Transverse Beam Dynamics

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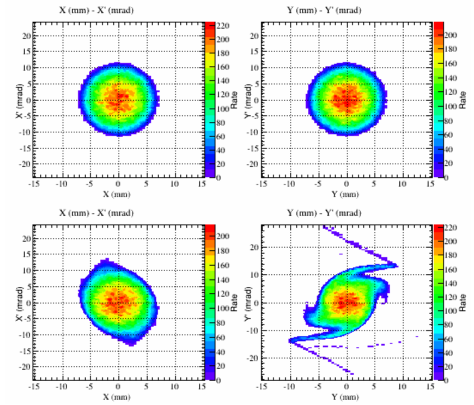
Particles dynamics specificity

Particles dynamics in an accelerator is usually treated in 2 specific formalisms :

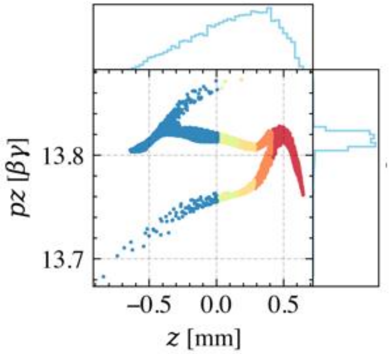
- Transverse beam dynamics
  - ➢ Associated velocities are relatively small
  - ➢ Beam visible « shape »
  - ➢ Beam guidance
  - ➢ Necessary to avoid losses in the walls

First course



- Longitudinal beam dynamics
  - ➢ Very high velocities
  - ➢ Acceleration

Second course



First course : content

I. Charged particle in electromagnetic field
Around Lorentz equation : $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

II. Guided and focalization magnets : dipoles, quadrupoles, multi-poles

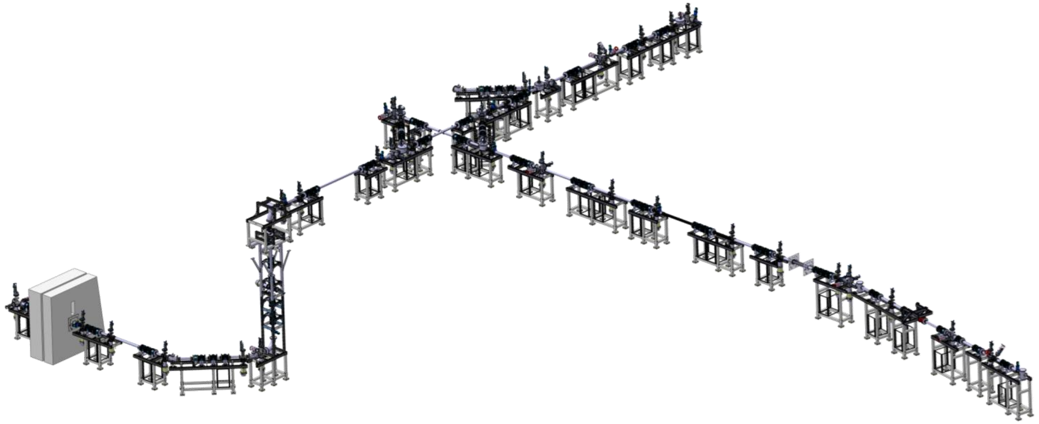
$$B_r(r,\varphi) = \sum_{n=1}^{\infty} r^{n-1}(b_n \sin n\varphi + a_n \cos n\varphi) \text{ et } B_\varphi(r,\varphi) = \sum_{n=1}^{\infty} r^{n-1}(b_n \cos n\varphi - a_n \sin n\varphi)$$

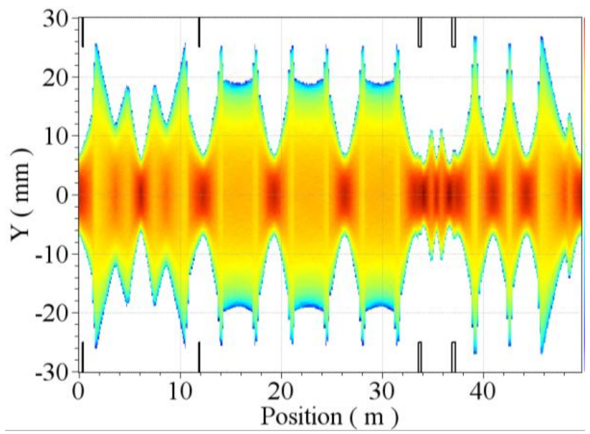
III. General development of magnetic field around the reference trajectory:

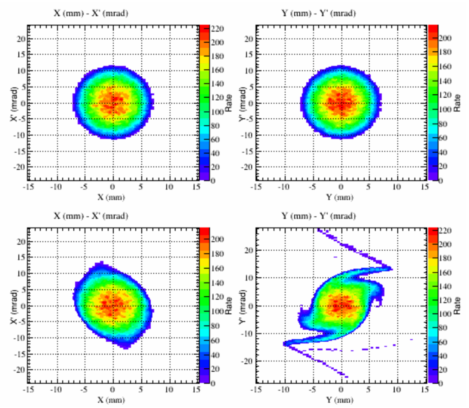
The magnetic field equation : $B_x(s) = h^{-1} B_{z_0} (-nh^2 z + 2\beta h^3 xz + \ldots)$

IV. Particles motion around the reference trajectory : $y'' + K_x(s) y = f(s)$

V. Beam envelop and emittance $\gamma_y y^2 + 2\alpha_y y' y + \beta_y y'^2 = \varepsilon_y / \pi$







References

1. CERN accelerator School: http://cas.web.cern.ch/cas
2. CERN Joint Universities Accelerator School (JUAS) : https://indico.cern.ch/event/356897/contributions, https://indico.cern.ch/event/286275/contributions
3. K.L. Brown, D.C. Carey, Ch. Iselin and F. Rothacker, Transport, a computer program for designing charged beam transport systems, CERN 80-04
4. F. Dupont, Cours élémentaire sur les transports de faisceau et les accélérateurs de particules, janvier 1988
5. Bernhard Holzer, CERN-LHC, Introduction to accelerator Physics
6. Marc Muñoz, CELLS-ALBA, Introduction to charged particle optics, Jan 2010
7. Stanley Humphries, Principles of Charged Particle Acceleration, Wiley-Interscience, 1986
8. A.P. Banford, The Transport of Charged Particle Beams, Spon, 1966
9. http://bc1.lbl.gov/CBP_pages/educational/WoB/home.htm
10. P.J. Bryant, A Brief History and review of accelerators, CERN, Geneva, Switzerland
11. E. Baron, Panorama des accélérateurs et de leurs utilisations, HAL Id: cel-0012288
12. A. R. Steere, A Timeline of Major Particle Accelerators, Michigan State University, Department of Physics and Astronomy, 2005
13. M. Sands, The Physics of Electron Storage Rings, an Introduction, SLAC-121, UC-28 (ACC), 1970

I – Charged particles in electromagnetic fields

First let us define some important values :

Einstein's mass-energy

$$E_0 = m_0 \times c^2$$

With: particle mass m_0

speed of light $c = 2.99.10^8\ m/s$

Typical masses : 511keV for e-

938.3MeV for protons

Total energy

$$E_{tot} = \gamma\, m_0 c^2$$

With: $\gamma = \frac{E_{tot}}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1-\beta^2}}$

$\beta = \frac{v}{c}$

Kinetic energy

$$E_{kin} = E_{tot} - E_0 = (\gamma - 1) m_0 c^2$$

For a rest particle : $\beta = 0, \gamma = 1$

For a non relativist particle : $\beta \ll 1, \gamma \approx 1$

For a ultra-relativist particle (close to speed light) : $\beta \to 1, \gamma \to \infty$

- Momentum : p $= m\, v = \gamma\, m_0 v = \beta\, \gamma\, m_0\, c$ (in MeV/c)
- $E_{tot}{}^2 - E_0{}^2 = (\gamma\, m_0 c^2)^2 - (m_0 c^2)^2 = (\gamma^2 - 1) m_0{}^2 c^4 = (\beta\, \gamma\, m_0\, c)^2 c^2 = p^2 c^2$

For $E_{cin} \ll E_0$, $\gamma = 1$, we have :

$$p^2 c^2 = E_{tot}{}^2 - E_0{}^2 = (E_{tot} - E_0)(E_{tot} + E_0) = E_{cin}(2E_0 + E_{cin}) \cong 2E_0 E_{cin}$$

Therefore : $E_{cin} = \frac{p^2 c^2}{2E_0} = \frac{\gamma^2 m_0{}^2 v^2 c^2}{2 m_0 c^2} = \frac{1}{2} m_0 v^2$

Lorentz force

The motion of a charged particle in a electro-magnetic field is given by:

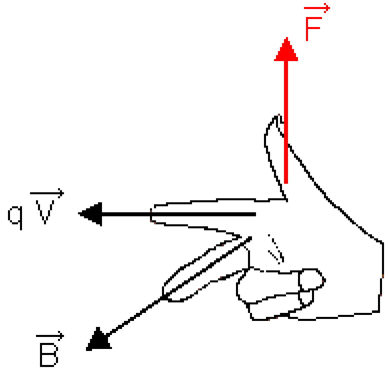
$$\frac{d\vec{p}}{dt} = \vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

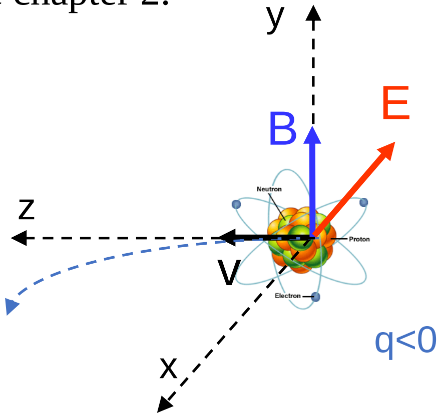
- F : lorentz force in Newton
- p : momentum in kg.m/s
- q : particle charge (±Ze) in Coulomb
- E, B : electric and magnetic induction (resp. V/m and T)

Remark : B is the magnetic induction, $\vec{B} = \mu\vec{H}$

H is the magnetic field (A/m)

μ is the permeability of the medium (the degree of magnetization of a material in response to a magnetic field) in henries per meter ($H.m^{-1}$). More explanation in the chapter 2.



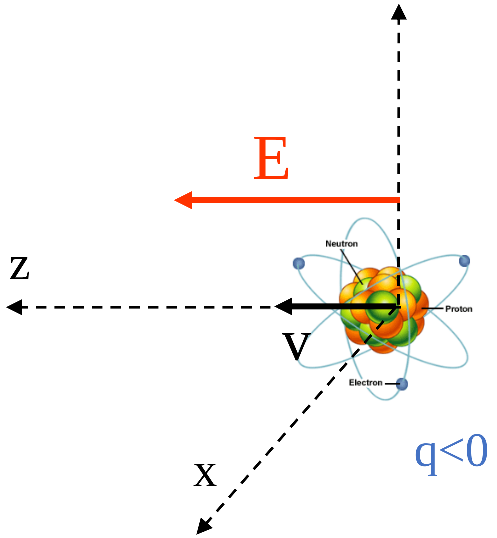


I – Charged particles in electromagnetic fields

Important remark : variation of energy in EM fields :

$$W = \int_{t1}^{t2} \vec{F} \cdot \vec{v}\, dt = \int_{t1}^{t2} q\left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \vec{v}\, dt$$

$\Rightarrow \frac{dE_{tot}}{dt} = q\, \vec{v} \,.\, \vec{E} \Rightarrow$ No acceleration with magnetic fields !!



To accelerate or increase the particle energy:

- Only electric field is useful
- If $\vec{E} \perp \vec{v}$, there is no acceleration
- There is acceleration only if $\vec{E} \,//\, \vec{v}$

Energy gain ΔE_{tot} in a static electric field is :

$\Delta E_{tot}\ (MeV) = qE \int vdt = q\ E\ \Delta x = q\ \Delta V$ with ΔV the applied potential in MV

An eV is the energy gained by an electron under 1V of potential difference

I – Charged particles in electromagnetic fields

Example :

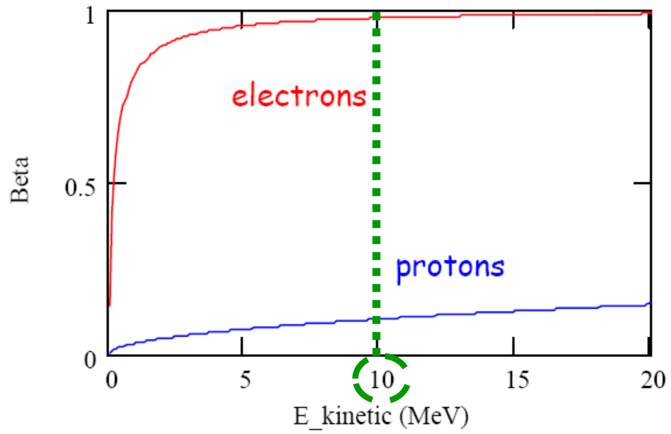
Considering an Electron ($q = -1$) and a Proton ($q = 1$) at $E_{initial} = 0$

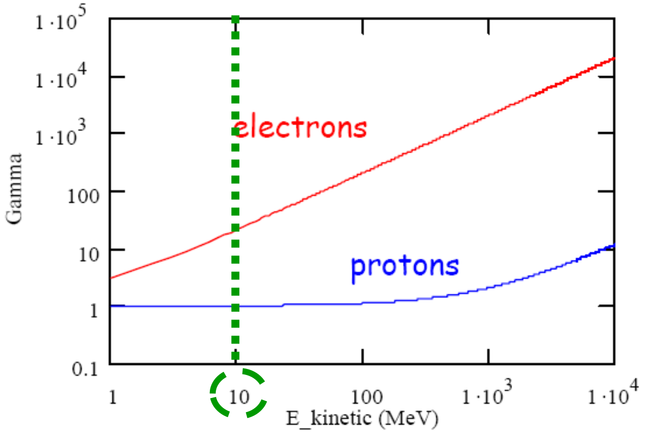
We apply a accelerating potential of 10MV

- For both particles, energy gain is 10MeV
- The speed gain will be:

For Electron : $\gamma_e = 1 + \frac{E_{cin}}{m_0 c^2} = 1 + \frac{10}{0.511} \approx 20.6$ and $\beta_e = \sqrt{1 - \frac{1}{\gamma_e{}^2}} \approx 0.9988$

For Protons : $\gamma_p = 1 + \frac{10}{938.3} \approx 1.0107$ and $\beta_p \approx 0.145$

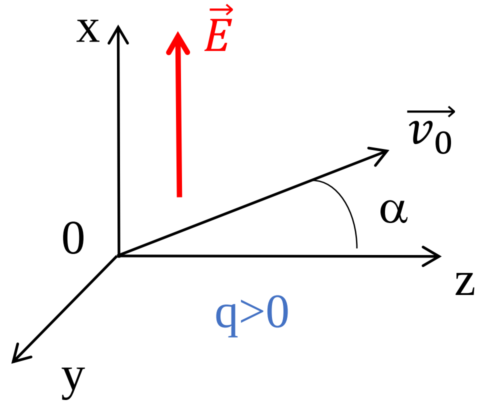




Accelerator and structures must be design according to particle characteristics

I – Charged particles in electromagnetic fields

- Particle motion in a transverse electric field : $\vec{F} = \frac{d\vec{p}}{dt} = q\vec{E}$



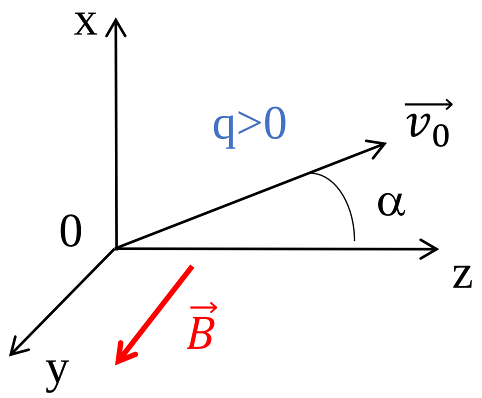
$\vec{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$, with : $\vec{v} = \begin{pmatrix} \dot{x}_0 \\ 0 \\ \dot{z}_0 \end{pmatrix}$, $\vec{E} = \begin{pmatrix} E_x \\ 0 \\ 0 \end{pmatrix}$, at $t = 0$: $\begin{pmatrix} x_0 = 0 \\ y_0 = 0 \\ z_0 = 0 \end{pmatrix}$

