Longitudinal Beam Dynamics
Charged beam acceleration

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Content

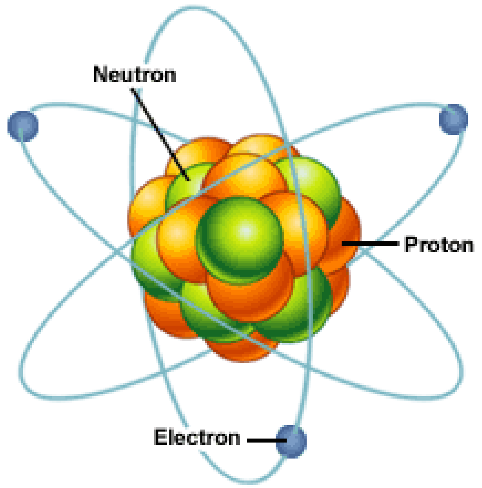
1. How to accelerate a charged particle ?
2. RF accelerating cavity basics
3. Synchronism & stability
4. Introduction to longitudinal dynamics

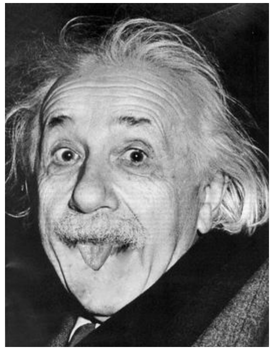
Content

1. How to accelerate a charged particle ?

1.1. Mass of a particle at rest

- **Rest mass** => $$E_0 = m_0 c^2$$ **(1905)**





avec m_0 : particle mass (kg)

$c \approx 2.998 \times 10^8$ m/s : speed light

ex : electrons : $E_0 \approx 511$ keV

protons : $E_0 \approx 938.3$ MeV

heavy ions : $E_0 \approx A \times$ uma

(1 uma ≈ 931.5 MeV, A : nb nuclei)

- **1 eV** (électron-volt) = energy gained by a particle with elementary charge $e=1.602 \times 10^{-19}$ C (ex: electron, proton) seeing a 1 Volt voltage

$$\underline{1 \text{ eV} \Leftrightarrow 1.602 \times 10^{-19} \text{ J}}$$

1.4. Lorentz force

- **Effect of the Lorentz force on the energy of a charged particle**

$$\vec{p}\cdot\frac{d\vec{p}}{dt} = \frac{1}{2}\frac{dp^2}{dt} = \frac{1}{2c^2}\frac{dE_{tot}^{\ 2}}{dt} = \frac{E_{tot}}{c^2}\frac{dE_{tot}}{dt} = \gamma m_0 \frac{dE_{tot}}{dt}$$

$$= \gamma m_0 \vec{v}\cdot q\left(\vec{E}+\vec{v}\wedge\vec{B}\right) = \gamma q m_0 \vec{v}\cdot\vec{E}$$

$$\boxed{\frac{dE_{tot}}{dt} = q\vec{v}\cdot\vec{E}}$$

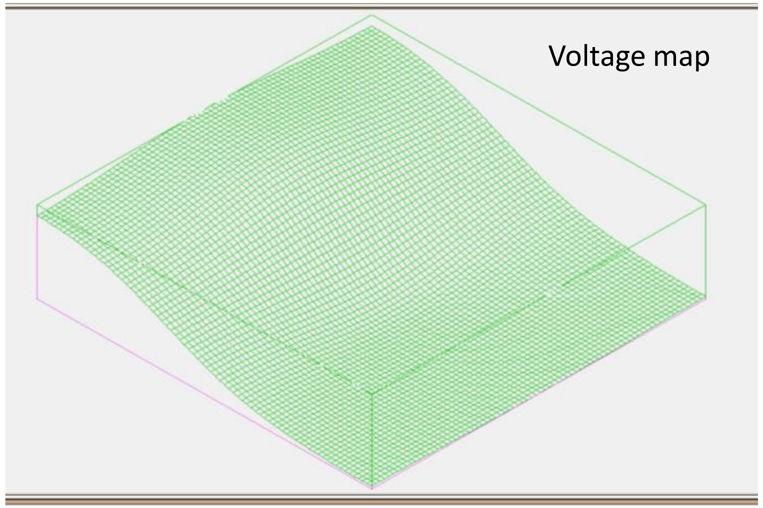
- **In order to accelerate / gain some energy:**
  - Only the <u>electric field</u> is useful
  - If $\vec{E}\perp\vec{v}$, NO acceleration
  - If $\vec{E}\ //\ \vec{v}$, optimum acceleration

=> Energy gain ΔE_{tot} in a static electric field :

$$\Delta E_{tot} = qE\int v dt = q\,E\,\Delta x = q\,\Delta V$$

(MeV) — Nb elementary charges (q) — ΔV applied voltage (MV)

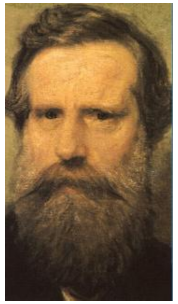
This is ELECTROSTATIC acceleration

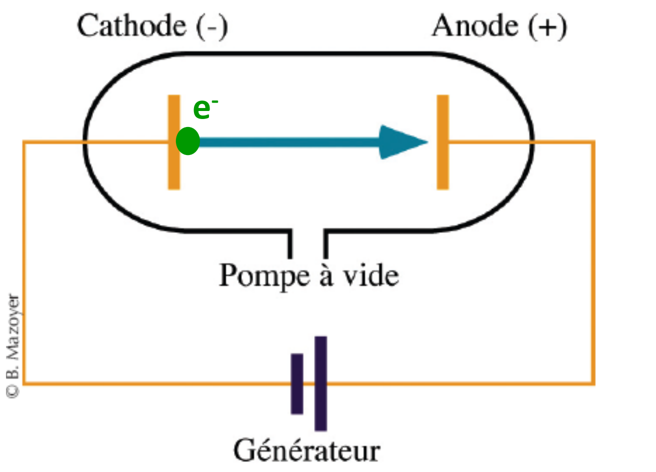


1.6. Early electrostatic experiments: vacuum tubes

1875: W. Crookes experiment

Study of the influence of air density on the voltage to be applied between two electrodes to create an electric discharge

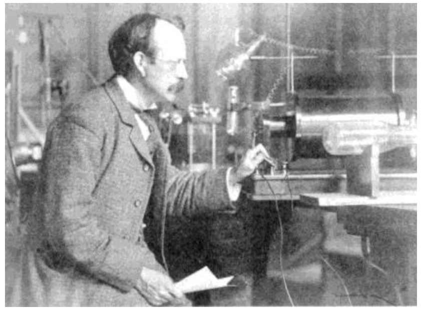


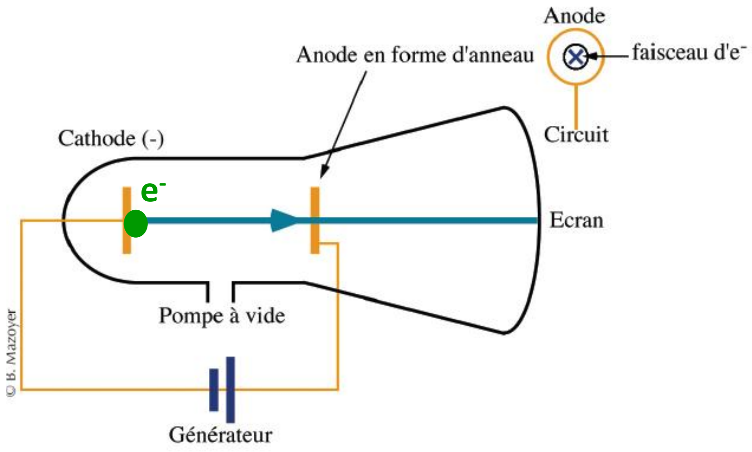


1897 : J.J. Thomson experiment

Perforated anode,
phosporescent screen,
& magnetic coils,
300 V voltage

=> Discovery of the electron !



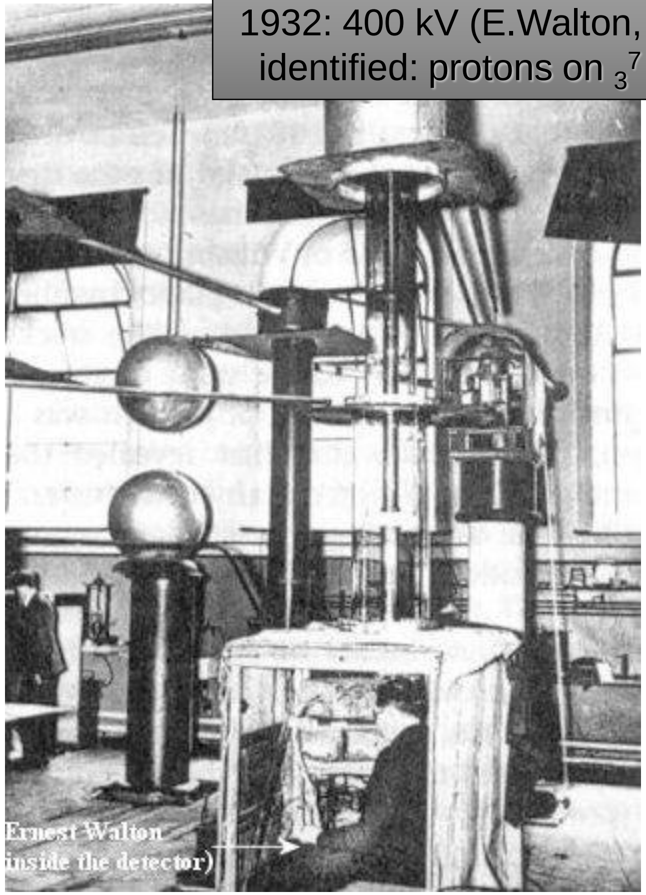


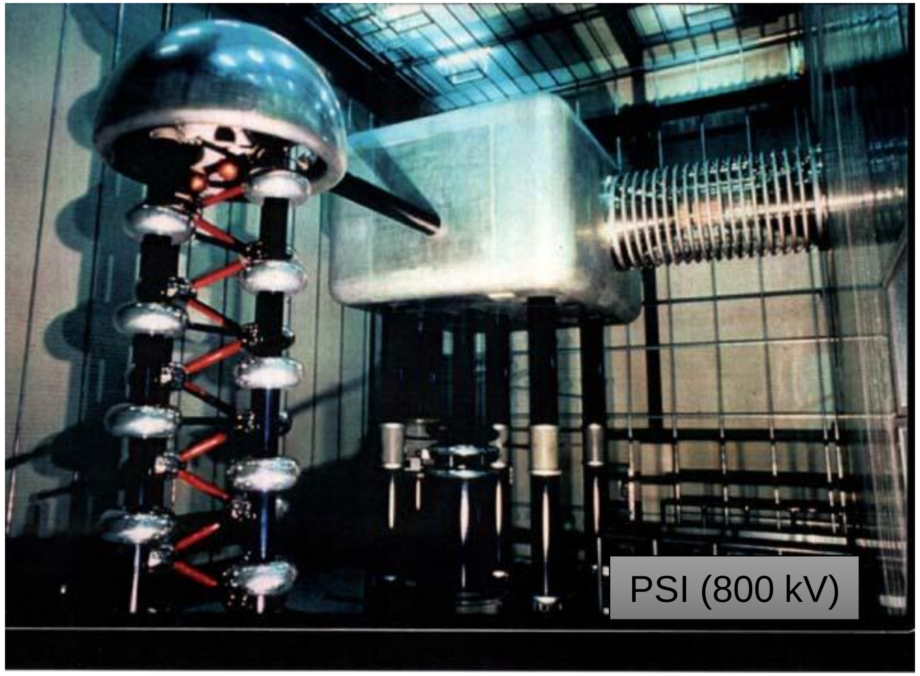

Crookes tube used by Thomson

1.7. Example of electrostatic accelerators (1)

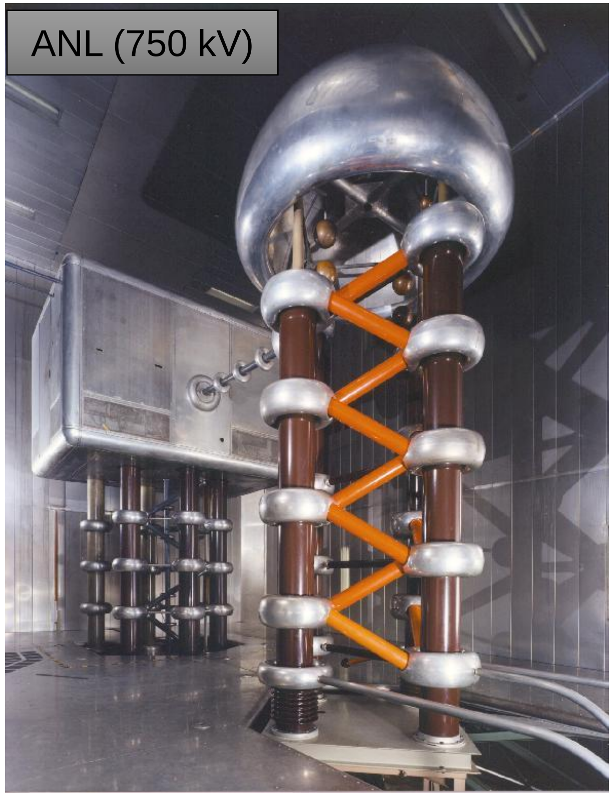
Cockcroft & Walton accelerators

1932: 400 kV (E.Walton, first fissions identified: protons on $_3^7$Li -> $_2^4$He)





PSI (800 kV)



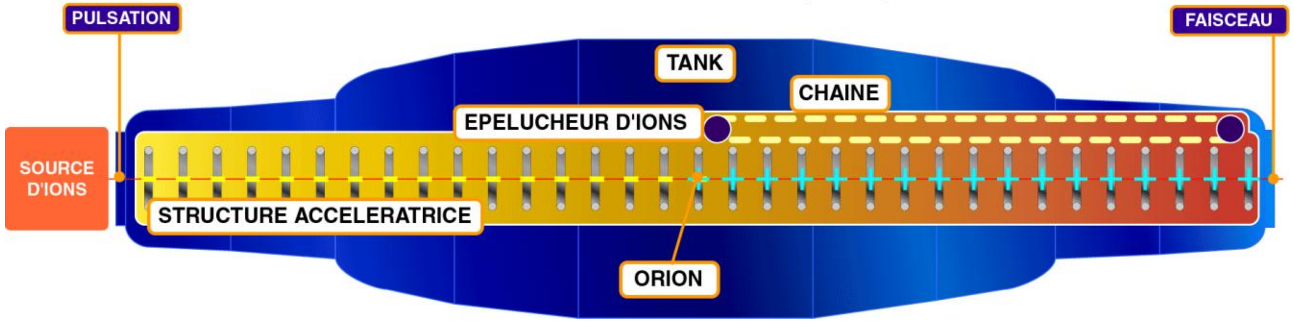
ANL (750 kV)

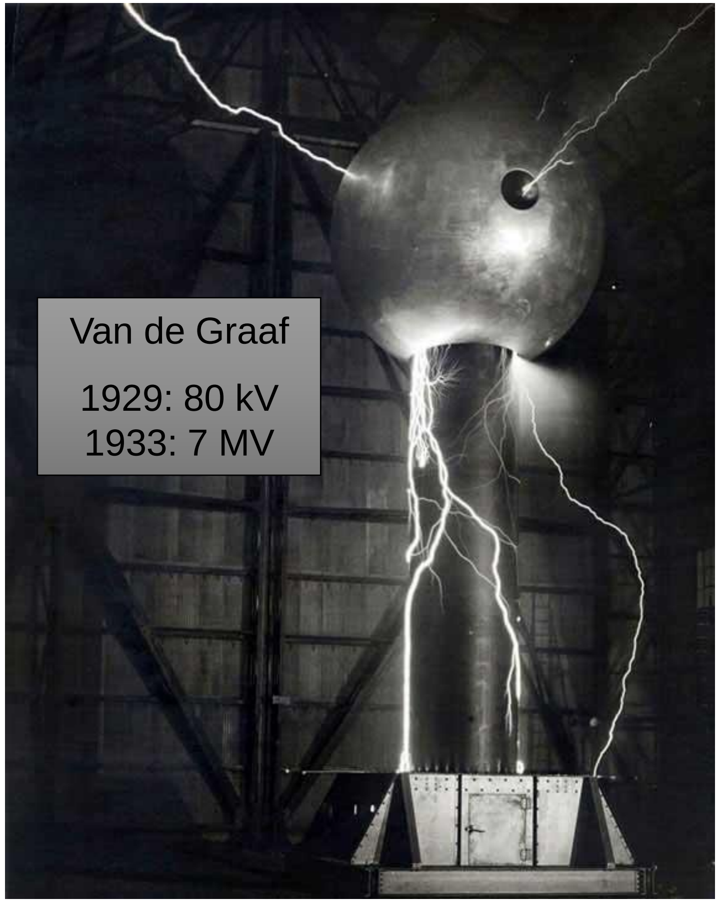
1.7. Example of electrostatic accelerators (2)

Van de Graaf accelerators & Tandems

Tandem Orsay (15 MV)

TANDEM : Schéma de principe





Van de Graaf

1929: 80 kV
1933: 7 MV

1.8. From electrostatic to RF acceleration

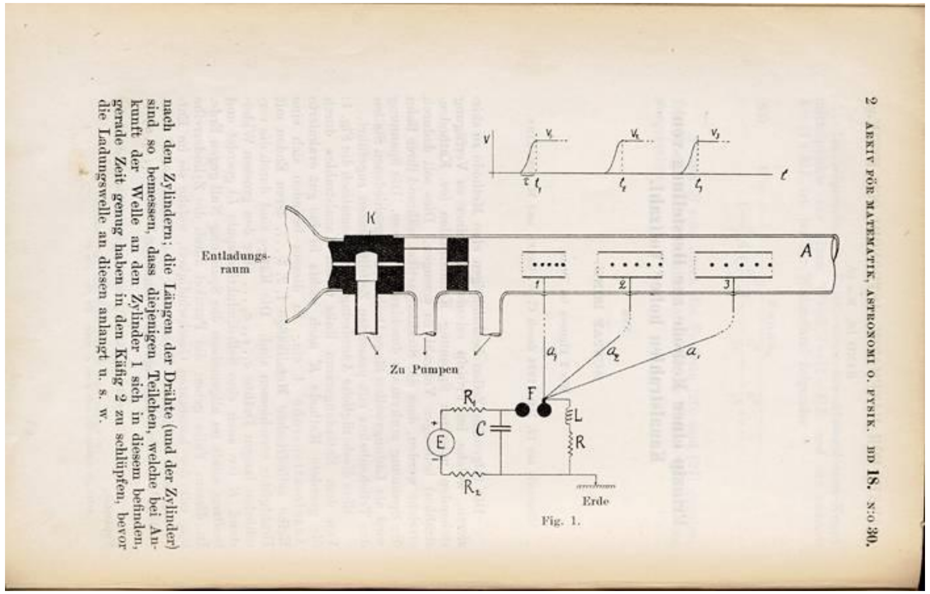
Limitation of electrostatic acceleration:

The beam energy gain is directly proportional to the voltage being applied between the 2 electrodes of the system. This concept is limited by electrostatic breakdown... that means up to a few MV in the best case...

1924: G. Ising paper

1st step towards RF acceleration: given the electrostatic limitations, Ising proposes to give the energy to the particle using several modest accelerations instead of one single large one. The concept of RF acceleration is born.

