Hadronic Vacuum Polarization in Three-Loop Chiral Perturbation Theory

GDR QCD workshop, Orsay

Based on [2510.12885] [2603.15252]

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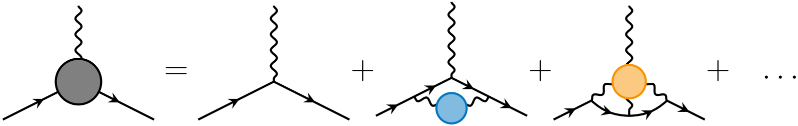
Introduction

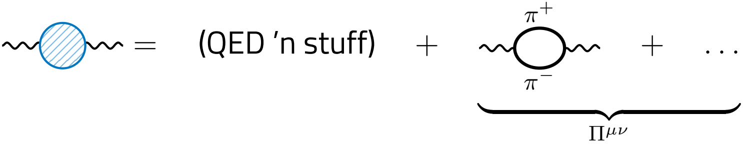
HVP recap

- Fermion-photon coupling (g-factor, etc.)



- ...from which the *vacuum polarization* 

- ...from which the *hadronic* vacuum polarization $\Pi^{\mu\nu}$



The lattice-ChPT symbiosis

- ChPT models HVP well
- ...but precise HVP takes Lattice QCD BMW/DMC [2407.10913]
- ...but there are strong finite-volume effects (FVEs)
- ChPT is much more accurate for FVEs than for the whole HVP

A rough outline

$$(\text{lattice QCD})_L + \underbrace{(\text{ChPT})_\infty - (\text{ChPT})_L}_{\substack{\text{no high-energy modes} \\ \text{fewer parameters} \\ \text{good for ChPT}}} \approx (\text{true QCD})_\infty$$

The challenge

PHYSICAL REVIEW D **102,** 094511 (2020)

Application of effective field theory to finite-volume effects in a_μ^{HVP}

Christopher Aubin,[1] Thomas Blum,[2] Maarten Golterman,[3] and Santiago Peris[4]

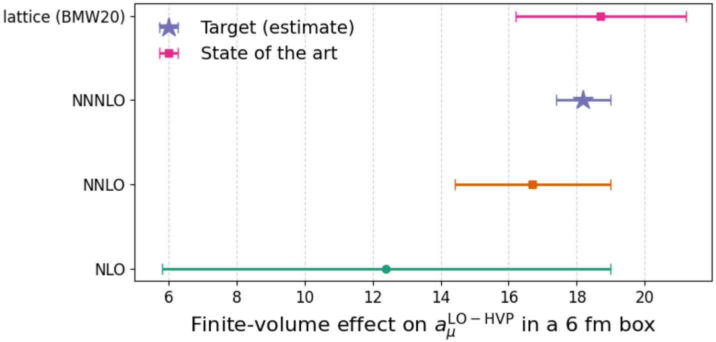
appear already at two loops, which is the minimum number. While it is unlikely that the full N^{3}LO analysis will ever be worked out in practice, it is important to establish the validity of the EFT framework for the study of a_μ^{HVP} in order to be assured that even the NNLO analysis of finite-volume effects carried out in Ref. [10] has a solid EFT basis. We believe that our discussion in this paper illustrates why indeed this is the case. We expect the same separation

Arthur C. Clarke's 1st law

"When a distinguished [...] scientist [...] states that something is impossible, he is very probably wrong."

The ultimate goal

Assuming a geometric progression of FVE errors...



(analysis & figure by Alessandro Lupo)

The collaboration

initiated by  **Kálmán Szabo**

implemented by  **Mattias Sjö**

loop integrals by  **Pierre Vanhove**

phenomenology by  **Alessandro Lupo**

under the guidance of  **Laurent Lellouch**

The ChPT setup

The ChPT Lagrangian

- 2 flavors (pions only), non-dynamic photon field
- Vertices with $2n$ pions and up to 2 photons, drawn from

$$\mathcal{L}_{\text{ChPT}} = \mathcal{L}_{\text{LO}} + \mathcal{L}_{\text{NLO}} + \mathcal{L}_{\text{NNLO}} + \mathcal{L}_{\text{N}^3\text{LO}} + \cdots$$

