

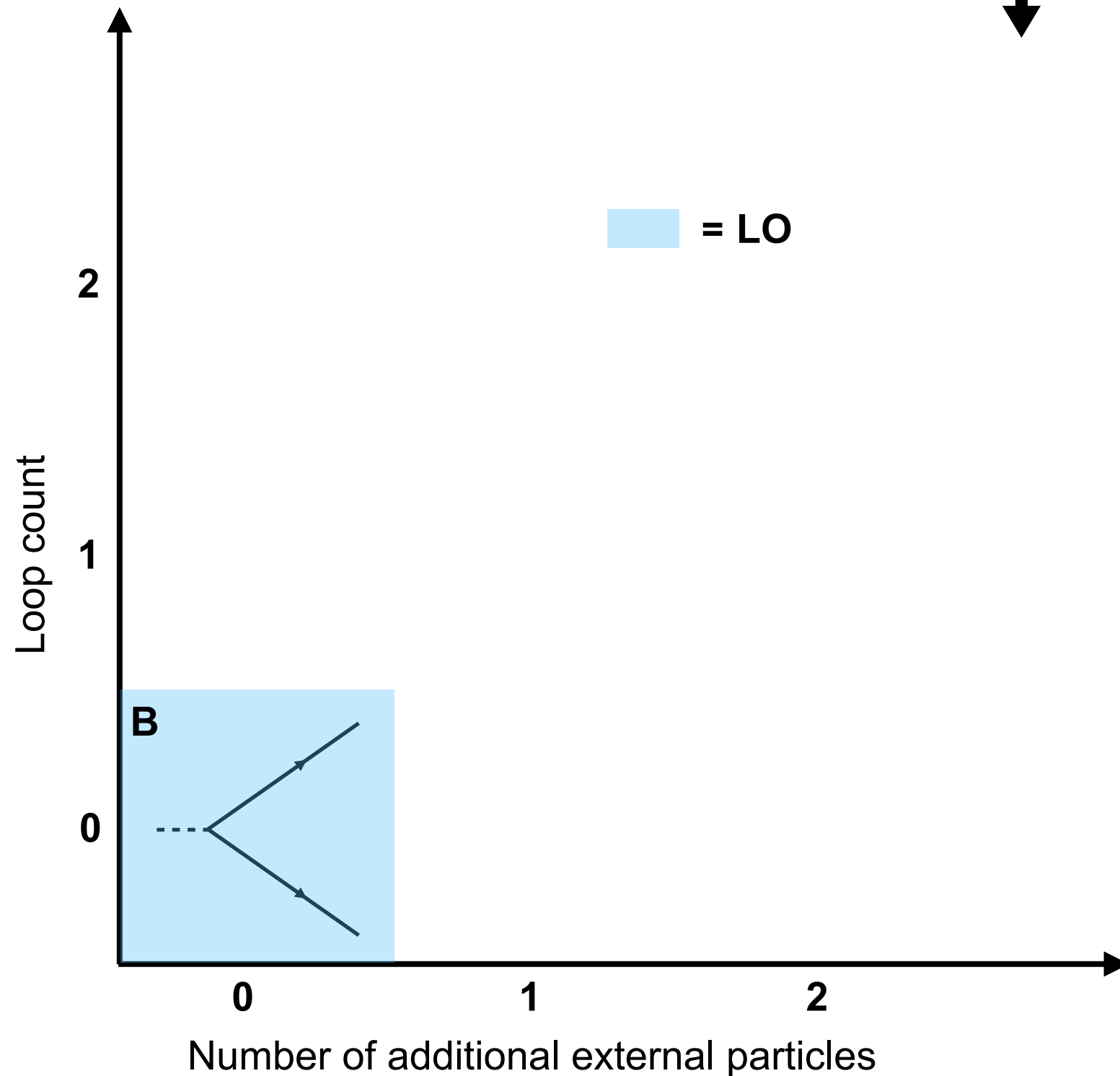
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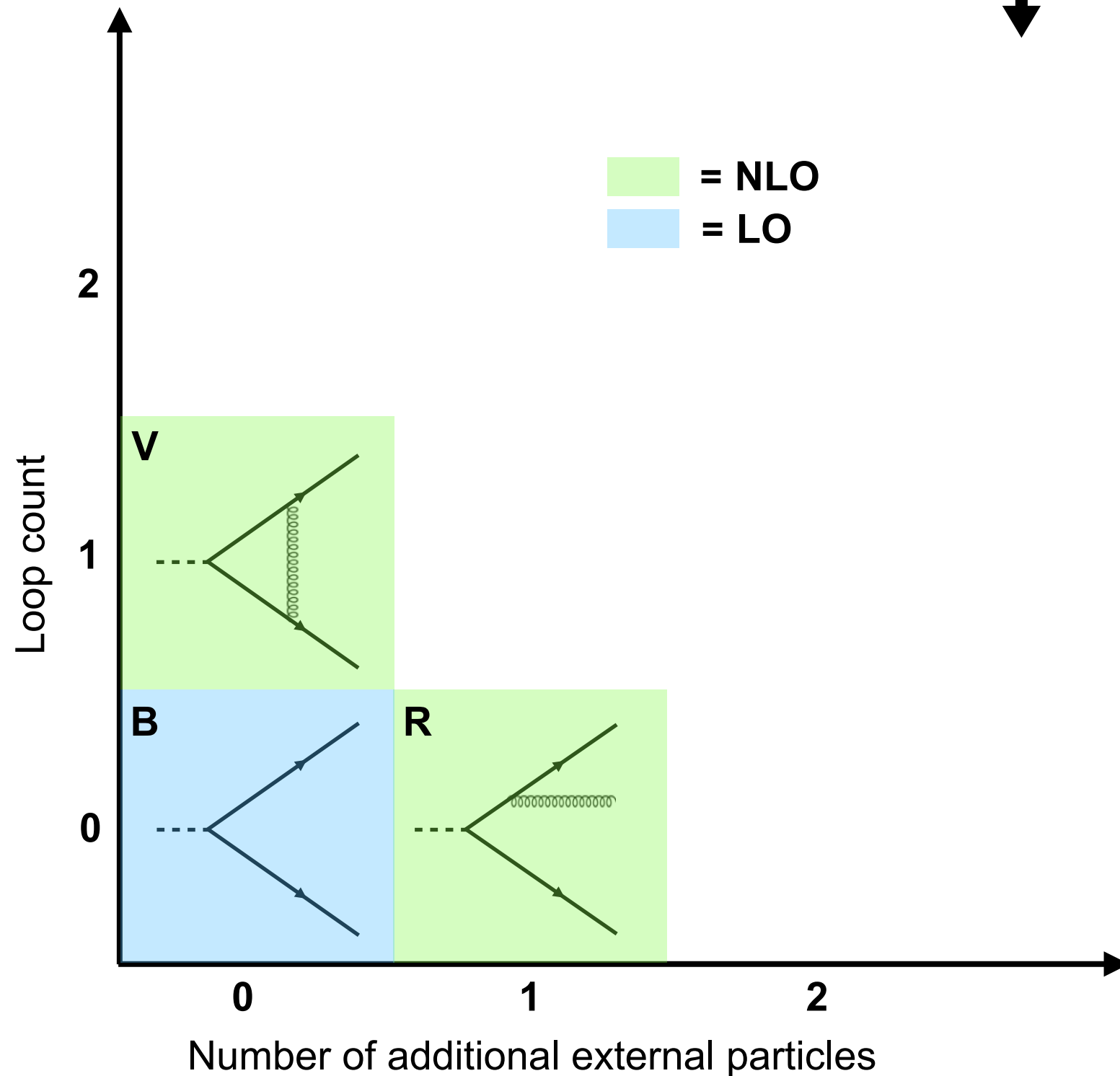
PERTURBATIVE EXPANSIONS

