Progress on the Microscopic Theory of Inhomogeneous Superconductors

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Motivation and State of the Art

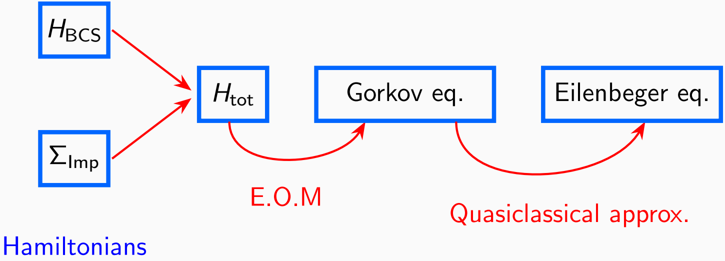
In the accelerator community one of the main objectives is to achieve the highest possible quality factor (Q) and highest superheating field (H_{sh}).

- Recent experimental results (arXiv:2509.18564, 2025, H. Hu, Y-K. Kim, &, D. Bafia) show how impurities may play a crucial role to achieve a high Q in cavities.
- There is already literature on this topic (Phys. R. Research, 2019, V. Ngampruetikorn & J. Sauls). But more theoretical results are lacking.
- Solving the eqs. from the microscopic theory is difficult, so far experimentalists resort to a macroscopic theory (London's eq.). We study how precise this approach is. (No justification so far!)
- We have that adding impurities enhances the gap; on the other hand impurities profile is inversely proportional to the mean free path profile. $R_s \propto A(l)e^{-\Delta/k_b T}$. There has to be a compromise between the 2.

Theory Brush-up

General context of microscopic Theory

It is important to comprehend the current theory and its limitations. For homogeneous impurities, London's equation is justified from Eliashberg's.



Gap function

$$[(\epsilon - \tfrac{1}{2}\hbar \boldsymbol{v}_F \cdot \boldsymbol{q})\hat{\tau}_3 - \hat{\Sigma}_{\text{imp}} + i\hat{\tau}_2\Delta, \hat{g}] = 0, \quad \hat{g}\cdot\hat{g} = 1$$

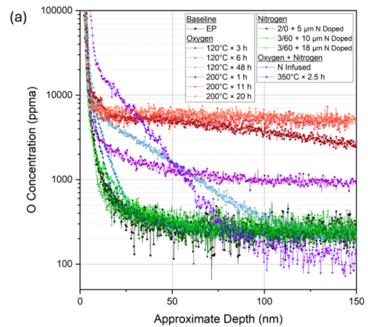
Energy   Fermi vel.   Superfluid momentum   Green's functions (2 × 2 matrix)

Applications to cavities

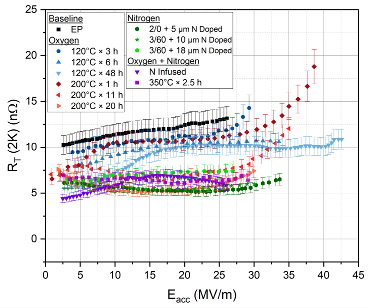
In our case, we can model non-magnetic impurities in the quasiclassical regime as:

$$\hat{\Sigma}_{\text{imp}} = -i\gamma(x)\langle\hat{g}\rangle$$

The experimental impurity profile inside the cavity is described by $\gamma(x)$.



(a) Experimental impurity profile of a cavity (arXiv:2509.18564, 2025, H. Hu, Y-K. Kim, &, D. Bafia)



(b) Impact of the impurity profile in the surface resistance R_s

Modified London eq.

Since impurities are not constant, penetration depth is not either, since

$$\lambda(x) = \lambda_0 \sqrt{1 + \frac{\xi_0}{l(x)}}, \quad l(x) \propto \frac{1}{\gamma(x)}$$

This implies that

$$\nabla^2(\lambda^2(x)\boldsymbol{B}(x)) = \boldsymbol{B}(x)$$

(J. Appl. Phys. 7 April 2024; 135 (13): 133902., E.M. Lechner, et. al.) Much simpler than Eilenberger's approach (Very easy to solve! Just a partial differential equation). But neglects interesting results that arise from the microscopic approach (DOS, Gap function enhancement, Anderson's Theorem, etc.). How precise will it be?

Results

Effects of impurities in the gap

From solving the Eilenberger equation, in the dirty and clean limits.