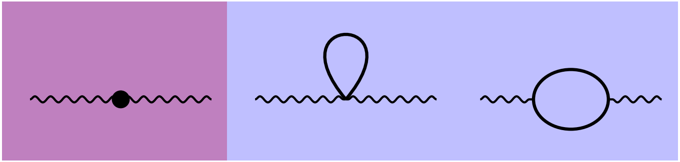
- $\mathcal{L}_{\text{N}^3\text{LO}}$ → 475^* $\mathcal{O}(p^8)$ counterterms
  Bijnens, (Herman-Trued)sson & Wang [1810.06384]
- $\mathcal{L}_{\text{NNLO}}$ → 57^* $\mathcal{O}(p^6)$ counterterms
  Bijnens, Colangelo & Ecker [hep-ph/9902437], [hep-ph/9907333]
- $\mathcal{L}_{\text{NLO}}$ → 7^* $\mathcal{O}(p^4)$ counterterms
  Gasser & Leutwyler (1984)
- $\mathcal{L}_{\text{LO}}$ → Basic $\mathcal{O}(p^2)$ vertex

Power counting (the simple way)

$$\text{N}^k\text{LO diagram:} \qquad k = (\#\text{ loops}) + \sum_{\text{vertices}} (\#\text{ N's})$$

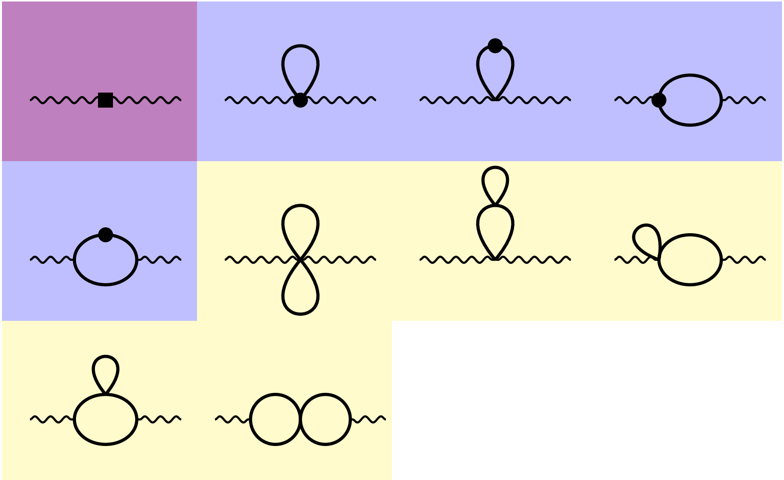
*These are 2-flavor numbers! It's 11,94,1254 for 3 flavors and 12,115,1862 for $\geq$ 4 flavors

NLO diagrams



$\bullet$ = NLO (unmarked) = LO

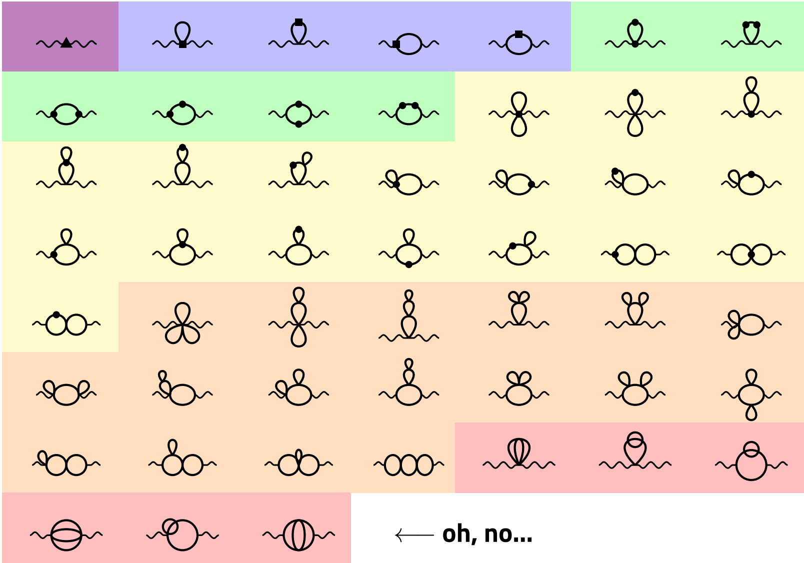
NNLO diagrams



$\blacksquare$ = NNLO $\bullet$ = NLO (unmarked) = LO

Note: no "genuine" 2-loop diagrams

N^3LO diagrams



⟵ **oh, no...**

ChPTlib

- New suite for ChPT calculations
  - "Automation of Hans Bijnens's spirit"
  - Written in FORM

    Vermaseren [math-ph/0010025]
  - Flexible implementation of $\mathcal{L}_{\text{ChPT}}$
  - Efficient extraction of Feynman rules
  - Simple input of Feynman diagrams
  - Soon to include FORM 5 diagram generation Vermaseren *et al.* [2601.19982]

- Git repository: `github.com/mssjo/ChPTlib`
- HVP application: `github.com/mssjo/HVP-3loop`