$$\langle p_1, \dots, p_n | S | q_1, \dots, q_m \rangle$$



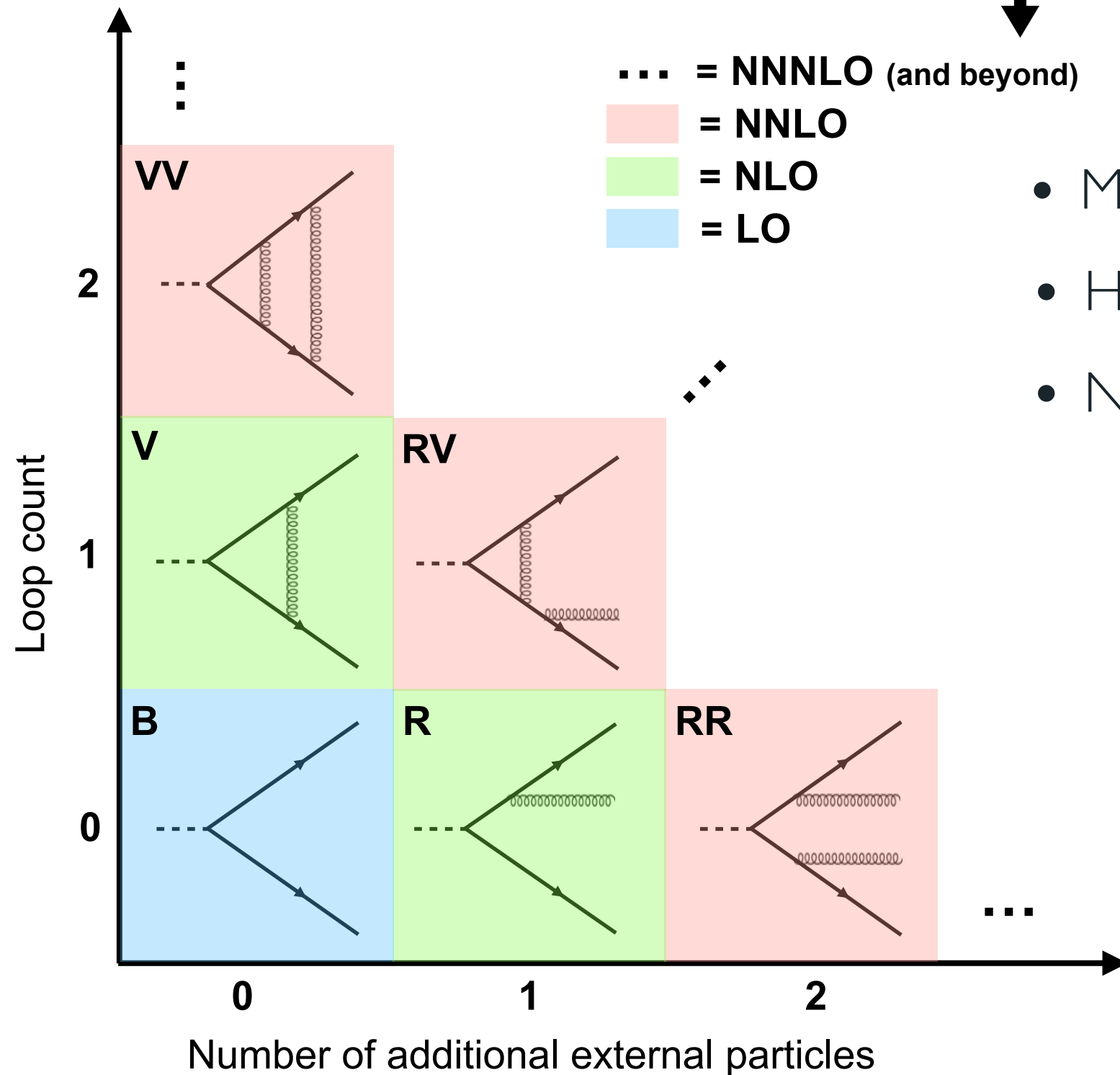
PERTURBATIVE EXPANSIONS

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PERTURBATIVE EXPANSIONS

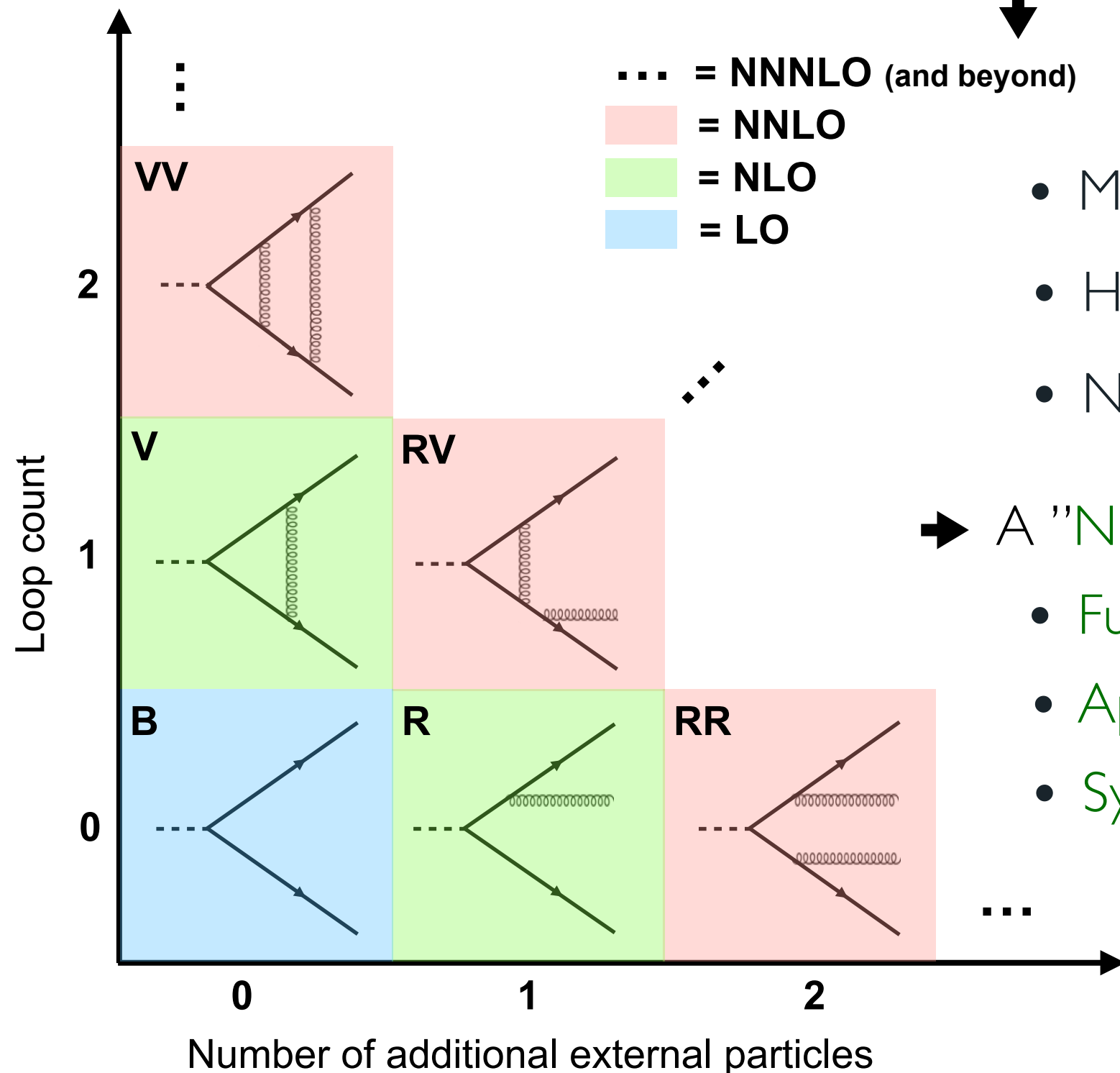
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- More complicated singularities
- Higher dimensionality
- New (unknown) analytic integrals

PERTURBATIVE EXPANSIONS

$$\langle p_1, \dots, p_n | S | q_1, \dots, q_m \rangle$$



- More complicated singularities
- Higher dimensionality
- New (unknown) analytic integrals

➔ A "Numerical Collider" ideally features:

- Fully numerical approach
- Applicable to all scatterings
- Systematic at all orders

OBSTACLES: INFRARED LIMITS

$$\sigma \sim \int d^4 p_1 \dots d^4 p_N |\mathcal{M}^{(L)}(\{p_i\})|^2 \delta p_1^2 \dots \delta p_N^2 \delta^4(p_1 + \dots p_N)$$