This concept is presently used in all modern large accelerators

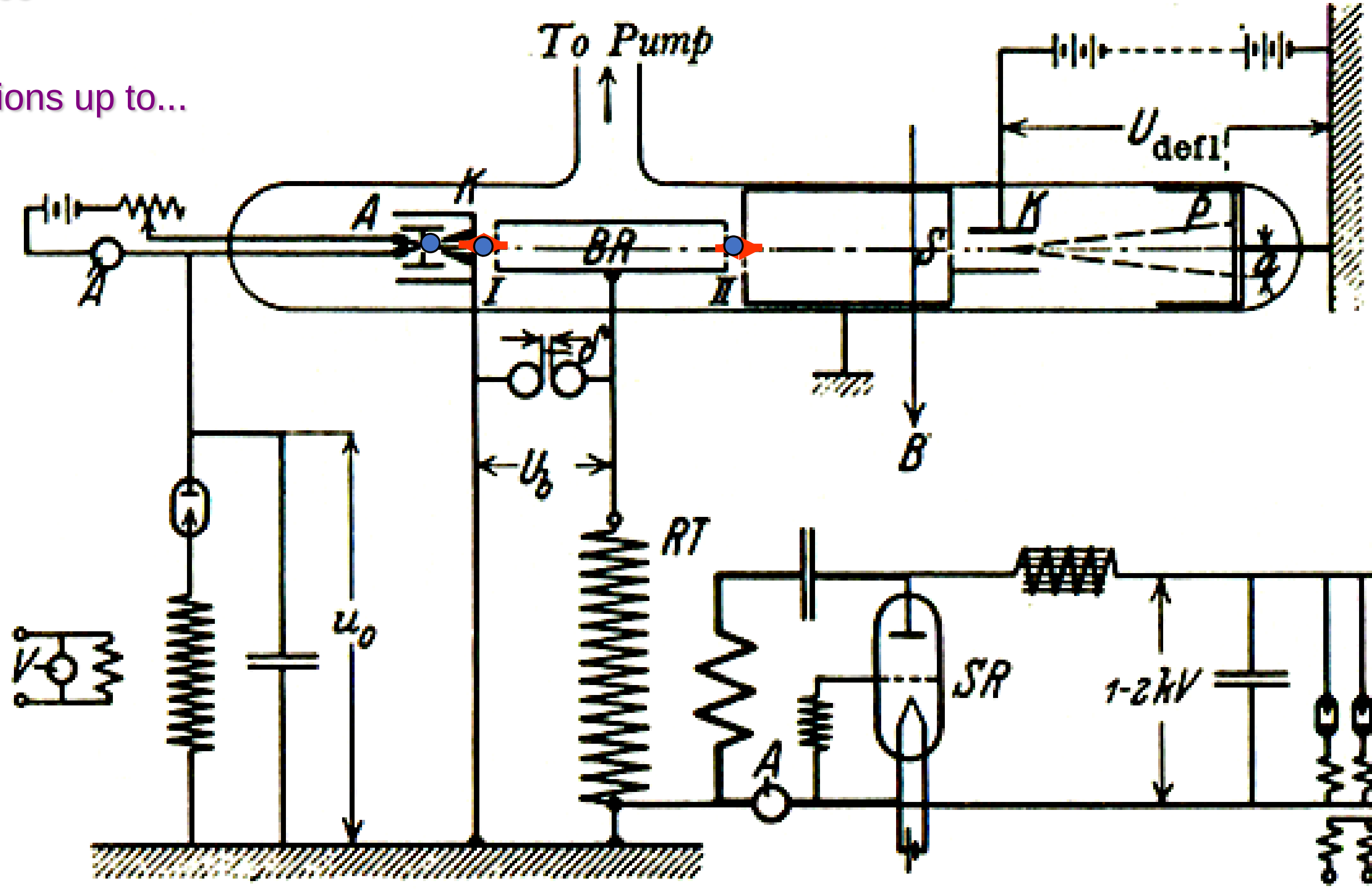


1.9. Wideröe experiment

1928: R. Wideröe experiment

First experimental proof of RF acceleration: an alternating AC voltage of 25kV is applied to a drift tube located between 2 grounded tubes

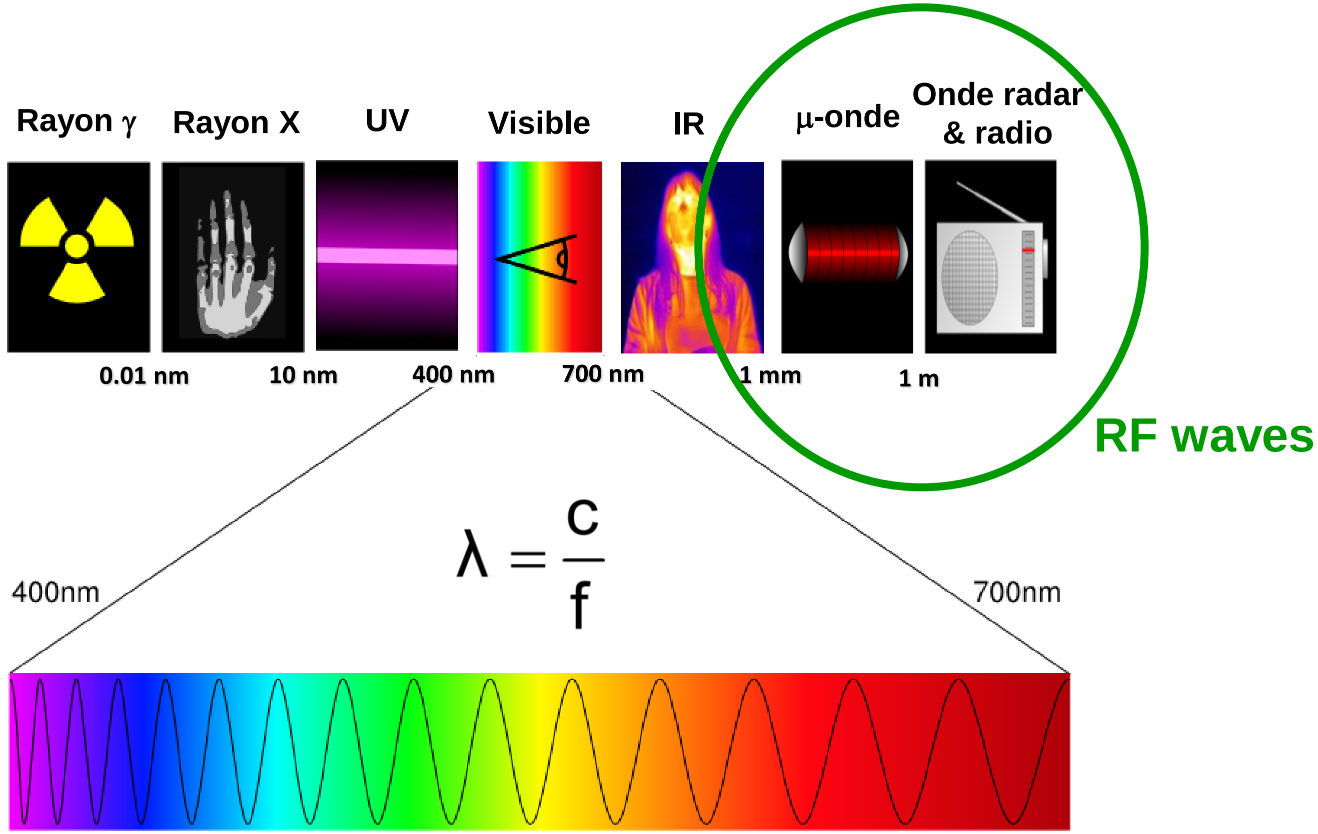
=> He suceeded in accelerating ions up to...
...2*25 = 50 keV !



2. RF accelerating cavity basics

2.1. RF waves

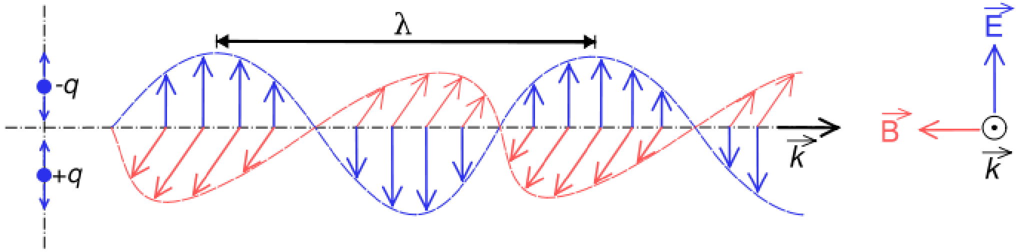
- **A radio-frequency (RF) wave** is an electromagnetic wave whose frequency (f) is, by convention, **between 9 kHz and 3000 GHz**, that corresponds to wavelengths (λ) between 33 km and 0,1 mm.



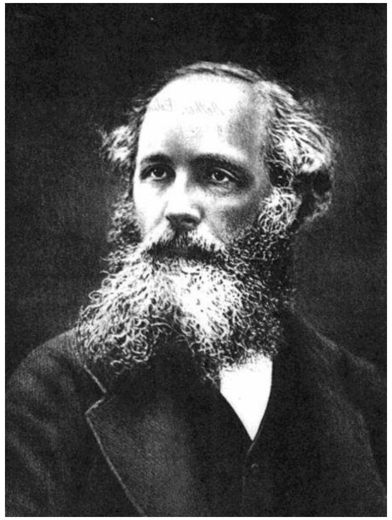
$$\lambda = \frac{c}{f}$$

2.2. Maxwell equations

- **Electromagnetic wave** = coupled oscillation of electric and magnetic fields, traveling in vacuum with the speed of light



- **The evolution of electric & magnetic fields are linked by Maxwell equations (1873)**



$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$ Gauss law (links electric charge & electric field)

$$\vec{\nabla}\cdot\vec{B} = 0$$ (no "magnetic charge", no single magnetic pole)

$$\vec{\nabla}\wedge\vec{B} = \mu_0\left(\vec{j} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$ Ampere law (links electric current & magnetic field)

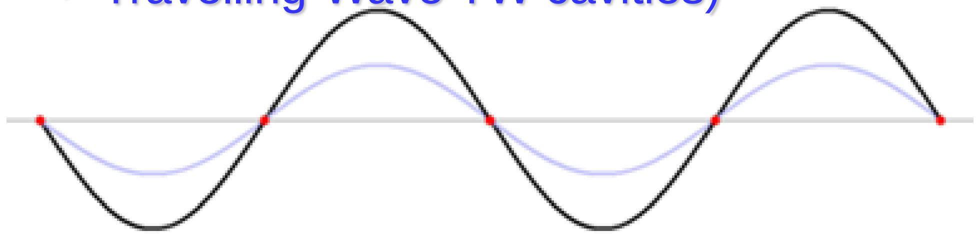
$$\vec{\nabla}\wedge\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$ Faraday law (interaction between electric & magnetic field)

where ρ : charge density (C/m^3) & j : current density (A/m^2)

2.3. Resonant RF cavities

- **<u>Resonant cavity</u>** = volume of dielectric material (usually = vacuum) surrounded by conductive walls, in which the electromagnetic waves are being reflected, creating an array of electromagnetic fields, oscillating at several frequencies with various spatial configurations
  => possibility to create **stationnary waves -> Standing-Wave SW cavities**

- (but note that progressive EM waves can also be used, especially for electrons -> Travelling-Wave TW cavities)



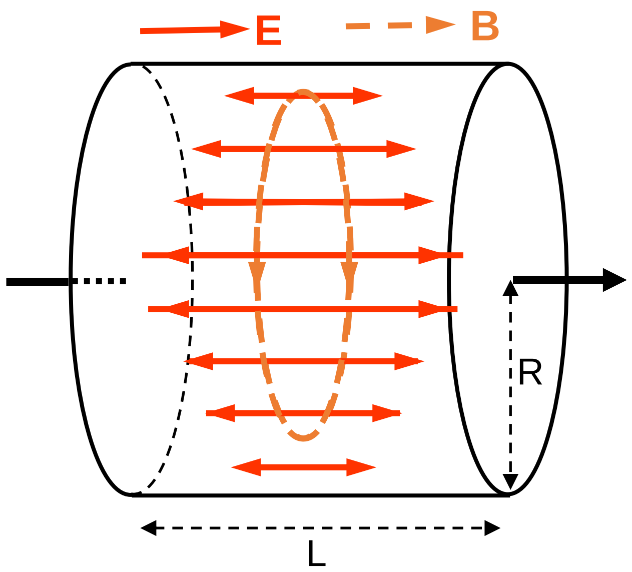
- **<u>Ex: "pillbox" cavity</u> of length L et radius R**

=> RF frequency of the "accelerating mode" (E on beam axis)

$$f_{TM010} = \frac{2.405\,c}{2\pi R}$$

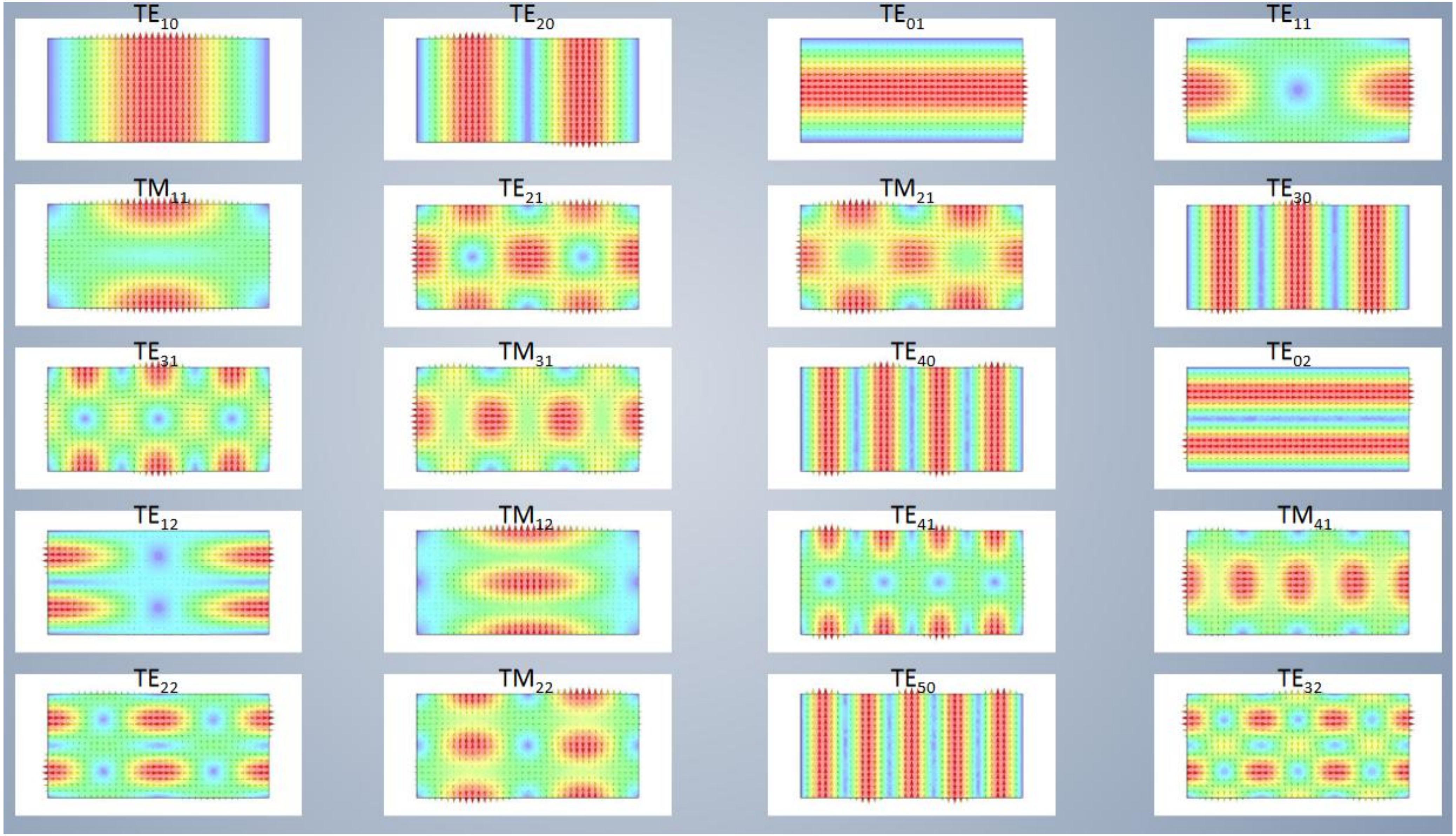
Ex : f = 700 MHz → R = 16.4 cm

LOW frequency = BIG cavity !



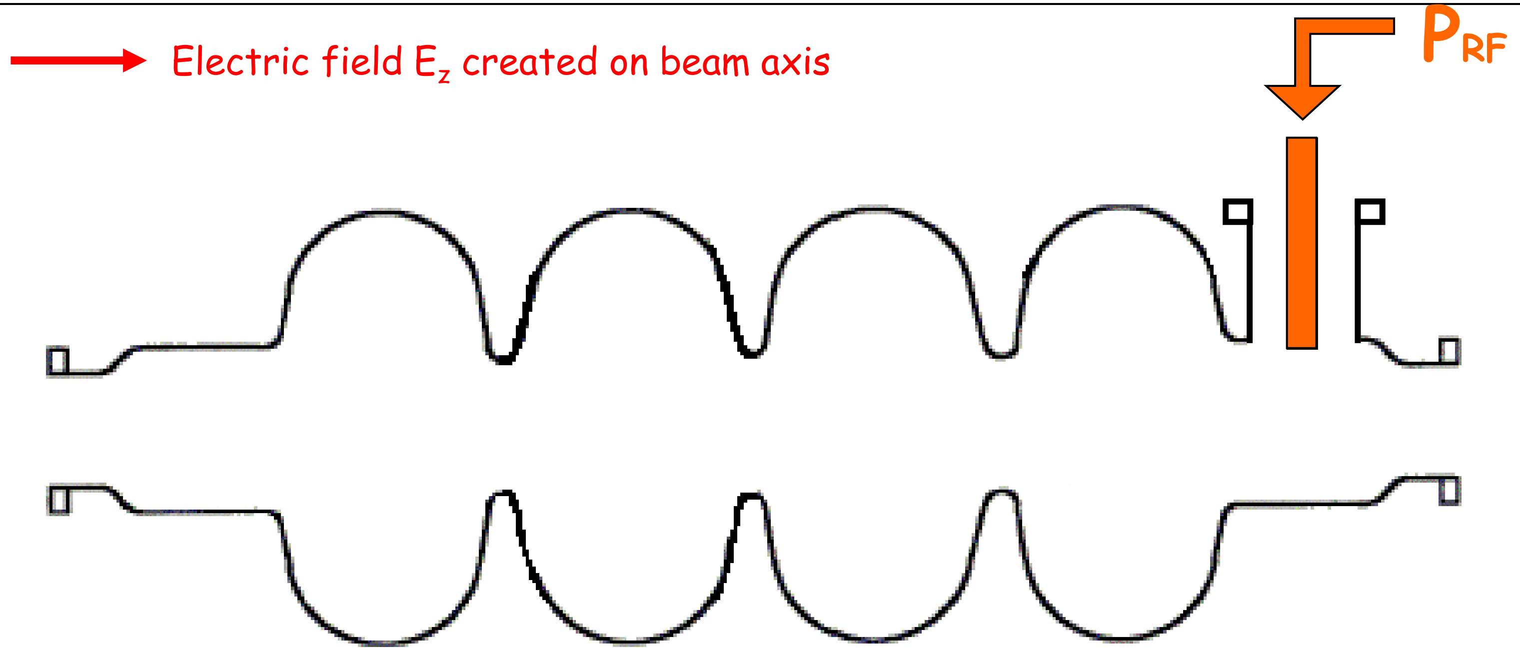
« Pill-box » cavity : TM 010 mode

Rectangular waveguide modes



2.4. Principle of an accelerating RF cavity (1)

(1) Creation of a radio-frequency (RF) electric field on beam axis, that can be used to accelerate charged particles



