Mellin-Barnes integrals, hypergeometric functions and Feynman integrals

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My talk at the 2022 GDR:

- Bestiary of (multivariate) hypergeometric functions: Gauss, Fox, Appell, Kampé de Fériet, Horn, Lauricella, Srivastava-Daoust, etc.
- A first step towards their numerical evaluation: Analytic continuations by two methods:
  Transformation theory and Mellin-Barnes (MB) representation.
- Advanced calculus of multifold MB integrals.
- Some applications to the analytic computation of Feynman integrals and hypergeometric functions.

Any progress since then?

Computing (scalar, master) Feynman integrals:

Current approach: differential equations.

→ Study of a (no-end?) tower of more and more complicated functions associated with various geometries:

- 1-loop: Riemann sphere (multiple polylogarithms).
- 2-loop: Elliptic and hyperelliptic curves, Calabi-Yau and Fano varietes. [P. Bargiela et al., 2026]
- 3-loop: ??
- etc.

Alternative approach: multivariate hypergeometric functions via Mellin-Barnes representation (runner-up, put aside since 2013).

Motivations

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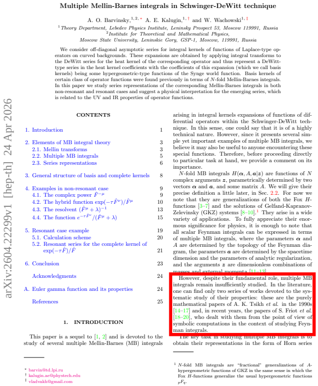

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Multiple Mellin-Barnes integrals in Schwinger-DeWitt technique

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We consider off-diagonal asymptotic series for integral kernels of functions of Laplace-type operators on curved backgrounds. These expansions are obtained by applying integral transforms to the DeWitt series for the heat kernel of the corresponding operator and thus represent a DeWitt-type series in the heat kernel coefficients with the coefficients of this expansion (which we call basis kernels) being some hypergeometric-type functions of the Synge world function. Basis kernels of certain class of operator functions were found previously in terms of N-fold Mellin-Barnes integrals. In this paper we study series representations of the corresponding Mellin-Barnes integrals in both non-resonant and resonant cases and suggest a physical interpretation for the emerging series, which is related to the UV and IR properties of operator functions.

CONTENTS

1. INTRODUCTION

This paper is a sequel to [1, 2] and is devoted to the study of several multiple Mellin-Barnes (MB) integrals arising in integral kernels expansions of functions of differential operators within the Schwinger-DeWitt technique. In this sense, one could say that it is of a highly technical nature. However, since it presents several simple yet important examples of multiple MB integrals, we believe it may also be useful to anyone encountering these special functions. Therefore, before proceeding directly to particular task at hand, we provide a comment on its importance.

N-fold MB integrals $H(\boldsymbol{\alpha}, A, \boldsymbol{a}|\boldsymbol{z})$ are functions of N complex arguments $\boldsymbol{z}$, parametrically determined by two vectors $\boldsymbol{\alpha}$ and $\boldsymbol{a}$, and some matrix A. We will give their precise definition a little later, in Sec. 2.2. For now we note that they are generalizations of both the Fox H-functions [3–7] and the solutions of Gelfand-Kapranov-Zelevinsky (GKZ) systems [8–10].[1] They arise in a wide variety of applications. To fully appreciate their enormous significance for physics, it is enough to note that all scalar Feynman integrals can be expressed in terms of multiple MB integrals, where the parameters $\boldsymbol{\alpha}$ and A are determined by the topology of the Feynman diagram, the parameters $\boldsymbol{a}$ are determined by the spacetime dimension and the parameters of analytic regularization, and the arguments $\boldsymbol{z}$ are dimensionless combinations of masses and external momenta [11–13].

However, despite their fundamental role, multiple MB integrals remain insufficiently studied. In the literature, one can find only two series of works devoted to the systematic study of their properties: these are the purely mathematical papers of A. K. Tsikh *et al.* in the 1990s [14–17] and, in recent years, the papers of S. Friot *et al.* [18–20], who dealt with them from the point of view of symbolic computations in the context of studying Feynman integrals.