Loop-wrangling

State-of-the art loop tools

IBP reduction

All integrals $\rightarrow$ small number of *master integrals*
Implemented with Roman Lee's `LiteRed2` [1212.2685]

Tarasov's dimensional shift

$(4-2\epsilon)$-dimensional integrals (divergent)
$\rightarrow (2-2\epsilon)$-dimensional integrals (mostly finite)

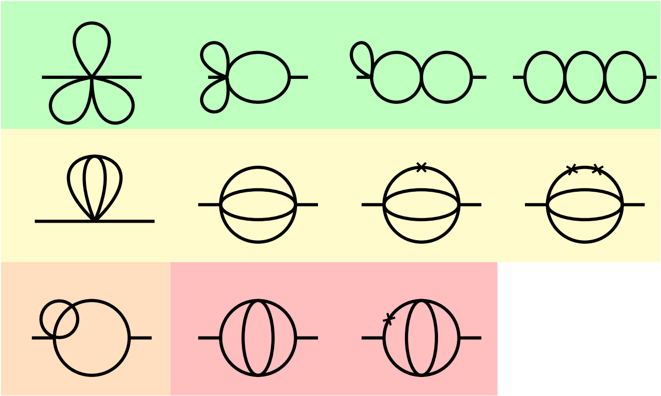
Schouten relations

Fix remaining divergences (novel!)

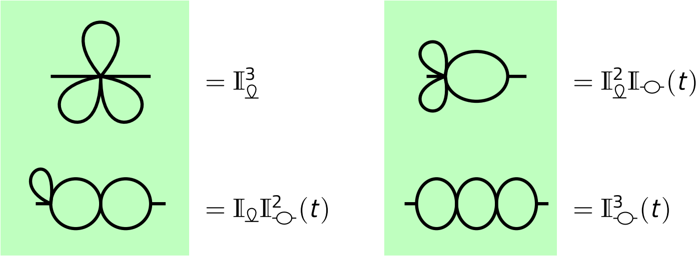
Evaluation of integrals

Analytic and high-precision semi-analytic

A tour of our masters

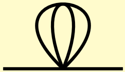


Masters: the easy ones

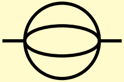


- Divergent (even in 2D) but simple
- Bubble integral required up to $\mathcal{O}(\epsilon^2)$
- Polylogarithms of the photon virtuality t

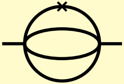
Masters: the beachball



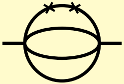
$$= \mathbb{I}_{\ominus}(0) = 7\zeta(3)$$



$$\equiv \mathbb{I}_{\ominus}(t)$$



$$= [\mathbb{I}_{\ominus}(t), \mathbb{I}'_{\ominus}(t)]$$



$$= [\mathbb{I}_{\ominus}(t), \mathbb{I}'_{\ominus}(t), \mathbb{I}''_{\ominus}(t)]$$

- Example of the well-studied *banana graphs*
- $\mathbb{I}_{\ominus}(t)$ known (!) in terms of elliptic functions

  Bloch, Kerr & Vanhove [1406.2664]
- The linchpin of our amplitude

The elliptic wonderland

Take q, $|q| < 1$ such that

$$t = -\left[\frac{\eta(q)\eta(q^3)}{\eta(q^2)\eta(q^6)}\right]^6,$$

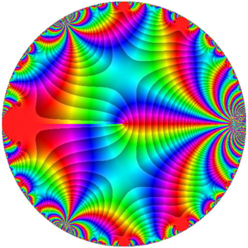
using the Dirichlet η function

Then

$$\mathbb{I}_{\ominus}(t) = \frac{\left[\eta(q^2)\eta(q^6)\right]^4}{\left[\eta(q)\eta(q^3)\right]^2}$$

$$\times\left(3(\log q)^3 - 16\zeta(3) + \sum_{n=1}^{\infty}\frac{\psi_{\ominus}(n)}{n^3}\frac{q^n}{1-q^n}\right)$$

$\psi_{\ominus} = \{1, -15, -8, -15, 1, 120\}$ repeated



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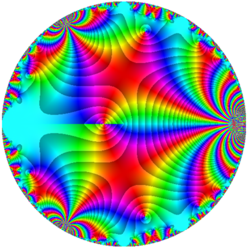

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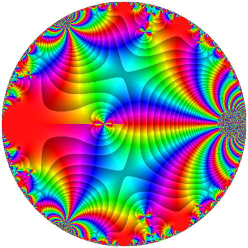

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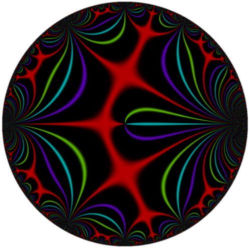
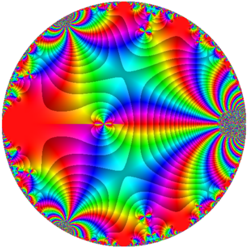




