dimensions are $d_1 = d_\psi = 1$ and $d_\sigma = \sqrt{2}$. The total quantum dimension (squared) is $\mathcal{D}^2 = d_1^2 + d_\psi^2 + d_\sigma^2 = 4$.

Braiding is given by specifying self-statistics (the exchange phases, or the topological spins, are usually arranged in a diagonal matrix known as the T-matrix) and the mutual statistics (the braiding phases of distinguishable particles are given in a filled matrix known as the S-matrix) [15]. Kitaev [3] has shown that there are 16 different anyon theories that are possible depending on ν mod 16 for a 2D spinless superconductor. This is known as the 16-fold way. They all have a total quantum dimension $\mathcal{D} = 2$ and they can be regrouped in 3 big families:

1. Toric code like

When $\nu = 0, \pm 4, 8$ (i.e. $\nu = 0$ mod 4). There are two different vortices called e and m. A *single* vortex hosts a complex CdGM state at finite energy that is either empty m or occupied e: these are the two different types of vortices. The 4 anyon types are $\{1, e, m, \psi\}$ such that $e \times e = 1 = m \times m$ and $e = m \times \psi$. They are Abelian anyons with $d_1 = d_e = d_m = d_\psi = 1$. The prototype is the toric code MTC corresponding to $\nu = 0$.

2. Ising like

When $\nu = \pm 1, \pm 3, \pm 5, \pm 7$ (i.e. ν is odd). There is a single type of vortex called σ (MZM) that behaves as a non-Abelian anyon with quantum dimension $d_\sigma = \sqrt{2} > 1$. The 3 anyon types are $\{1, \sigma, \psi\}$ such that $\sigma \times \sigma = 1 + \psi$. A *pair* of σ's code for a single complex fermionic state that can be empty or occupied. THe vortex or MZM is a Non-Abelian anyons with $d_\sigma = \sqrt{2}$, while $d_1 = d_\psi = 1$. The prototype is the Ising MTC corresponding to $\nu = 1$.

3. $\mathbb{Z}_4$ like

When $\nu = \pm 2, \pm 6$ (i.e. $\nu = 2$ mod 4). There are two different vortices called a and $\bar{a}$. A single π vortex hosts a complex CdGM state at finite energy. The 4 anyon types are $\{1, a, \bar{a}, \psi\}$ such that $a \times \bar{a} = 1$ and $a \times a = \psi = \bar{a} \times \bar{a}$. They are Abelian anyons with $d_1 = d_a = d_{\bar{a}} = d_\psi = 1$. The prototype is the $\mathbb{Z}_4^{(1/2)}$ MTC corresponding to $\nu = 2$.

The twists (i.e. $\theta = e^{i\varphi_{\text{ex}}}$ where φ_{ex} is the phase acquired during an exchange of two indistinguishable particles) are $\theta_1 = 1 = e^{i0}$ (boson), $\theta_\psi = -1 = e^{i\pi}$ (fermion) and $\theta_{\text{vortex}} = e^{i\pi\nu/8}$ that shows that ν matters modulo 16 [3]. Let's try to see why $\theta_{\text{vortex}} = e^{i\pi\nu/8}$. Let us stack two identical copies of $p_x + ip_y$ SC with Chern number ν. This is equivalent to an IQHE with Chern number ν so that the Hall conductivity is $\sigma_{xy} = \nu e^2/h = \nu/(2\pi)$. Threading a SC flux of $\Delta\phi = \pi$ across the two layers binds a charge of $\Delta Q = \sigma_{xy}\Delta\phi = \nu/2$ (Streda formula) due to the quantum Hall effect. Now braiding this π flux with another such π flux gives an Aharonov-Bohm phase of $\pi\nu/2$ (moving a charge of $\nu/2$ around a flux of π), so that the exchange phase is $\pi\nu/4$. It means that for each SC layer, the exchange phase of two vortices is indeed $\pi\nu/8$ [9].

The modular data for the toric code ($\nu = 0$) is

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and for the Ising MTC ($\nu = 1$) is:

$$S = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\pi/8} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

A system that has a consistent anyon theory is said to possess topological (quantum) order, as defined by Xiao-Gang Wen.

V. TWO FAMILIES OF GAPPED TOPOLOGICAL PHASES

There are two families of gapped topological phases of matter:

A. Topological bands

Topological band are also known as integer topological phases or symmetry-protected topological (SPT) phases. It concerns non-interacting or weakly interacting electrons (or spins or bosons). Single-particle bands carry topological invariants (such as Chern number of winding number). There is no fractionnalization (quasiparticles are not exotic). The ground state has no long range entanglement. Examples are IQHE, Chern insulator, quantum spin Hall insulator (à la Kane and Mele), and "textbook" superconductors.

B. Topological order

Topological quantum order is also known as fractional topological phases. It concerns strongly interacting electrons (or spins or bosons). Quasiparticles are fractionnalized, meaning that they carry quantum numbers (charge, spin, statistics, etc) that are not integer multiple of that of the microscopic constituents and they are usually called anyons (emphasizing that the most spectacular fractionalization is that of quantum statistics and of internal space). The ground state has long range entanglement, as revealed by a non-zero topological entanglement entropy. Examples are FQHE, gapped quantum spin liquids and "real-life" superconductors. Topological order is characterized by giving the modular data (S,T) of its anyon types and the chiral central charge c_- that characterizes its edges. From the bulk (S,T) one can compute c_- modulo 8, that is known as the topological central charge.