The key task in studying multiple MB integrals is to obtain their representations in the form of Horn series

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[1] N-fold MB integrals are "fractional" generalizations of A-hypergeometric functions of GKZ in the same sense in which the Fox H-functions generalize the usual hypergeometric functions ${}_pF_q$.

$$\boxed{\text{However, despite their fundamental role, multiple MB integrals remain insufficiently studied.}}$$

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This is not exactly true:

T. Riemann, M. Ochman `MBsums.m` (straight contours, iterative)

From the physics side (mainly before 2013):

- M. Czackon (`MB.m`, `MBasymptotics.m`)
- J. Gluza, K. Kajda, T. Riemann (`AMBRE.m`)
- V.A. Smirnov, A.V. Smirnov, A.V. Belitsky (`MBresolve.m`, `MBcreate.m`)
- D. Kosower (`Barnesroutine.m`)

Our contribution: `MBConicHulls.wl`

Outline of the talk:

Progress has been made in the following directions:

- Straight contours of integrations in multiple MB integrals
- Speed improvement of `MBConicHulls.wl`
- Inclusion of polygamma functions in the MB integrand
- Considering fractional coefficients of the integration variables in the arguments of Γ functions

Corresponding works done in collaboration with Sumit Banik:

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e-Print: 2512.19803 (accepted in PRD)

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One-fold case:

$$I_1 = \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \; x^z \frac{\prod_{i=1}^{k} \Gamma^{a_i}(e_i z + g_i)}{\prod_{j=1}^{l} \Gamma^{b_j}(f_j z + h_j)}$$

N-fold case:

$$I_N = \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \cdots \int_{-i\infty}^{+i\infty} \frac{dz_N}{2\pi i} \; x_1^{z_1} \cdots x_N^{z_N} \frac{\prod_{i=1}^{k} \Gamma^{a_i}(\vec{\mathrm{e}_i} \cdot \vec{\mathrm{z}} + g_i)}{\prod_{j=1}^{l} \Gamma^{b_j}(\vec{\mathrm{f}_j} \cdot \vec{\mathrm{z}} + h_j)}$$

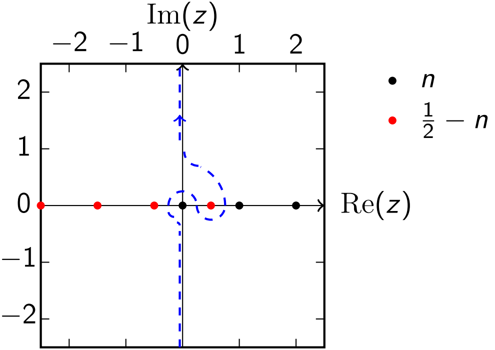
where a_i, b_j, k, l and N are positive integers and e_i, f_j, g_i, h_j are real numbers.

- $\vec{\mathrm{e}_i}$, $\vec{\mathrm{f}_j}$ are N-dimensional coefficient vectors and $\vec{\mathrm{z}} = (z_1, ..., z_N)$.

Simple one-fold example:

$$I_1 = \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i}(-x)^z\,\Gamma(-z)\Gamma(-1/2+z)$$

The integrand has poles at $z = 0, 1, 2, \cdots$ (due to $\Gamma(-z)$) and at $z = 1/2, -1/2, -3/2, \cdots$ (from $\Gamma(-1/2+z)$).



What is a Mellin-Barnes integral?

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- Closing the contour to the right takes into account the poles at $z = 0, 1, 2...$ so the series representation of the MB integral is:

$$\sum_{n=0}^{\infty} \Gamma(-1/2+n)\frac{x^n}{n!} \qquad |x| < 1$$

- Closing to the left gives us

$$(-x)^{1/2} \sum_{n=0}^{\infty} \Gamma(-1/2+n)\frac{x^{-n}}{n!} \qquad |x| > 1$$

Both series can be resummed, yielding $I_1 = -2\sqrt{\pi}\sqrt{1-x}$ which is a particular case of the well-known MB representation used in QFT

$$\frac{1}{(A+B)^{\alpha}} = \frac{1}{\Gamma(\alpha)} \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \Gamma(-z)\Gamma(\alpha+z) A^{-\alpha-z} B^{z}$$

Let us come back to the N-fold case:

$$I_N = \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \cdots \int_{-i\infty}^{+i\infty} \frac{dz_N}{2\pi i}\; x_1^{z_1} \cdots x_N^{z_N} \frac{\prod_{i=1}^{k} \Gamma^{a_i}(\vec{e_i} \cdot \vec{z} + g_i)}{\prod_{j=1}^{l} \Gamma^{b_j}(\vec{f_j} \cdot \vec{z} + h_j)}$$

Straight contours fix $\Re(z_i)$.

Interesting *per se* but can also appear when solving ϵ singularities.