We have : $m_0 \frac{d^2x}{dt^2} = qE_x$, $m_0 \frac{d^2y}{dt^2} = 0$, $m_0 \frac{d^2z}{dt^2} = 0$, therefore : $\dot{x} = \frac{q}{m_0} E_x t + \dot{x}_0$, $\dot{y} = 0$, $\dot{z} = \dot{z}_0$, then $x = \frac{q}{2m_0} E_x t^2 + \dot{x}_0 t$, $y = 0$, $z = \dot{z}_0 t$

With : $\dot{x}_0 = v_0 \sin\alpha$ and $\dot{z}_0 = v_0 \cos\alpha$, then $\begin{cases} x = \frac{q}{2m_0} E_x t^2 + v_0 \sin\alpha \, t \\ y = 0 \\ z = v_0 \cos\alpha \, t \end{cases}$

Particle trajectory is parabolic : $x = \frac{qE_x}{2m_0} \frac{z^2}{(v_0 \cos\alpha)} + z \tan\alpha$

- Particle motion in a transverse magnetic field :



$$\vec{F} = \frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B},\ \vec{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \text{ with : } \vec{v} = \begin{pmatrix} \dot{x_0} \\ 0 \\ \dot{z_0} \end{pmatrix}, \vec{B} = \begin{pmatrix} 0 \\ B_y \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{d^2x}{dt^2} \\ \frac{d^2y}{dt^2} \\ \frac{d^2z}{dt^2} \end{pmatrix} = \frac{q}{m}\vec{v} \times \vec{B} = \frac{q}{m}\begin{pmatrix} \dot{y}B_z - \dot{z}B_y \\ \dot{z}B_x - \dot{x}B_z \\ \dot{x}B_y - \dot{y}B_x \end{pmatrix} = \frac{q}{m}\begin{pmatrix} -\dot{z}B_y \\ 0 \\ \dot{x}B_y \end{pmatrix} = \begin{pmatrix} -\omega\dot{z} \\ 0 \\ \omega\dot{x} \end{pmatrix} \text{ with } \omega = \frac{qB_y}{m}$$

With $\dot{z} + i\dot{x}$: $\frac{d(\dot{z}+i\dot{x})}{dt} = \omega(\dot{x} - i\dot{z}) = -i\omega(\dot{z} + i\dot{x})$, then $\frac{d(\dot{z}+i\dot{x})}{(\dot{z}+i\dot{x})} = -i\omega dt$

Solution is : $\dot{z} + i\dot{x} = Ze^{-i\omega t} = (Z_r + iZ_i)(\cos\omega t - i\sin\omega t)$

At $t = 0$: $\dot{z_0} + i\dot{x} = Z_r + iZ_i = v_0\cos\alpha + iv_0\sin\alpha \Rightarrow \begin{cases} \dot{x} = v_0\sin(\omega t - \alpha) \\ \dot{z} = v_0\cos(\omega t - \alpha) \end{cases}$

We can verify that velocity is constant : $\dot{z}^2 + \dot{x}^2 = {v_0}^2$

Finally : $\begin{cases} x = \frac{v_0}{\omega}\cos(\omega t - \alpha) - \frac{v_0\cos\alpha}{\omega} \\ 0 \\ z = \frac{v_0}{\omega}\sin(\omega t - \alpha) + \frac{v_0\sin(\alpha)}{\omega} \end{cases}$

I – Charged particles in electromagnetic fields

Particle motion in a transverse magnetic field is a circle

$$\left(z - \frac{v_0 \sin\alpha}{\omega}\right)^2 + \left(z - \frac{v_0 \cos\alpha}{\omega}\right)^2 = \frac{{v_0}^2}{\omega^2} = \rho^2$$ With radius $\rho = \frac{v_0}{\omega} = \frac{P}{qB_y}$ centered in : $\begin{cases} x_c = \frac{v_0 \cos\alpha}{\omega} \\ z_c = \frac{v_0 \sin\alpha}{\omega} \end{cases}$

The cyclotron frequency is $\omega = \frac{qB_y}{m}$

The revolution period is then $T = \frac{2\pi}{\omega} = \frac{2\pi\rho}{v_0} = \frac{2\pi\, m}{q\, B_y}$

The magnetic rigidity is : $\mathrm{B}\rho = \frac{P}{q}$

Numerically : $\mathrm{B}\rho(T.m) = \frac{10^9}{c}\frac{P\,(GeV/c)}{q} = 3.3356\,\frac{P\,(GeV/c)}{q}$

In the same way, we speak also about the electric rigidity of the beam with :

$$E\rho\,(MV) = \frac{v\,P}{c\,q} = \beta c\frac{B\rho\,(T.m)}{10^6} = \beta\frac{10^3\,P\,(GeV/c)}{q}$$

Example : $^{12}C^{6+}$ at 95MeV/u : $E_{tot} = 1140MeV, B\rho = 2.8772T.m, v = 12.6\,cm/ns$

$^{12}C^{1+}$ at 60keV : $B\rho = 0.1222T.m, v = 0.098\,cm/ns$

Protons at LHC : 7TeV $B\rho = 23352.6T.m, v = 29.979\,cm/ns \approx c$

Parenthesis : mass separation, spectrometers

(Non-relativistic approx.)

$$B\rho = \frac{mv}{q} = \frac{p}{q}$$

$$E_k = \frac{1}{2}mv^2 \Rightarrow v^2 = 2\frac{E_k}{m}$$

$$B\rho = \frac{\sqrt{2\,\boldsymbol{m}\,\boldsymbol{E}}}{q}$$

$$E\rho = \frac{\gamma m v^2}{q}$$

$$E_k = \frac{1}{2}mv^2 \Rightarrow v^2 = 2\frac{E_k}{m}$$

$$E\rho = \frac{2E_k}{q}$$

No mass dependance on electric fields !

Particle mass separators (or spectrometers) need a magnetic field !

I – Charged particles in electromagnetic fields

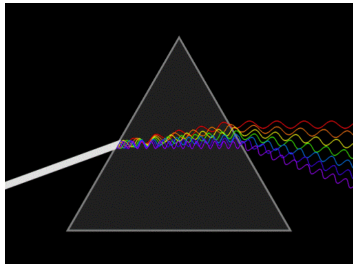
In a more general approach, electric and magnetic fields are described by the Maxwell equations

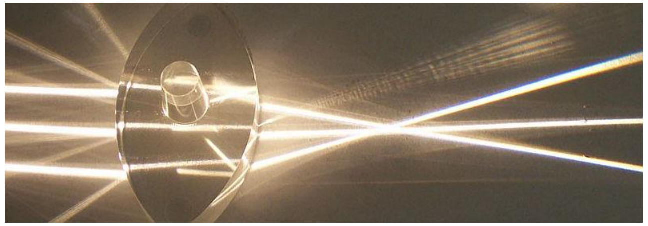
Maxwell Equations :

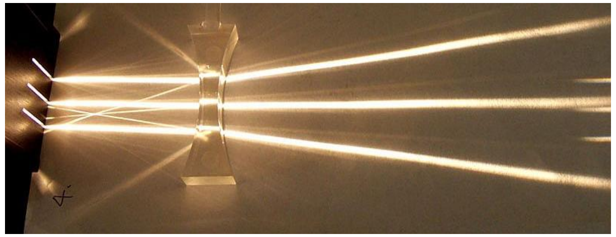
1. Divergence of the electric field $\vec{E}$ equals charge density ρ divided by ϵ_0 : $div\ \vec{E} = \frac{\rho}{\epsilon_0}$
2. Divergence of the magnetic field is zero : $div\ \vec{B} = 0$
3. Curl ($\overrightarrow{rot}$) of the electric field is minus the rate of change of the magnetic field : $\overrightarrow{rot}\vec{E} = -\left(\frac{\partial \vec{B}}{\partial t}\right)$
4. Curl of the magnetic field $\vec{B}$ equals μ_0 times current density $\vec{J}$, plus the rate of change of electric field divided by c^2 : $\overrightarrow{rot}\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$

Generalities

In beam optic, we apply a analogy with the geometrical optic where light beams are deflected by prisms and focus by using focusing or defocusing lenses.







- Same approach is applied in corpuscular optic.
- Optic structures are designed in order to induce bending and focalization of the charged particles.
- Bending and focalization can be separated or combined.
- Systems with electric and/or magnetic fields around a central trajectory are realized.
- Systems ensure the transverse dimensions of the beam (transverse = orthogonal plane of the beam direction)

Use of magnetic and electric fields :

- Acceleration is only possible with electric fields
- Bending/manipulations can be done with both electric or magnetic fields

From Lorentz equation, we can deduce : $\left|\frac{F_E}{F_B}\right| = \left|\frac{q\,\vec{E}}{q\,\vec{v}\times\vec{B}}\right| = \frac{|\vec{E}|_{V/m}}{\beta\, c_{m/s}\,|\vec{B}|_T}$

Q : In which case do we use electric or magnetic fields for bending ?

Answer :

- Magnetic fields at high energy (high β)
- Electric fields at low energy (low β)

In any case, feasibility and cost have to be taken into account

Ex : A Tin beam 122Sn50, 1+ of energy 60keV has a magnetic rigidity of 0.3894Tm, electrostatic is more interesting. For 1GeV protons, Brho=5.65838Tm, magnetic is more interesting.

- In most circular accelerators, conventional electro-magnets (with iron) inducing magnetic fields to $|\vec{B}|_{max} \sim 1.8\ T$ at room temperature are used.
- In protons or heavy ions machines at very high energy ($\beta \sim 1$) like at LHC, we use superconducting magnets inducing fields up to 10T.

I – Guiding and focalizing magnets : dipoles

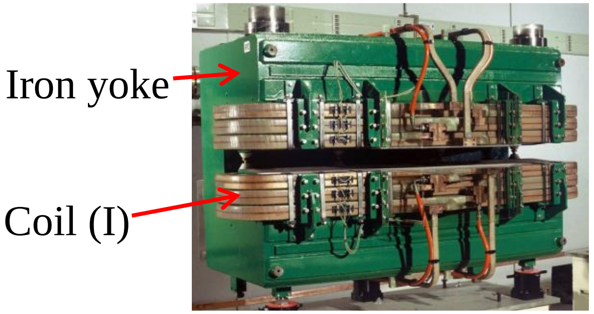
Magnetic optical element n°1 : the bending magnet, or dipole

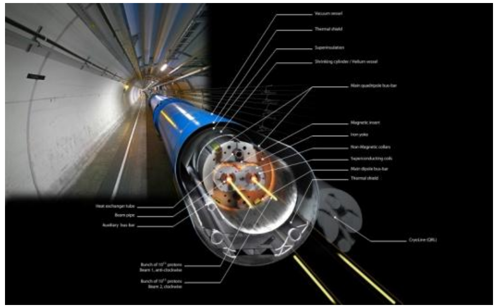
- Dipole, means a north and a south pole : 2 poles
- It results a constant magnetic field ($B = Cste$)
- Used to guide the central trajectory of a beam

Magnet dipole

Bend with flat and parallel poles create a field B_e uniform at the center in the gap

Bend can be small (for e- beam at few 100 keV) or huge (15m at LHC for protons at 7TeV)

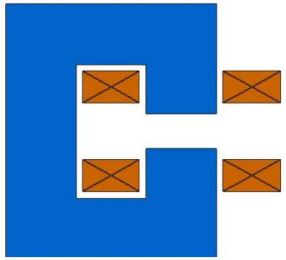




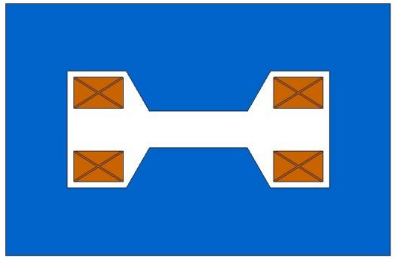
LHC Bending magnet
ρ=2804m, L=15m
N=1232
Bρ=23352.6Tm
B=8.33T

Structure of the yoke give the name of bends

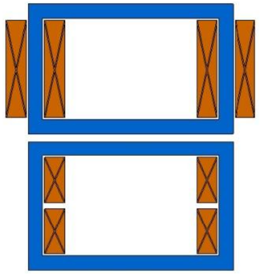
C-shape



H-shape

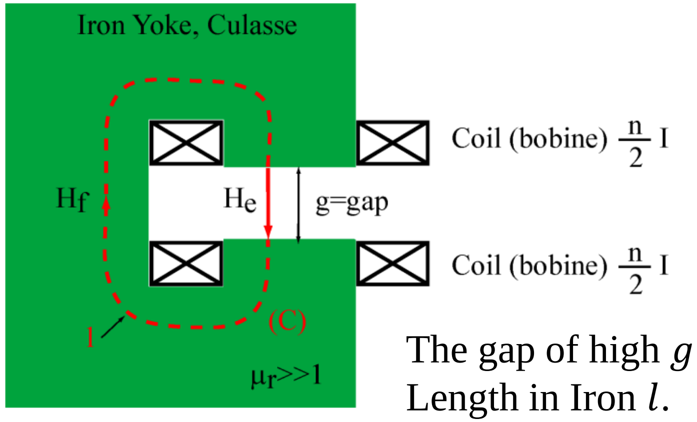


Window shape



I – Guiding and focalizing magnets : dipoles

Determination of the magnetic field B_e is obtained by the application of the Ampère theorem at (C) circuit surrounded the two excitation coils designed by $N/2$ conductors in which circulate a current I.



The gap of high g.
Length in Iron l.

- In the gap, induction is $H_e = \frac{B_e}{\mu_0}$
- In the Iron, $H_f = \frac{B_f}{\mu_f} = \frac{B_f}{\mu_r \, \mu_0}$ where μ_0 is the vacuum permeability $(4\pi \, 10^{-7} \, T.A^{-1}m)$ and μ_r is the relative permeability of Iron

Ampère theorem : $\sum I = \oint_C \vec{H}.d\vec{l} = NI = \oint_{Gap} H_e.dl + \oint_{Iron} H_f.dl = gH_e + l \, H_f$

$$NI = g\frac{B_e}{\mu_0}\left(1 + \frac{B_f}{\mu_r B_e}\frac{l}{g}\right)$$

With $B_f \sim B_e$ (continuity of the orthogonal part of $\vec{B}$) and $\mu_r \sim 10^3$ (outside saturation) :

Ampère-turns is $NI \sim g\frac{B_e}{\mu_0}$, for g ↑ , NI ↑ , cost ↑

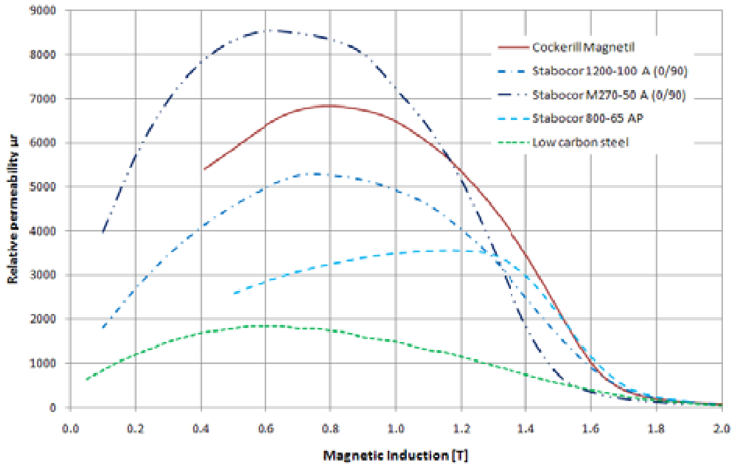
Ex: LISE spectrometer at GANIL: $NI = 0.1 \times \frac{1.7}{\mu_0} \sim 1.35 \, 10^5 \, A.t$ for $B_{max} = 1.7T$ with $N = 160 \, spires$, $I_{max} \sim 850A$

I – Guiding and focalizing magnets : dipoles

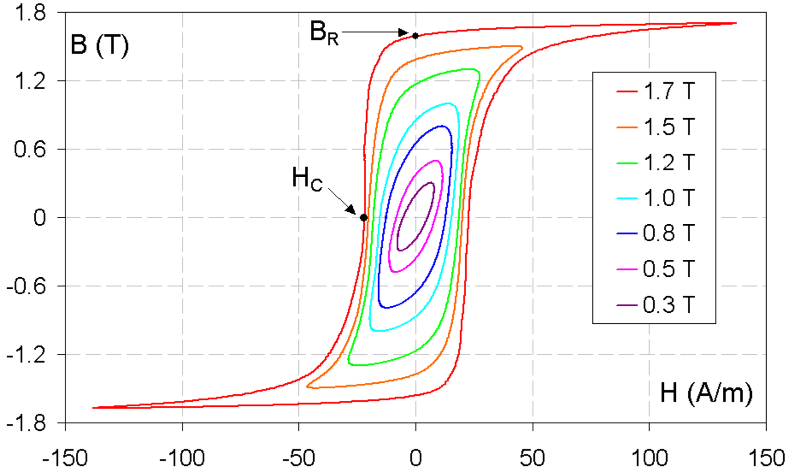
At room temperature, $B_e = f(NI)$ is not linear due to circulation of H in Iron.