This electric field is time-varying, with RF period T

If f = cavity resonance frequency, the RF period is T = 1 / f

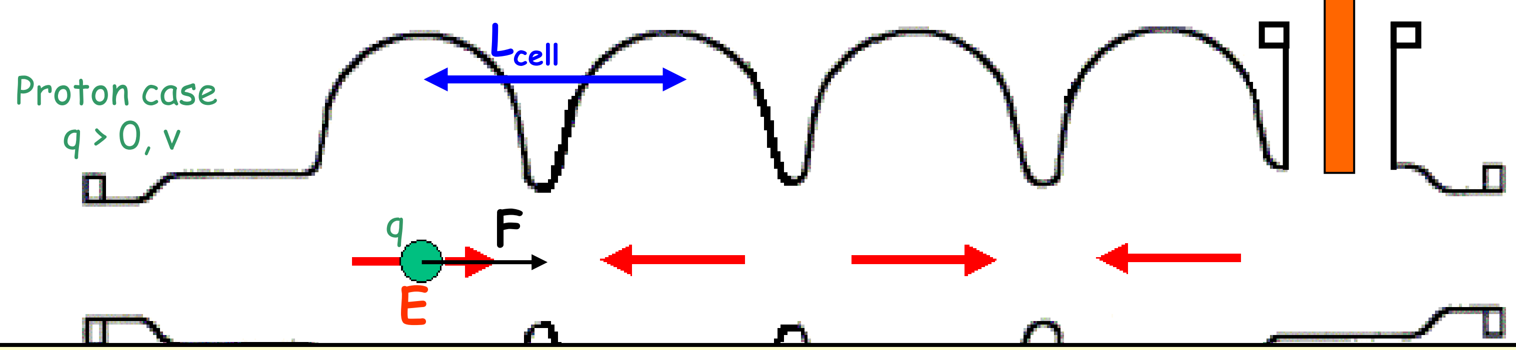
Ex : f = 700 MHz → T = 1,43 ns

Example of an elliptical cavity: accelerating mode « TM 010 -π »

2.4. Principle of an accelerating RF cavity (2)

(2) <u>A charged particle is arriving</u> : for an efficient acceleration, electromagnetic fields need to be correctly synchronised with the particle we want to accelerate

The force leading to acceleration $\vec{F} = q \, . \, \vec{E}$ needs to be correctly oriented



<u>Synchronization condition</u> :

The charged particle needs to travel from one cell to another cell in $T_{RF}/2$ ⇔ $\frac{L_{cell}}{v} = \frac{1}{2f}$

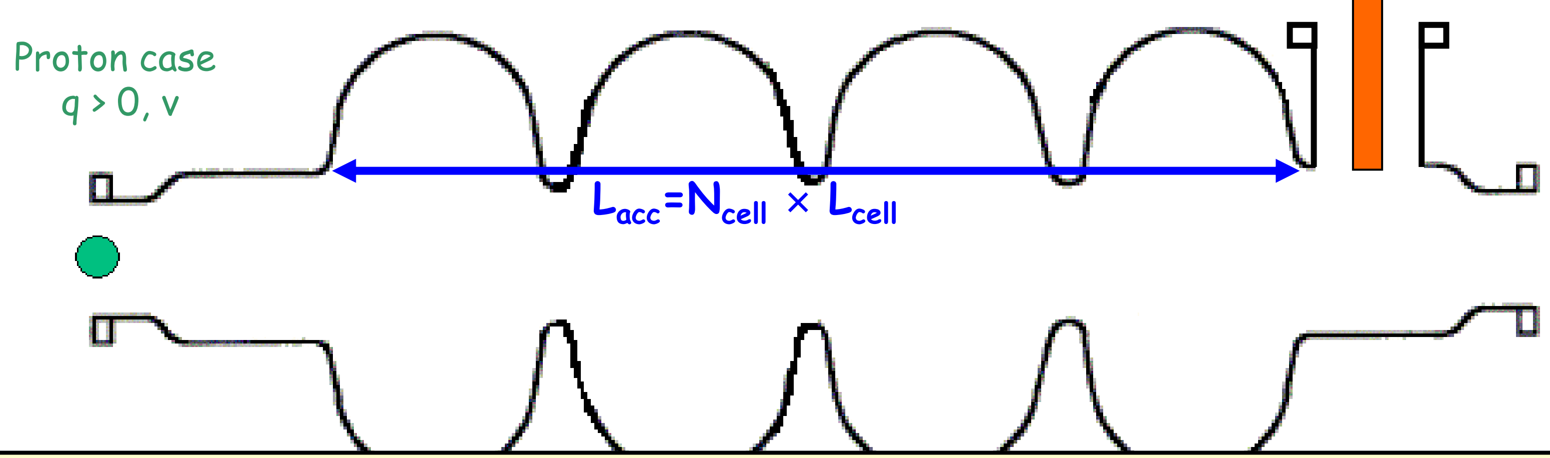
The cell length has therefore to verify: $L_{cell} = \frac{v}{2f} = \frac{\beta c}{2f}$ or $L_{cell} = \frac{\beta\lambda}{2}$

« The lower the particle speed, the lower L_{cell} and/or f »

Ex: f = 700MHz & $\beta = 1$ ⇒ L_{cell} = 21,4 cm (half for $\beta = 0{,}5$)

2.4. Principle of an accelerating RF cavity (3)

Example of an elliptical cavity: accelerating mode « TM 010 -π »



Energy given to the particle :

$$\Delta W = q \times \int_{t_{entrée}}^{t_{sortie}} \vec{E}.\vec{v}dt \quad \text{or} \quad \boxed{\Delta W = q \times E_{acc} \times L_{acc} \times \cos(\varphi_s)}$$

E_{acc}: accelerating field seen by the particle (for a particle with speed v)

L_{acc}: accelerating length of the cavity

φ_s : synchronous phase particle / RF wave

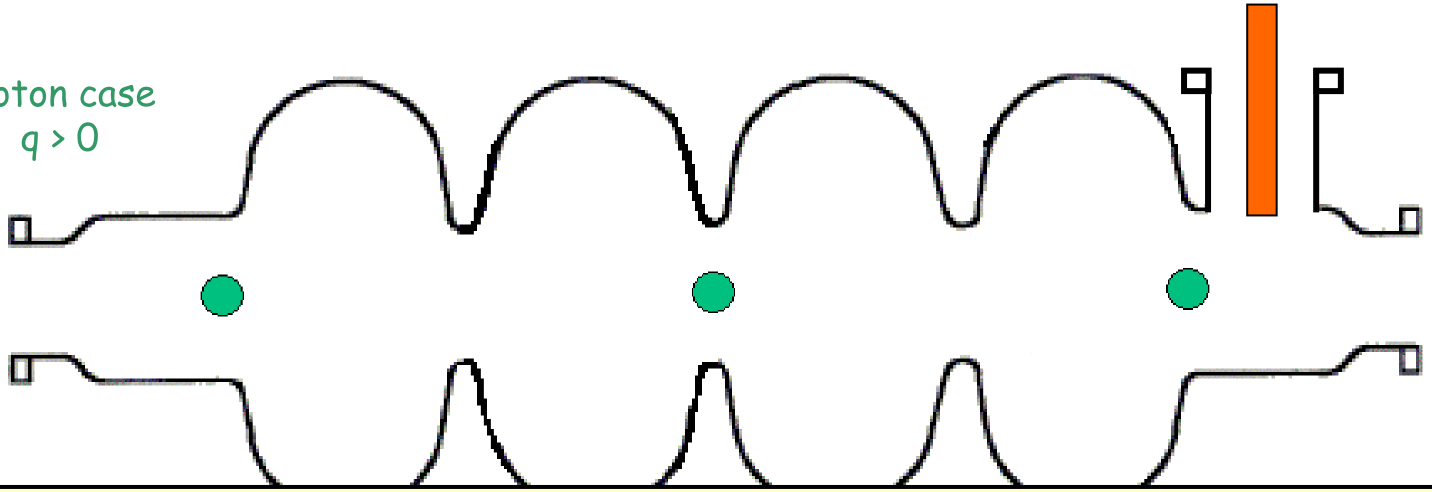
Ex : f = 700MHz ; 5-cell cavity β = 0.65 ; E_{acc}= 10MV/m ; φ_s = -30°

$\Rightarrow$ *Energy gained* : *ΔW = 1 eV × 10MV/m × 0,7 m × 0.87 =* ***6 MeV***

2.4. Principle of an accelerating RF cavity (4)

(3) Beam acceleration : particles need to be grouped in bunches, correctly synchronized with the RF frequency

Proton case
q > 0



The time left between 2 bunches has to fit with the RF period (or with a multiple of it)

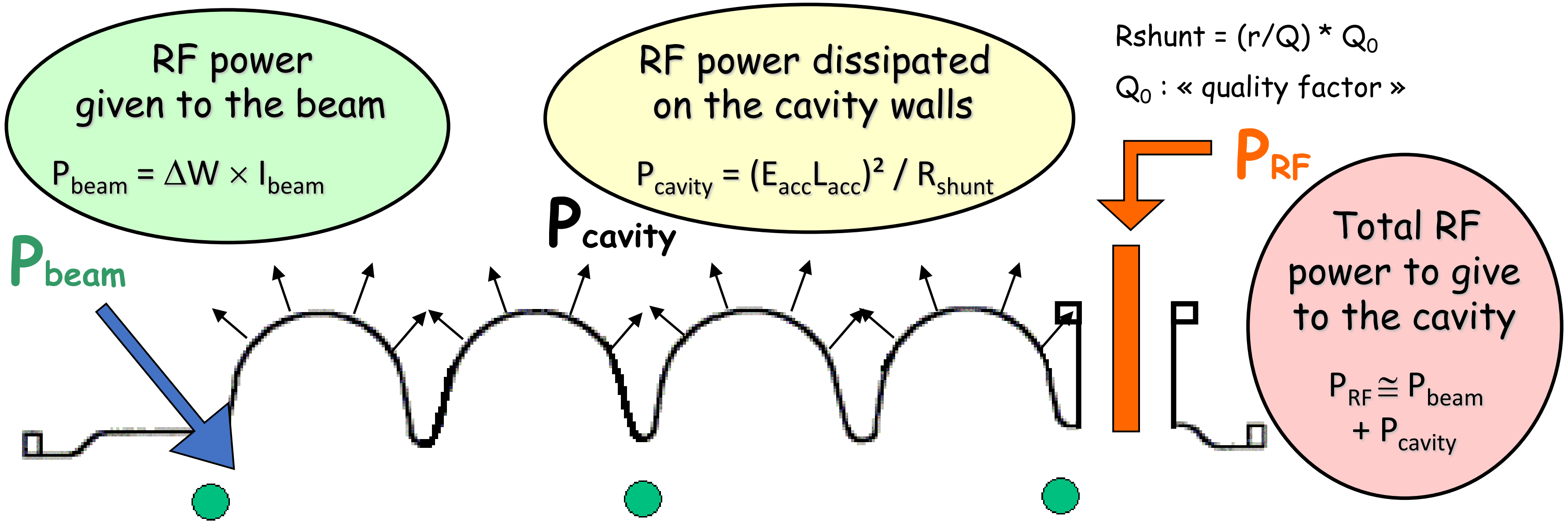
$T_{beam} = n\ T_{RF}$ (n=1,2,3…)

« The RF frequency of a resonant cavity needs to be an integer multiple of the repetition frequency of the beam it has to accelerate »

Ex: if f_{beam}=350 MHz (T_{beam}=2,86ns), thus the cavity needs to resonate with:

f = 350 MHz (T_{RF}=2,86ns), or f = 700 MHz (T_{RF}=1,43ns), or f = 1050 MHz (T_{RF}=0,95ns), etc.

2.5. Power balance in an accelerating cavity



Orders of magnitude *(cavity 700 MHz - β = 0.65 - 5 cells - 10MV/m - φ = -30° - proton beam 10 mA)*

SC cold cavity (Q_0 ~ 10^{10}) : P_{beam} = 6 MeV × 10 mA = **60 kW**

P_{cavity} ≈ **16 W** *to be evacuated at 4K (equiv. 4kW) or 2K (equiv. 10kW)*

Room-temp. cavity (Q_0 ~ 3.10^4) : P_{beam} = 60 kW also

P_{cavity} ≈ **5,5 MW !!!** ← *impossible in CW mode...!!!!*

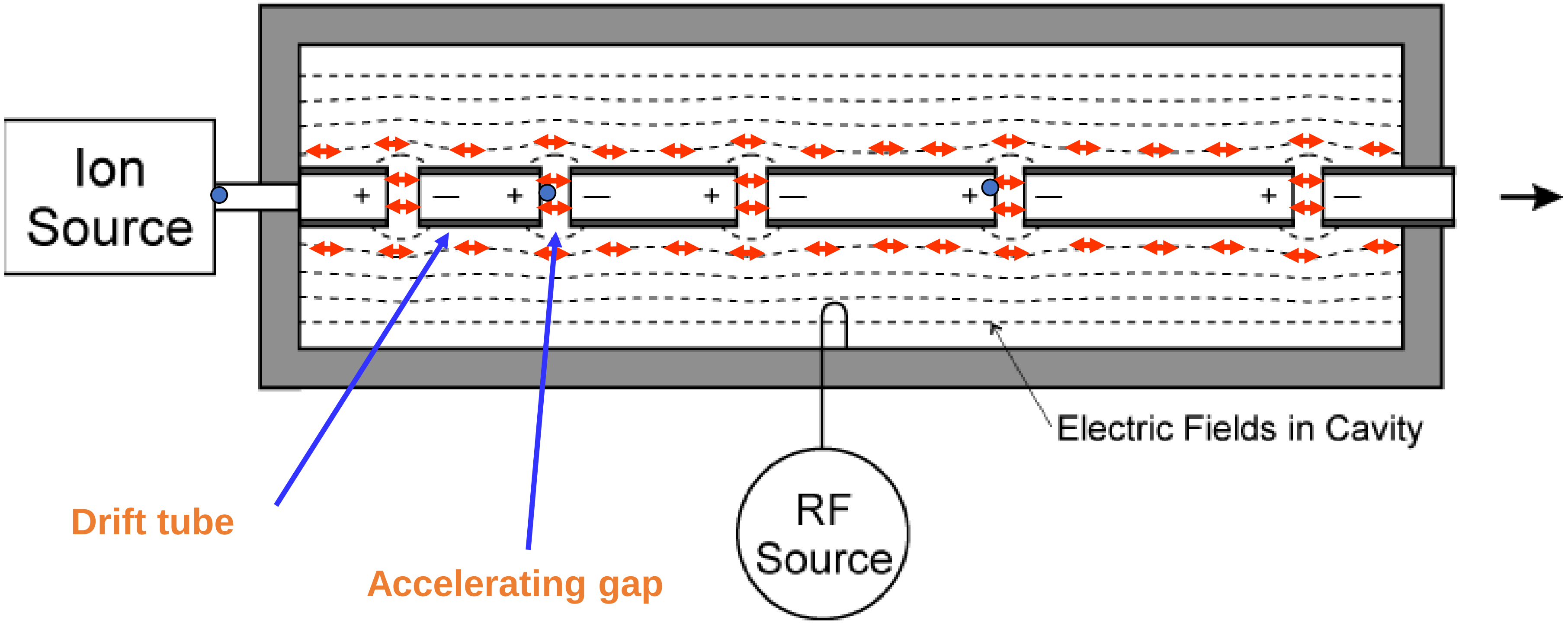
=> Need to limit the field or/and the RF duty cycle in room-temperature cavities

2.6. The first RF accelerating cavity

1946 : 1st proton linac by L.W. Alvarez

Alvarez DTL = long « pill-box » cavity where are inserted drift-tubes to hide the wrong polarity electric field from the beam; it operates on the $TM_{010\text{-}2\pi}$ mode

Possible thanks to new RF power sources developped during WW2 for military applications at f > 10MHz



2.7. RF structures Vs particle speed

Example :

Let's consider an electron (-1 eV)
& a proton (+1 eV) at rest, and let's give them
a net accelerating voltage of 10 MV

- **Energy gain** => ?????
- **Proton ?**
- **Electron ?**

2.7. RF structures Vs particle speed

Example :

Let's consider an electron (-1 eV)
& a proton (+1 eV) at rest, and let's give them
a net accelerating voltage of 10 MV

- **Energy gain** => 10 MeV in each case

- **Speed gain ????????**

2.7. RF structures Vs particle speed

Example :

Let's consider an electron (-1 eV)
& a proton (+1 eV) at rest, and let's give them
a net accelerating voltage of 10 MV

- **Energy gain** => 10 MeV in each case

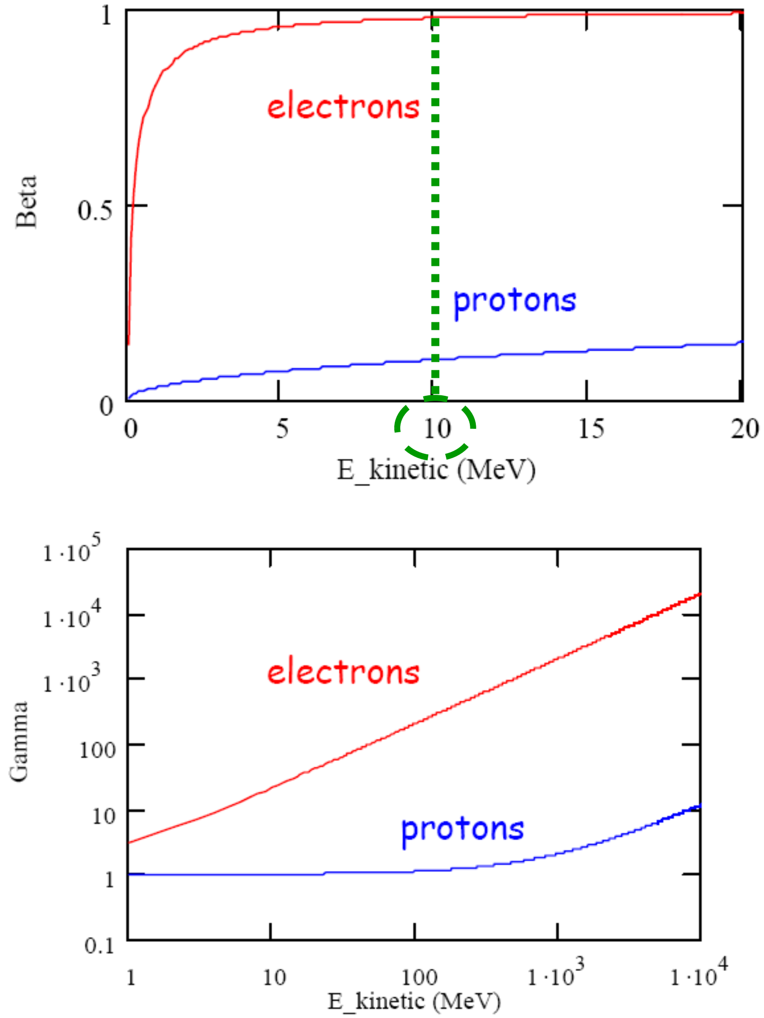
- **Speed gain**

electron $$\gamma_e = 1 + \frac{E_{cin}}{m_0 c^2} = 1 + \frac{10}{0.511} \approx 20.6$$

$$\beta_e = \sqrt{1 - \frac{1}{\gamma^2}} \approx 0.9988$$

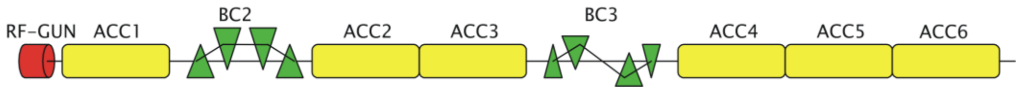
proton $$\gamma_p = 1 + \frac{10}{938.3} \approx 1.01 \qquad \beta_p \approx 0.145$$

The accelerator structures have to be designed to match the beta-profile of the particle we want to accelerate !!!