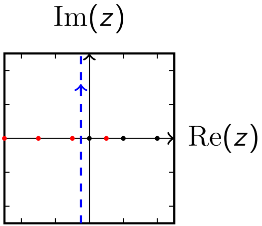
Equivalent to non-straight contours only if $\Re(\vec{e_i} \cdot \vec{z} + g_i) > 0, \forall i$.

Simple example again (suppose straight contour $\Re(z) = -1/4$):

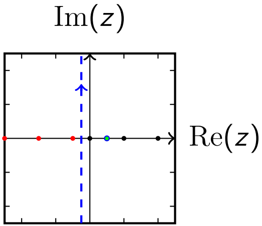
$$I_2 = \int_{-1/4-i\infty}^{-1/4+i\infty} \frac{dz}{2\pi i} (-x)^z \, \Gamma(-z)\Gamma(-1/2+z)$$

Conic hulls method does not apply here.
Using $\Gamma(s-n) = \frac{\Gamma(s)\Gamma(1-s)(-1)^n}{\Gamma(n+1-s)}$ one gets

$$I_2 = -\int_{-1/4-i\infty}^{-1/4+i\infty} \frac{dz}{2\pi i} (-x)^z \, \Gamma(-z) \frac{\Gamma(1/2+z)\Gamma(1/2-z)}{\Gamma(3/2-s)}$$



$\Rightarrow$



Conic hulls method can be applied

The conic hulls method is very efficient, but still too slow when dealing with (very) complicated objects.

Regular triangulations of particular point configurations are dual to the relevant intersections of conic hulls.

Much faster way to get the results.

New version of `MBConicHulls.wl` which now incorporates the `TOPCOM` software.

More complicated MB integrals can be computed.

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The necessary configurations of points are readily obtained from the integrand of the MB integral, after putting it in the canonical form

$$I_N = \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \cdots \int_{-i\infty}^{+i\infty} \frac{dz_N}{2\pi i}\; x_1^{z_1} \cdots x_N^{z_N} \;\prod_{i=1}^{N} \Gamma(-z_i) \frac{\prod_{i=N+1}^{k'} \Gamma^{a'_i}(\vec{e'}_i \cdot \vec{z} + g'_i)}{\prod_{j=1}^{l} \Gamma^{b'_j}(\vec{f'}_j \cdot \vec{z} + h'_j)}$$

These are

$$P_1 = e'_{i1} \;, \qquad P_2 = e'_{i2} \;, \qquad \cdots \qquad P_N = e'_{iN}$$

and $\sum_{i=N+1}^{k'} a'_i$ additional points corresponding to the unit vectors of dimension $\sum_{i=N+1}^{k'} a'_i$.

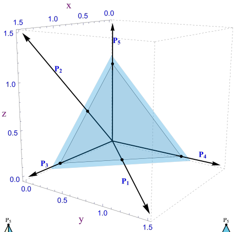
Simple two-fold example:
Let us take the Appell F_1 MB representation.

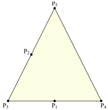
$$F_1 \sim \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz_2}{2\pi i} (-x)^{z_1} (-y)^{z_2}$$
$$\times \Gamma(-z_1)\Gamma(-z_2) \frac{\Gamma(a+z_1+z_2)\Gamma(b_1+z_1)\Gamma(b_2+z_2)}{\Gamma(c+z_1+z_2)}$$

The corresponding point configuration is

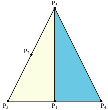
$$P_{F_1} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

The five regular triangulations of the configuration of points P_{F_1}

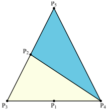




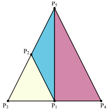
(a) Simplex
$\{3, 4, 5\}$



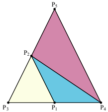
(b) Simplices
$\{1, 4, 5\}, \{1, 3, 5\}$



(c) Simplices
$\{2, 3, 4\}, \{2, 4, 5\}$



(d) Simplices
$\{1, 4, 5\}, \{1, 2, 5\}, \{1, 2, 3\}$



(e) Simplices
$\{1, 2, 4\}, \{1, 2, 3\}, \{2, 4, 5\}$

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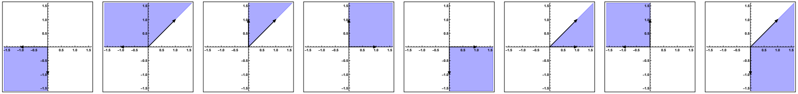
The complement in $\{1, \ldots, 5\}$ of each simplex in the 5 triangulations above yields the following sets of conic hulls :

$$\{C_{1,2}\}, \{C_{2,3}, C_{2,4}\}, \{C_{1,5}, C_{1,3}\}, \{C_{2,3}, C_{3,4}, C_{4,5}\}, \{C_{1,3}, C_{3,5}, C_{4,5}\}$$

which correspond to the 5 series representations of Appell F_1.