Relative permeability μ_r of Iron is a function to B_f ($\mu_r \to 1$ when $B_f \uparrow$).

Permeability dependence to induction
for various steel types



Hysteresis curve
$|B_{max}|$ level depend to μ_r



- Yoke which channeling the magnetic flux can be realized in massive Iron or by stack of bonded plates in order to reduce the Foucault currents produce by the B dependence to time (useful in synchrotron)
- Ampere-turn NI are realized by the appropriated number of spires surrounded upper and lower pole of the bend. In the precedent case, we have $2x160$ conductors (Copper) are carrying by the maximum current to 850A (equivalent maximum magnetic rigidity of the beam $B\rho = 4.42Tm$ with dipole radius to $\rho = 2.6m$).

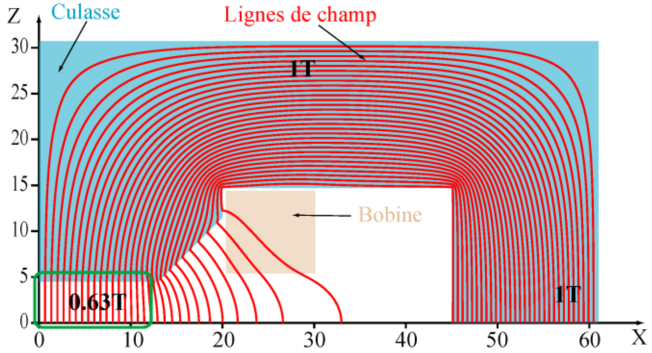
I – Guiding and focalizing magnets : dipoles

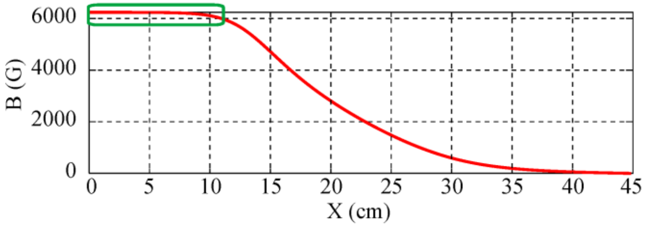
Building a bending magnet :

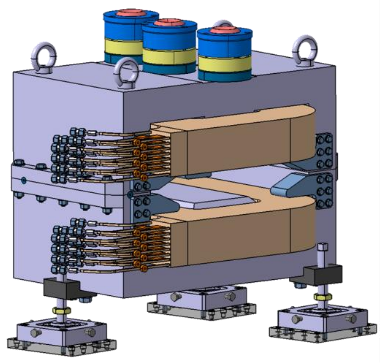
1. Specifications are fixed by beam dynamics : which beams ? Which deviation ? Which beam size ? …
   ex.: $B\rho_{max} = 0.84Tm$, deviation angle $\theta = 17.2°$, curvature radius ρ $= 1137.5mm$, therefore $B_{max} = 0.63T$, $Gap = 90mm$, extension of the good field zone= $\pm 35mm$
1. Magnetic study using calculation codes 2D (POISSON, OPERA2D), 3D (OPERA3D, ANSYS)
2. Optimization (shimming of the return yoke, coils)
3. Mechanical conception (CATIA…), detailed drawings
4. Call for tender; building in company of the system
5. Magnetic measurements (conformity)
6. On-site installation and alignments (by surveyors)

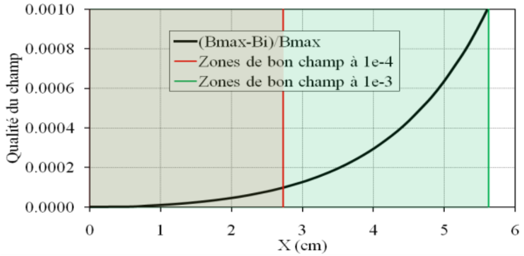
- Yokes : 560kG of Iron
- 2 coils : 105kG of Copper

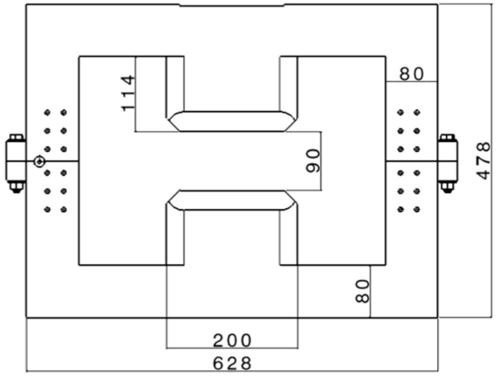
- $I_{max} = 153.8A$
- Spires $N = 156$

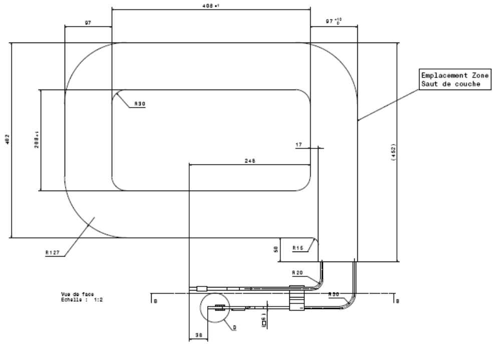






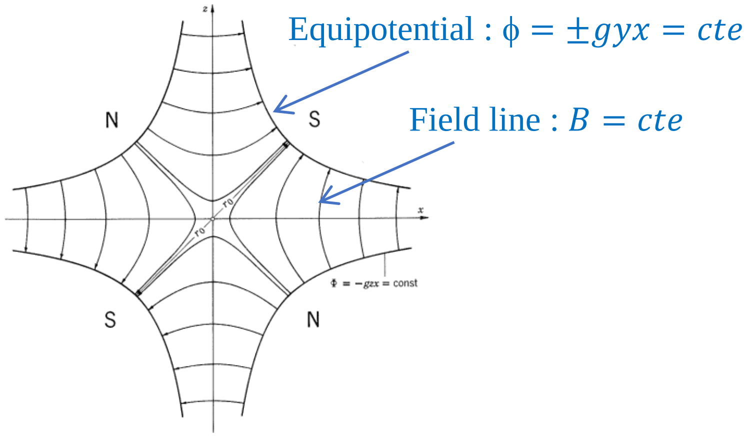


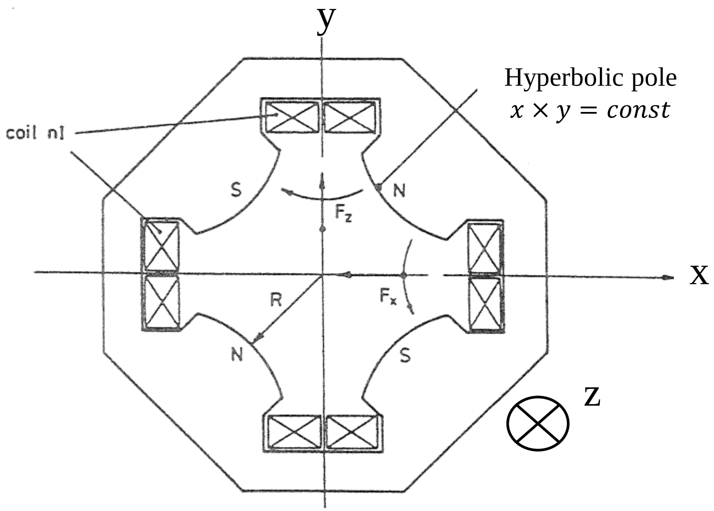


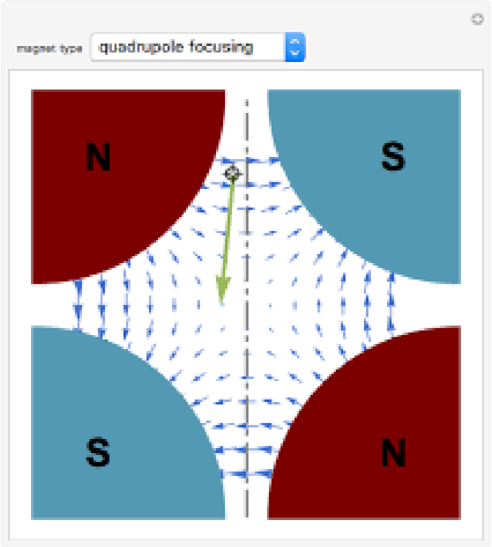


Magnetic optical element n°2 : the focaization magnet, or quadrupole

- Quadrupole, means a north, south, north and south poles : 4 poles
- Magnetic field evolves linearly with distance to the center ($B \propto r$)
- Used to focus or defocus the beam envelop







Q : What is the effect of this quad on the beam ?

Magnetic optical element n°2 : the focaization magnet, or quadrupole

- Quadrupole, means a north, south, north and south poles : 4 poles
- Magnetic field evolves linearly with distance to the center ($B \propto r$)
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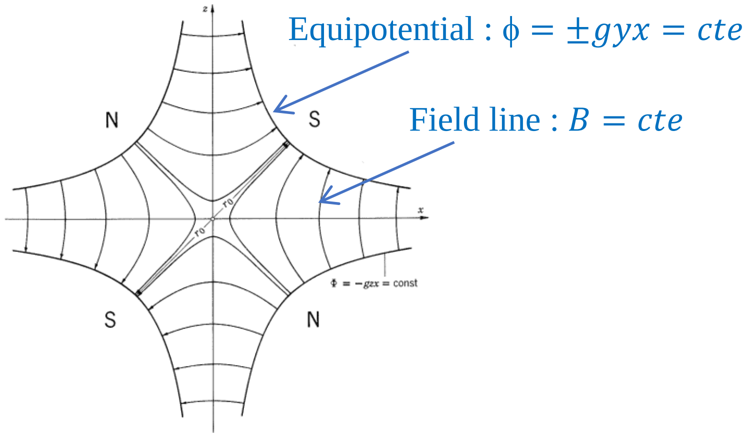
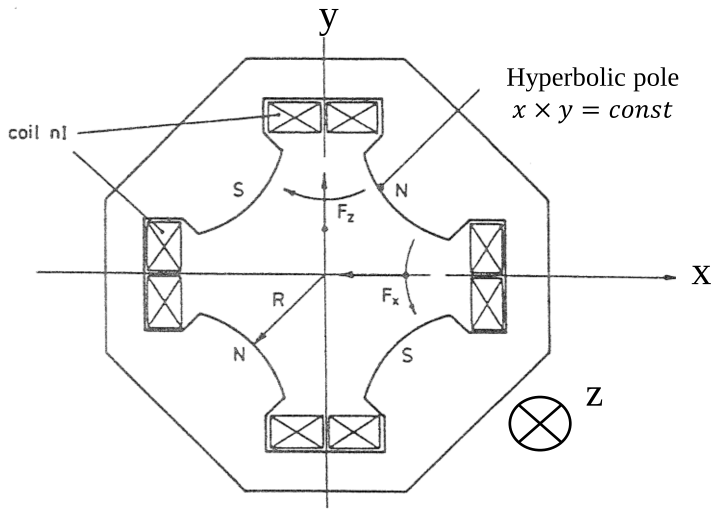
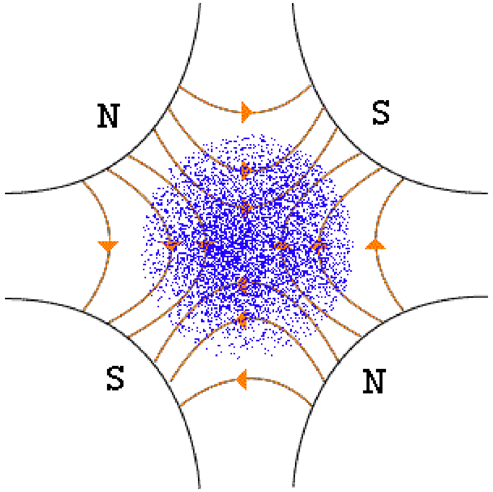
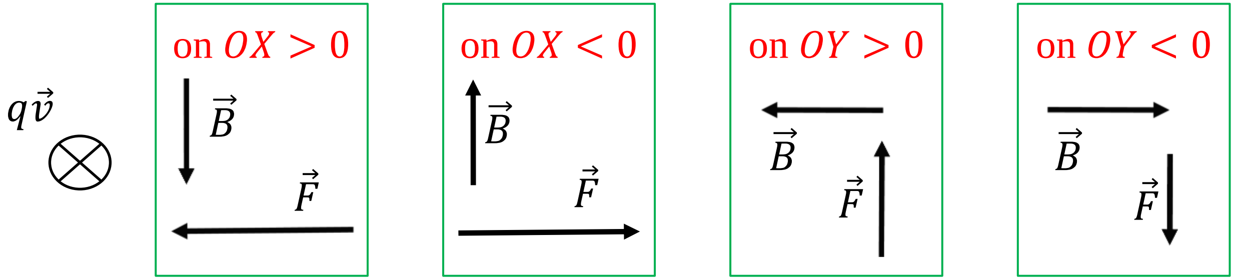




Using $\vec{F} = q\vec{v} \times \vec{B}$ with $q\vec{v}$ in the $0\vec{z}$.

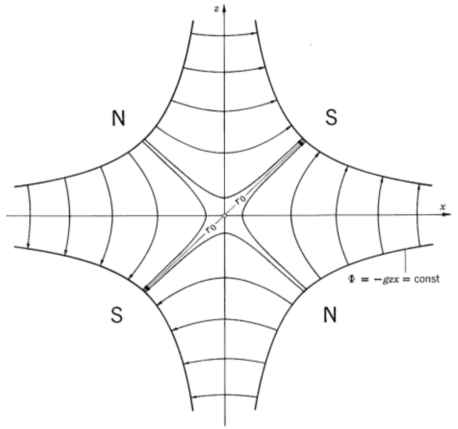
- Horizontal component of the Lorentz force bring back particles in the $0yz$
- Vertical component eject particles of the plan Oxz

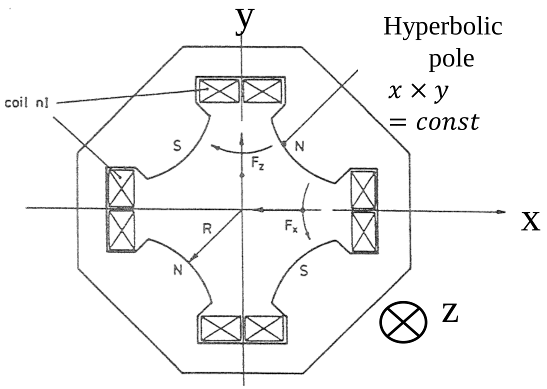


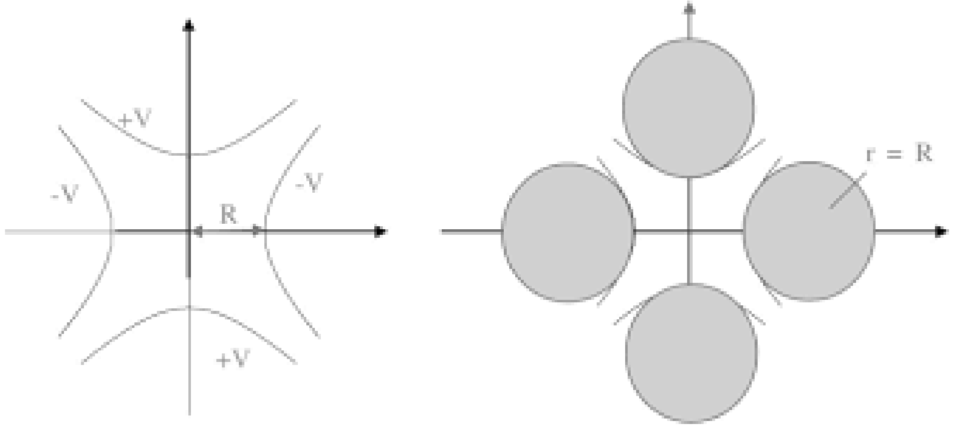


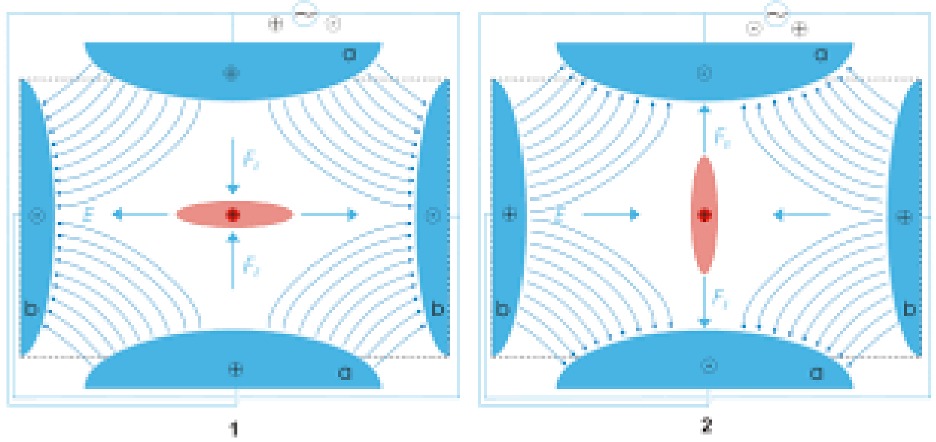
I – Guiding and focalizing magnets : quadrupoles

Electrostatic quadrupole :
- Q1 : where are the poles ?
- Q2 : what voltage sign on the poles ?