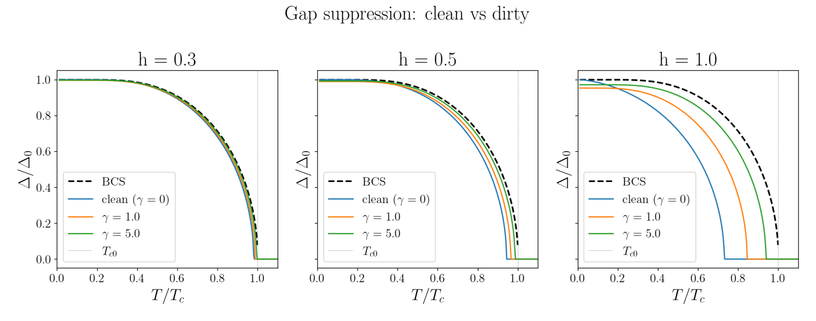
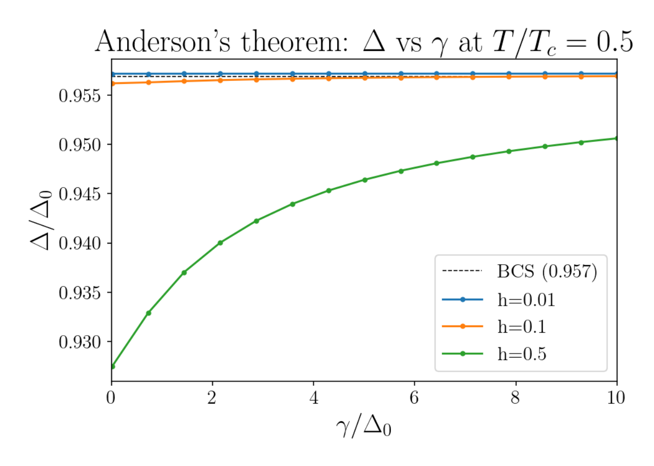


Figure 2: Comparison between the gap for a different number of impurities. $h \propto \boldsymbol{A}$ (Vector potential).

When we are applying high fields, the dirty limit protects the gap more. Meaning we can reach a higher E_{acc}. But at low temperatures, the clean limit has lower surface resistance.

Anderson's theorem

This lack of contribution of non-magnetic impurities in the absence of an electromagnetic field is known as Anderson's theorem. It can be checked as well with the gap. $\sigma_1 \propto e^{-\Delta/k_B T}$



Impurity profiles I

We have seen how impurities affect the gap. But they also affect the magnetic field inside the superconductor.

$$R_s \propto \int_0^\infty \mathrm{d}x\, \sigma_1(x) |E(x)|^2, \quad \sigma_1(x) \propto e^{-\Delta(x)/k_B T}$$

The gap barely changes, no matter the impurity profile. So it can be taken as constant.

What is the optimal concentration of impurities? Which impurity profile is better? As we need to determine the magnetic field inside the superconductor, we can compare the microscopic and macroscopic approaches.

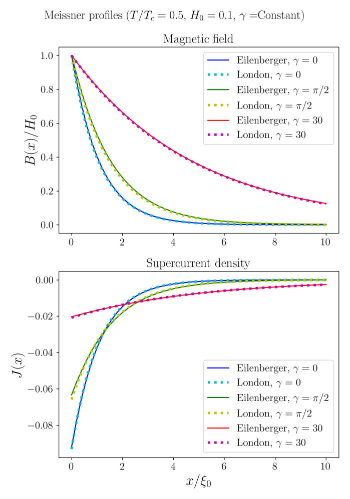


Figure 3: Impurity profile, magnetic field, and supercurrent density for different constant impurity profiles.

There is no difference between London's and Eilenberger's solution for a constant profile ✓.

For a constant profile, we can see the difference between the clean limit ($l \ll \xi_0$), the dirty limit ($l \gg \xi_0$), and what happens when $l \simeq \xi_0$.

Let's see what happens when we introduce very localized impurities using a Lorentzian distribution.

Impurity profiles II

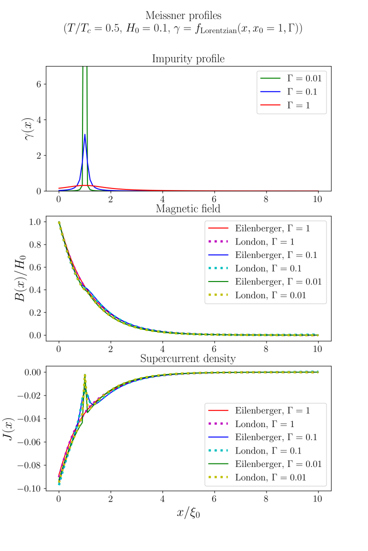


Figure 4: Lorentzian distribution results.

1. Despite having an impurity profile, London's eq. remains accurate (Up to the superheating field).
2. Introducing very localized impurities barely changes the magnetic field, but there is a significant gradient in the current, which can be used to protect the cavities against vortices penetrating (Effective SIS).

Conclusions and Future Prospects

- We developed a better understanding of how the impurities affect the gap, the Green's functions, and the fields through the Eilenberger equation.
- We fully validated London's solution with inhomogeneous impurities. Meaning London's equation can be used as a tool to find out the magnetic profile at low fields. Useful for experimentalists
- Next steps are to calculate the superheating field, and study the behaviour of the Superconductor when a profile of inhomogeneous impurities is added.

Thank you!

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