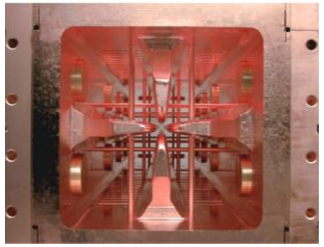


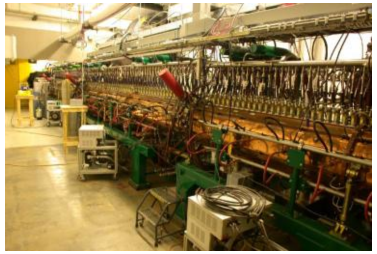
2.8. Electron Vs Proton linear accelerators

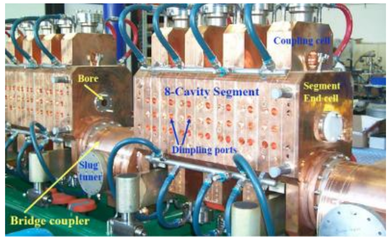
FLASH at DESY: 1.3 GeV electron linac, <100m



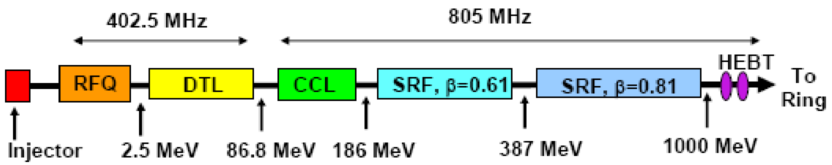
All RF cavities are practically similar:
1.3GHz β=1 elliptical cavities







SNS at ORNL: 1.0 GeV proton linac, >300m

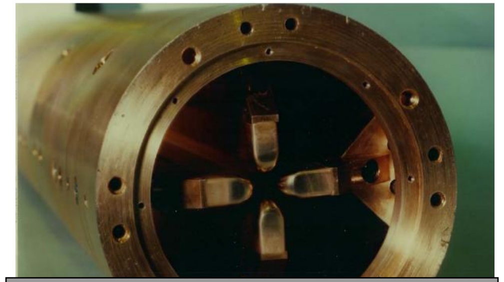


1 RFQ
6 DTL Tanks
4 CCL Modules

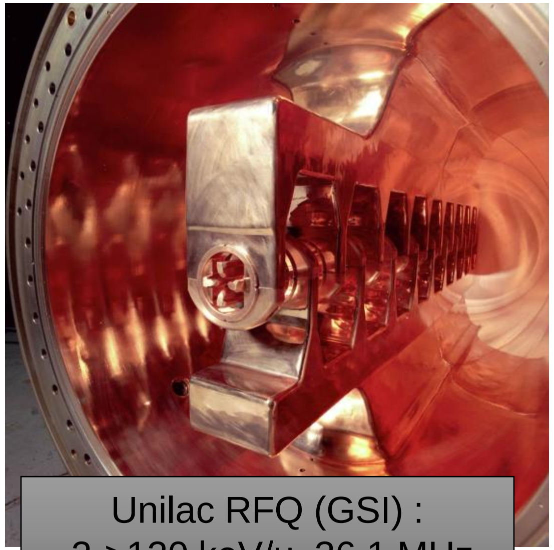
11 Medium-β Cryomodules - 3 Nb cavities each
12 High-β Cryomodules - 4 Nb cavities each

A lot of different RF structures…!!

2.9. Examples of RF accelerating cavities (1)

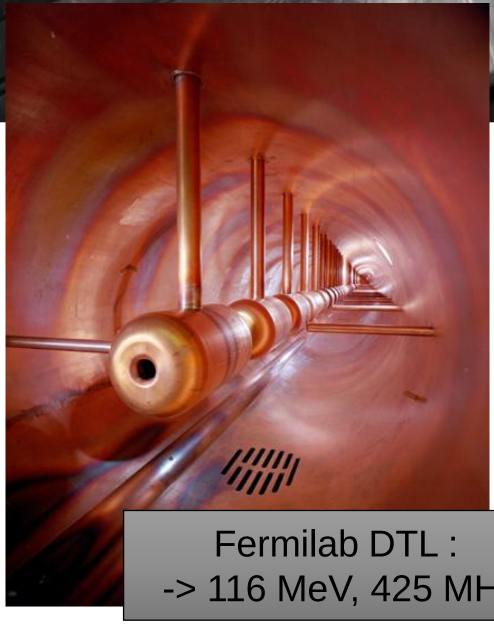


1st RFQ (LANL, 1980) :
p 100->640 keV, 425 MHz



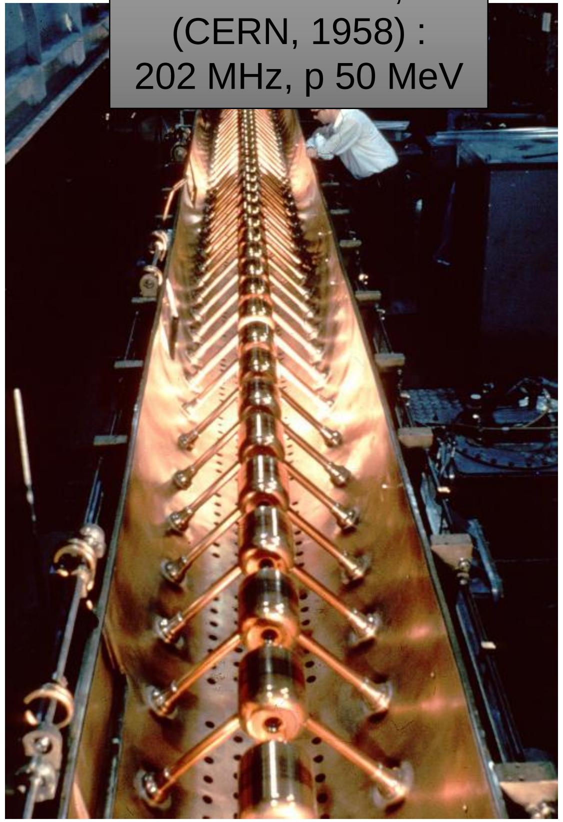
Unilac RFQ (GSI) :
2->120 keV/u, 36.1 MHz

The original Alvarez DTL (Berkeley, 1946) :
200 MHz, p 4->32 MeV, 12 m

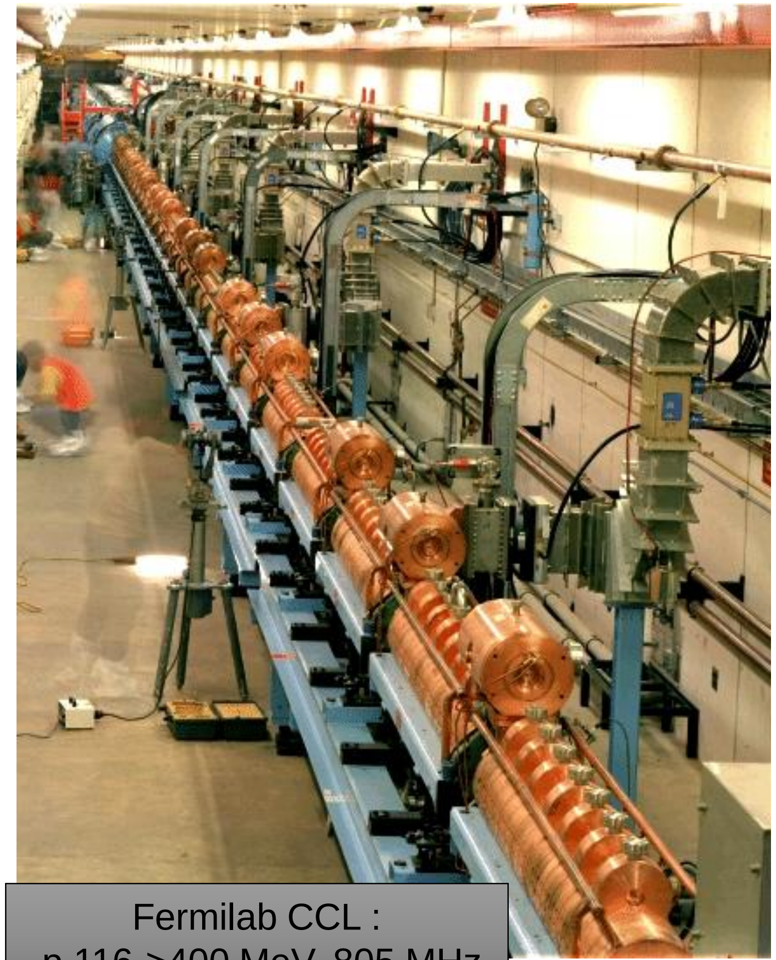


Fermilab DTL :
-> 116 MeV, 425 MHz

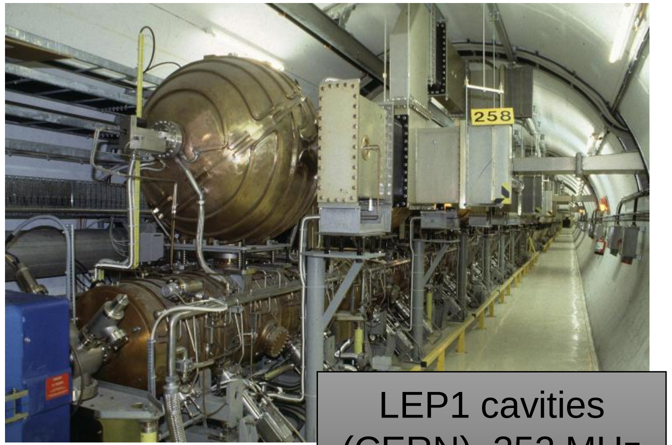
DTL Linac 1,
(CERN, 1958) :
202 MHz, p 50 MeV



2.9. Examples of RF accelerating cavities (2)

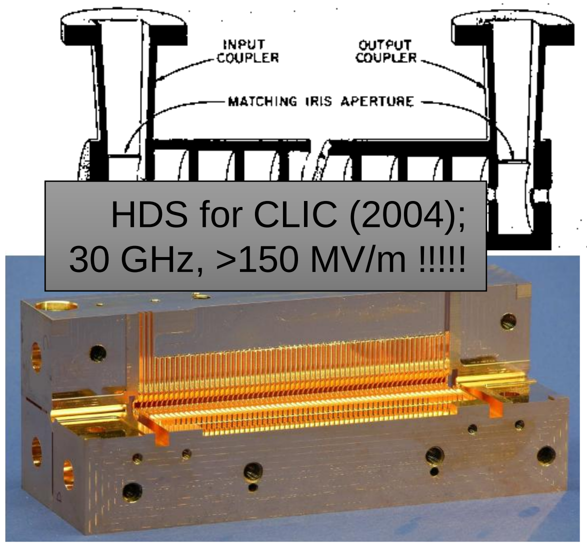


Fermilab CCL :
p,116->400 MeV, 805 MHz

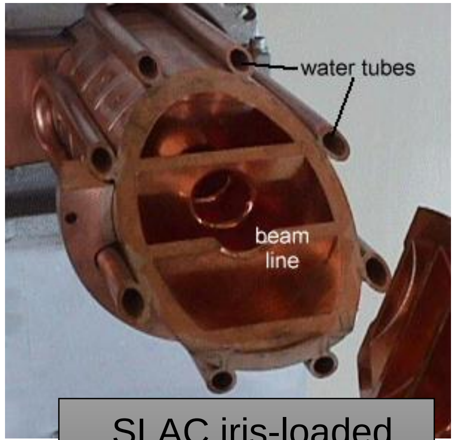


LEP1 cavities
(CERN), 352 MHz

Split-ring re-buncher
(TRIUMF), 35 MHz



HDS for CLIC (2004);
30 GHz, >150 MV/m !!!!!



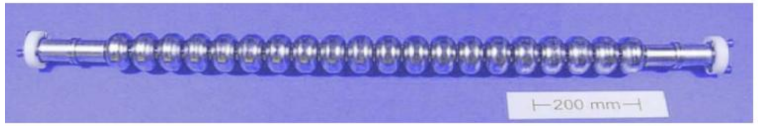
SLAC iris-loaded
structure, 2.86 GHz

2.9. Examples of RF accelerating cavities (3)

Niobium elliptical cavities: size Vs frequency

352 MHz
LEP

1300 MHz
TESLA

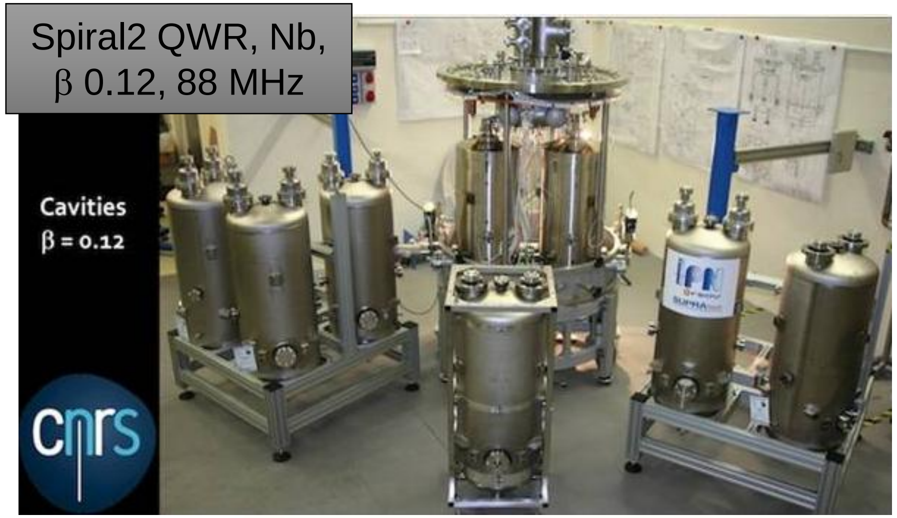


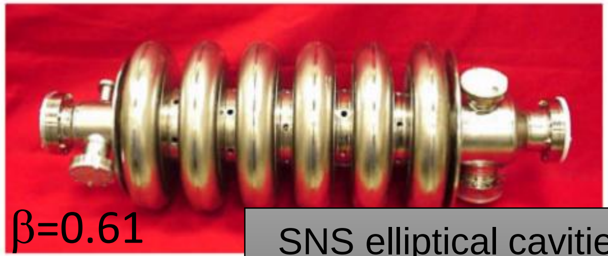
3000 MHz
S-DALINAC

ANL Spoke cavity, Nb,
β =0,4, 345 MHz

Nb CESR-III cavity
(Cornell), 500 MHz

Spiral2 QWR, Nb,
β 0.12, 88 MHz





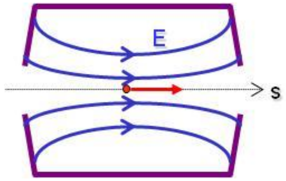
β=0.61

SNS elliptical cavities,
Nb, 805 MHz, 15MV/m

β=0.82

2.10. Energy gain model

The energy gained by a particle in a cavity is:



$$\Delta W = \int qEz(s)\cdot\cos(\phi(s))\cdot ds$$

$\phi(s)$ is the RF phase when particle is at s: $\phi(s) = \phi_0 + \omega\cdot t = \phi_0 + \frac{\omega}{c}\int_{s_0}^{s}\frac{ds}{\beta_z(s)}$

We define $V_0 = \int |Ez(s)|\cdot ds$ ⟶ The cavity **maximum voltage**

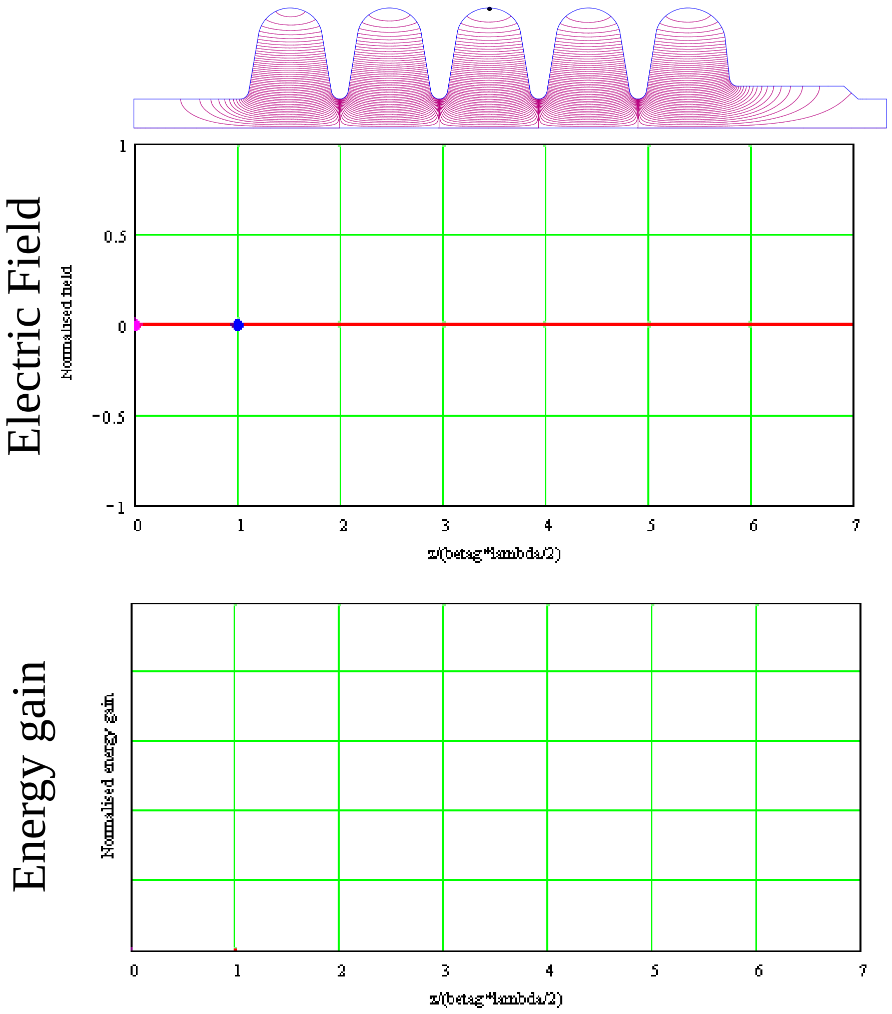
$$T = \frac{1}{V_0}\left|\int Ez(s)\cdot e^{j\phi(s)}\cdot ds\right|$$ ⟶ The **transit time factor** (= Ez field ratio seen by the particle)

$$\varphi_s = \arctan\left(\frac{\int Ez(s)\cdot\sin(\phi(s))\cdot ds}{\int Ez(s)\cdot\cos(\phi(s))\cdot ds}\right)$$ ⟶ The **synchronous phase** (= particle average phase with respect to RF)

Energy gain model: $\Delta W = |q|\cdot V_0\cdot T(\bar{\beta})\cdot\cos\varphi_s$

$V_0T(\beta)$ is what is often called the accelerating voltage V_{acc} (= $E_{acc}L_{acc}$)

2.11. Ex: Transit time factor in a multi-cell cavity

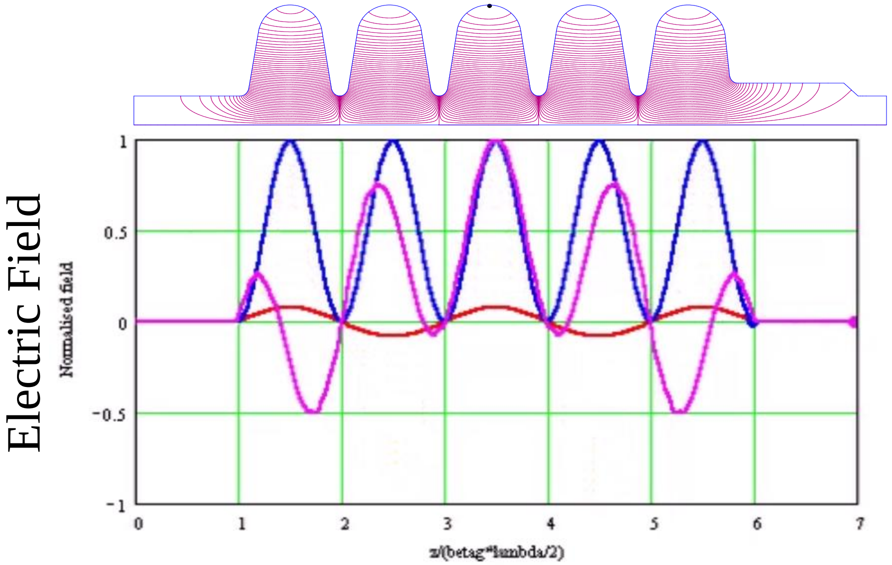


Let's suppose the synchronous phase is chosen for maximum acceleration, i.e. φ_s=0.
Thus $\Delta W = |q|\, V_0 \cdot T\left(\bar{\beta}\right)$

- Cavity Oscillating field (red)
- Field seen by a "synchronous" particle (blue)
- Field seen by a non synchronous (too fast!) particle (magenta)