Numerical

LO

$$d\bar{d} \rightarrow Z$$

{ }

OBSTACLES: INFRARED LIMITS

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Numerical

LO

$$d\bar{d} \rightarrow Z$$

NLO

$$d\bar{d} \rightarrow Zg$$

{

| L0 : ((S(4),),)
 | L1 : ((C(1,4),),)
 | L2 : ((C(2,4),),)
 | L3 : ((C(S(4),1),),)
 | L4 : ((C(S(4),2),),)

OBSTACLES: INFRARED LIMITS

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NNLO

$$d\bar{d} \rightarrow Zgg$$

| L0 : ((S(4),),)
 | L1 : ((S(5),),)
 | L2 : ((C(4,5),),)
 | L3 : ((C(1,4),),)
 | L4 : ((C(2,4),),)
 [... 71 limits ...]
 | L69 : (((C(S(4),2),S(5),),),)
 | L70 : (((S(C(S(4),5),),2),),)
 | L71 : ((C(S(C(S(5),4),),2),),)

OBSTACLES: INFRARED LIMITS

$$\sigma \sim \int d^4 p_1 \dots d^4 p_N |\mathcal{M}^{(L)}(\{p_i\})|^2 \delta p_1^2 \dots \delta p_N^2 \delta^4(p_1 + \dots p_N)$$