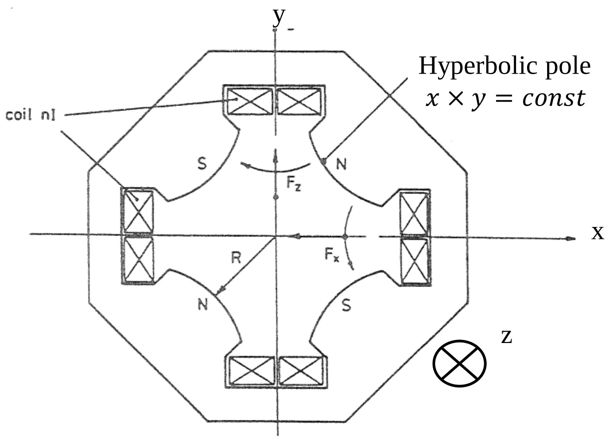




I – Guiding and focalizing magnets : quadrupoles

Quad is focusing in the horizontal plane and de-focusing in the vertical plane.

If the quadrupole is rotated by 90° or if the polarities are inverted, the opposite effect is obtained.



For mechanical and electromagnetic symmetry reasons :

- $B_y(x, -y) = B_y(x,y)$
- $B_y(-x, y) = -B_y(x,y)$

Or B_y even in y, odd in x

- $B_x(x, -y) = -B_x(x,y)$
- $B_x(-x, y) = B_x(x,y)$

Or B_x odd in y, even in x

Therefore $B_y = B_x = 0$ at $x = y = 0$. A centered beam on the Oz axis is not deflected.

We can develop the transverse B_y and B_x magnetic field (see Taylor expansions) :

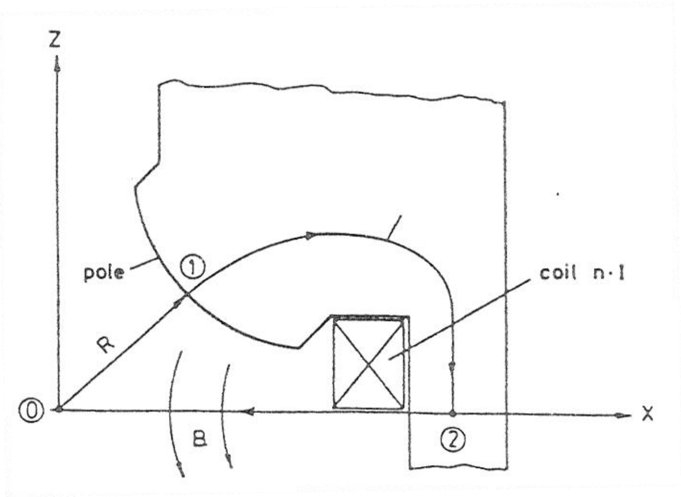
$$B_y(x,y) = 0 + \frac{\partial B_y}{\partial x} x + \text{higher orders}$$

$$B_x(x,y) = 0 + \frac{\partial B_x}{\partial y} y + \text{higher orders}$$

Field increase linearly with distance to the center

Field gradient G as a function of Ampere-turns (the current) in the quadrupole coils is determined by using the Ampère theorem to $\mathcal{C}$.

$$\sum I = \oint_{\mathcal{C}} \vec{H}.d\vec{l}$$



$$NI = \oint_{0\ to\ 1\ (R)} H(r).dr + \oint_{1\ to\ 2\ (Iron)} \vec{H}_I.d\vec{l} + \int_{2\ to\ 0} \vec{H}.d\vec{l}$$

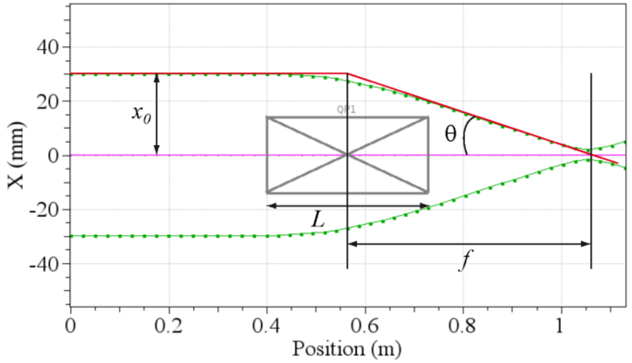
- From 0 to 1 : $H(r) = \frac{B(r)}{\mu_0} = \frac{\sqrt{B_x^2+B_y^2}}{\mu_0} = \frac{G}{\mu_0}\sqrt{x^2+y^2} = \frac{Gr}{\mu_0}$
- From 1 to 2 : $H_I = \frac{B_I}{\mu_0\mu_r} \sim 0$ because $\mu_r \gg 1$
- From 2 to 0 : $\vec{H} \perp d\vec{l}$, therefore $\vec{H}.d\vec{l} = 0$

$NI \sim \frac{G}{\mu_0}\int_0^R r\ dr \sim \frac{G\ R^2}{2\ \mu_0}$ and the gradient $G = \frac{2\ \mu_0\ NI}{R^2}$ with R the quadrupole radius.

Particle travelling into a magnetic quadrupole of length L at a distance x_0 from the central axis is deflected by :

$$\Delta\theta = \frac{1}{P/q}\int B\ dl = \frac{1}{B\rho}\int B\ dl = \frac{G\ L\ x_0}{B\rho}$$

The quadrupole focal length is : $\frac{1}{f} = \frac{x_0}{\tan\theta} = \frac{x_0}{\Delta\theta} = \frac{G\ L}{B\rho}$ with $\tan\theta \sim \Delta\theta$



I – Guiding and focalizing magnets : dipoles, quadrupoles and multipoles

Summarize : HEBT quadrupoles at SPIRAL2 Caen :

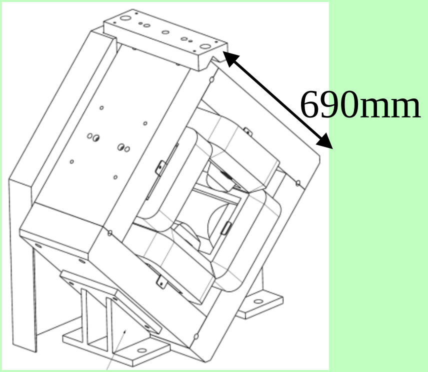
$G_{max} = 13T/m$, $L_m = 330mm$, aperture radius= $64mm$

4 coils to 25160A.turn.

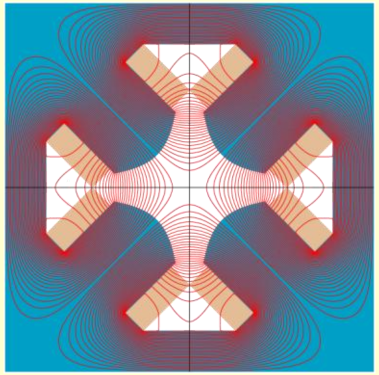
Yoke Iron weight : 750kg, 4 Coils Copper weight : 132kg.

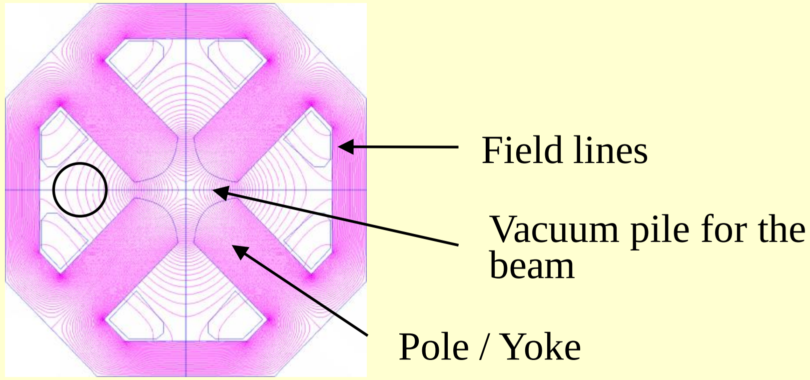
$I_{max} = 370A$, $P_{max} = 16.3kW$ (need water cooling)

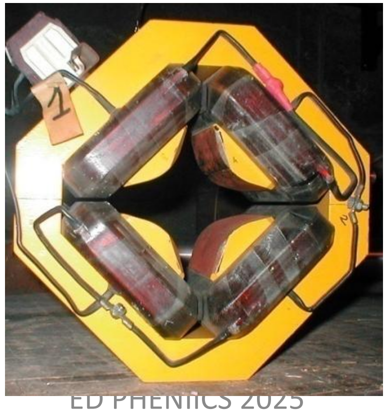
68 turns (Copper length 1.1m for One turn), water cooled along 75m circuit.



Few examples :
Calculation using Poisson code of the magnetic field in a warm quadrupole.

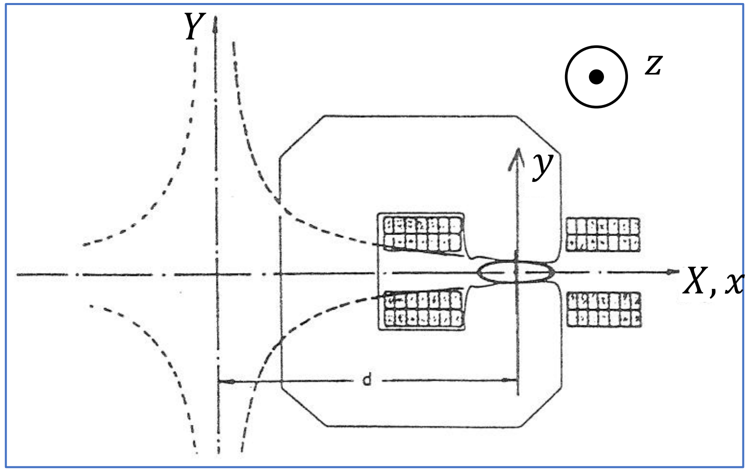


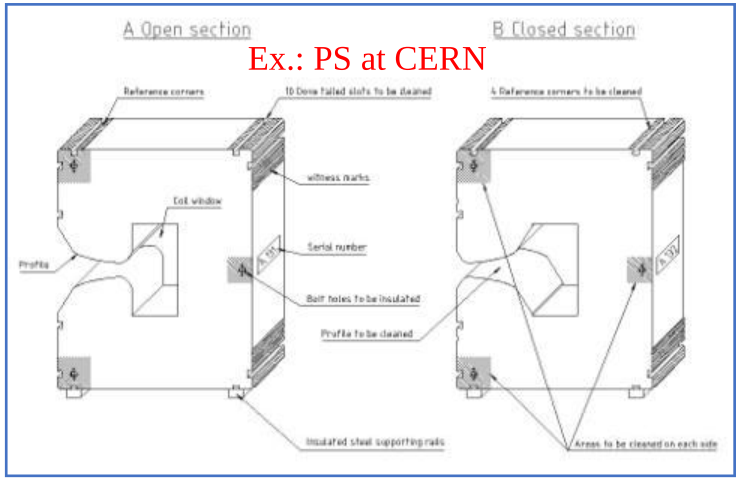




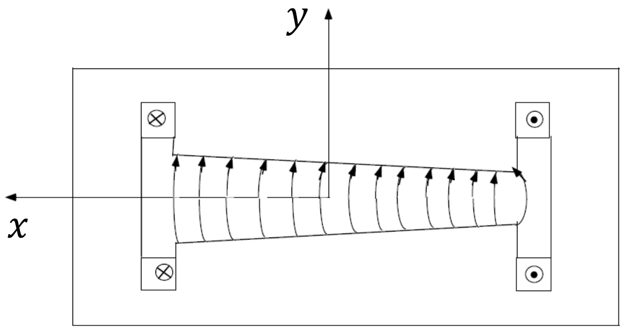
Bend with combined function

- Used in synchrotron with strong focusing
- Combination of deflection and focalization using hyperbolic poles





Such bend can be considered like a quadrupole at a distance d from the center.



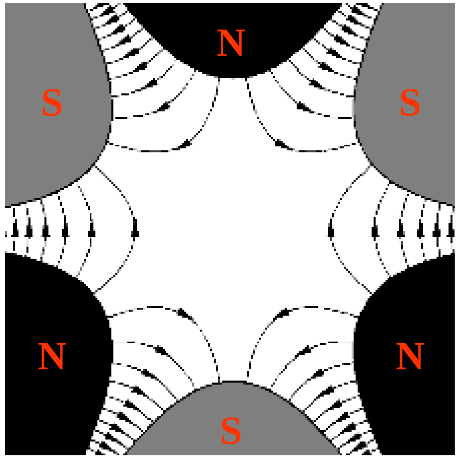
We can deduce B_X and B_Y component of the field :

Using the new reference (x, y) to (X, Y) : $Y = y$ and $X = x + d$

- $B_X = GY = Gy = B_x$
- $B_Y = GX = G\ (x + d) = B_0 + gx = B_0 + B_y$ with $B_0 = Gd$

Magnetic optical element n°3 : the sextupole

- Sexupole, means 6 poles (alternate south and north)
- Magnetic field evolves quadratically with distance to the center ($B \propto r^2$)
- Used to correct higher order effects (chromatic aberrations)



Using Ampère-theorem, the turn-numbers NI are function to the S force by :

$$NI = \oint \vec{H}.d\vec{l} = \int_0^R H_r\, dr = \int_0^R \frac{B_r}{\mu_0} dr = \int_0^R \frac{1}{\mu_0} S r^2 dr$$

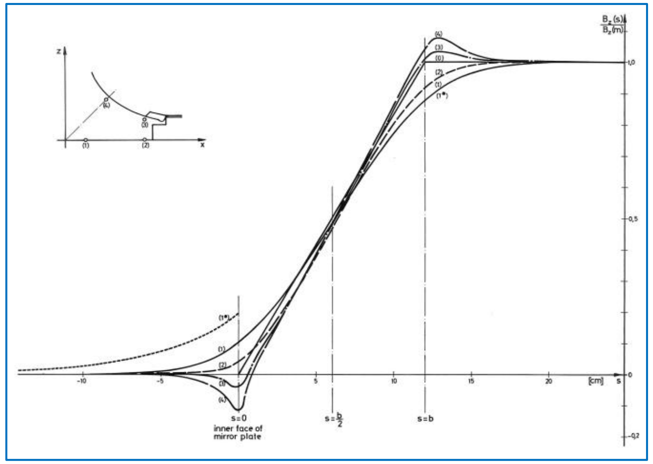
$NI = \frac{SR^3}{3\mu_0}$ with R the sextuple aperture radius

Multipolar lenses ($2 \times n$ poles)

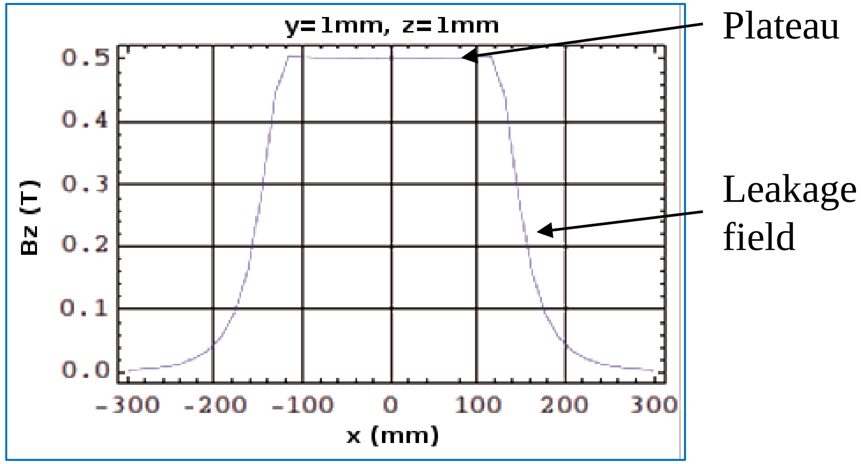
Strictly, expressions of the magnetic field components already seen are valid only close to the center of the gap in a bend. Higher order of the Taylor expansion of the field B have been neglected and transverses position of particles were small.