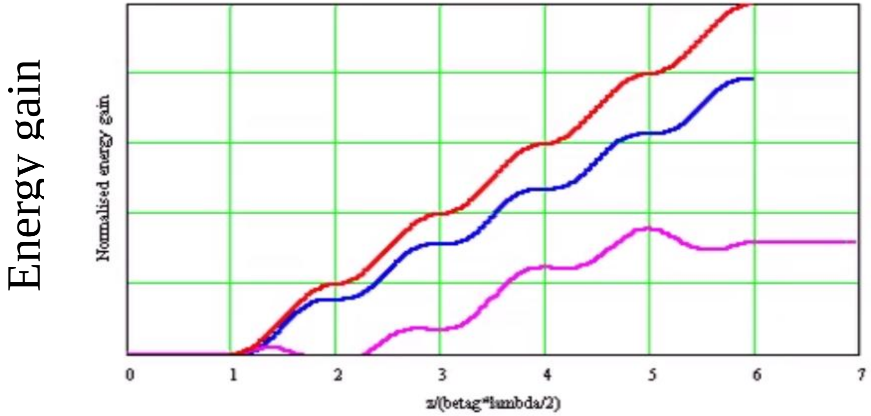
- Hypothetical max energy gain qV_0 (red)
- Energy gained by the "synchronous" particle : the transit time factor is T ~ 0.8 (blue)
- Energy gained by a non synchronous particle : the transit time factor is reduced (magenta)

2.11. Ex: Transit time factor in a multi-cell cavity



Let's suppose the synchronous phase is chosen for maximum acceleration, i.e. φ_s=0.
Thus $\Delta W = |q|\, V_0 \cdot T\left(\bar{\beta}\right)$

- Cavity Oscillating field
- Field seen by a "synchronous" particle
- Field seen by a non synchronous (too fast!) particle

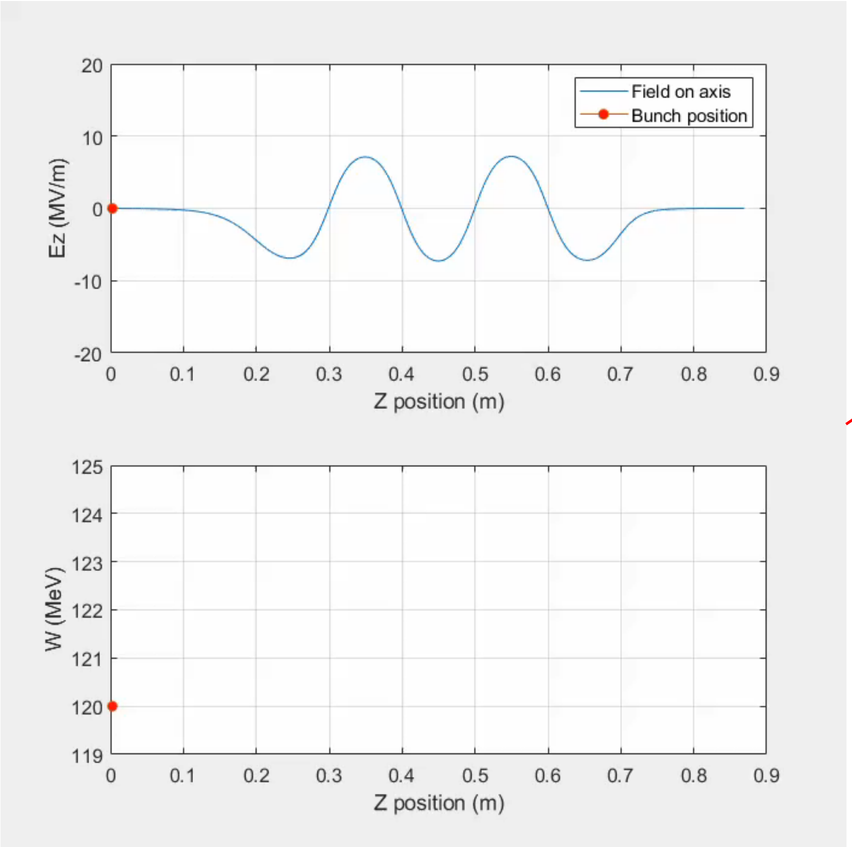


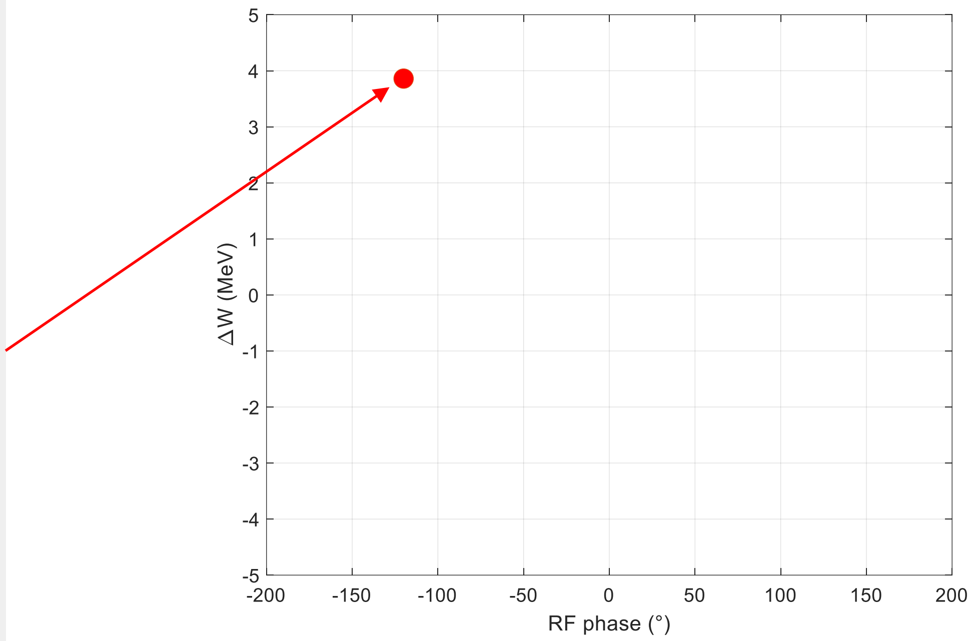
- Hypothetical max energy gain qV_0
- Energy gained by the "synchronous" particle : the transit time factor is T ~ 0.8
- Energy gained by a non synchronous particle : the transit time factor is reduced

Acceleration – Deceleration process

- Ex : 5-cell elliptical cavity (β_{opt} = 0.5) accelerating a proton beam
  - W_{in} = 120 MeV

$$\varphi_{RF} = -120°$$

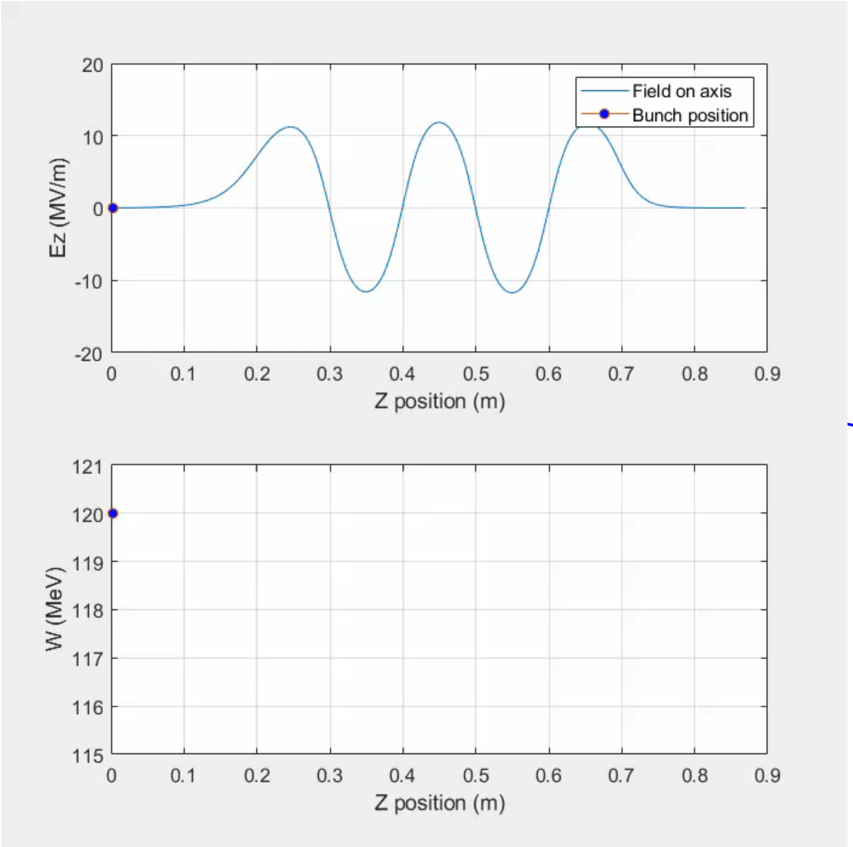


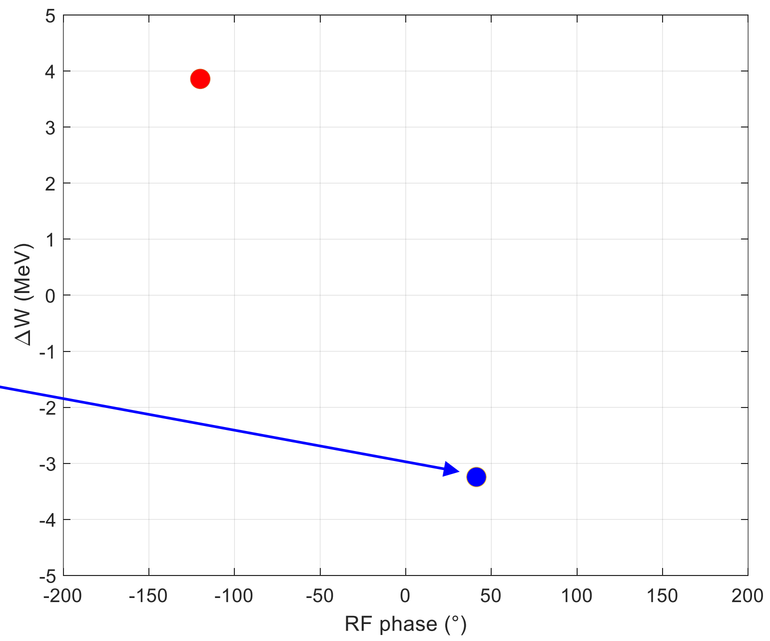


Acceleration – Deceleration process

- Ex : 5-cell elliptical cavity (β_{opt} = 0.5) accelerating a proton beam
  - W_{in} = 120 MeV

$$\varphi_{RF} = 41.3°$$





3. Synchronism & stability

3.2. Examples of cyclotrons (1)

1930: 1st cyclotron by E.O. Lawrence

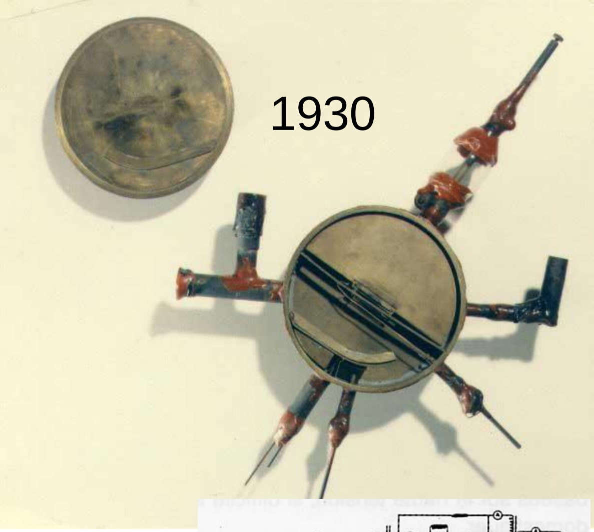
11 cm diameter, 80 keV

1931: 28 cm, 1 MeV

1932: 69 cm, 4.8 MeV

1939: 1.5 m, 19 MeV

…



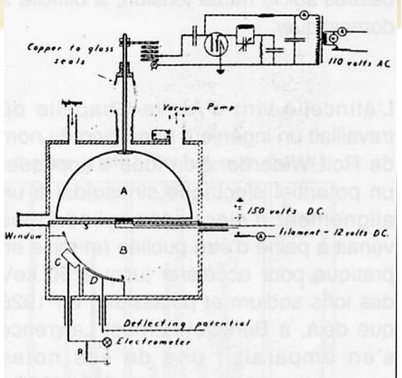
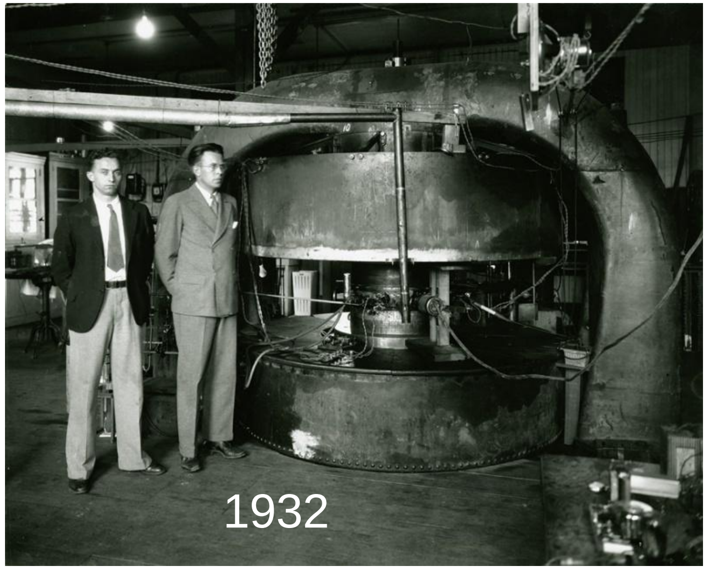
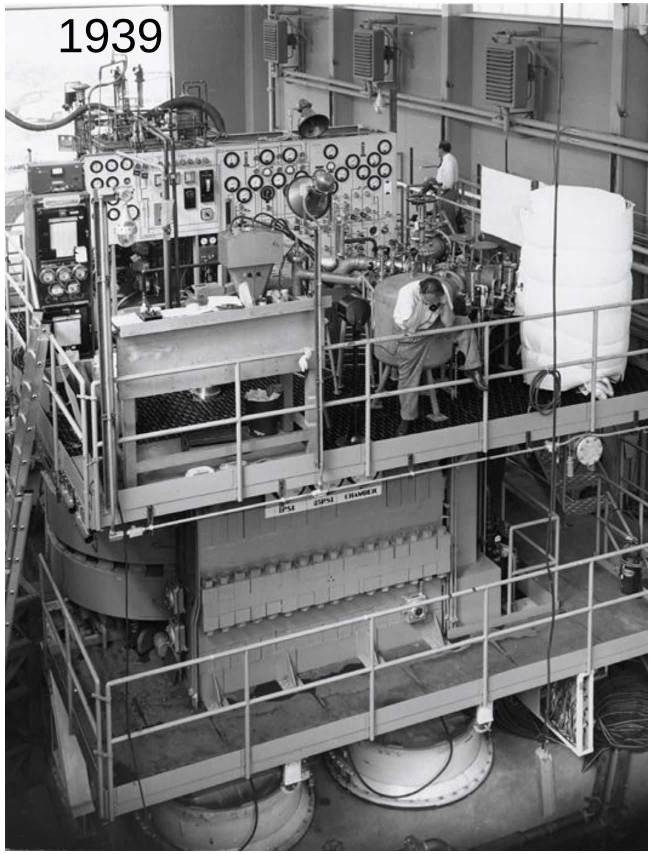


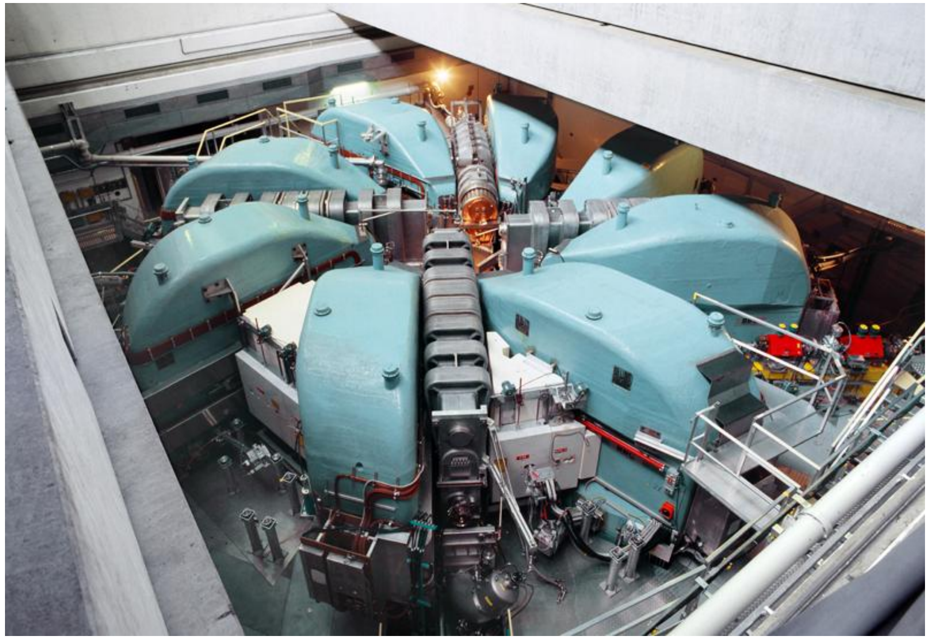
Figure 4
Schéma du premier cyclotron
Le diamètre du dee est de 5 pouces

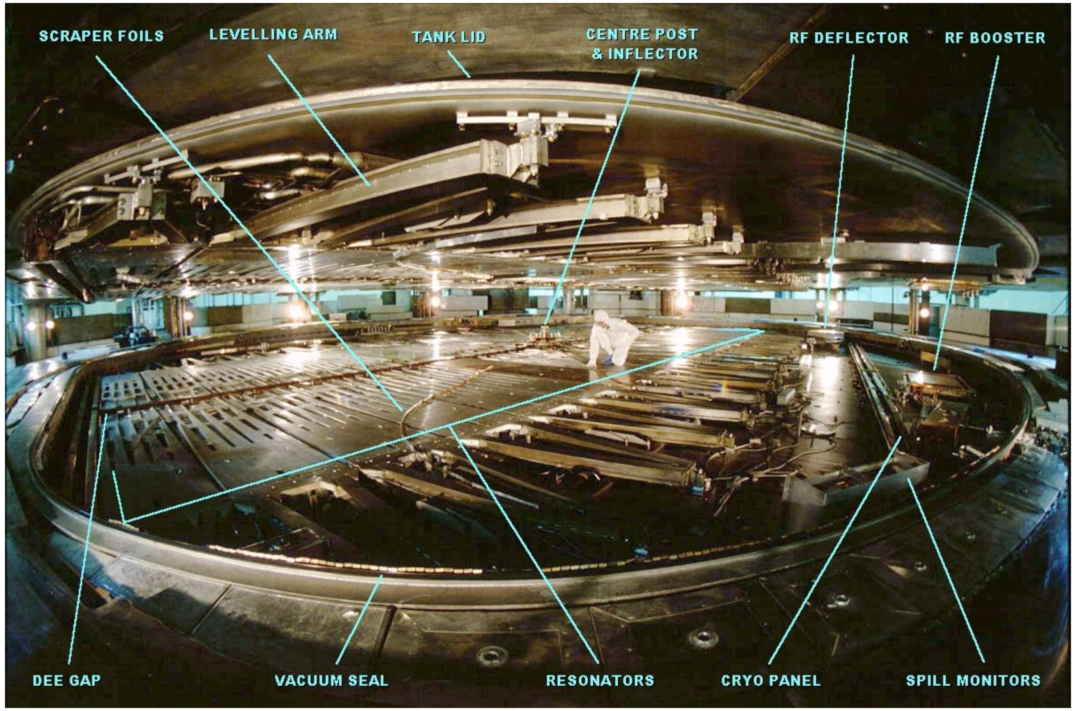




3.2. Examples of cyclotrons (2)

View of the most powerful cyclotron worlwide at PSI (H+, 590MeV, 1.3MW), f=50MHz, 1T (1974)





Inside view of the biggest cyclotron worldwide at TRIUMF (H-,520MeV) f=23 MHz, 0.6T (1974)