Numerical

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| L0 : ((S(4),),)
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| L2 : ((C(2,4),),)
| L3 : ((C(S(4),1),),)
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NNLO

$$d\bar{d} \rightarrow Zgg$$

| L0 : ((S(4),),)
| L1 : ((S(5),),)
| L2 : ((C(4,5),),)
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[... 71 limits ...]
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| L71 : ((C(S(C(S(5),4),),2),),)

N3LO

$$d\bar{d} \rightarrow Zggg$$

| L0 : ((S(4),),)
| L1 : ((S(5),),)
| L2 : ((S(6),),)
| L3 : ((C(4,5),),)
| L4 : ((C(4,6),),)
[... 1396 limits ...]
| L1394 : ((C(S(C(S(C(S(6),4),),5),),2),),)
| L1395 : ((C(S(C(S(C(S(5),6),),4),),2),),)
| L1396 : ((C(S(C(S(C(S(6),5),),4),),2),),)

...

OBSTACLES: INFRARED LIMITS

$$\sigma \sim \int d^4 p_1 \dots d^4 p_N |\mathcal{M}^{(L)}(\{p_i\})|^2 \delta p_1^2 \dots \delta p_N^2 \delta^4(p_1 + \dots p_N)$$

Numerical

LO	NLO	NNLO	N3LO	...
$d\bar{d} \rightarrow Z$	$d\bar{d} \rightarrow Zg$	$d\bar{d} \rightarrow Zgg$	$d\bar{d} \rightarrow Zggg$	
{	L0 : ((S(4),),) L1 : ((C(1,4),),) L2 : ((C(2,4),),) L3 : ((C(S(4),1),),) L4 : ((C(S(4),2),),)	L0 : ((S(4),),) L1 : ((S(5),),) L2 : ((C(4,5),),) L3 : ((C(1,4),),) L4 : ((C(2,4),),) [... 71 limits ...] L69 : (((C(S(4),2),S(5),),),) L70 : (((S(C(S(4),5),),2),),) L71 : (((S(C(S(5),4),),2),),)	L0 : ((S(4),),) L1 : ((S(5),),) L2 : ((S(6),),) L3 : ((C(4,5),),) L4 : ((C(4,6),),) [... 1396 limits ...] L1394 : (((S(C(S(C(S(6),4),),5),),2),),) L1395 : (((S(C(S(C(S(5),6),),4),),2),),) L1396 : (((S(C(S(C(S(6),5),),4),),2),),)	

On top of :

- $R^k V^{k-n}$ proliferation of contributions
- Convolution with PDF counterterms
- Growth of number of partonic channels



Very challenging at
and beyond NNLO...

OBSTACLES: MULTI-LOOPS MES

$$\sigma \sim \int d^4 p_1 \dots d^4 p_N |\mathcal{M}^{(L)}(\{p_i\})|^2 \delta p_1^2 \dots \delta p_N^2 \delta^4(p_1 + \dots p_N)$$

Numerical

$$|\mathcal{M}^{(L)}(\{p_i\})|^2 \sim \int d^d k_1 \dots \int d^d k_L I(\{p_i\}, \{k_i\})$$

Analytical ?

Timings

Tree-level

MG5aMC

[\[arXiv:1405.0301\]](#)

$$d\bar{d} \rightarrow ZZ \quad 7 \mu\text{s}$$

$$d\bar{d} \rightarrow ZZg \quad 35 \mu\text{s}$$

$$d\bar{d} \rightarrow ZZgg \quad 220 \mu\text{s}$$

OBSTACLES: MULTI-LOOPS MES

$$\sigma \sim \int d^4 p_1 \dots d^4 p_N |\mathcal{M}^{(L)}(\{p_i\})|^2 \delta p_1^2 \dots \delta p_N^2 \delta^4(p_1 + \dots + p_N)$$

Numerical

$$|\mathcal{M}^{(L)}(\{p_i\})|^2 \sim \int d^d k_1 \dots \int d^d k_L I(\{p_i\}, \{k_i\})$$

Analytical ?

Timings	Tree-level MG5aMC [arXiv:1405.0301]	One-loop MadLoop [arXiv:1103.0621]
$d\bar{d} \rightarrow ZZ$	7 μs	0.6 ms
$d\bar{d} \rightarrow ZZg$	35 μs	38 ms
$d\bar{d} \rightarrow ZZgg$	220 μs	1200 ms

OBSTACLES: MULTI-LOOPS MES

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Numerical

$$|\mathcal{M}^{(L)}(\{p_i\})|^2 \sim \int d^d k_1 \dots \int d^d k_L I(\{p_i\}, \{k_i\})$$

Analytical ?

Timings	Tree-level MG5aMC [arXiv:1405.0301]	One-loop MadLoop [arXiv:1103.0621]	Two-loop VVamp [arXiv:1503.08835]
$d\bar{d} \rightarrow ZZ$	7 μs	$\times 10^2 \rightarrow$	0.6 ms $\times 10^3 \rightarrow$ $\mathcal{O}(\sim 1\text{s})$
$d\bar{d} \rightarrow ZZg$	35 μs	$\times 10^3 \rightarrow$	38 ms $\xrightarrow{?}$ N/A
$d\bar{d} \rightarrow ZZgg$	220 μs	$\times 10^4 \rightarrow$	1200 ms $\xrightarrow{\quad}$ N/A

NUMERICAL COLLIDER

Wishlist for a numerical collider:

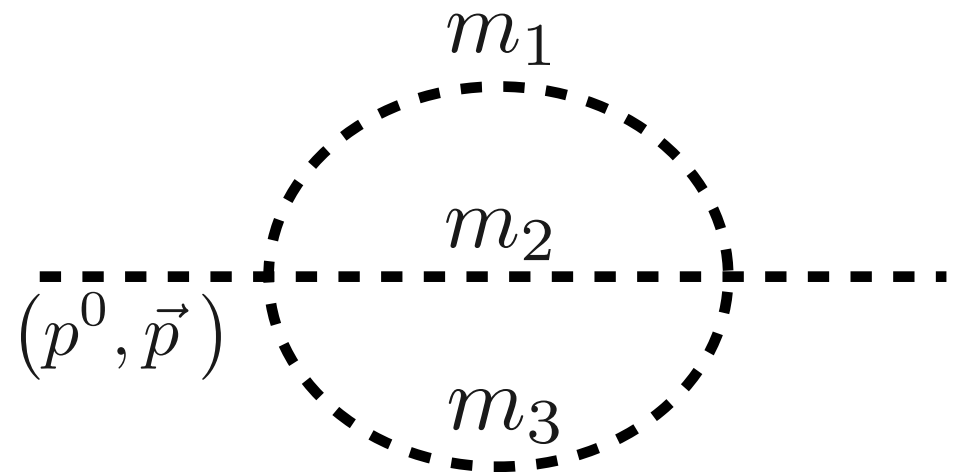
- Computational resources may be the only limiting factor.
- Full automation guarantees reproducibility and correctness.
- Implementation is flexible and easy-to-use, rallying many in the effort.
- Renormalisation in the $\overline{\text{MS}} + \text{OS}$ scheme happens automatically.
- Yields fully differential predictions (e.g. event generator).

In particular :

$$\mathcal{O}(\text{p p} \rightarrow X + (2\textcolor{red}{n} \times j)) \sim \mathcal{O}(\text{p p} \rightarrow X @ \textcolor{red}{n} \times \text{loops})$$

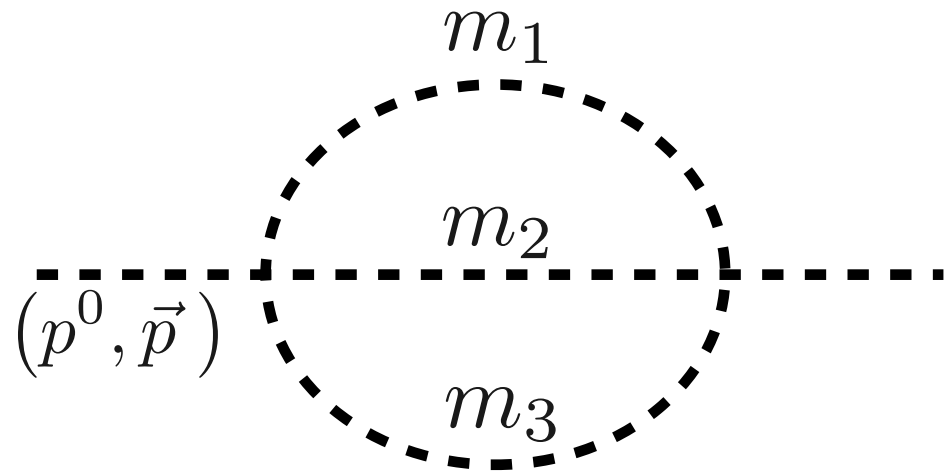
That is : aim for the loop count of the contributing forward scattering graphs to be the main complexity parameter.

ANALYTICS FOR LOOPS



"The Sunrise"

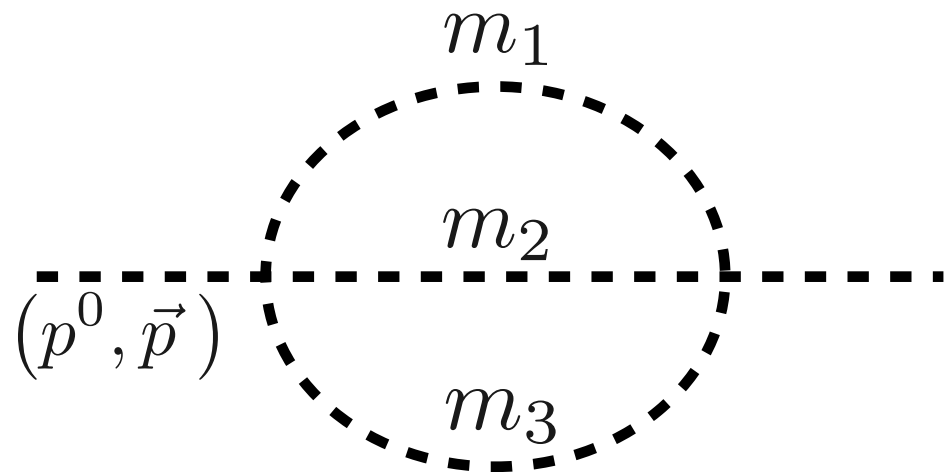
ANALYTICS FOR LOOPS



"The Sunrise"

$$= \int_{-\infty}^{+\infty} dk^0 d\vec{k} dl^0 d\vec{l} \frac{1}{(k^0 - |\vec{k}|^2 - m_1^2)(l^0 - |\vec{l}|^2 - m_2^2)(k^0 + l^0 + p^0 - |\vec{k} + \vec{l} + \vec{p}|^2 - m_3^2)}$$

ANALYTICS FOR LOOPS



"The Sunrise"

$$= \int_{-\infty}^{+\infty} dk^0 d\vec{k} dl^0 d\vec{l} \frac{1}{(k^0 - |\vec{k}|^2 - m_1^2)(l^0 - |\vec{l}|^2 - m_2^2)(k^0 + l^0 + p^0 - |\vec{k} + \vec{l} + \vec{p}|^2 - m_3^2)}$$