Therefore, it exist high order (non linear terms) terms due to finite dimensions of the pole surface :

Extension of the poles in the transverse plane (section of hyperbole in quadrupole)



Finite length of bend which induce a leakage field

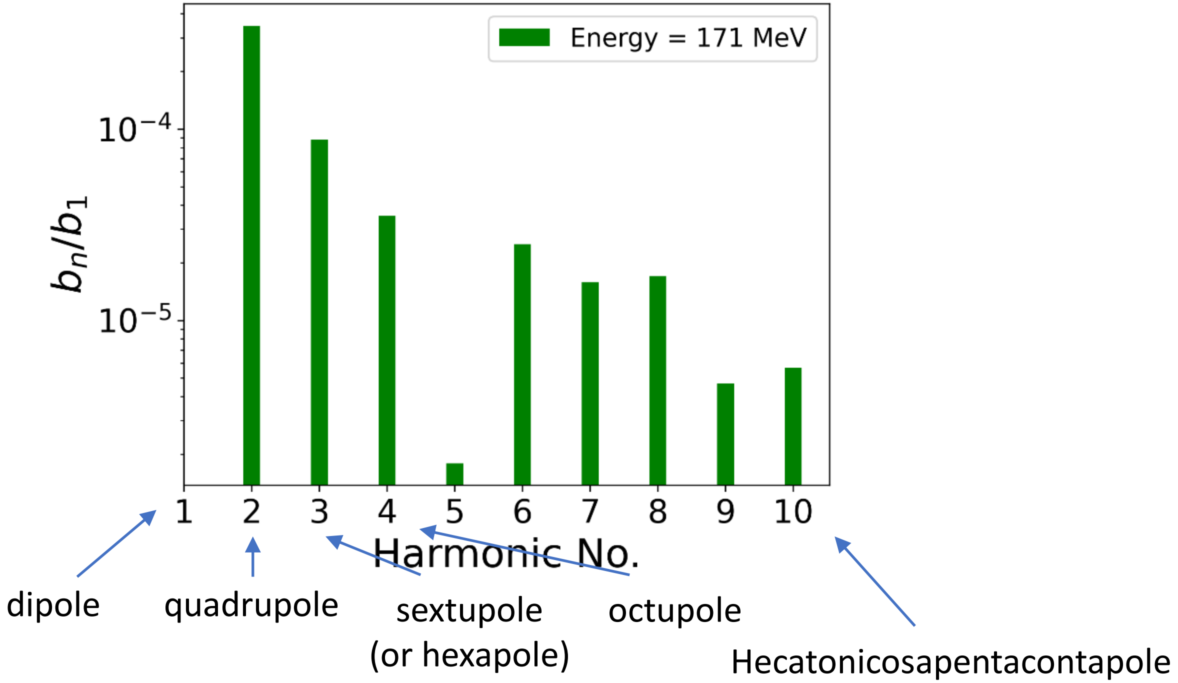


- According amplitudes of non linearity (effects on the beam), the compensation of these defects can justify the use of multipolar lenses. These lens create non linear fields at a given order.
- These elements correct induce aberrations.
- The much common lens use is the sextuple (or hexapoles).

I – Guiding and focalizing magnets : multipoles

A magnetic dipole, is meant to apply a… dipolar field on the beam
A perfect dipole would, but reality hits sometimes :

Harmonic content of a magnetic dipole (for illustration)

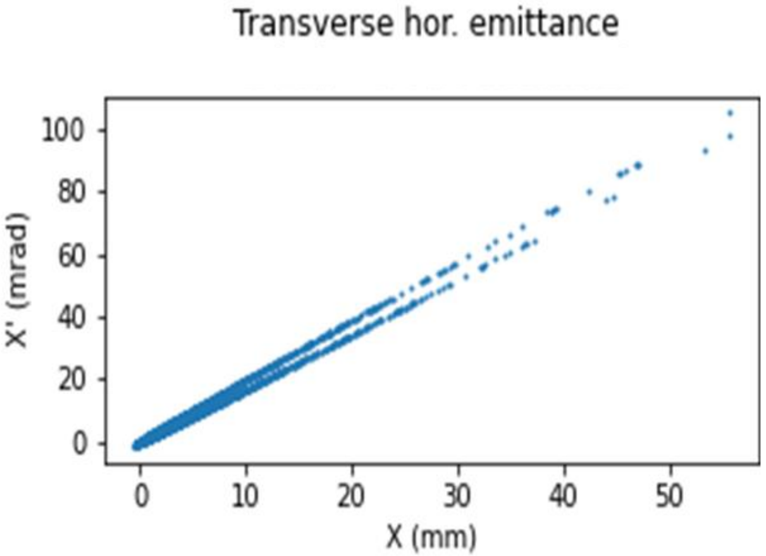


Can be corrected with higher order multipole :

Example of a 48 poles electric multipole

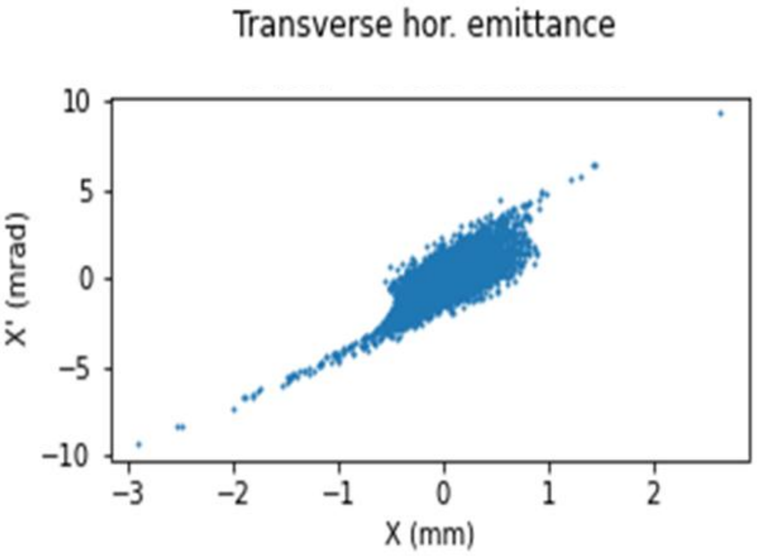
Higher-order fields will affect the shape of the beam : this is called optical aberrations

I – Guiding and focalizing magnets : dipoles, quadrupoles and multipoles



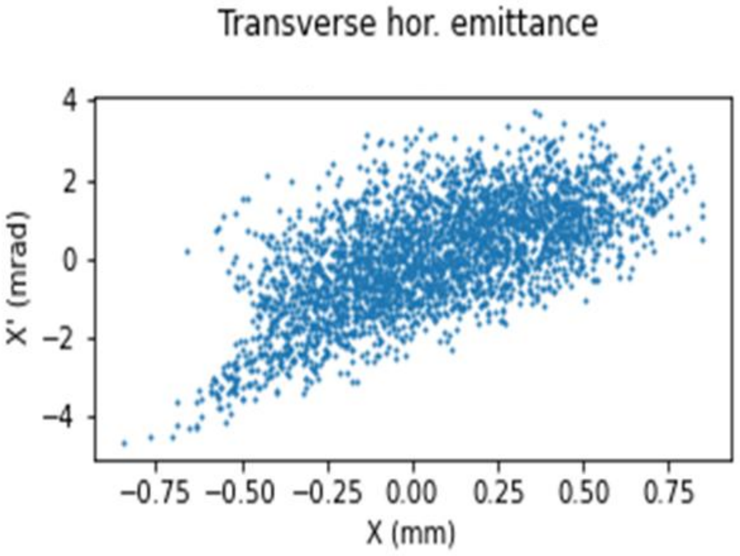
Hexapolar aberration (2^{nd} order)
is dominant

Typically « C-shaped »



Octupolar aberration (3^{rd} order)
Hexapolar corrected

Typically « S-shaped »

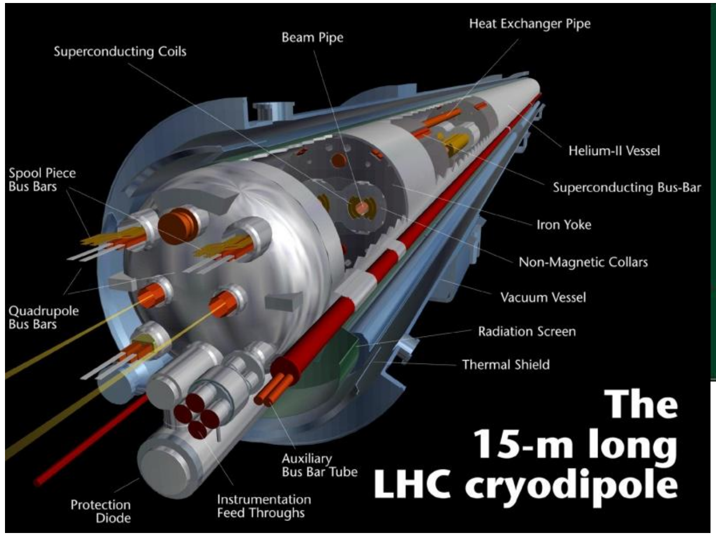


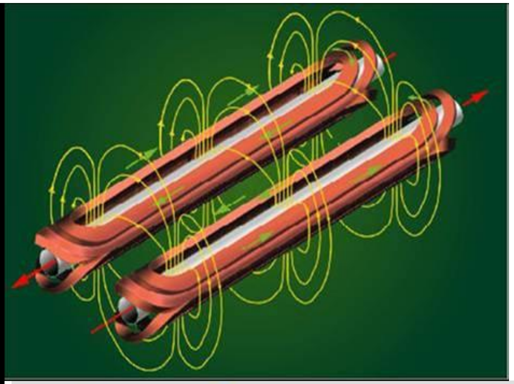
Higher order aberration ($> 3^{rd}$ order)
Octupolar corrected

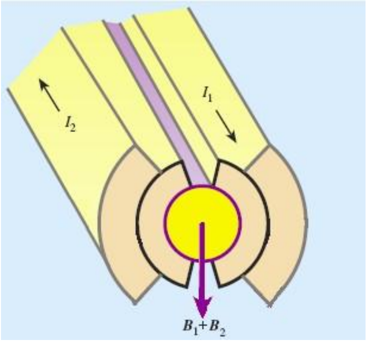
Optical aberrations tend to increase the beam size and need to be corrected

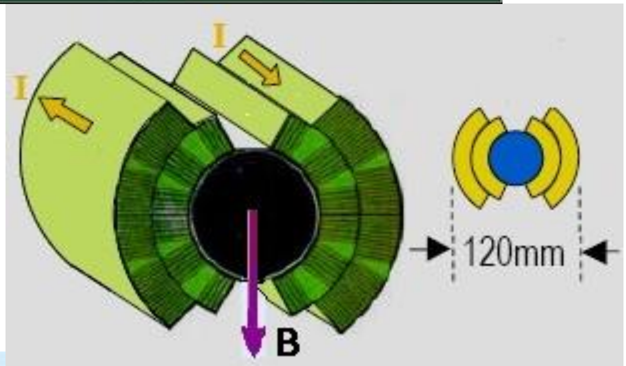
II – Guiding and focalizing magnets : dipoles, quadrupoles and multipoles