VI. CONVENTIONAL SUPERCONDUCTORS ARE TOPOLOGICALLY ORDERED

When considered with a dynamical gauge field, conventional (s-wave singlet, TRS, $\nu = 0$) superconductors are topologically-ordered. With TO of the toric code kind. The four anyon types are: $1 =$ Cooper pair condensate, $m =$ empty SC vortex, $e =$ occupied SC vortex, $\psi =$ BQP. Fractionalization occurs in the sense that the vortex carries a half-flux quantum $h/(2e) = \pi$ that is not an integer multiple of the flux quantum $h/e = 2\pi$. Also there is spin-charge separation: the BQP carries the spin (it is a fermion) but no charge. And the condensate carries the charge ($-2e$) but not the spin (singlet). This is described in great detail in a paper by Hanssson, Oganesyan and Sondhi [11] following an original idea of X.-G. Wen 1991.

VII. CONCLUSION

The main messages are:

- MF is different from MZM. The first is a fundamental particle, a real fermion, it need not be at zero energy, it carries no charge, it is its own antiparticle. It has half the degrees of freedom of a complex fermion. The second is a composite object that only exists as a collective mode in a 2D (or 1D) system. It is made of a MF at zero energy bound to a topological defect (such as a vortex or a domain wall). In 2D, this composite object ("MF + topo defect + zero energy") behaves as a non-Abelian anyon. A MZM is also known as Majorana bound state (MBS), an unpaired Majorana fermion or a Majorino. Note that speaking of "Majoranas" is very ambiguous as one does not know if it means MF or MZM. MF are ubiquitous in BdG superconductors. MZM are much rarer. Those are the ones that are eagerly searched for in experiments and that could be useful for topological quantum computation.

- Topological bands versus topological quantum order: **"textbook SC" (static background electromagnetic field: fermions only) versus "real-life SC" (dynamical quantum fluctuating electromagnetic field: BQP and vortices)**. Even non-chiral non topological ($\nu = 0$) SC are topologically-ordered when considered as "real-life SC". They have the same topological order as the toric code (also known as $\mathbb{Z}_2$ topological order, that of $\mathbb{Z}_2$ lattice gauge theory).

- 2D spinless superconductors admit several classifications: $\mathbb{Z}$, $\mathbb{Z}_2$ and $\mathbb{Z}_{16}$. In the first two bullets below, they are seen as topological bands or SPT phases. In the last two bullets, they are seen as topological order.
• they can be seen as belonging to the D-class of the 10-fold periodic table of topo insulators/superconductors. As such they have a bulk Chern number (topo bands) ($\nu \in \mathbb{Z}$).
• because of bulk-edge correspondence, they necessarily host chiral-Majorana edge states (with chiral central charge $c_- = \nu/2$). There are as many edge channels as the bulk Chern number.
• there is an even/odd effect for vortices. When ν is even/odd, there is a CdGM/MZM state.
• there are 16 different bulk anyon theories (ν mod 16). The anyon types are: condensate (1, vacuum), BQP (ψ, fermion), vortex (e, m or σ or a, $\bar{a}$, anyon that is Abelian when ν is even and non-Abelian when ν is odd).

[1] P. B. Pal, "Dirac, Majorana and Weyl fermions", arXiv:1006.1718
[2] A. Y. Kitaev, "Unpaired Majorana fermions in quantum wires", arXiv:cond-mat/0010440
[3] A. Y. Kitaev, "Anyons in an exactly solved model and beyond", arXiv:0506438
[4] C. Chamon, R. Jackiw, Y. Nishida, S.-Y. Pi, and L. Santos, "Quantizing Majorana fermions in a superconductor", arXiv:1001.2760
[5] C. W. J. Beenakker, "Annihilation of Colliding Bogoliubov Quasiparticles Reveals their Majorana Nature", arXiv:1312.2001
[6] A. P. Schnyder, S. Ryu, A. Furusaki and A. W. W. Ludwig, "Classification of Topological Insulators and Superconductors", arXiv:0905.2029
[7] A. Y. Kitaev, "Periodic table for topological insulators and superconductors", arXiv:0901.2886
[8] A. B. Bernevig, "Topological insulators and topological superconductors" (Princeton university press 2013). Chapter 16 by T.L. Hughes.
[9] A. B. Bernevig and T. Neupert, "Topological superconductors and category theory", arXiv:1506.05805
[10] Steve H. Simon, "Topological quantum" (Oxford University Press, 2023).
[11] T. H. Hansson, Vadim Oganesyan, S. L. Sondhi, "Superconductors are topologically ordered", arXiv:cond-mat/0404327

[12] In 3+1, a Dirac fermion has 4 components – spin up/down and particle/antiparticle – and can have both a mass and a charge. A Majorana fermion has only 2 components and is necessarily charge neutral, it is its own antiparticle, but it can have a mass. A Weyl fermion has only 2 components, it must have zero mass but it can be charged. It has a definitive chirality. In short Dirac = complex, Majorana = real and Weyl = chiral.
[13] Actually the second homotopy group concerns maps from $S^2 \to \mathcal{M}$. Here we do as if there was no difference between classifying maps $S^2 \to S^2$ and $T^2 \to S^2$.
[14] As parity is conserved, one actually needs $N = 4$ MZMs to encode one topological qubit.
[15] These S and T matrices are not the ones from scattering theory. They are actually the generator of the modular group.