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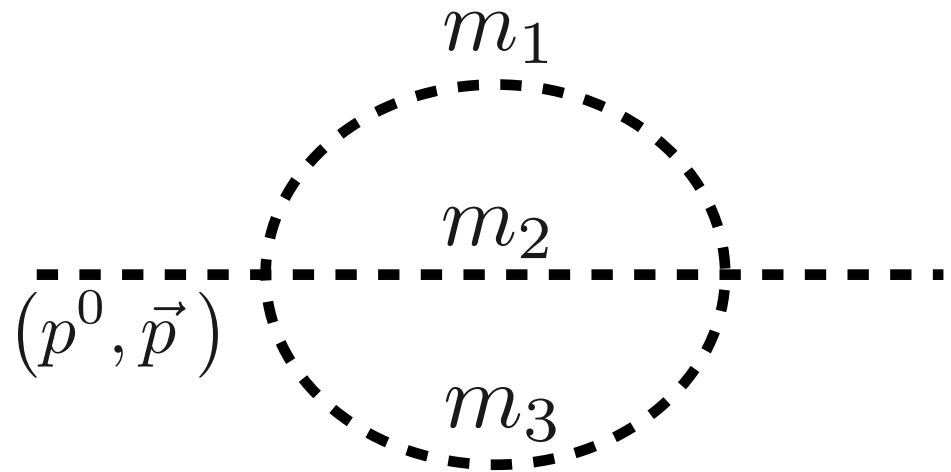
The cosmic Galois group, the sunrise Feynman integral, and the relative completion of $\Gamma_1(6)$ #1

[Matija Tapušković](#) (Mar 30, 2023)

e-Print: [2303.17534](#) [math.AG]

pdf cite claim reference search 2 citations

ANALYTICS FOR LOOPS



"The Sunrise"

$$= \int_{-\infty}^{+\infty} dk^0 d\vec{k} dl^0 d\vec{l} \frac{1}{(k^0 - |\vec{k}|^2 - m_1^2)(l^0 - |\vec{l}|^2 - m_2^2)(k^0 + l^0 + p^0 - |\vec{k} + \vec{l} + \vec{p}|^2 - m_3^2)}$$

Loop-Tree Duality for Multiloop Numerical Integration

#12

Zeno Capatti (ETH, Zurich (main)), Valentin Hirschi (ETH, Zurich (main)), Dario Kermanschah (ETH, Zurich (main)), Ben Ruijl (ETH, Zurich (main)) (Jun 14, 2019)

Published in: *Phys.Rev.Lett.* 123 (2019) 15, 151602 • e-Print: [1906.06138](#) [hep-ph]

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DOI

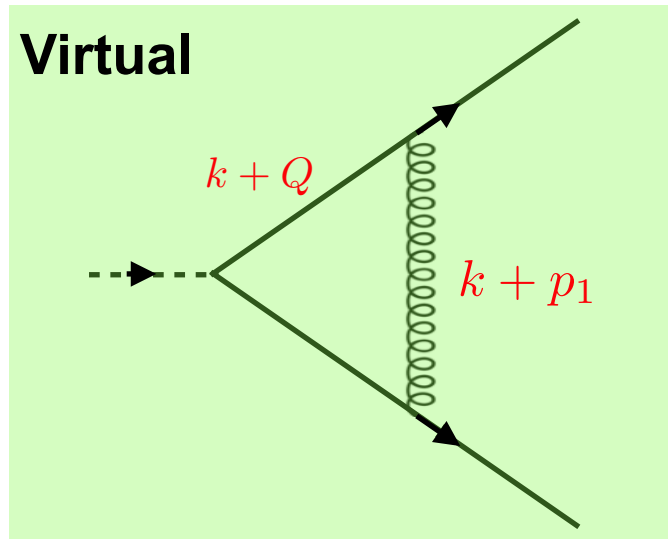
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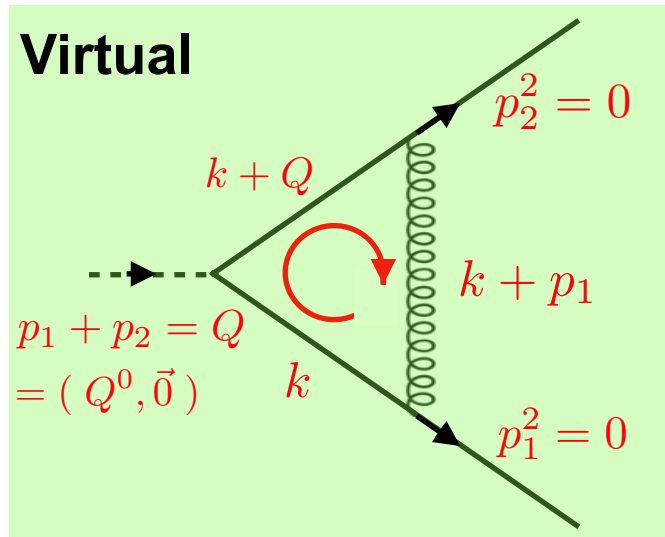
reference search