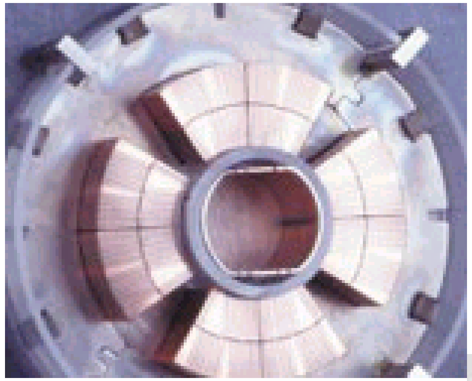
LHC at CERN

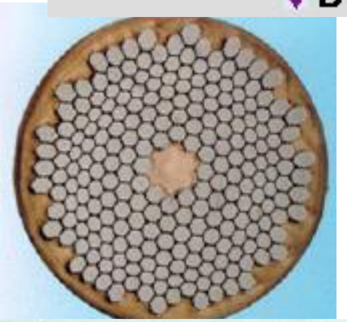


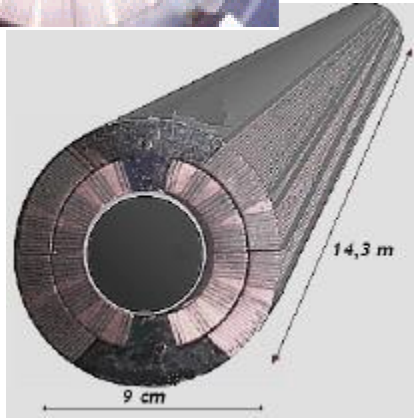


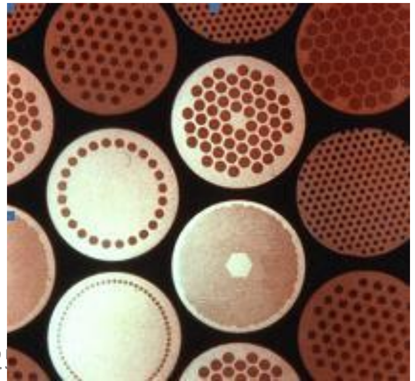


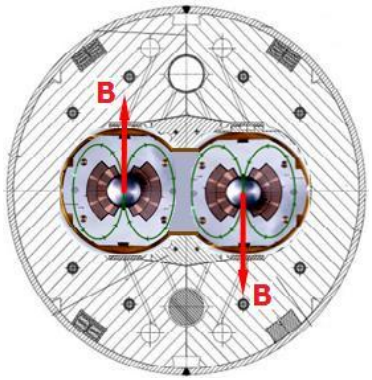




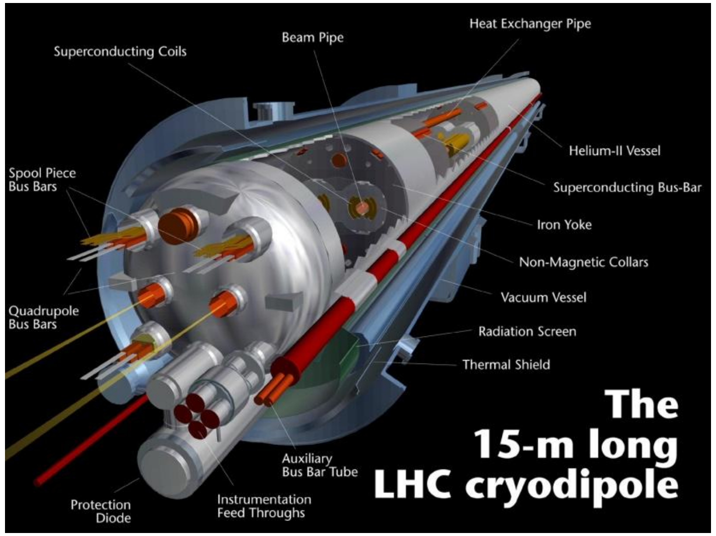




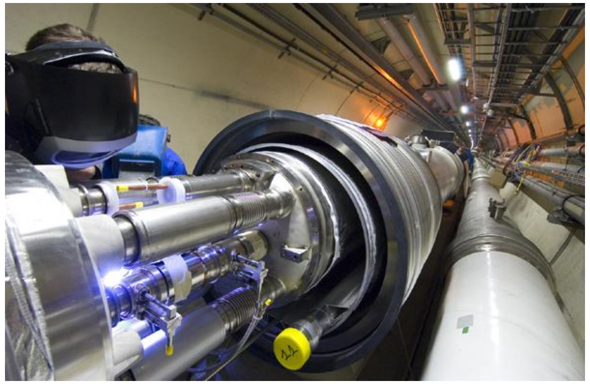


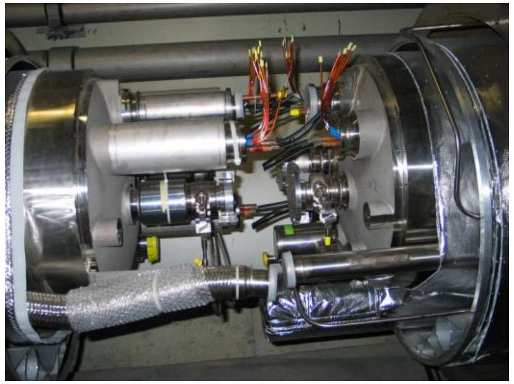


II – Guiding and focalizing magnets : dipoles, quadrupoles and multipoles



LHC at CERN





III – Magnetic field around the reference trajectory

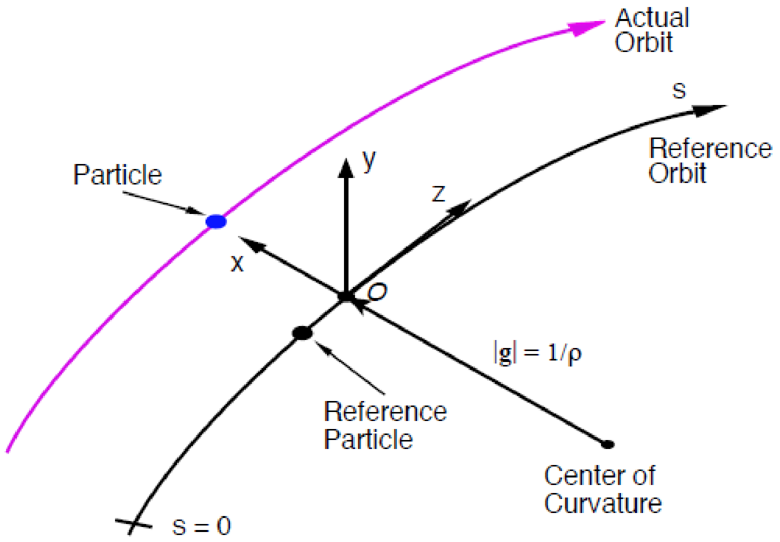
III.1 Coordinate systems

- Accelerators are structured by a succession of bending magnets and multipolar lenses (quadrupoles, sextupoles …).
- In practice, we try to put these elements in the same horizontal plane which is call mid-plane.
- For the calculation of the beam (a set of particles) motion, we describe the system in the single referential.
- By convention, we use the system (x, y, z) with a reference trajectory (C) associated to a particle (impulsion p_0)
- We use also (x,y,s) system instead of (x, y, z).

Beam vector is described in 3D by :

$$\left\{x, x', y, y', z, \delta = \frac{\Delta E}{E_0}\right\}$$

Transverse horizontal (x, x') — Transverse vertical (y, y') — Longitudinal (z, δ)



It can be extended to mass and charge:

$$\left\{x, x', y, y', z, \frac{\Delta E}{E_0}, \frac{\Delta m}{m_0}, \frac{\Delta q}{q_0}\right\}$$ *but not often used*

III.2 Field Development

The (magnetic) fields can be developed with a taylor expansion :

In plane $y = 0$: $B_y(y=0) = a_{00} + a_{01}x + a_{02}x^2 + \cdots = B_{y0}\left(1 + \frac{1}{B_{y0}}\left(\frac{\partial B_y}{\partial x}\right)_0 x + \frac{1}{2B_{y0}}\left(\frac{\partial^2 B_y}{\partial x^2}\right)_0 x^2 + \cdots\right)$

$B_y(y=0) = B_{y0}(1 - nhx + \beta h^2 x^2 + \cdots)$

Field index : $n = -\frac{1}{hB_{y0}}\left(\frac{\partial B_y}{\partial x}\right)_0$, sextupolar terme : $\beta = \frac{1}{2h^2 B_{y0}}\left(\frac{\partial^2 B_y}{\partial x^2}\right)_0$

and $h = h(s) = \frac{1}{\rho(s)} = -\frac{q}{P_0}B_{y0}(s)$ (from $B_0\rho = \frac{P_0}{q}$)

From this taylor expansion we can deduce dipole, quadrupole, sextupole… terms

Particle dynamics is calculated using the « Newton-Lorentz » equation :

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}[\boldsymbol{m\gamma v}] = \boldsymbol{q} \cdot (\boldsymbol{E} + \boldsymbol{v} \times \boxed{\boldsymbol{B}})$$ ← B field expression injected here

In the horizontal plane x : equation is : $x'' + K_x(s)x = x'' + (1-n)h^2x = h(s)\delta = f(s)$
The complete solution is :

$$x(s) = x_0C_x(s) + x_0'S_x(s) + \delta\left[S_x(s)\int_0^s \frac{C_x(s)}{\rho(s)}ds - C_x(s)\int_0^s \frac{S_x(s)}{\rho(s)}ds\right] = x_0C_x(s) + x_0'S_x(s) + D_x(s)\delta$$

$$x'(s) = x_0C_x'(s) + x_0'S_x'(s) + D_x'(s)\delta$$

Finally we have a simple equation for $\boldsymbol{x_1(x_0, x_0', \delta)}$
Where x : position, x' : angle, δ : energy

In the vertical plane y : motion equation is : $y'' + K_y(s)y = y'' + nh^2y = 0$
The complete solution is :

$$y(s) = y_0C_y(s) + y_0'S_y(s)$$

$$y'(s) = y_0C_y'(s) + y_0'S_y'(s)$$

x_0, x_0', y_0, y_0' and δ are initial particles characteristics in $s = 0$.
Functions $C(s), S(s)$ and $D(s)$ are called principal trajectories.
$D(s)$ function characterise chromatic properties of the system (dispersion function).

IV – Particles motion around the reference trajectory

Particle dynamics is calculated using the « Newton-Lorentz » equation :

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}[\boldsymbol{m\gamma v}] = \boldsymbol{q} \cdot (\boldsymbol{E} + \boldsymbol{v} \times \boxed{\boldsymbol{B}})$$

B field expression injected here

It is usually sufficient to describe beam dynamics at 1st order.

➔ Development of equations along reference path at 1st order, as a function of the 3D(+) phase space.

$$\left\{x, x', y, y', z, \frac{\Delta E}{E_0}, \frac{\Delta m}{m_0}, \frac{\Delta q}{q_0}\right\}$$

This can be expressed under the form of matrix transport :

General transport matrix :

$$\begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{bmatrix}_{final} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} \\ R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66} \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{bmatrix}_{init}$$

IV – Particles motion around the reference trajectory

Particle dynamics is calculated using the « Newton-Lorentz » equation :

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}[\boldsymbol{m\gamma v}] = \boldsymbol{q} \cdot (\boldsymbol{E} + \boldsymbol{v} \times \boxed{\boldsymbol{B}})$$

B field expression injected here

It is usually sufficient to describe beam dynamics at 1st order.
➔ Development of equations along reference path at 1st order, as a function of the 3D(+) phase space. $\left\{x, x', y, y', z, \frac{\Delta E}{E_0}, \frac{\Delta m}{m_0}, \frac{\Delta q}{q_0}\right\}$
This can be expressed under the form of matrix transport :

General transport matrix :

$$\begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{bmatrix}_{final} = \begin{bmatrix} R_{11} & R_{12} & & & & \\ R_{21} & R_{22} & & & & \\ & & R_{33} & R_{34} & & \\ & & R_{43} & R_{44} & & \\ & & & & & \\ & & & & & \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{bmatrix}_{init}$$

(horizontal: $R_{11}, R_{12}, R_{21}, R_{22}$; vertical: $R_{33}, R_{34}, R_{43}, R_{44}$)

But at first order, horizontal and vertical motions are generally independant

Matrix formalism :

We can write these equations in a matrix system :

x axis : $$\begin{pmatrix} x(s) \\ x'(s) \\ \delta \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) & D_x(s) \\ C_x'(s) & S_x'(s) & D_x(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0(s) \\ x_0'(s) \\ \delta \end{pmatrix} = T_x \begin{pmatrix} x_0(s) \\ x_0'(s) \\ \delta \end{pmatrix}$$

y axis : $$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_y(s) & S_y(s) \\ C_y'(s) & S_y'(s) \end{pmatrix} \begin{pmatrix} y_0(s) \\ y_0'(s) \end{pmatrix} = T_y \begin{pmatrix} y_0(s) \\ y_0'(s) \end{pmatrix}$$

With $\det T_x = \det T_y = 1$

T_x and T_y are called matrix transfer between two plane.

In practice, beam lines are structured by various optical elements (dipoles, quadruples, drift, …).

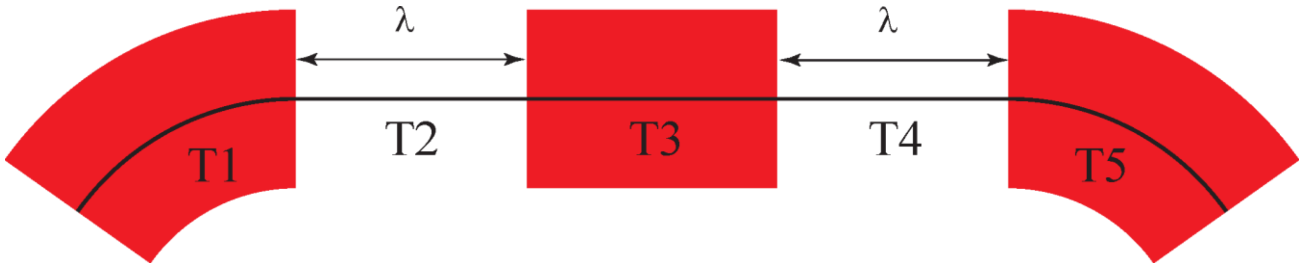
We calculate the transfer matrix T_x, T_z of each single elements.

Total matrix transfer M_x, M_z is the product of each single matrix $T_{x,z}$.

Example :

System with 5 elements : from left to right :

dipole (T_1), drift (T_2), quadruple (T_3), drift (T_4), dipole (T_5) :



The full system matrix with 5 elements is $M_{x,y} = T_{5/x,y}T_{4/x,y}T_{3/x,y}T_{2/x,y}T_{1/x,y}$

$$\begin{cases} x(s) = x_{\text{final}} = T_{x11}x_0 + T_{x12}x_0' + T_{x13}\delta \\ x'(s) = x_{\text{final}} = T_{x21}x_0 + T_{x22}x_0' + T_{x23}\delta \\ y(s) = y_{\text{final}} = T_{y11}y_0 + T_{y12}y_0' \\ y'(s) = y_{\text{final}} = T_{y11}y_0 + T_{y12}y_0' \end{cases}$$

IV – Particles motion around the reference trajectory

Matrix of a quadrupole (x focusing)	Matrix of a drift length L:	Matrix of a dipole magnet with length
. where L is its length and $k = G / B\rho_{ref}$ with $G = dB_y / dx$	γ is the relativistic factor for the reference trajectory	L & deviation ϕ $L = \phi R$ and $R_{16} = 1 - \cos(kL)$, with $k_x = 1/R$
$R = \begin{bmatrix} \cos\sqrt{k}L & \frac{\sin\sqrt{k}L}{\sqrt{k}} & 0 & 0 & 0 & 0 \\ -\sqrt{k}\sin\sqrt{k}L & \cos\sqrt{k}L & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh\sqrt{k}L & \frac{\sinh\sqrt{k}L}{\sqrt{k}} & 0 & 0 \\ 0 & 0 & \sqrt{k}\sinh\sqrt{k}L & \cosh\sqrt{k}L & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ *horizontal focusing part* *Vertical defocusing part*	$R = \begin{bmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	$R = \begin{bmatrix} \cos k_x L & \frac{\sin k_x L}{k_x} & 0 & 0 & 0 & R_{16} \\ -k_x \sin k_x L & \cos k_x L & 0 & 0 & 0 & R_{26} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -R_{16} & -R_{26} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ dispersive part

IV – Particles motion around the reference trajectory

For spectrometers :

$$\begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{bmatrix}_{final} = \begin{bmatrix} R_{11} & 0 & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & 0 & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{bmatrix}_{init}$$

Dispersion (sometimes expressed in cm/%)

Point-to-point focusing

Magnification (minimize if possible)

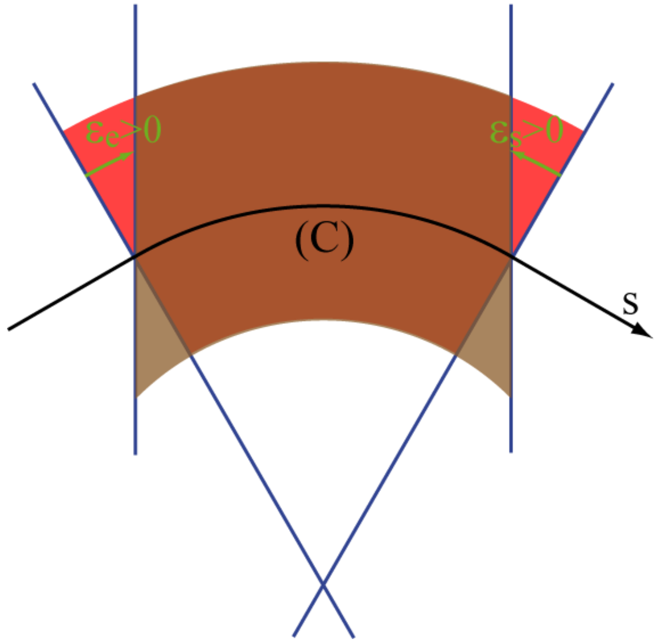
Time of fligth (bunched beams)

No energy change

$$\Rightarrow \; x_1 = R_{11} \cdot x_0 + R_{16} \cdot \delta_0$$

The final position of a particle depends on the magnification and the dispersion

d- Contribution of the dipole with entrance and exit face angle



Transfert matrix associated to the turned face :

$$T_x = \begin{pmatrix} 1 & 0 & 0 \\ \tan\varepsilon/\rho & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } T_z = \begin{pmatrix} 1 & 0 \\ -\tan\varepsilon/\rho & 1 \end{pmatrix}$$

, with ε angle of the entrance or exit face.

For $\varepsilon > 0$, beam is defocusing in horizontal and focusing in vertical plane.

2nd order effects can be added by using parabolic faces

e- Transfer matrix of the quadruple

At 1st order, motion equations are :

$$x'' + K_x(s)x = 0 \quad \text{with } K_x(s) = \frac{G}{B\rho}$$

$$z'' + K_z(s)z = 0 \quad \text{with } K_z(s) = -\frac{G}{B\rho}$$

For G>0, $K_x(s) = K = \frac{G}{B\rho} > 0$. With $\varphi = \sqrt{K}L$, L is the magnetic quadrupole length.

$$T_x = \begin{pmatrix} \cos\varphi & \sin\varphi/\sqrt{K} & 0 \\ -\sqrt{K}\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_z = \begin{pmatrix} \cosh\varphi & \sinh\varphi/\sqrt{K} \\ \sqrt{K}\sinh\varphi & \cosh\varphi \end{pmatrix}$$

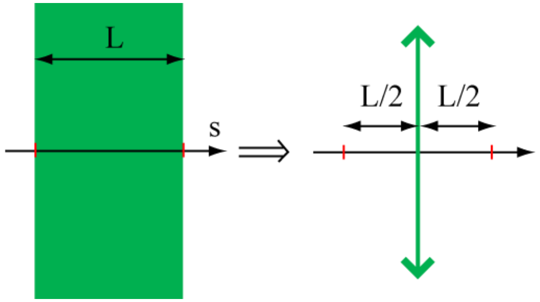
For $G > 0$ quadruple is beam focusing on horizontal plane and defocusing in vertical plane.