3.3. Synchronism in synchrotrons

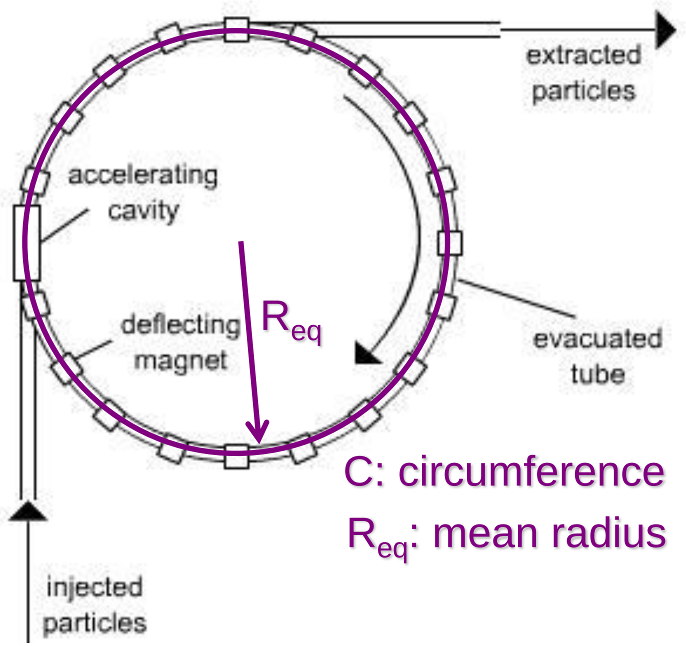
Synchrotron concept

-> Acceleration at constant curvature radius ρ

$B \neq cte$ $f_{RF} \neq cte$ $\rho = cte$

$$B = \frac{p}{\rho q} = \frac{\beta\gamma \cdot m_0 c}{\rho q}$$

-> B must be increased during acceleration



Condition for synchronism

$$h \cdot T_{RF} = \frac{C}{v} = \frac{2\pi R_{eq}}{\beta c} \rightarrow f_{RF} = \frac{\beta c}{2\pi R_{eq}} \cdot h$$

h: harmonic number

f_{rev}: revolution frequency

-> f_{RF} must be increased during acceleration

-> but when the particle is ultra-relativistic $\beta \simeq 1$, f_{RF} is kept practically constant

3.7. Particularity of circular machines

The criteria for stability is changed for synchroton machines because a particle with higher energy will not always arrive earlier

-> higher energy = higher speed β, so it travels faster (ok...!^^)

-> BUT higher energy = higher momentum p, so its trajectory is longer

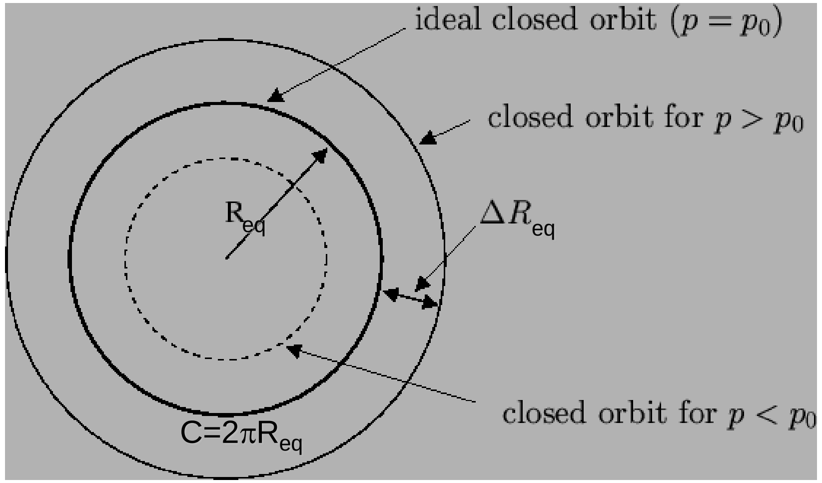
$$B\rho = \frac{p}{q}$$

To know if a particle with higher energy will arrive earlier or later, we look at the sign of **parameter η**

$$\eta = \frac{1}{\gamma^2} - \alpha$$

Where α is the momentum compaction

$$\alpha = \frac{dC/C}{dp/p} = \frac{dR_{eq}/R_{eq}}{dp/p}$$

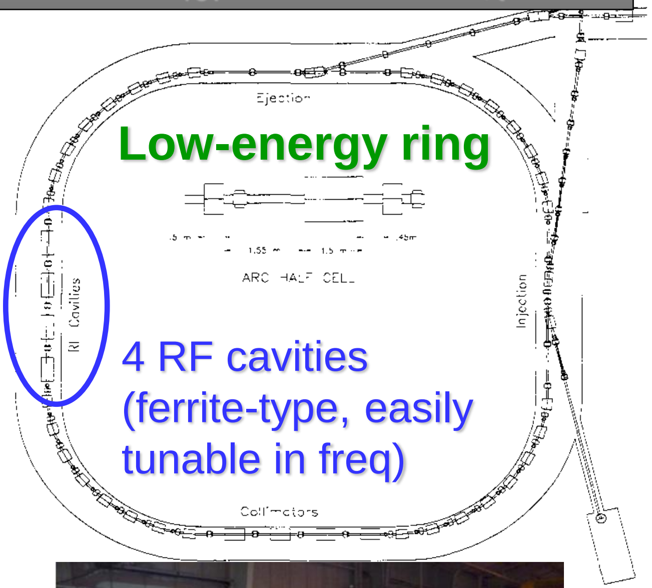


$$\eta = 0 \text{ when } \gamma = \frac{1}{\sqrt{\alpha}} = \gamma_t$$

γ_t = transition energy of the synchrotron

3.4. Examples of synchrotrons

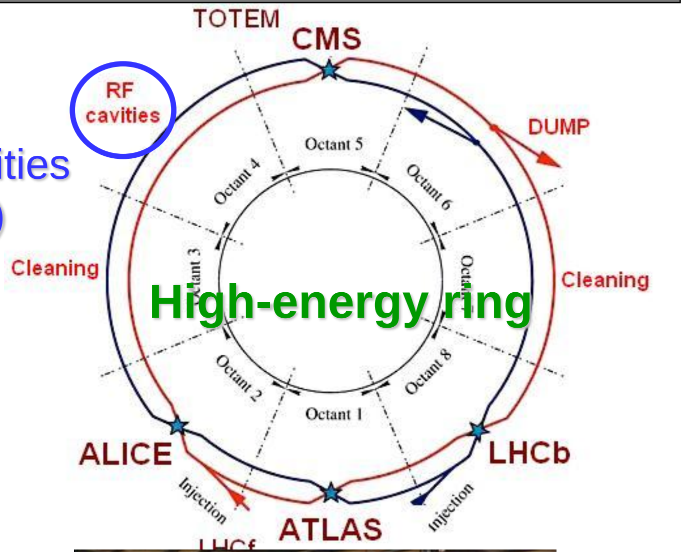
SNS accumulator ring (ORNL):
proton, 1 GeV (γ=2.06), 0.79T
C=248m, f_{rev}=1.058 MHz, γ_t=5.23




SNS RCS ferrite cavities
1.058 MHz (h=1), 25kV/m

LHC collider (CERN): proton,
450 GeV (γ=480) -> 7 TeV (γ=7460), 8.33T
C=26.7km, f_{rev}=11.25 kHz, γ_t=57.5

8 RF cavities
(SC-type)




LHC superconducting cavities
400.8 MHz (h=35640), 5 MV/m

3.5. Synchronism in linear accelerators (linacs)

Linac concept

-> A linac is a set of accelerating cells or of independent cavities along a linear path

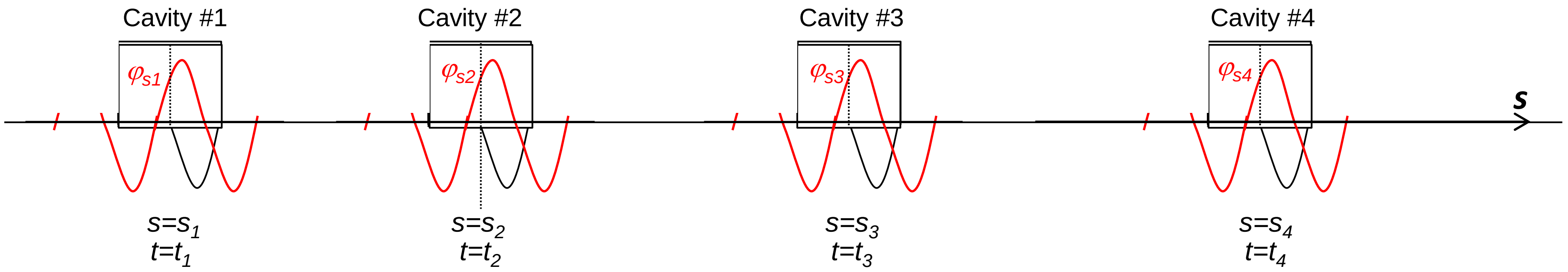
Condition for synchronism

If cavities phases are coupled (RFQ, DTL, multi-cell cavities...)
→ distances are adjusted for synchronism in the cavity design.

$$\frac{T_{RF}}{k} = \frac{L_i}{v_i} \quad \rightarrow L_i = \frac{\beta_i c}{k f_{RF}} = \frac{\beta_i \cdot \lambda_{RF}}{k}$$

L_i : distance between cell i & i+1
v_i: particle speed between cell i & i+1
k: defines the RF mode (mode 2pi/k)

If cavities phases are independent (between DTL tanks, SC cavities linac...)
→ Cavities are phased for synchronism according to the distance between cavities



4. Introduction to longitudinal dynamics

4.1. Longitudinal coordinates (φ,W)

Particles 6D vector coordinates

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ \varphi \\ W \end{pmatrix}$$

Transverse coordinates

- (x, y) are the particle transverse coordinates
- (x', y') are the particle transverse slopes $x' = \frac{dx}{ds}$ $y' = \frac{dy}{ds}$

Longitudinal coordinates

- φ is the phase at which the particle reaches position *s* ($\varphi = 2\pi . f_{RF} \times t$)
- W is the particle kinetic energy at position s

Particle phase varies like: $$d\varphi(s) = \frac{2\pi f_{RF}}{\beta(s)\,c} \cdot ds$$

phase relative to the RF phase in cavities

Particle energy varies like: $$dW(s) = qE_z(s) \cdot \overset{\cancel{\cos}}{\sin}(\varphi(s) - \varphi_{RF}) \cdot ds$$

Longitudinal electric field in cavities

4.2. Reference to synchronous particle

The synchronous particle is a perfect virtual particle that travels on the reference trajectory and is always in perfect synchronism with cavities' RF fields

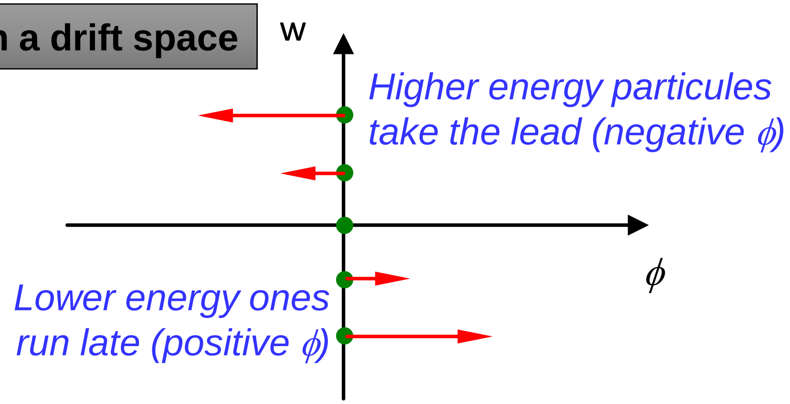
- The phase & energy of the synchronous particle are set by design: $\varphi_s(s)$, $W_s(s)$
- Beam particles are therefore referred to this synchronous particle

$$\begin{cases} \phi(s) = \varphi(s) - \varphi_s(s) \\ w(s) = W(s) - W_s(s) \end{cases}$$

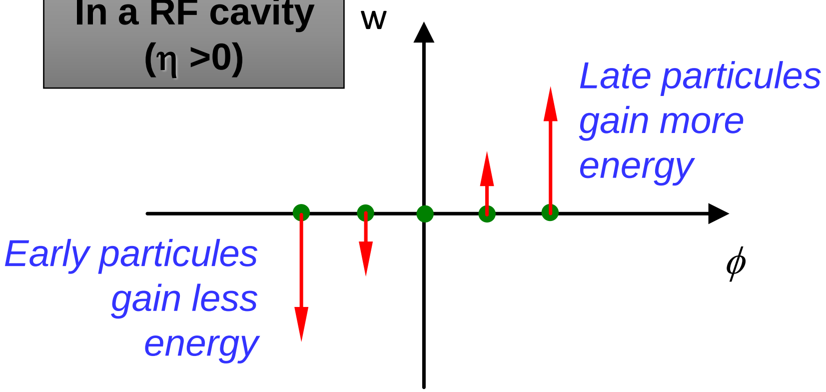
(ϕ,w) is the longitudinal phase space

Examples of trajectories in the longitudinal phase space

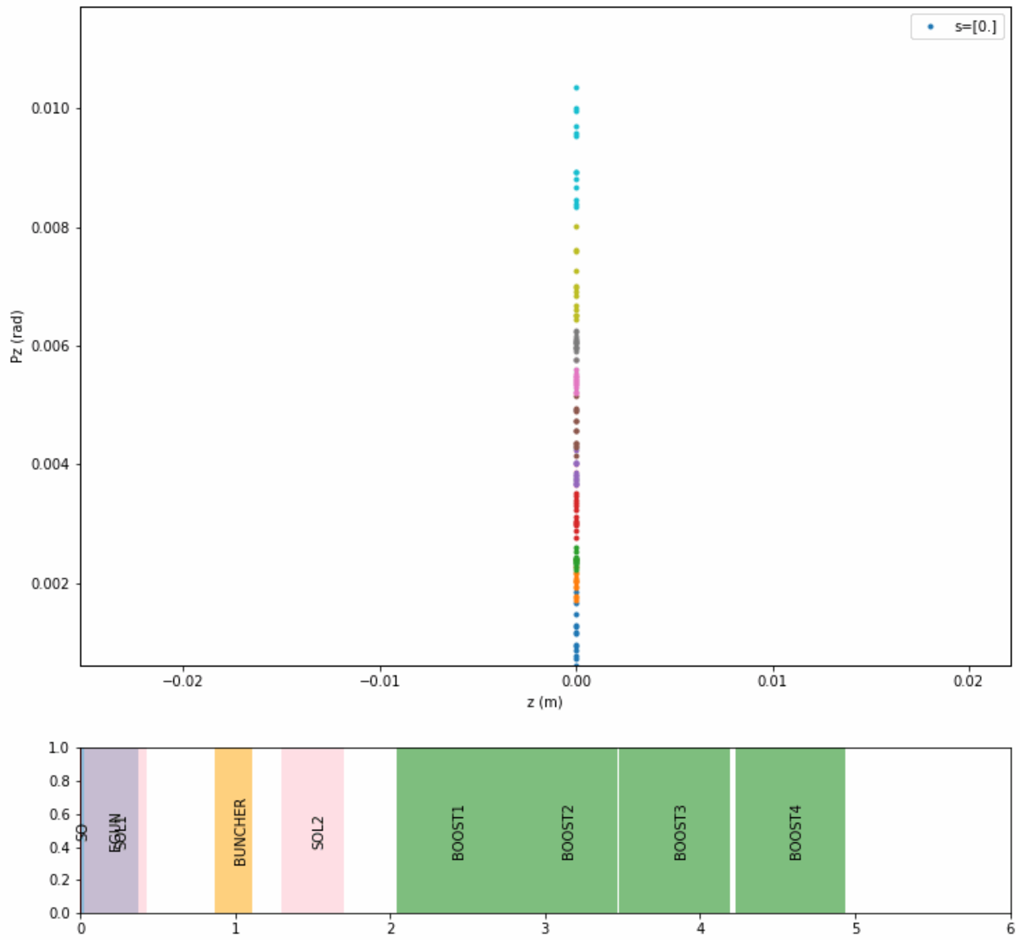
In a drift space



In a RF cavity (η >0)

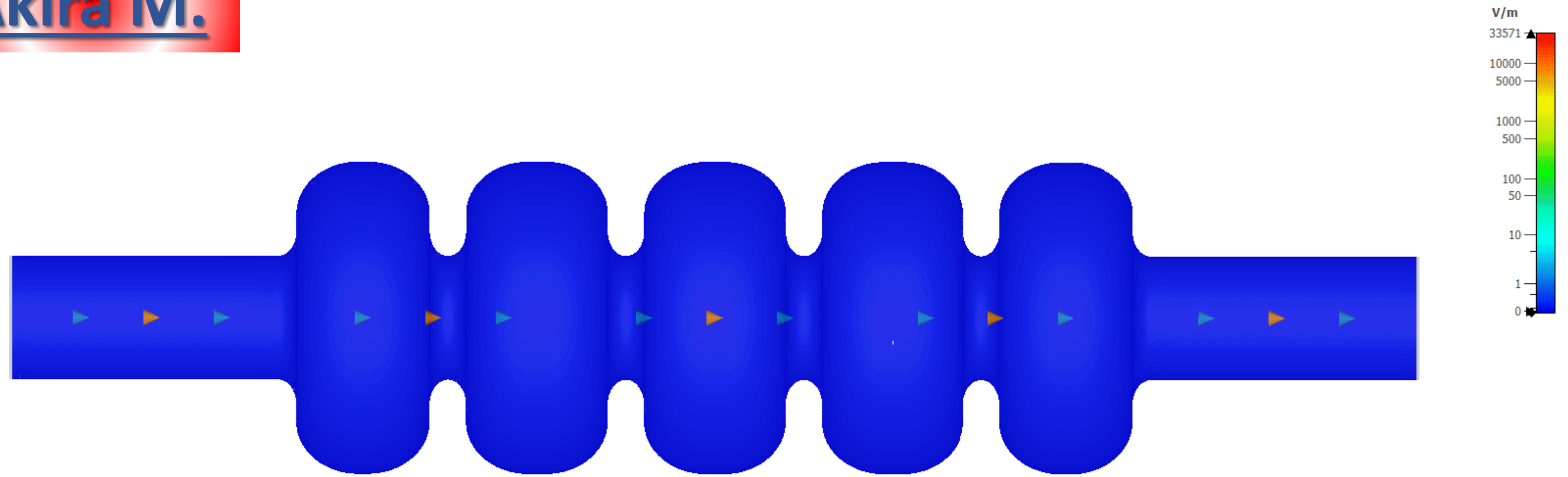


Example of longitudinal beam dynamics



Bonus (if time lef) : energy recuperation

Akira M.



e-field (t=0..end(0.02)) [pb]