The road to wonderland

$$q(t) = i\frac{\varpi_c(\tilde{t})}{\varpi_r(\tilde{t})}$$

$$\tilde{t} = \frac{5t - 32 + 4\sqrt{(z-4)(z-16)}}{t}$$

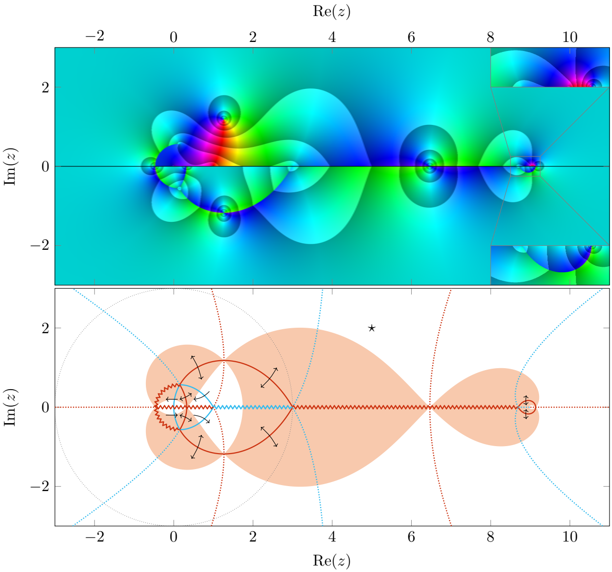
$$\varpi_c(z) = \frac{2\pi\rho(z)}{\sqrt[4]{z-3}\sqrt[4]{z^3-9z^2+3z-3}}$$

$$\varpi_r(z) = \frac{12\sqrt{3}}{z-9}\,\varpi_c\left(\frac{9z-9}{z-9}\right)$$

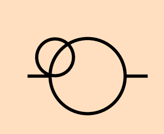
$$\rho(z) = {}_2F_1\left(\left.\begin{matrix} 1/12, 5/12 \\ 1 \end{matrix}\right| \frac{1728(z-9)(z-1)^3z^2}{(z-3)^3(z^3-9z^2+3z-3)^3}\right)$$

...except not really $(\sqrt{x^2} \neq x)$

The odyssey of monodromy

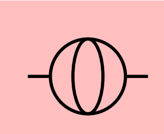


Masters: the irrelevant one

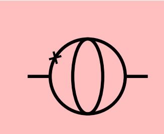


- Less well-studied, no explicit solution known before
- **Vanishes** under dimensional shift
  (up to $[\mathbb{I}_{\bigcirc}, \mathbb{I}_{\ominus}(t), \mathbb{I}'_{\ominus}(t), \mathbb{I}''_{\ominus}(t)]$)

Masters: the cat's eye



$\equiv \mathbb{I}_{\oplus}(t)$



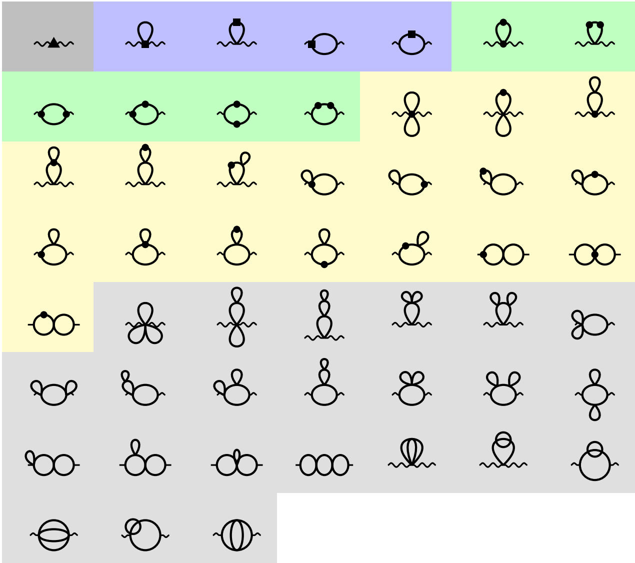
$\sim \mathbb{I}'_{\oplus}(t)$

- Very difficult
- Seemingly breaks renormalization:

$$\mathbb{I}_{\ominus}^{(d=4-2\epsilon)} \sim \overbrace{\frac{1}{\epsilon^3} + \frac{[\mathbb{I}_{\circ}]}{\epsilon^2} + \frac{[\mathbb{I}_{\circ}]}{\epsilon}}^{\text{managed by counterterms}} + \overbrace{\bar{\mathbb{I}}_{\ominus}}^{\text{finite}}$$

$$\mathbb{I}_{\oplus}^{(d=4-2\epsilon)} \sim \frac{1}{\epsilon^3} + \overbrace{\frac{[\mathbb{I}_{\circ}, \mathbb{I}_{\ominus}, \mathbb{I}_{\oplus}]}{\epsilon^2} + \frac{[\mathbb{I}_{\circ}, \mathbb{I}_{\ominus}, \mathbb{I}_{\oplus}]}{\epsilon}}^{\textbf{no counterterm!}} + \overbrace{\bar{\mathbb{I}}_{\oplus}}^{\text{finite}}$$