76 citations

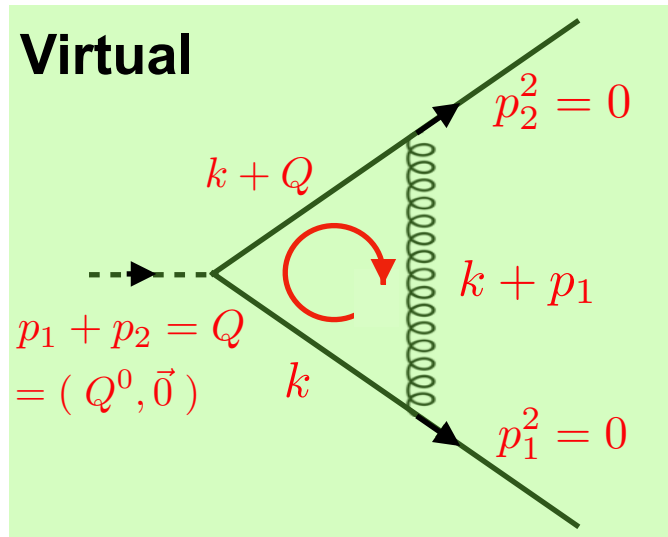
LOOP INTEGRAND SINGULARITIES



LOOP INTEGRAND SINGULARITIES

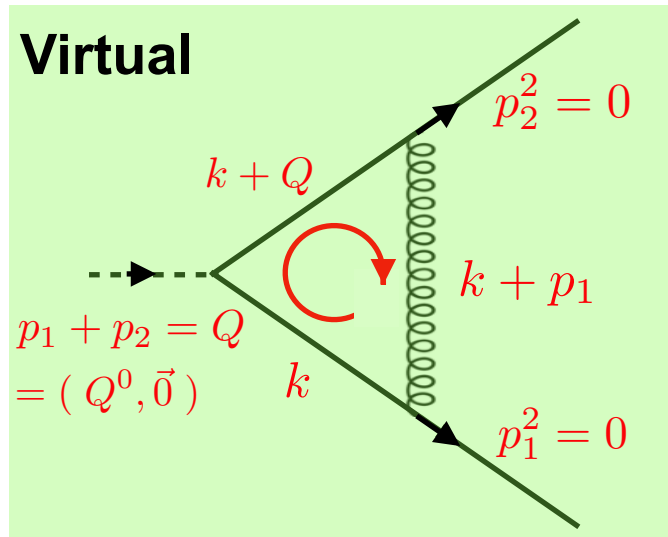


LOOP INTEGRAND SINGULARITIES



$$\longrightarrow \int dk^0 d^3 \vec{k} \frac{\bar{v}_{p_2} \gamma^\mu \not{k} (\not{k} + \not{Q}) \gamma_\mu u_{p_1}}{k^2 (k + Q)^2 (k + p_1)^2}$$

LOOP INTEGRAND SINGULARITIES

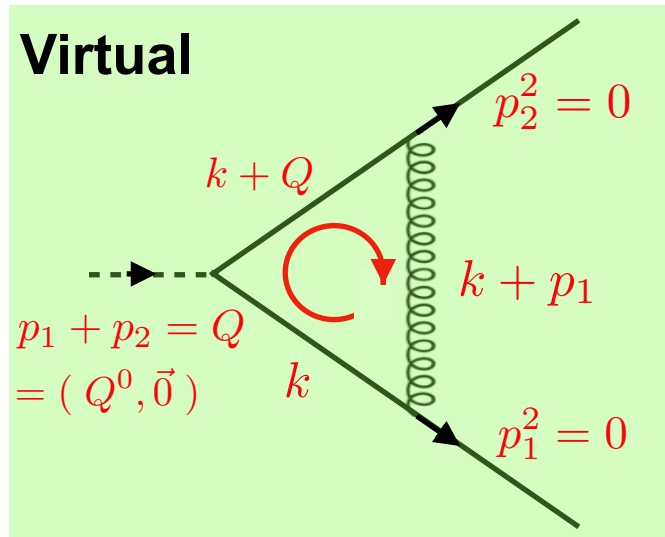


$$\int dk^0 d^3 \vec{k} \frac{\bar{v}_{p_2} \gamma^\mu \not{k} (\not{k} + \not{Q}) \gamma_\mu u_{p_1}}{k^2 (k + Q)^2 (k + p_1)^2}$$

Loop-Tree Duality
LTD / 3D rep

$$\int d^3 \vec{k} \left[\right]$$

LOOP INTEGRAND SINGULARITIES

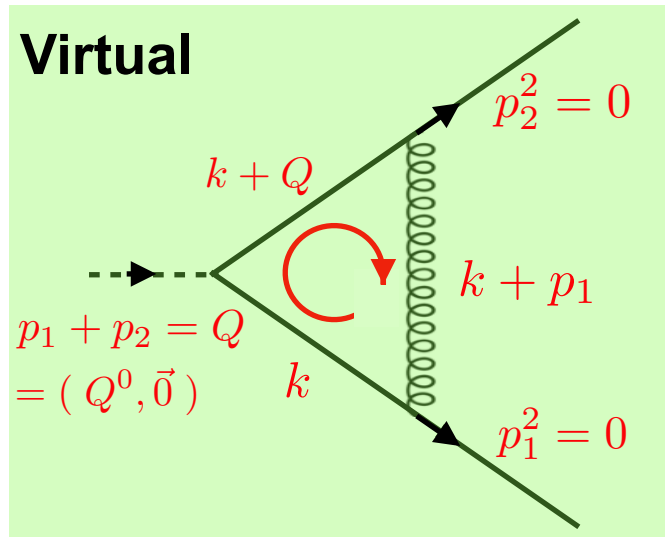


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Loop-Tree Duality
LTD / 3D rep

$$\int \equiv \delta^+(q_i^2) \int d^3 \vec{k} \left[\text{Diagram} \right]$$

LOOP INTEGRAND SINGULARITIES

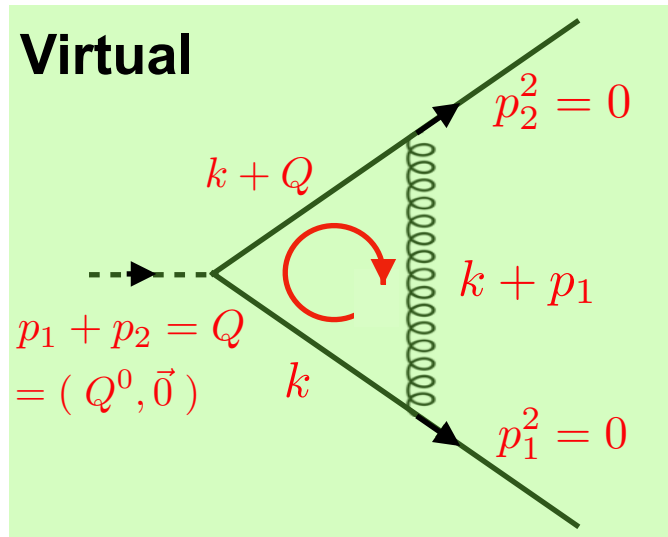


$$\int dk^0 d^3 \vec{k} \frac{\bar{v}_{p_2} \gamma^\mu \not{k} (\not{k} + \not{Q}) \gamma_\mu u_{p_1}}{k^2 (k + Q)^2 (k + p_1)^2}$$