Conic hulls $C_{1,2}, C_{1,3}, C_{3,5}, C_{4,5}, C_{2,4}, C_{3,4}, C_{1,5}, C_{2,3}$

We applied the method on various Feynman integrals having MB representations with a high number of folds.

- hexagon and double-box (9-fold MB) $\rightarrow$ 194160 and 243186 different series representations.
- Simplest series representations are made of linear combinations of 25 series (instead of resp. 26 and 44)
- TOPCOM limits: off-shell one loop scalar massless 15-point Feynman integral (whose MB has 104 folds).

The class of integrals we consider in this work is

$$\int_{-i\infty}^{+i\infty}\frac{dz_1}{2\pi i}\cdots\int_{-i\infty}^{+i\infty}\frac{dz_N}{2\pi i}\; x_1^{z_1}\cdots x_N^{z_N}\;\frac{\prod_{i=1}^{k}\Gamma^{a_i}(\vec{e_i}\cdot\vec{z}+g_i)\prod_{k=1}^{n}\psi^{c_k}(m_k,\vec{i_k}\cdot\vec{z}+j_k)}{\prod_{j=1}^{l}\Gamma^{b_j}(\vec{f_j}\cdot\vec{z}+h_j)}$$

Polygamma functions of order m are defined as

$$\psi(m,z)=\frac{\mathrm{d}^m}{\mathrm{d}z^m}\psi(0,z)$$

where $\psi(0,z)=\frac{d}{dz}\ln\Gamma(z)$ is the digamma function.

At all non positive integers, polygamma functions of order m have poles of order $m+1$.

A simple example: the **J011**(1,2,2) sunset [Davydychev and Kalmykov 2001]

Sunsets (or sunrises) are two-loop Feynman integrals of the type

$$H_{\sigma,\nu_1,\nu_2}(m_1, m_2, m_3; p^2) \equiv$$
$$\int \frac{d^d k_1 d^d k_2}{\left[(k_1 - p)^2 - m_1^2 + i0\right]^{\nu_1} \left[k_2^2 - m_2^2 + i0\right]^{\sigma} \left[(k_1 - k_2)^2 - m_3^2 + i0\right]^{\nu_2}}$$

where $d = 4 - 2\epsilon$.

In the present case,
$$\mathbf{J011}(1, 2, 2) = (i\pi^{2-\epsilon}\Gamma(1+\epsilon))^{-2} H_{1,2,2}(m, 0, m; m^2)|_{m\to 1}.$$

Davydychev and Kalmykov have shown that

$$\mathbf{J011}(1,2,2) = \frac{1}{3\epsilon^2}\left(\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)} - 1\right)$$

whose ϵ expansion reads

$$\mathbf{J011}(1,2,2) = \frac{\pi^2}{9} + \frac{1}{3}\psi(2,1)\epsilon + \frac{\pi^4}{30}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

which can also be written as

$$\mathbf{J011}(1,2,2) = \frac{2}{3}\zeta_2 - \frac{2}{3}\zeta_3\epsilon + 3\zeta_4\epsilon^2 + \mathcal{O}(\epsilon^3)$$

Mellin-Barnes approach

Find the MB representation of **J011**(1,2,2) using AMBRE [Gluza et al. 2007]

$$\mathbf{J011}(1,2,2) = \frac{\sqrt{\pi}}{2^{1+2\epsilon}} \frac{\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)^2}$$

$$\times \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \left(-\frac{1}{4}\right)^{z} \frac{\Gamma(-z)\Gamma(1+z)\Gamma(1+\epsilon+z)\Gamma(1+2\epsilon+z)}{\Gamma\left(\frac{3}{2}+\epsilon+z\right)\Gamma(2-\epsilon+z)}$$