Masters: the cat’s eye



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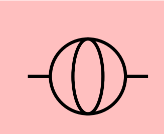
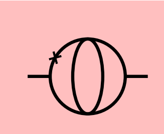
 $\equiv \mathbb{I}_{\oplus}(t)$  $\sim \mathbb{I}'_{\oplus}(t)$

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Schouten relations

Gram determinant — Schouten identity

$$\mathbb{G}(u,v,w) = \begin{pmatrix} u^2 & u\cdot v & u\cdot w \\ v\cdot u & v^2 & v\cdot w \\ w\cdot u & w\cdot v & w^2 \end{pmatrix} \quad \Rightarrow \quad |\mathbb{G}(u,v,w)| = 0 \text{ in } d=2$$

- Put $\mathbb{G}(\ell_1, \ell_2, \ell_3)$ as numerator in suitable integral
  $\Rightarrow$ linear combination $[\mathbb{I}_{\circ}, \mathbb{I}_{\ominus}, \mathbb{I}_{\oplus}]$ vanishing at $d=2$
  $\Rightarrow$ renormalization restored
- Effectively eliminates master integral $\mathbb{I}'_{\oplus}(t)$ in 2D
  $\Rightarrow$ non-IBP relation!
- Novel here, but known in 2-loop sunset integrals with 3 independent masses Remiddi & Tancredi [1311.3342]

Solving the cat's eye

Using $t = \frac{4}{1-\beta^2}$,

$$\beta^2 \left[\frac{\mathrm{d}^2}{\mathrm{d}\beta^2} - \frac{2\beta^2}{\beta^2 - 1} \right] \mathbb{I}^{(d=2)}(\beta) = S(\beta)\,, \qquad S(\beta) \sim [\mathbb{I}, \mathbb{I}]$$

Solutions without the RHS:

$$g_1(\beta) = \beta^2 - 1 \qquad g_2(\beta) = \frac{\beta^2 - 1}{4} \log\left(\frac{\beta + 1}{\beta - 1}\right) - \frac{\beta}{2}$$

Full Wrońskian solution:

$$\begin{aligned} \mathbb{I}^{(d=2)}(\beta) = g_1(\beta) &\left[c_1 - \int_{\xi_1}^{\beta} S(\xi) g_2(\xi) \frac{\mathrm{d}\xi}{\xi^2} \right] \\ &+ g_2(\beta) \left[c_2 + \int_{\xi_2}^{\beta} S(\xi) g_1(\xi) \frac{\mathrm{d}\xi}{\xi^2} \right] \end{aligned}$$

Must **integrate** $\mathbb{I}$!

...with some effort

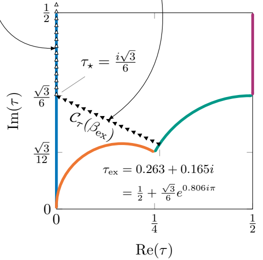
$$\mathcal{I}^{(1)}_{\text{rat}}(\beta_{\text{ex}}) = 21.034 - 2.937i \qquad \mathcal{I}^{(2)}_{\text{rat}}(\beta_{\text{ex}}) = 2.394 - 1.602i \qquad \mathcal{I}_{\text{div}}(\beta_{\text{ex}}) = 47.842 - 8.179i$$

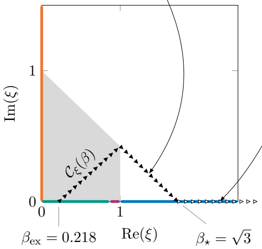
$$\mathcal{I}^{(1)}_{E}(\beta_\star) = (-6.020)_{<\chi} - (0.095)_{>\chi} = -6.115$$
$$\mathcal{I}^{(2)}_{E}(\beta_\star) = (\ \ 2.826)_{<\chi} + (0.001)_{>\chi} = +2.827$$

$$\mathcal{I}^{(1)}_{J}(\beta_\star) = (-0.897)_{<\chi} - (2\cdot 10^{-4}\ )_{>\chi} = -0.897$$
$$\mathcal{I}^{(2)}_{J}(\beta_\star) = (\ \ 0.008)_{<\chi} + (1\cdot 10^{-17})_{>\chi} = 0.008$$