For $G < 0$ quadruple is beam defocusing on horizontal plane and focusing in vertical plane.

f- Thin lens

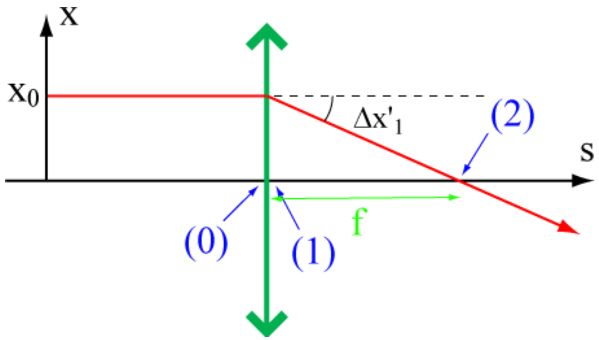
Thin lenses are use at the very beginning of a project.

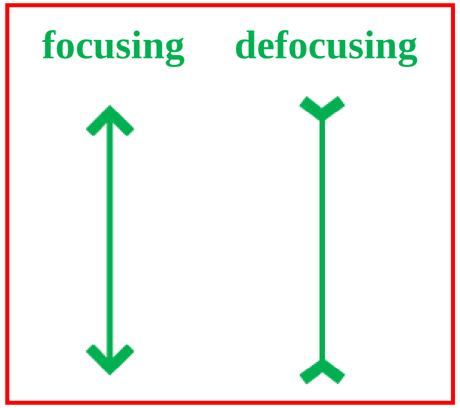
A quadruple (length L) is a thin lens with 2 drift space to length.



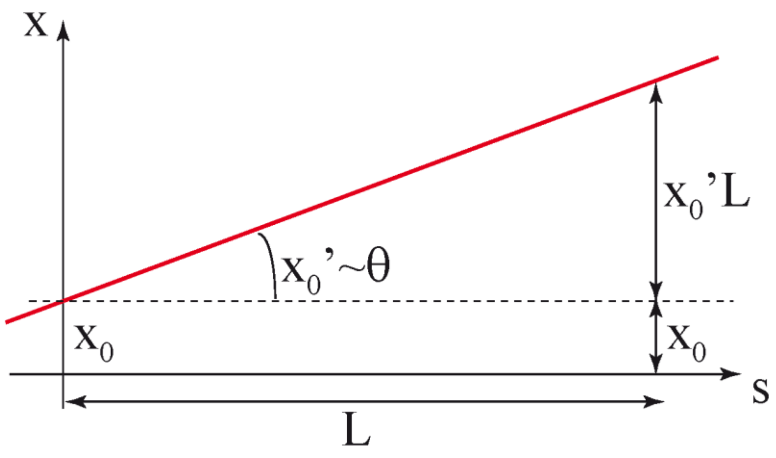
$$T_x = \begin{pmatrix} 1 & 0 & 0 \\ -KL & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_z = \begin{pmatrix} 1 & 0 \\ KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \text{ with } f = \frac{1}{KL} = \text{focal distance}$$



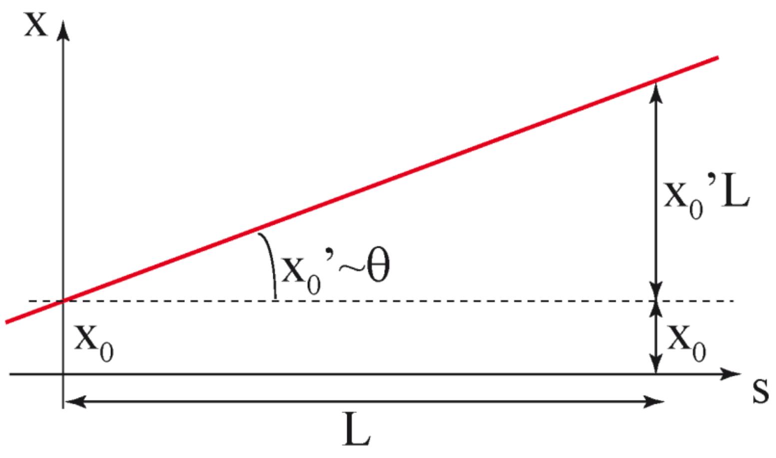


g- Matrix of the drift space to length L



Q : transfer matrix of a drift ?

g- Matrix of the drift space to length L



Motion equations in a drift are :

$$x''(s) = 0 \quad \text{and} \quad y''(s) = 0$$

with $K_x(s) = K_y(s) = 0$ and $h\delta = 0$ because $\rho = 1/h \rightarrow \infty$

$$T_x = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } T_y = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

h- Some keys words

Dispersion : $D_x(s) = T_{x13}(s) =$ position dispersion. Images position on x becomes $\Delta x(s) = T_{x13}(s)\dfrac{\Delta p}{p_0}$

$D_x'(s) = T_{x23}(s) =$ angular dispersion

Achromatic system : * Achromatic in position if $T_{x13} = D_x = 0$

* Achromatic in angle if $T_{x23} = D_x' = 0$

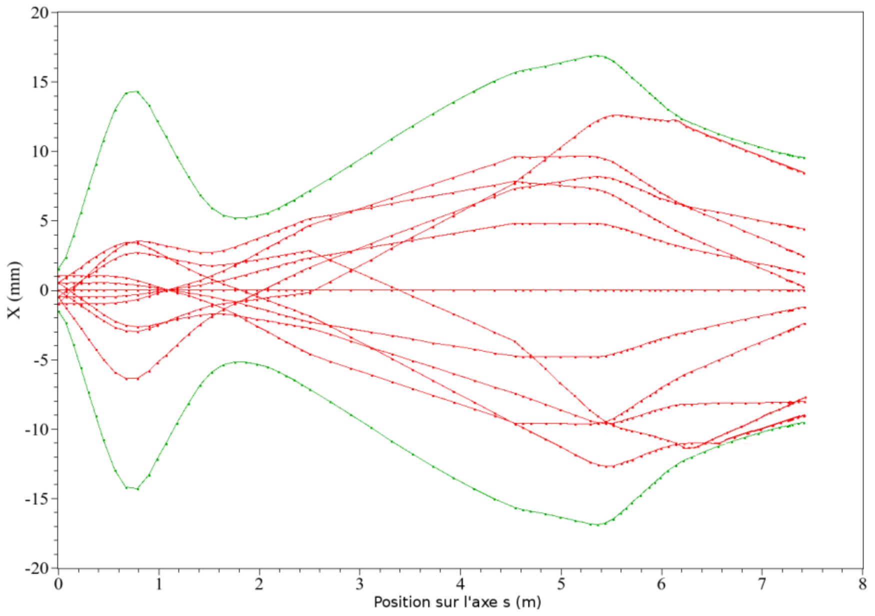
* Fully achromatic for $T_{x13} = T_{x23} = 0$

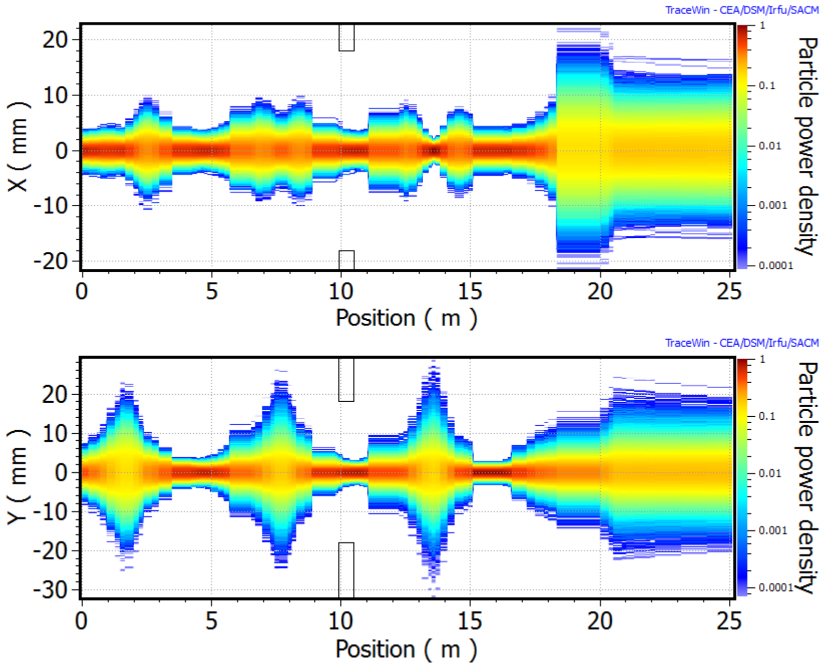
System resolution : In an point to point system ($T_{x12} = 0$), resolving power is :

$R = \dfrac{P}{\Delta P} = \left|\dfrac{T_{x13}}{2x_0 T_{x11}}\right|$. $2x_0$ is the transverse beam extension (size).

V.0 Introduction

A beam is a set of particles with different initials conditions values (x_0, x'_0, y_0, y'_0 ,z and δ). What happened to this set of particles along a line in terms of trajectories. We take about beam envelope.





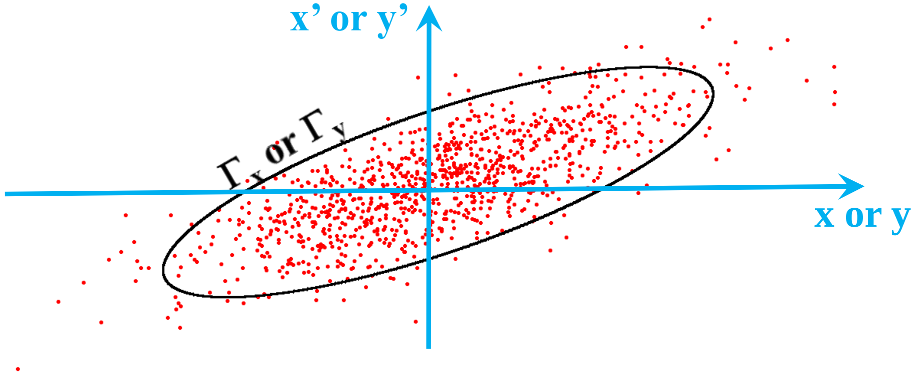
At first order, horizontal and vertical motion are decoupled.

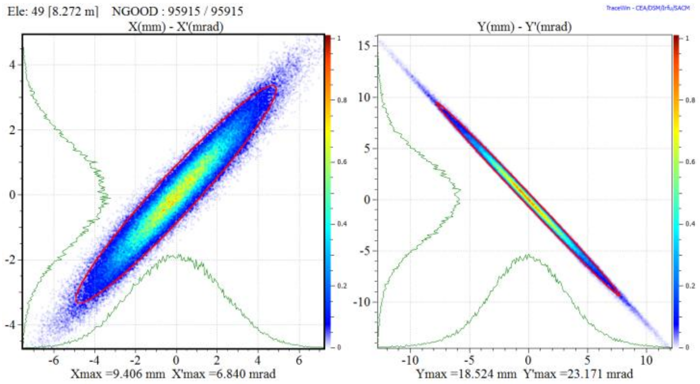
V.1 Phase space and emittance ellipse

For each particle of the beam, we give a point in the phase space (x,x') et (y,y').

In each plane, the surface occupied by all particles define the phase extension or beam emittance.

In the general case, complete phase space is 6 dimensions x, x', y, y', z, δ where l is the trajectory length difference of the beam particles.





The ellipse area, called the (RMS) beam **emittance**, is a quantity which is constant along the beam trajectory

We introduce the normalized emittance $\varepsilon_{norm} = \beta\gamma\varepsilon_{géométrique}$ =constante,

Where represente the surface $A_x = \pi\varepsilon_{geometrique\ x}$ in one plane

$\varepsilon_x = \dfrac{cte}{\beta\gamma}$ $\Rightarrow$ beam emittance decrease when beam speed increase.

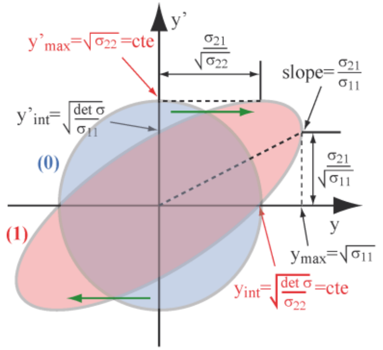
V – Beam envelop and emittance

1 – Ellipse transformation in a drift space to length L :

The drift space transfer matrix is $T = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$

$$V_{y1} = TV_{y0} \Rightarrow \begin{cases} y_1 = y_0 + Ly'_0 \\ y'_1 = y'_0 \end{cases} \Rightarrow \begin{cases} \sigma_{11}(s_1) = \sigma_{11}(s_0) + 2L\sigma_{21}(s_0) + L^2\sigma_{22}(s_0) \\ \sigma_{21}(s_1) = \sigma_{21}(s_0) + L\sigma_{22}(s_0) \\ \sigma_{22}(s_1) = \sigma_{22}(s_0) \end{cases}$$

We have $y_{int} = \sqrt{\frac{\det\sigma}{\sigma_{22}}} = cte$ and $y'_{max} = \sqrt{\sigma_{22}} = cte$

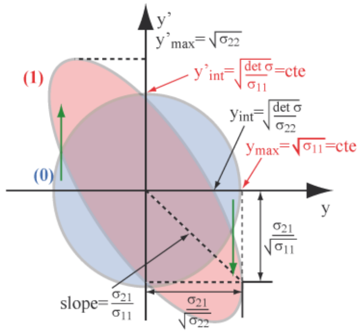


2 – Ellipse transformation in a focusing lens (same for quadrupole) :

Quadrupole transfer matrix is $\begin{pmatrix} \cos\varphi & \sin\varphi/\sqrt{K} \\ -\sqrt{K}\sin\varphi & \cos\varphi \end{pmatrix}$ or thin lense $\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$

$$V_{y1} = TV_{y0} \Rightarrow \begin{cases} y_1 = y_0 \\ y'_1 = -\frac{1}{f}y_0 + y'_0 \end{cases}$$

We have $y'_{int} = \sqrt{\frac{\det\sigma}{\sigma_{11}}} = cte$ and $y_{max} = \sqrt{\sigma_{11}} = cte$

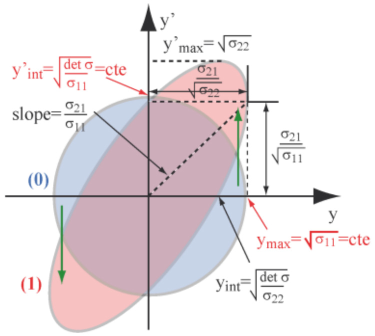


3 – Ellipse transformation in a defocusing lens (same for quadrupole) :

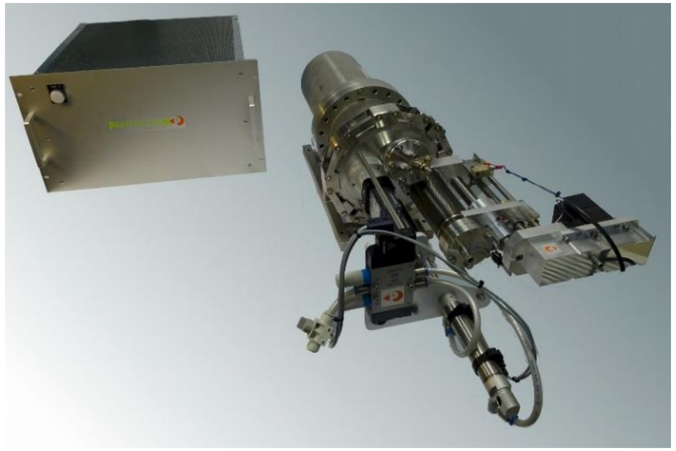
Quadrupole transfer matrix is $\begin{pmatrix} \cosh\varphi & \sinh\varphi/\sqrt{K} \\ \sqrt{K}\sinh\varphi & \cosh\varphi \end{pmatrix}$ or thin lense $\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$