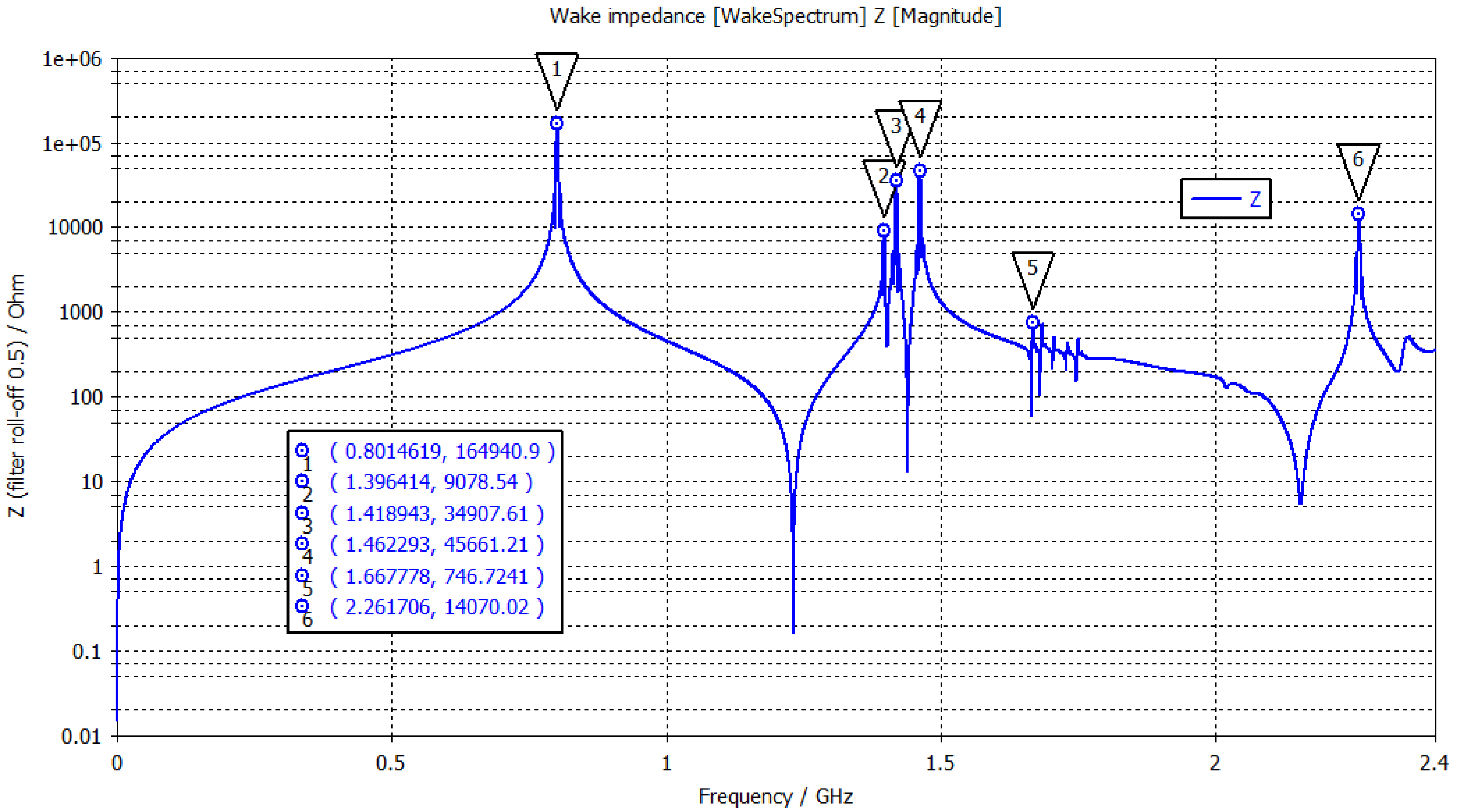
Component	Abs
Sample	1/456
Time	0 ns
Cross section	A
Cutplane at X	0.000 mm
Maximum on Plane (Sample)	0 V/m
Maximum (Sample)	0 V/m
Maximum (Global)	65584.9 V/m

$\sigma = 30\ mm \gg 3\ mm$ for demonstration

→ Frequency of $(2\pi c/\sigma)$ 1.6 GHz covers most of the interesting trapped modes

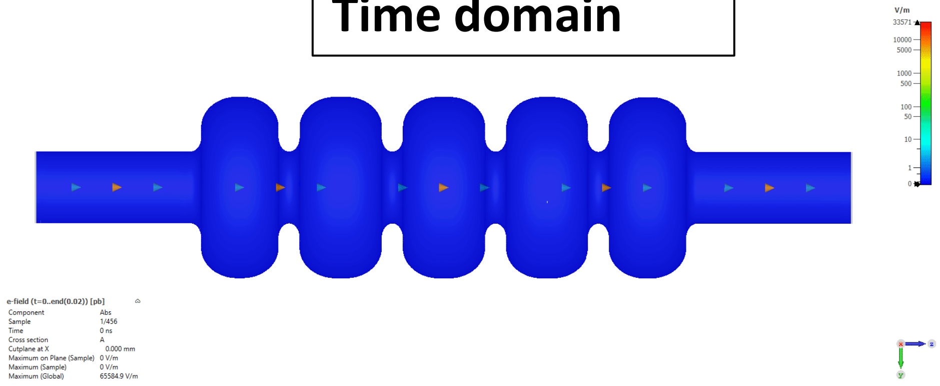
Beam goes through at exactly the center along the cavity beam axis

Bonus (if time lef) : energy recuperation

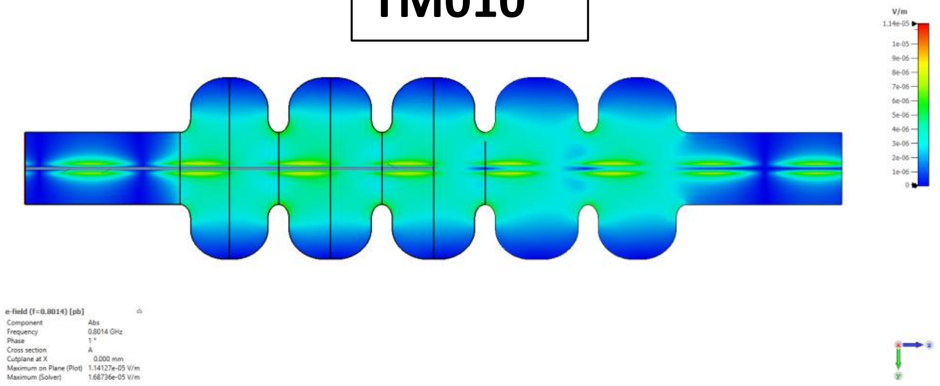


Bonus (if time lef) : energy recuperation

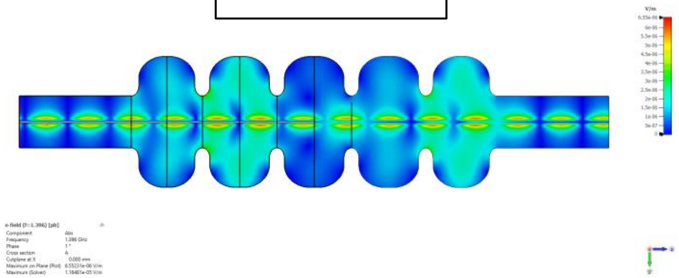
Time domain



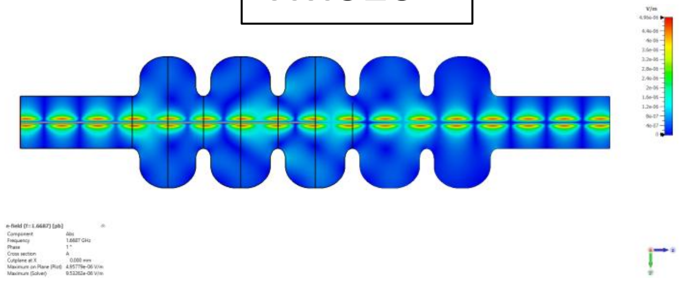
TM010



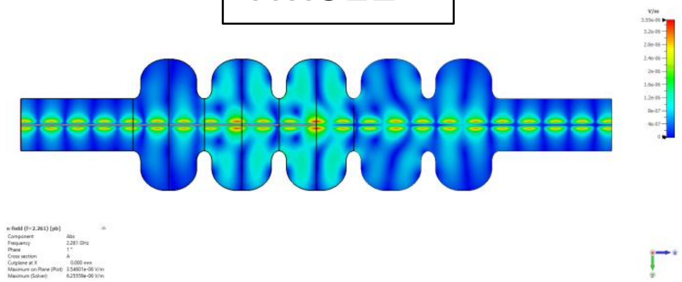
TM011



TM020



TM012



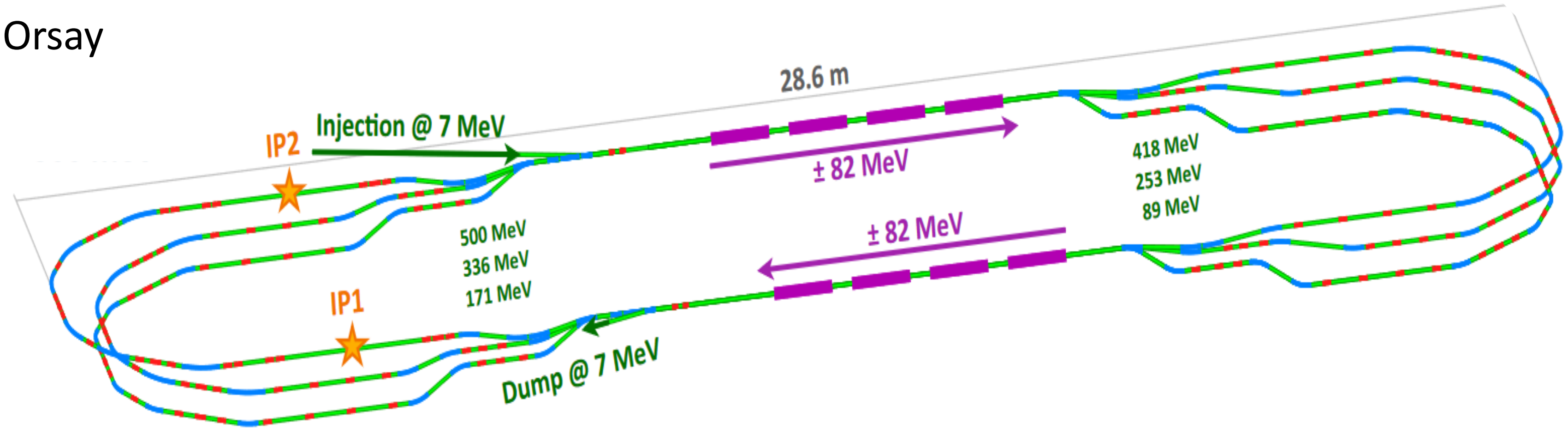
$$\vec{E}(t,r) = \sum_n c_n \vec{e}_n e^{i(\omega_n t + \phi_n)} \rightarrow U \propto \left|\vec{E}\right|^2 = \sum_n c_n^2$$

??? Only with volume integral

$$\left(\leftarrow \sum_n (\vec{e}_m^{\,*} \cdot \vec{e}_n) e^{i(\omega_n t + \phi_n - \omega_m - \phi_m)} = \delta_{nm} \right)$$

Energy recovery linacs

PERLE@Orsay



Thank you for listening

4.3. Synchrotron equations

Considering a continuous accelerating channel, the evolution of the variables ϕ and w can be derived:

$$\eta = \frac{1}{\gamma^2} - \alpha$$

$$\begin{cases} \dfrac{d\phi}{ds} = -\dfrac{2\pi \cdot \eta \cdot f_{RF}}{\beta_s{}^3 \gamma_s m_0 c^3} \cdot w \\ \dfrac{dw}{ds} = qE_0T \cdot (sin\varphi_s \cdot (cos\phi - 1) + cos\varphi_s \cdot sin\phi) \end{cases}$$

Synchrotron equations

= Equation of a non-linear oscillator

These equations can be rewritten as:

$$\begin{cases} \dfrac{d\phi}{ds} = -\dfrac{\partial H}{\partial w} \\ \dfrac{dw}{ds} = \dfrac{\partial H}{\partial \phi} \end{cases}$$

Particles follow curves for which *H*=Cst

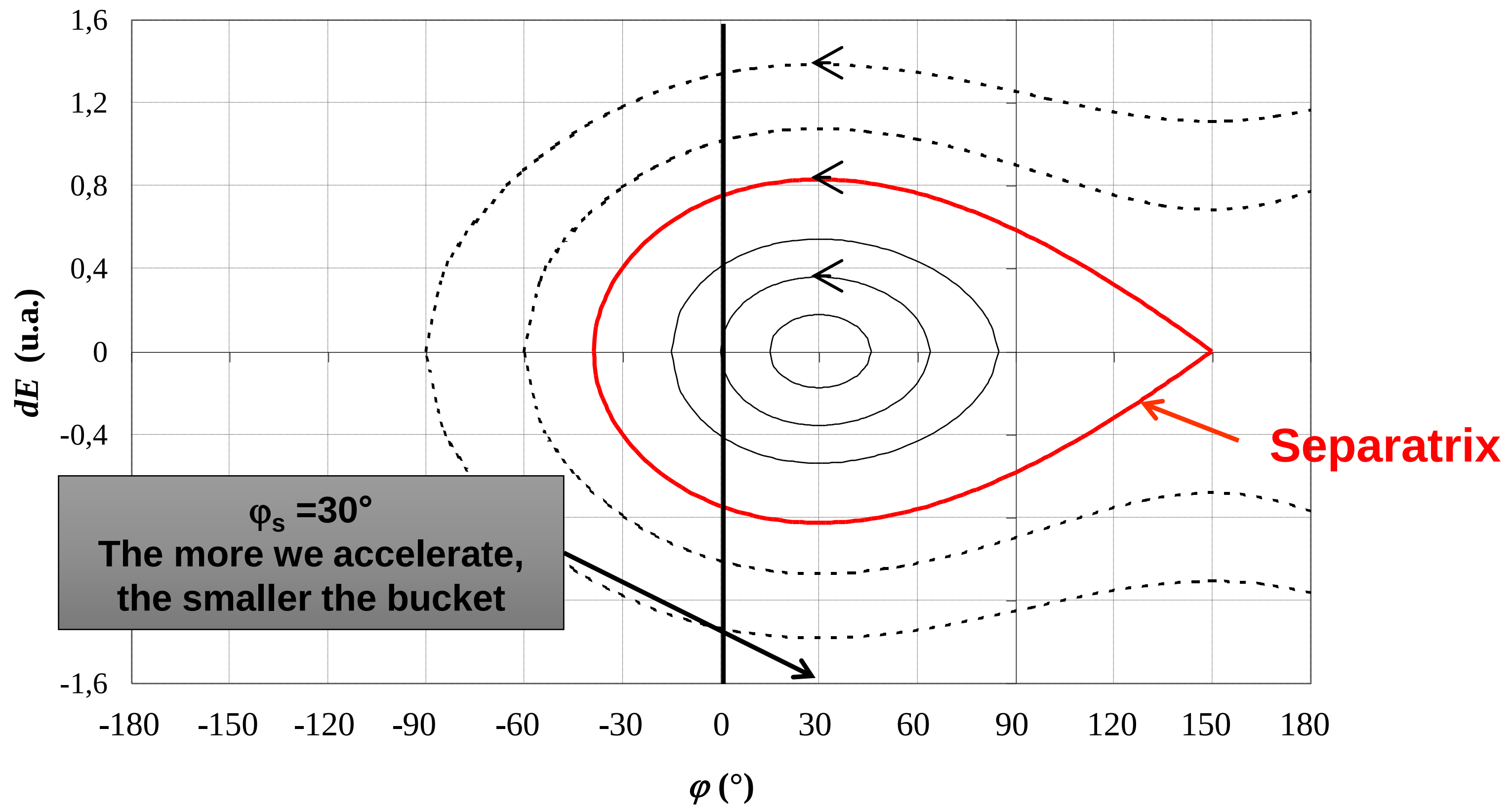
Where H is the longitudinal Hamiltonian of the movement, which is an <u>invariant</u>

$$H(\phi, w) = \frac{\pi \cdot \eta \cdot f_{RF}}{\beta_s{}^3 \gamma_s m_0 c^3} \cdot w^2 - qE_0T \cdot (sin\varphi_s \cdot (\phi - sin\phi) - cos\varphi_s \cdot (1 - cos\phi))$$

4.4. Trajectories in the longitudinal phase space (2)

Trajectories for a 30° synchronous phase (sinus « ring-convention ») **and η>0**

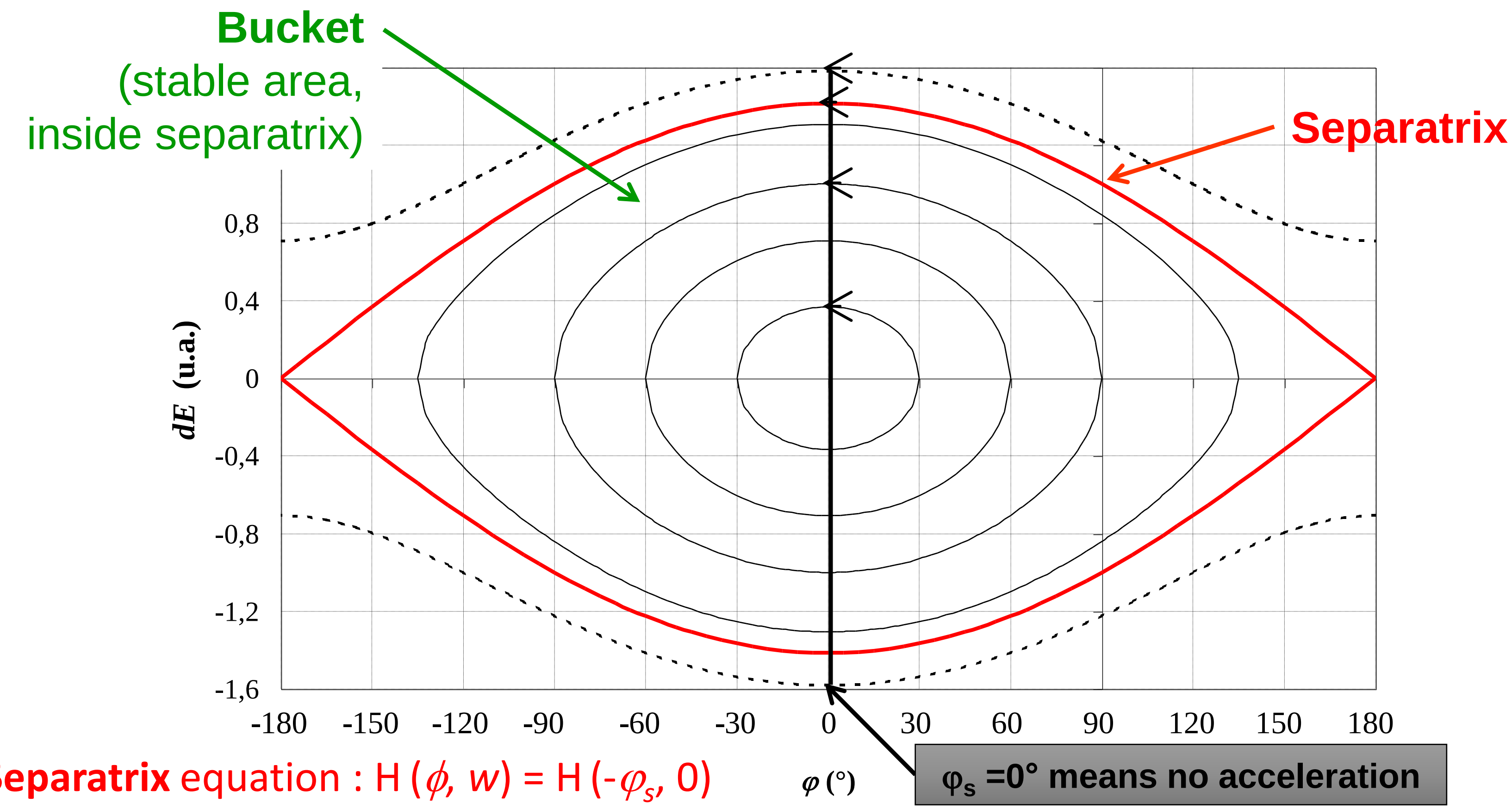
(-60° in cosinus « linac-convention »)



4.4. Trajectories in the longitudinal phase space (1)

Trajectories for a 0° synchronous phase (sinus « ring-convention ») **and η>0**

(-90° in cosinus « linac-convention »)