$$\int_{C_\tau(\beta_{\text{ex}})} h_1 \dots \mathrm{d}\tau = 44.172 + 16.284i$$
$$\int_{C_\tau(\beta_{\text{ex}})} h_2 \dots \mathrm{d}\tau = \ 8.462 - 67.465i$$

$$\int_{\mathcal{C}_\xi(\beta_{\text{ex}})} g_1 \dots \mathrm{d}\xi = 70.061 - 21.617i$$
$$\int_{\mathcal{C}_\xi(\beta_{\text{ex}})} g_2 \dots \mathrm{d}\xi = \ 2.283 - 45.262i$$





$$\begin{aligned} G_{15}(\beta_{\text{ex}}) &= -0.952 \\ G_{25}(\beta_{\text{ex}}) &= -0.215 + 0.748i \\ B_5(\beta_{\text{ex}}) &= 12.853 \\ \mathcal{I}[\beta_{\text{ex}}; G_{15}(\beta_{\text{ex}}), G_{25}(\beta_{\text{ex}})] &= -10.294 + 18.144i \\ \boldsymbol{\bar{E}_5(4, t_{\text{ex}})} &= \mathbf{2.559 + 18.144}\boldsymbol{i} \end{aligned}$$

$$\begin{aligned} G_{16}(\beta_{\text{ex}}) &= 0 \\ G_{26}(\beta_{\text{ex}}) &= 0.546 \\ B_6(\beta_{\text{ex}}) &= -48.299 + 0.841i \\ \mathcal{I}[\beta_{\text{ex}}; G_{16}(\beta_{\text{ex}}), G_{26}(\beta_{\text{ex}})] &= 96.069 - 8.974i \\ \boldsymbol{\bar{E}_6(4, t_{\text{ex}})} &= \mathbf{47.770 - 8.134}\boldsymbol{i} \end{aligned}$$

Results

HVP overview

Parameters

M_π, F_π	Pion mass & decay constant
$t = q^2/M_\pi^2$	Normalized photon virtuality
$l_i, c_i, \tilde{c}_i$	LECs (l_i well known, c_i poorly known, $\tilde{c}_i$ unknown)

- **Transverse** due to Ward identity
- **Finite** after $\overline{MS}$ renormalization (ChPT variant)
- **Invariant** under NG manifold reparametrization
- **Subtracted** at $t = 0$

Power counting layout

$$16\pi^2\left[\Pi_T(t) - \Pi_T(0)\right] = \bar{\Pi}^{\mathrm{NLO}}(t) + \xi\bar{\Pi}^{\mathrm{NNLO}}(t) + \xi^2\bar{\Pi}^{\mathrm{N^3LO}}(t) + \mathcal{O}(\xi^3)$$

(convergent since $\xi = M_\pi^2/(16\pi^2 F_\pi^2) \approx 0.03$)

Lower orders

- Leading order Gasser & Leutwyler (1985)

$$\bar{\Pi}^{\text{NLO}}(t) = \; +2B(t) + \tfrac{2}{3}$$

- Next-to-leading order Golowich & Kambor (1995)

$$\bar{\Pi}^{\text{NNLO}}(t) = \; + tB(t)^2 \; - 4tB(t)l_6 \; - 8c_{56}t$$

Function of t only	LECs known	LECs TBD	LECs unknown
Dispersive+FVE	FVE only (?)	No Dispersive/FVE	

Bubble function

$$B(t) = -\frac{1}{9} + \frac{4-t}{3t}\int_0^1 \log\left[1 - x(1-x)t\right]\,\mathrm{d}x$$

Three-loop order

- Next-to-next-to-next-to-leading order This work (2025)

$$
\begin{aligned}
\bar{\Pi}^{\mathrm{N^3LO}}(t) = \; & + [\bar{\mathbb{I}}, \bar{\mathbb{I}}, \bar{\mathbb{I}}', \bar{\mathbb{I}}'', \bar{\mathbb{I}}, \bar{\mathbb{I}}'] \\
& + tB(t)^2 \left[t(l_2 - 2l_1 - 2l_6) + 2l_4 \right] \\
& + 2tB(t) l_6 (4l_4 - l_6 t) \\
& + 4tB(t) \left(r_{V1} + r_{V2} t \right) \\
& - 16 l_4 c_{56} t \\
& + 16t(\tilde{c}_{332} - \tilde{c}_{333}) - 8t^2 \tilde{c}_{459}
\end{aligned}
$$