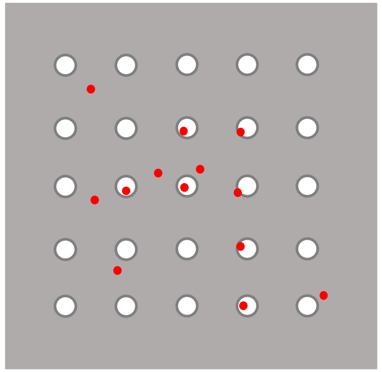
$$V_{y1} = TV_{y0} \Rightarrow \begin{cases} y_1 = y_0 \\ y'_1 = \frac{1}{f}y_0 + y'_0 \end{cases}$$

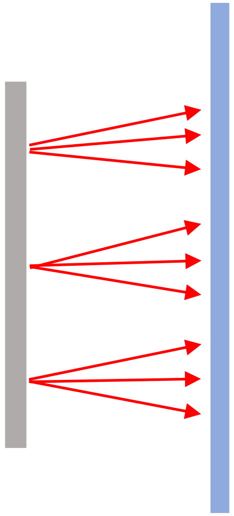
We have $y'_{int} = \sqrt{\frac{\det\sigma}{\sigma_{11}}} = cte$ and $y_{max} = \sqrt{\sigma_{11}} = cte$



V – Beam envelop and emittance





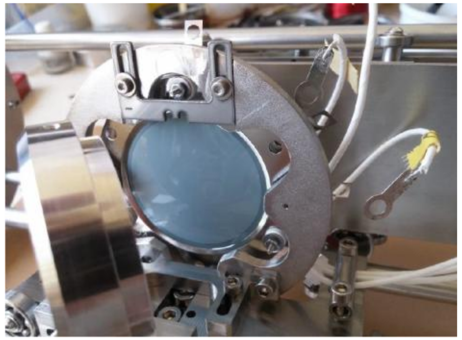


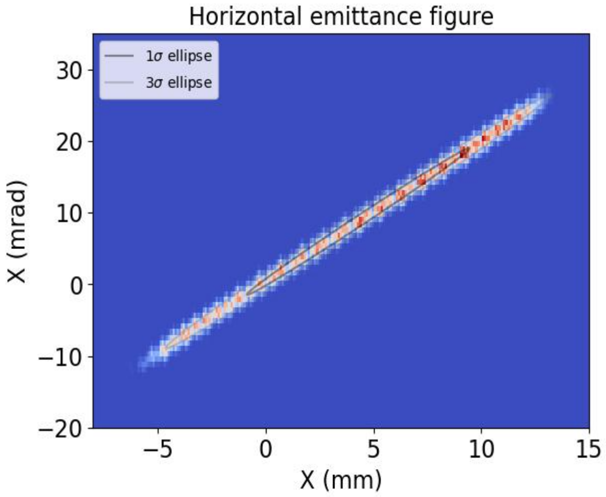
Front view:
tantal mask

Side view:
MCP + Phosphore screen

Front view:
CCD camera

Pepperpot
Emittance-meter



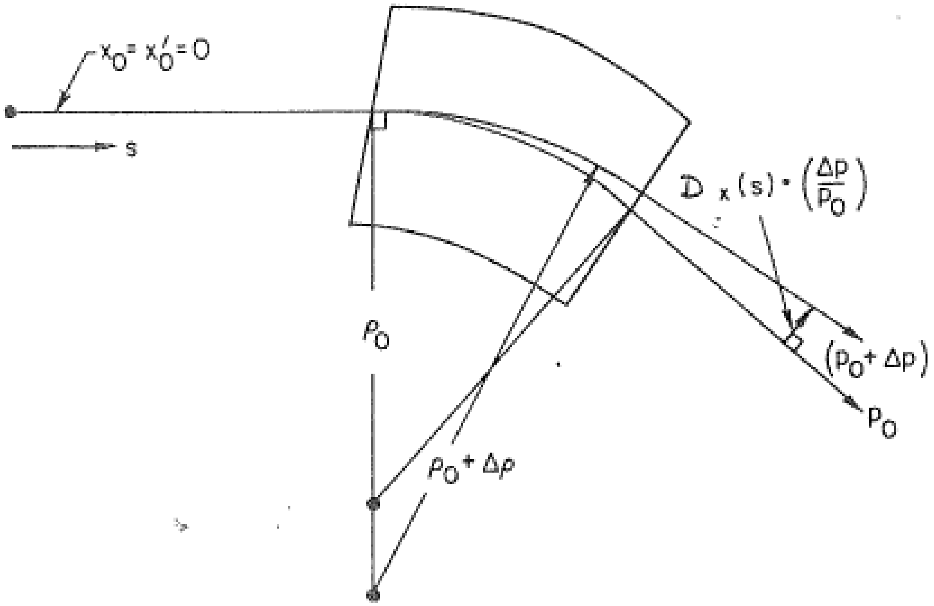


V.3 Effect of the dispersion energy

For 2 individual particles

$$\text{Particle 1 : } x_{01} = x'_{01} = 0 \text{ et } \delta_1 = 0 \Rightarrow p_1 = p_0 \text{ therefore } V_{01} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Particle 2 : } x_{02} = x'_{02} = 0 \text{ et } \delta_2 = \delta \Rightarrow p_2 = p_0\left(1+\delta\right) \text{ therefore } V_{02} = \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix}$$



For azimuth s after a bend :

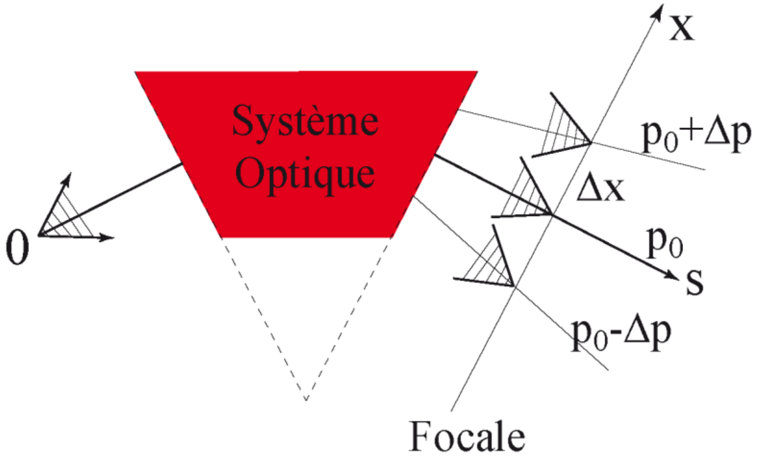
$$\begin{cases} x(s) = C_x(s)x_0 + S_x(s)x'_0 + D_x(s)\delta \\ x'(s) = C'_x(s)x_0 + S'_x(s)x'_0 + D'_x(s)\delta \end{cases}$$

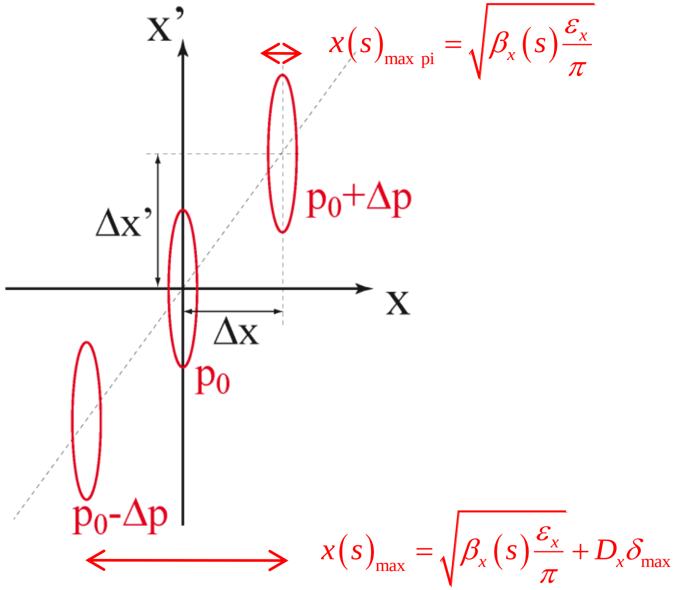
Particle 1 : $x_1(s) = x'_1(s) = 0 \ \forall s$

Particule 2 : $x_2(s) = D_x(s)\delta$ and $x'_2(s) = D_x{}'(s)\delta$

2 particles spread to Δx proportionally to δ and local dispersion $D_x(s)$ in the matrix transfer.

V.3 Effect of the dispersion energy





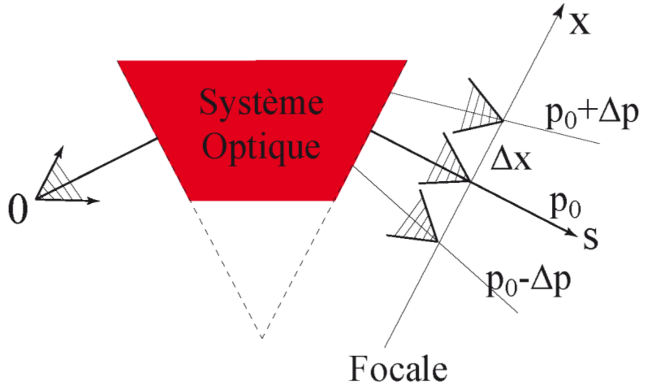
For a given value of p_i , beam extension is identical $x(s)_{\max\ pi} = \sqrt{\beta_x(s)\frac{\varepsilon_x}{\pi}}$

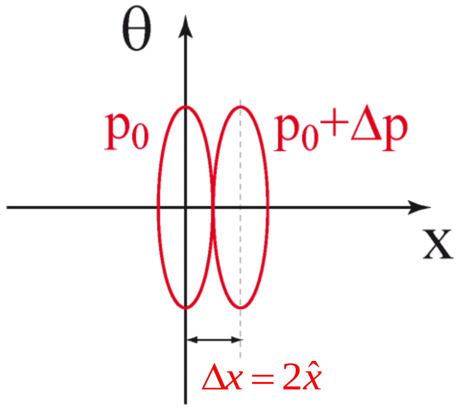
Position spread of the beam center for each p_i is $\Delta x = D_x \Delta p / p_0$

Total extension of the complete beam is : $x(s)_{\max} = \sqrt{\beta_x(s)\frac{\varepsilon_x}{\pi}} + D_x\delta_{\max}$

Resolving power

We can design an optical system with :
dispersion function $(T_{13} \neq 0)$ and focalisation at a given azimuth s $(T_{12} = 0)$.





In this dispersive plane, images center is : $\Delta x_{max} = D_x \dfrac{\Delta p_{max}}{p_0} = D_x \delta_{max} = T_{13} \dfrac{\Delta p_{max}}{p_0}$

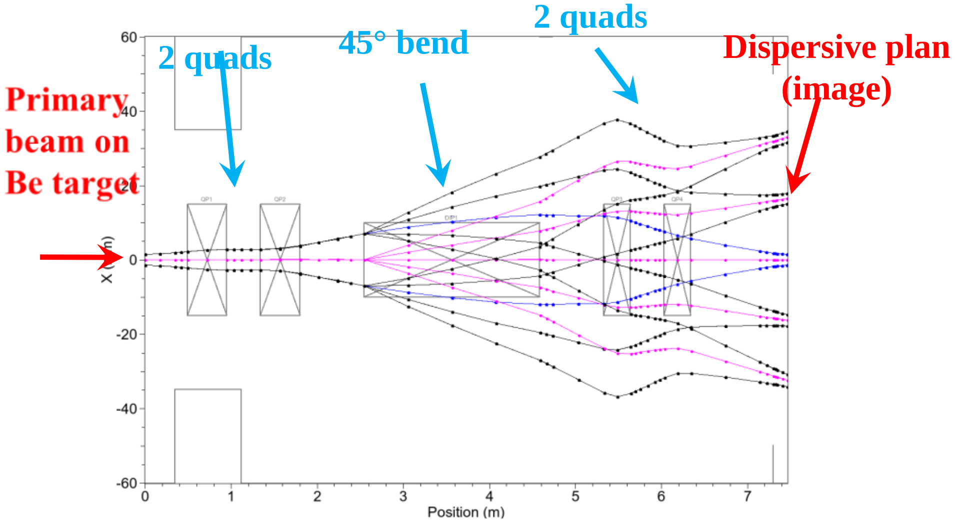
It existe a quantity $\delta_r = \dfrac{\Delta p}{p_0}$ where images are juxtaposed,

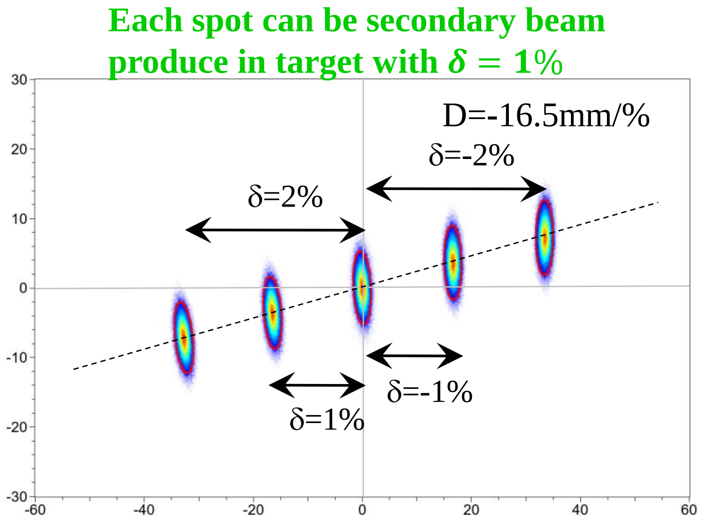
instead $\Delta x = T_{13}\delta_r = 2\hat{x}$, where $\hat{x}$ is the FWHM of the monochromatic image.

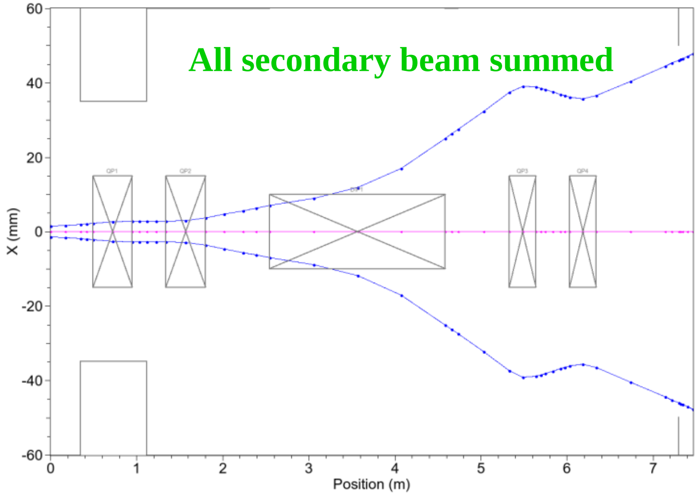
δ_r is called resolution power of the system : $\delta_r = R = \dfrac{2\hat{x}}{T_{13}} = \dfrac{2\sqrt{\beta_x \dfrac{\varepsilon_x}{\pi}}}{T_{13}}$

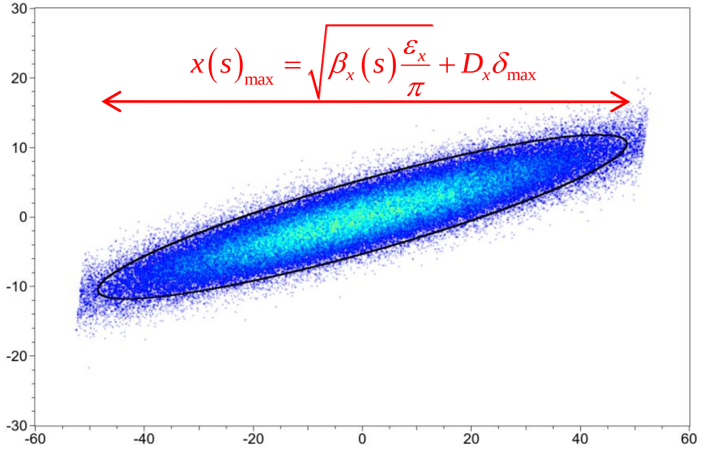
V – Beam envelop and emittance

Example with the dispersion section of the LISE (Ligne d'Ions Super Epluchés) spectrometer at GANIL in Caen.









Resolving power of this spectrometer is $\delta_r = R = \frac{2\hat{x}}{T_{13}} = \frac{2\times 0.6528}{-16.535} = 0.079\% \approx 0.1\%$

Thank you for listening !