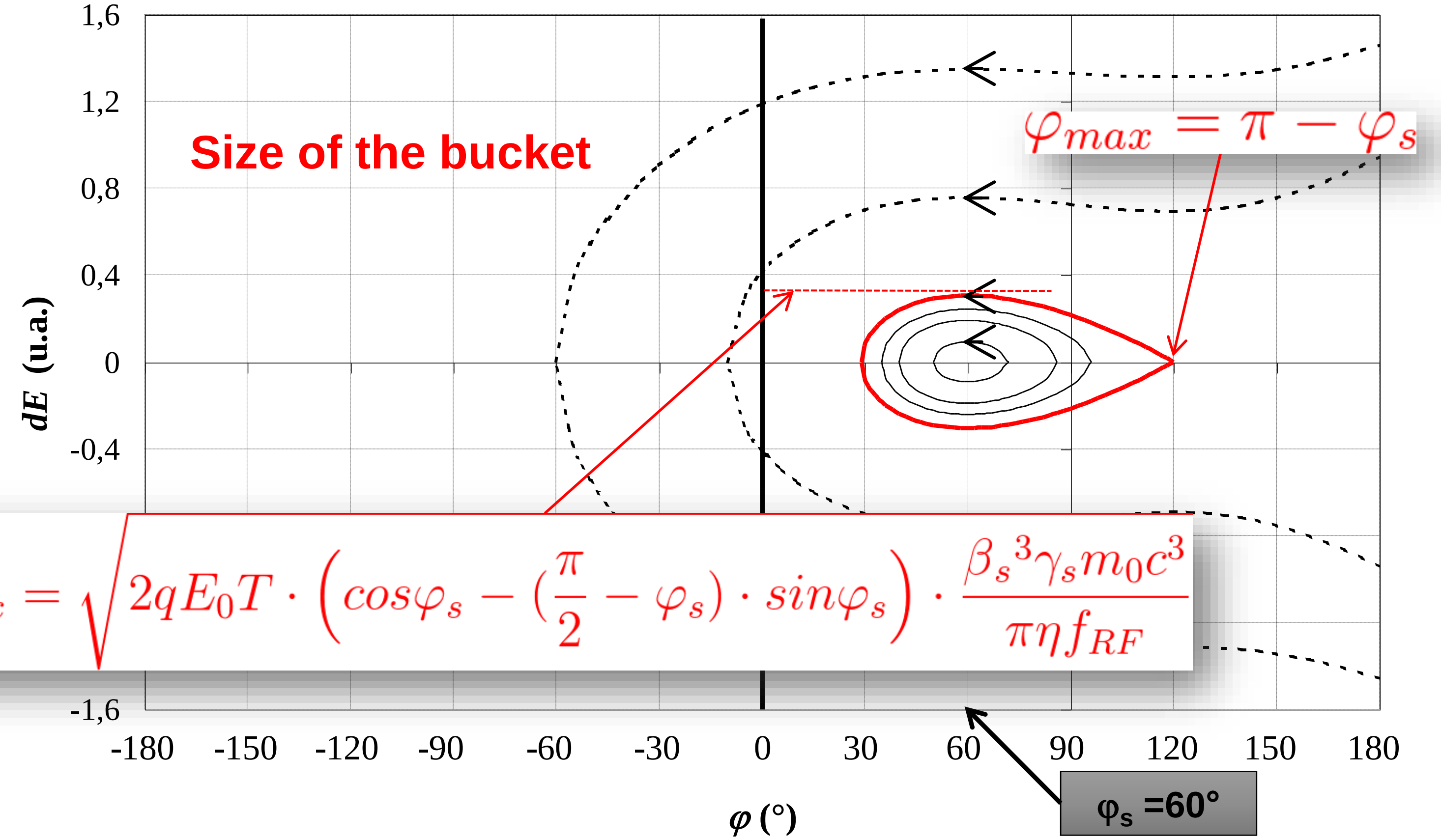


Separatrix equation : H (ϕ, w) = H (-φ_s, 0)

4.4. Trajectories in the longitudinal phase space (3)

Trajectories for a 60° synchronous phase (sinus « ring-convention ») **and η>0**

(-30° in cosinus « linac-convention »)



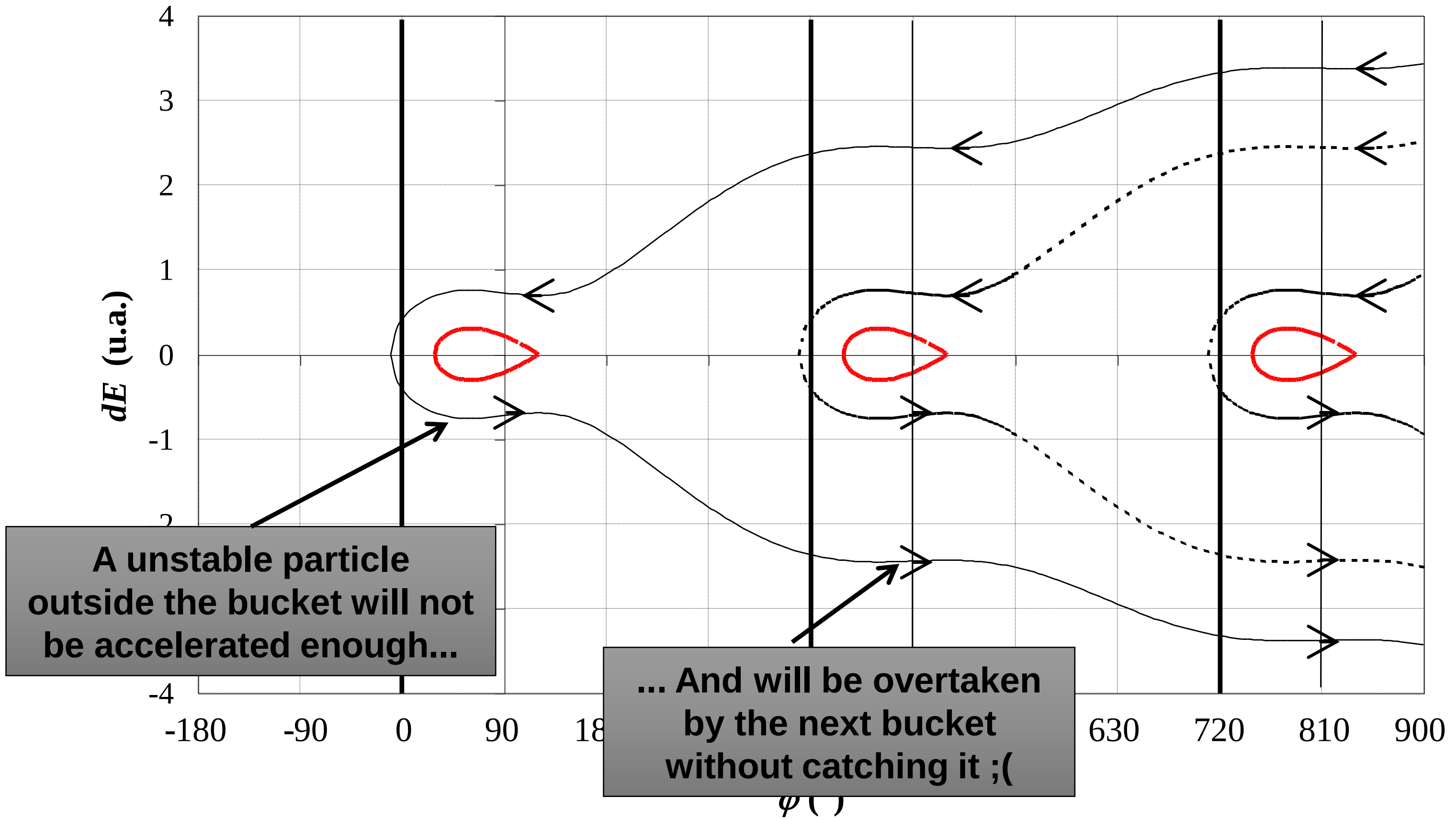
$$\varphi_{max} = \pi - \varphi_s$$

$$w_{max} = \sqrt{2qE_0T \cdot \left(cos\varphi_s - \left(\frac{\pi}{2} - \varphi_s\right) \cdot sin\varphi_s\right) \cdot \frac{{\beta_s}^3\gamma_s m_0 c^3}{\pi\eta f_{RF}}}$$

4.4. Trajectories in the longitudinal phase space (3)

Trajectories for a 60° synchronous phase (sinus « ring-convention ») **and η>0**

(-30° in cosinus « linac-convention »)



4.5. Treatment at small amplitudes

For small amplitudes (i.e. φ <<1: in the center of the bucket), the synchroton equations can be linearized:

$$\frac{d\phi}{ds} = -\frac{2\pi\cdot\eta\cdot f_{RF}}{{\beta_s}^3\gamma_s m_0 c^3}\cdot w = -\frac{\partial H}{\partial w} \qquad \frac{dw}{ds} = qE_0T\cdot\phi\cdot cos\varphi_s = \frac{\partial H}{\partial\phi}$$

It comes:

$$\frac{d^2\phi}{ds^2} = -\frac{2\pi\cdot\eta\cdot f_{RF}\cdot qE_0T\cdot cos\varphi_s}{{\beta_s}^3\gamma_s m_0 c^3}\cdot\phi$$

= Equation of a linear oscillator

-> Particles **oscillate around the synchronous particle** with angular frequency:

$$k_0 = \sqrt{\frac{2\pi f_{RF}\cdot\eta\cdot qE_0T\cdot cos\varphi_s}{{\beta_s}^3\gamma_s m_0 c^3}}$$

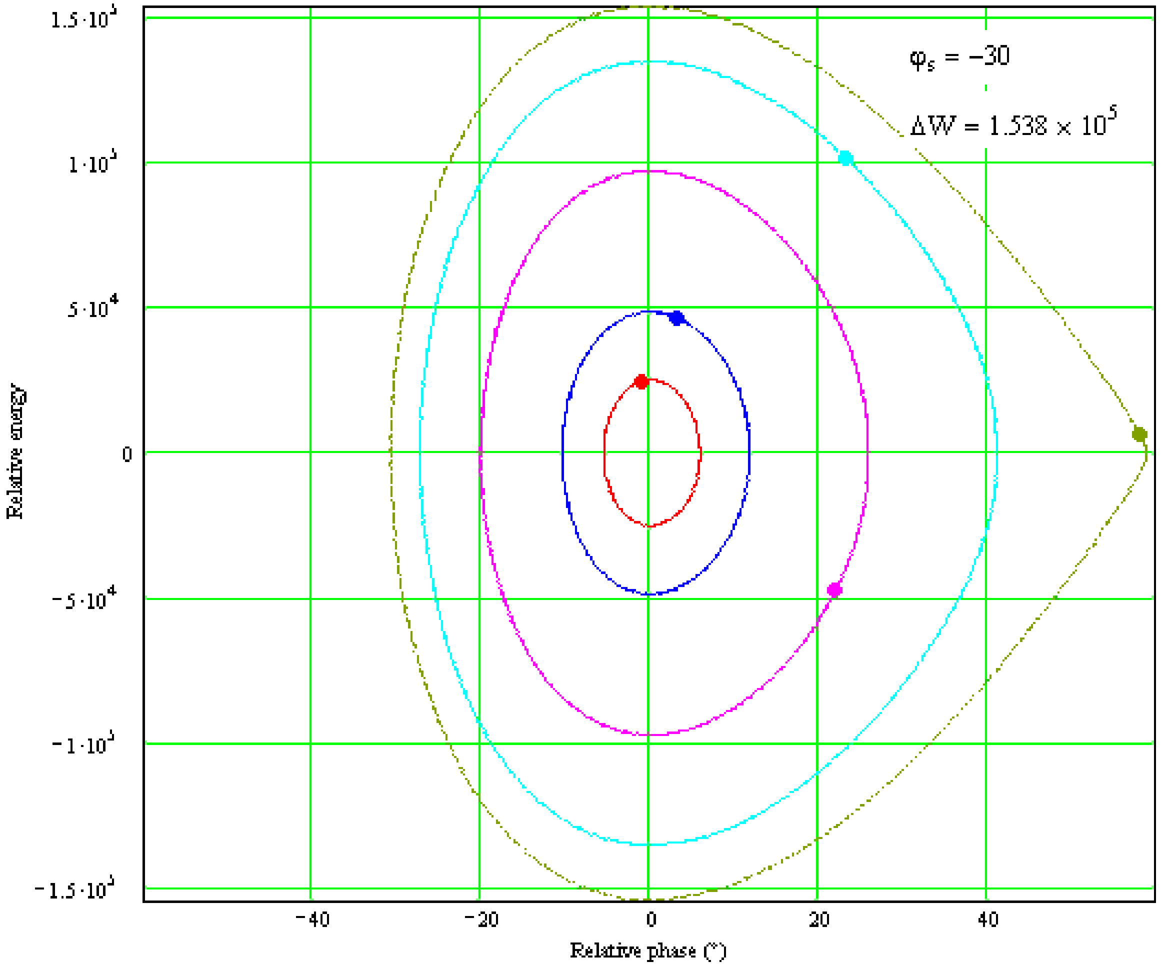
Synchrotron angular frequency

or synchrotron phase advance

-> Curves for which *H* = Cst become **ellipses** in the longitudinal phase space (ϕ,w)

$$\frac{\pi\cdot\eta\cdot f_{RF}}{{\beta_s}^3\gamma_s m_0 c^3}\cdot w^2 + \frac{qE_0T\cdot cos\varphi_s}{2}\cdot\phi^2 = Cte$$

4.6. Synchroton oscillations (at constant βγ)



1st order matrix model is valid for φ<<1

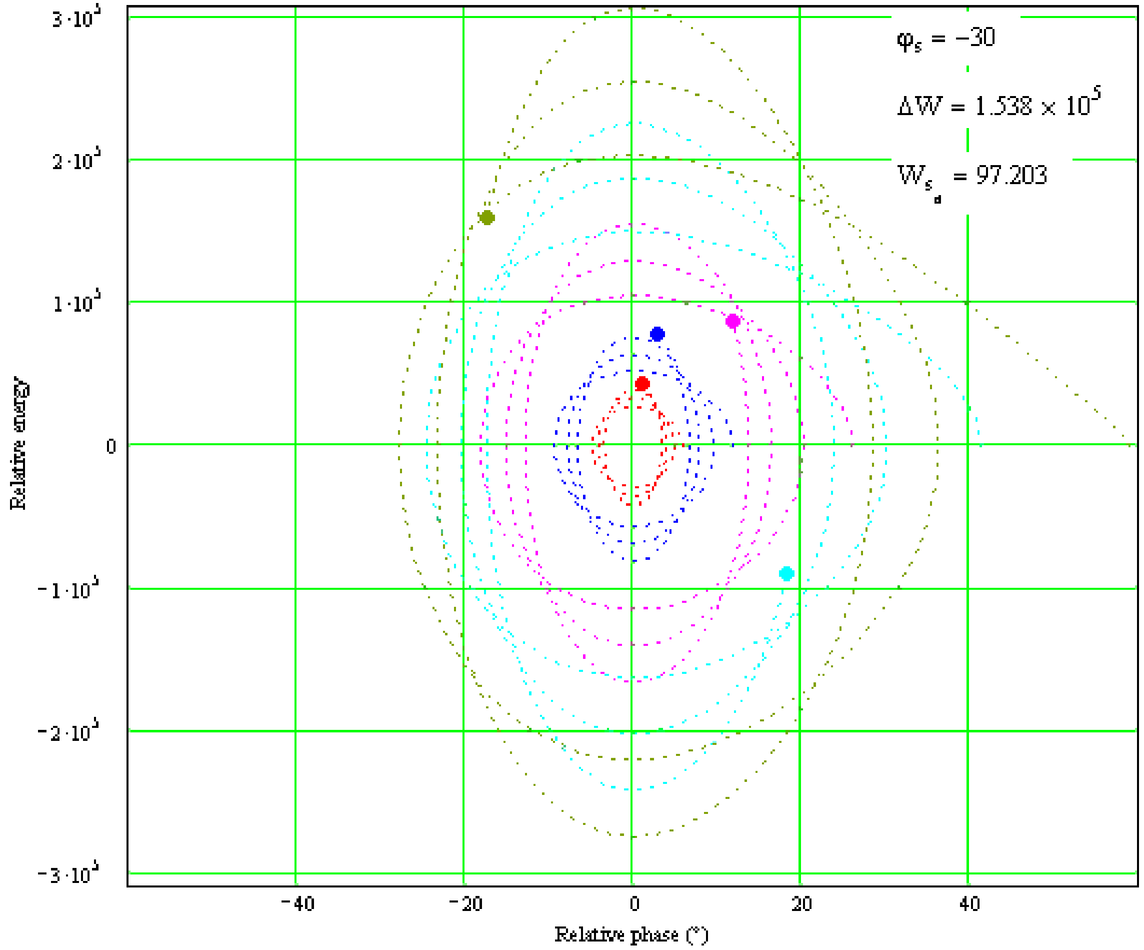
In the inner part of the bucket (φ<<1):
Behaviour of a linear oscillator at the synchrotron angular frequency

$$k_0 = \sqrt{\frac{2\pi f_{RF} \cdot \eta \cdot qE_0T \cdot cos\varphi_s}{{\beta_s}^3 \gamma_s m_0 c^3}}$$

In the outer part of the bucket:
Non linear behaviour, the particles oscillates more & more slowly

We assume here E_z = Cst, φ_s =-30° and $\beta_s\gamma_s$ = Cst

4.7. Synchroton oscillations (w/ increasing energy)



Increasing βγ, the amplitude of the oscillations are:
-> REDUCED in phase
-> INCREASED in energy
(by a factor $(\beta\gamma)^{3/4}$)

= adiabatic damping

BUT the surface of each H=cte trajectory is kept constant
(that means that the longitudinal emittance is a priori kept constant with increasing energy)

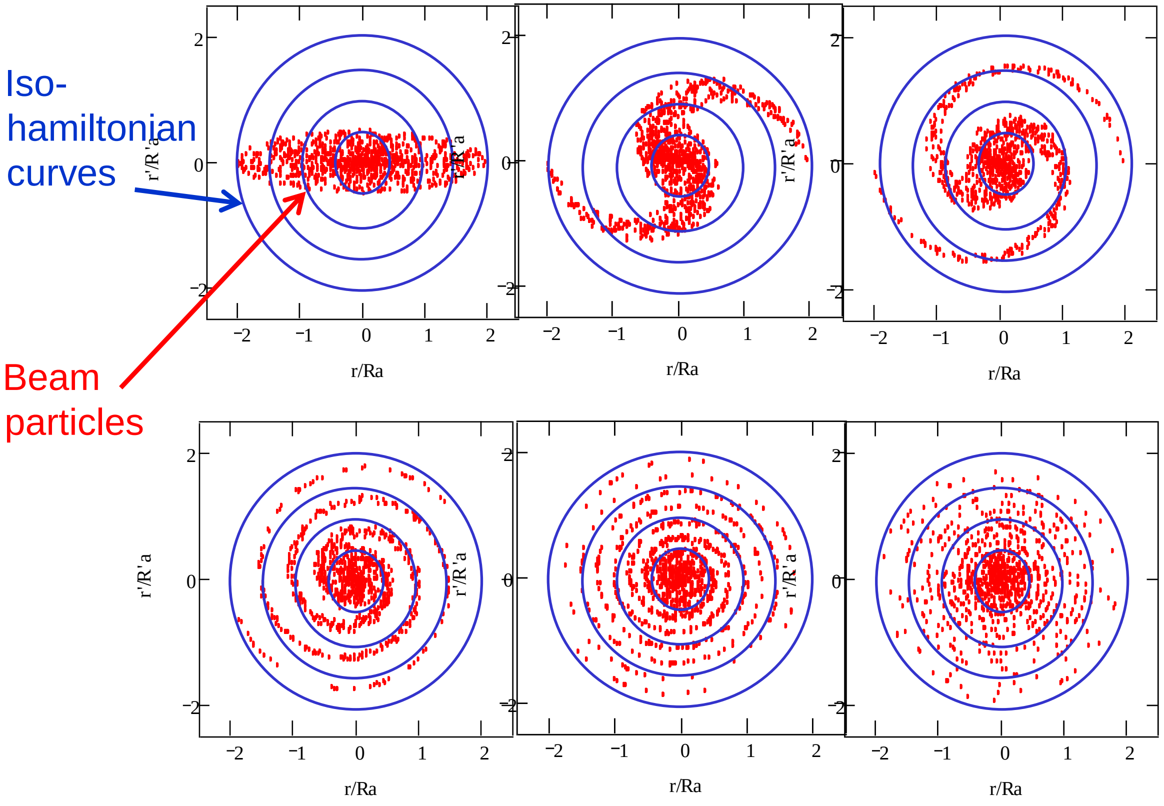
We assume here $E_z = Cst$, $\varphi_s = -30°$ and $\beta_s \gamma_s \neq Cst$

4.8. Filamentation in longitudinal phase space

All the beam particles are thus traveling on the H = Cst curves in the (ϕ,w) space

= trajectories at frequency k_0 if the forces are linear (inner bucket)

= trajectories at lower frequency when the forces become non-linear (outer bucket)



*=> If the beam is not correctly matched to H=cte curves, **filamentation** is quickly observed*

The beam **longitudinal rms emittance is then increasing**