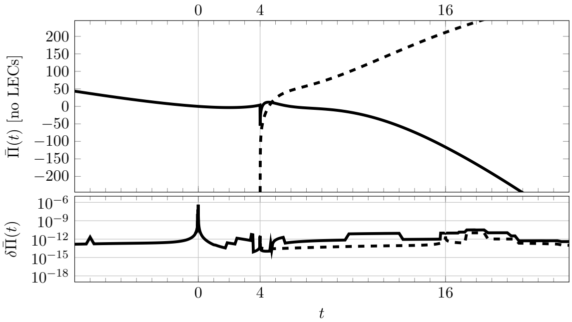
Function of t only | LECs known | LECs TBD | LECs unknown

Dispersive+FVE | FVE only (?) | No Dispersive/FVE

Pion vector form factor NNLO LECs

$$
r_{V1} = -4(2c_{53} + 2c_{35} + 4c_6)\,, \qquad r_{V2} = 4(c_{53} - c_{51})
$$

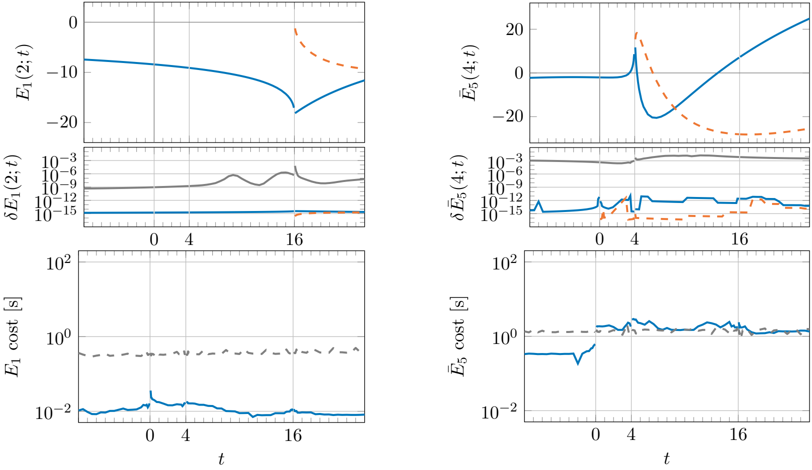
The LEC-independent part



Evaluation of master integrals

- Stable over entire complex plane
- 10^3–10^{12} times more precise than `pySecDec`
  Borowka et al. [1703.09692]
- 1–1000 times faster than `pySecDec`

Benchmarking against pySecDec



Loop tools in finite volume

Loop integrals (Minkowski)

$$\int_\ell \frac{1}{\prod_j (k_j^2 - m_j^2 + i\varepsilon)^{\nu_j}}$$

Integration-by-parts reduction

All integrals $\to$ small number of *master integrals*
Implemented with Roman Lee's `LiteRed2` [1212.2685]

Tarasov's dimensional shift + Schouten relations

$(4 - 2\epsilon)$-dimensional integrals (divergent)
$\to (2 - 2\epsilon)$-dimensional integrals (finite)

Loop tools in finite volume

Loop integroids (Euclid)

$$\hat{\mathcal{O}}_\ell \frac{1}{\prod_j (k_j^2 + m_j^2)^{\nu_j}}$$

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Operator-agnostic reduction

All integroids $\to$ small-ish number of *master integroids*
Novel method based on momentum rerouting

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Operator-agnostic reduction

All integroids $\rightarrow$ small-ish number of *master integroids*
Novel method based on momentum rerouting

Divergence rearrangement

Integroids (divergent, possibly non-factorizable)
$\rightarrow$ Integroids (finite, non-factorizable) + (divergent, factorizable)

Summary and Outlook

- Amplitude calculation published
- Loop integrals on arXiv and under review
- Low-hanging phenomenological fruit being picked
- A big step toward FVEs — to be continued…

Thank you!

The LECs

Pion form factor and charge radius

$$F_V^\pi(s) = 1 + \tfrac{1}{6}\langle r^2\rangle_V^\pi s + c_V^\pi s^2 + \mathcal{O}(s^3)$$

$$\langle r^2\rangle_V^\pi = \xi[l_6] + \xi^2\left([l_1, l_2, l_4, l_6] + 6\boldsymbol{r_{V1}}\right) + \mathcal{O}(\xi^3)$$

$$c_V^\pi = \frac{\xi}{60} + \xi^2\left([l_1, l_2, l_4, l_6] + \boldsymbol{r_{V2}}\right) + \mathcal{O}(\xi^3)$$

Vector meson dominance

$$r_{V1} \approx -6.2 \qquad r_{V2} \approx +6.5$$

Data (old)

$$r_{V2} = 4.0 \pm 1.2$$

Bijnens, Colangelo & Talavera [hep-ph/9805389]