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Performing the ϵ expansion at the integral level, we get, for the first three terms:

$$
\begin{aligned}
\mathbf{J011}(1,2,2) = \frac{\sqrt{\pi}}{2}\int_{-i\infty}^{+i\infty}\frac{dz}{2\pi i}\left(-\frac{1}{4}\right)^{z}\frac{\Gamma(-z)\Gamma(1+z)^{\mathbf{3}}}{\Gamma\left(\frac{\mathbf{3}}{\mathbf{2}}+z\right)\Gamma(2+z)} \\
\times\Bigg[1+\epsilon\left(-2+4\gamma_E+\psi\left(0,\frac{3}{2}\right)+3\psi(0,1+z)-\psi\left(0,\frac{3}{2}+z\right)+\psi(0,2+z)\right)+ \\
+\epsilon^{\mathbf{2}}\Bigg(2-8\gamma_E+8\gamma_E^{\mathbf{2}}-\frac{\pi^{\mathbf{2}}}{12}-(2-4\gamma_E)\psi\left(0,\frac{3}{2}\right)+\frac{1}{2}\psi\left(0,\frac{3}{2}\right)^{\mathbf{2}}+\left[-2+4\gamma_E+\psi\left(0,\frac{3}{2}\right)\right] \\
\times\left[3\psi\left(0,1+z\right)-\psi\left(0,\frac{3}{2}+z\right)+\psi\left(0,2+z\right)\right]+\frac{9}{2}\psi\left(0,1+z\right)^{\mathbf{2}}-3\psi\left(0,1+z\right)\psi\left(0,\frac{3}{2}+z\right) \\
+\frac{1}{2}\psi\left(0,\frac{3}{2}+z\right)^{\mathbf{2}}+3\psi\left(0,1+z\right)\psi\left(0,2+z\right)-\psi\left(0,\frac{3}{2}+z\right)\psi\left(0,2+z\right)+\frac{1}{2}\psi\left(0,2+z\right)^{\mathbf{2}} \\
+\frac{5}{2}\psi\left(1,1+z\right)-\frac{1}{2}\psi\left(1,\frac{3}{2}+z\right)-\frac{1}{2}\psi\left(1,2+z\right)\Bigg)+\mathcal{O}(\epsilon^{\mathbf{3}})\Bigg]
\end{aligned}
$$

In a similar way as the relation

$$\Gamma(z-n) = \frac{\Gamma(z+1)\Gamma(1-z)(-1)^n}{z\,\Gamma(n+1-z)}$$

is used to compute residues in the usual MB integral case, the relation

$$\begin{aligned}\psi(m,z-n) =& \frac{(-1)^{m+1}m!}{z^{m+1}} + \psi(m,z+1) \\ &+ (-1)^{m+1}\psi(m,1-z) + (-1)^m\psi(m,1+n-z)\end{aligned}$$

is used when polygamma functions also appear in the integrand.

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In the case of straight contours:

$$\psi(0,z) = \lim_{a\to 0} \frac{d}{da}\left[\frac{\Gamma(z+a)}{\Gamma(z)}\right]$$

$$\psi(m,z) = \lim_{a\to 0}\lim_{b\to 0} \frac{d^m}{db^m}\frac{d}{da}\left[\frac{\Gamma(z+a+b)}{\Gamma(z+b)}\right] \quad \text{for} \quad m \geq 1$$

The computation of N-fold MB integrals with polygamma functions and straight or non-straight contours can now be performed automatically with `MBConicHulls.wl`.

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In QFT and hypergeometric function theory one is in general concerned with MB integrals having integer coefficients in front of the integration variables which appear in the arguments of the various Γ and ψ functions of the integrand.

Fractional values of these coefficients provide an interesting generalization of the latter, in the same way as Fox functions generalize ${}_pF_q$ generalized hypergeometric functions.

Although our initial algorithm is in principle valid for these cases also, not enough checks have been performed.

$\rightarrow$ This is work in progress.

The *Mathematica* package `MBConicHulls.wl` has been improved to allow the computation of multifold MB integrals of high complexity:

- Both the straight and non-straight contours cases can be handled.
- A triangulation approach has been implemented to increase the speed of the package for the computation of higher-fold MB integrals.
- The inclusion of polygamma functions of arbitrary order in the MB integrand is now possible.
- Fractional values of the coefficients can be considered.

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There is still a lot of work to be done:

- Improve the method for Feynman integrals having less scales than integration variables. [Souvik and Sumit, e-Print: 2605.30216 [hep-ph]]
- Develop a systematic way to reach the "white zones". [Souvik and Sumit, e-Print: 2605.30216 [hep-ph]]
- Better understand the properties of the master series.
- Study non-degenerate MB integrals deeper (links with asymptotics in several variables, resurgence, etc.).
  $\rightarrow$ work in progress