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Vector meson dominance

Data

Lattice QCD

Potentially better precision (efforts underway)

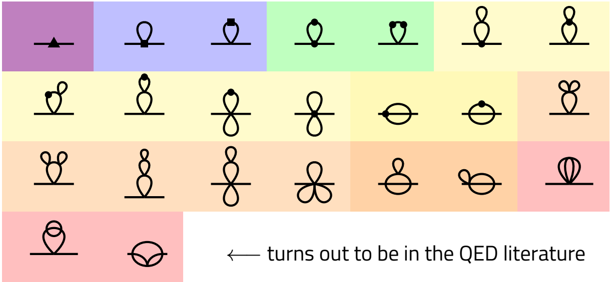
Access to pion mass dependence

Colangelo, Hofericher, Kubis, Niehus, de Elvira [2110.05493]

Bijnens, Colangelo & Talavera [hep-ph/9805389]

Pion self-energy for comparison

Bijnens, (Herman-Trued)sson & Wang [1810.06384]



⟵ turns out to be in the QED literature

More on dimensional shift

IBP reduction

Integration by parts

$$\mathbb{I}_k = \left[\prod_i \int \frac{\mathrm{d}^d \ell_i}{\pi^{d/2}}\right] \mathbb{J}_k \quad \Rightarrow \quad 0 = \left[\prod_i \int \frac{\mathrm{d}^d \ell_i}{\pi^{d/2}}\right] \frac{\partial}{\partial \ell_j^\mu} \left(q^\mu \mathbb{J}_k\right)$$

$\ell_j =$ any loop momentum, $q =$ any momentum

- Gives solvable relations $0 = \sum_k \alpha_k \mathbb{I}_k$ Laporta [hep-ph/0102033]
- Yields small set of irreducible integrals
  — **master integrals** —
  (all further efforts can be limited to these)
- We used Roman Lee's `LiteRed2` [1212.2685] for reduction
  (requires Mathematica, but other reducers struggled)

Symanzik graph polynomials

Spanning trees

Connect all vertices without forming loops:

$$\mathcal{U} = \sum_{\text{trees } T} \prod_{j \notin T} \alpha_j \,,$$

Spanning 2-forests

Make cut, form spanning tree for each half:

$$\mathcal{F} = \sum_{\text{2-forests } F} (-p_{\text{cut}}^2) \prod_{j \notin F} \alpha_j - \mathcal{U} \sum_j \alpha_j m_j^2 \,.$$

Schwinger form

$$\mathbb{I}_{\vec{\nu}} = \int_{\alpha} \frac{e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{\boldsymbol{d}/2}}$$

- **Schwinger form** of a loop integral:
- Derivative w.r.t. **mass** just drops down α:

$$\frac{\partial}{\partial m_j^2} \frac{e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{d/2}} = \frac{\partial}{\partial m_k^2} \frac{e^{(\dots)+\sum_i \alpha_i m_i^2}}{\mathcal{U}^{d/2}} = \alpha_j \frac{e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{d/2}}$$

- Meanwhile, in the propagator picture:

$$\frac{\partial}{\partial m_i^2} \frac{1}{D_j^{\nu_j}} = \frac{\nu_j \delta_{ij}}{D_j^{\nu_j+1}} \qquad \Rightarrow \qquad \frac{\partial}{\partial m_j^2} \mathbb{I}_{\vec{\nu}} = \nu_j \mathbb{I}_{\vec{\nu}+\hat{j}}$$

Dimensional shift

Tarasov [hep-th/9606018]

- Assemble facsimile of $\mathcal{U}$ from derivatives:

$$\mathcal{U} \equiv \sum_{\text{trees } T} \prod_{j \notin T} \alpha_j \qquad \longleftrightarrow \qquad \mathcal{D} \equiv \sum_{\text{trees } T} \prod_{j \notin T} \frac{\partial}{\partial m_j^2}$$

- Propagator form: just change $\vec{\nu}$:

$$\mathcal{D}\mathbb{I}_{\vec{\nu}}(d) = \sum_{\text{trees } T} \mathbb{I}_{\vec{\nu}+\vec{\nu}_T}(d) \prod_{j \notin T} \nu_j \,, \qquad \vec{\nu}_T = \sum_{j \notin T} \hat{j}$$

- Schwinger form: formally change the dimension:

$$\mathcal{D}\mathbb{I}_{\vec{\nu}}(d) = \int_\alpha \frac{\mathcal{U} e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{d/2-1}} = \mathbb{I}_{\vec{\nu}}(d-2)$$

- Allows $d = 2 - 2\epsilon$ (generally nicer than $d = 4 - 2\epsilon$)

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