Loop-Tree Duality
LTD / 3D rep

$$\int \equiv \delta^+(q_i^2) \int d^3 \vec{k} \left[\text{Diagram 1} + \text{Diagram 2} \right]$$

LOOP INTEGRAND SINGULARITIES

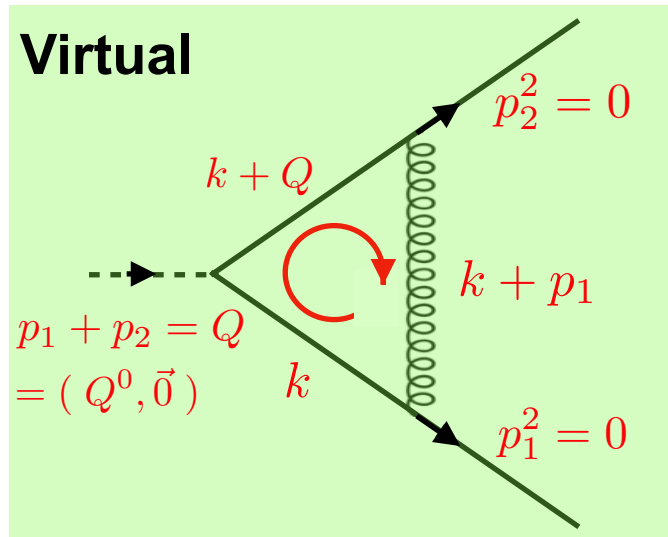


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Loop-Tree Duality
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$$\int \equiv \delta^+(q_i^2) \int d^3 \vec{k} \left[\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \right]$$

LOOP INTEGRAND SINGULARITIES



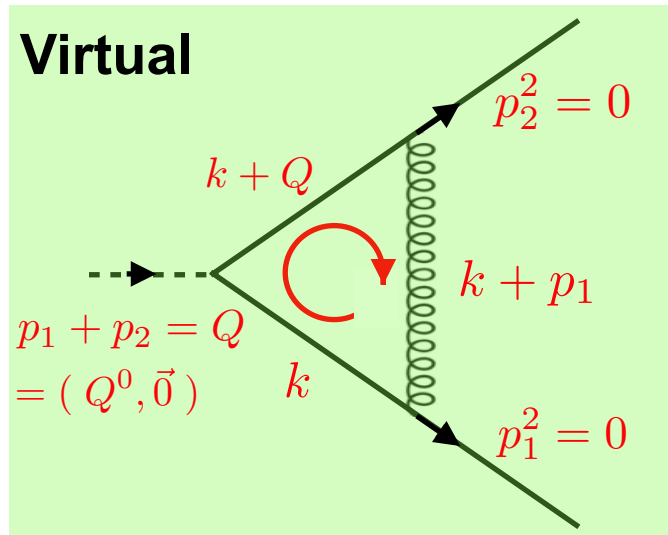
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Loop-Tree Duality
LTD / 3D rep

$$\int \equiv \delta^+(q_i^2) \int d^3 \vec{k} \left[\text{diagram 1} + \text{diagram 2} + \text{diagram 3} \right]$$

$$\text{cLTD} \supset \text{CFF} \int d^2 \Omega \int_0^\infty |\vec{k}|^2 d|\vec{k}| \frac{a|\vec{k}|^2 + b|\vec{k}| + c}{8|\vec{k}|^2 |\vec{k} + \vec{p}_1|} \frac{1}{|\vec{k}| + |\vec{k} + \vec{p}_1| - |\vec{p}_1|} \frac{1}{|\vec{k}| + |\vec{k} + \vec{Q}| - Q^0 + i\delta}$$

LOOP INTEGRAND SINGULARITIES



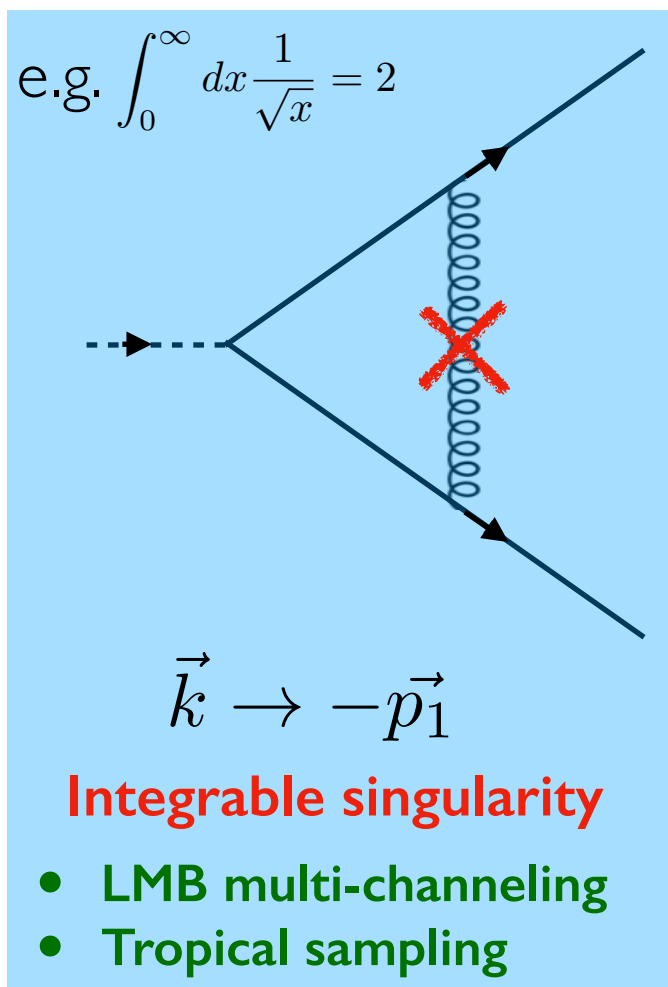
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LTD / 3D rep

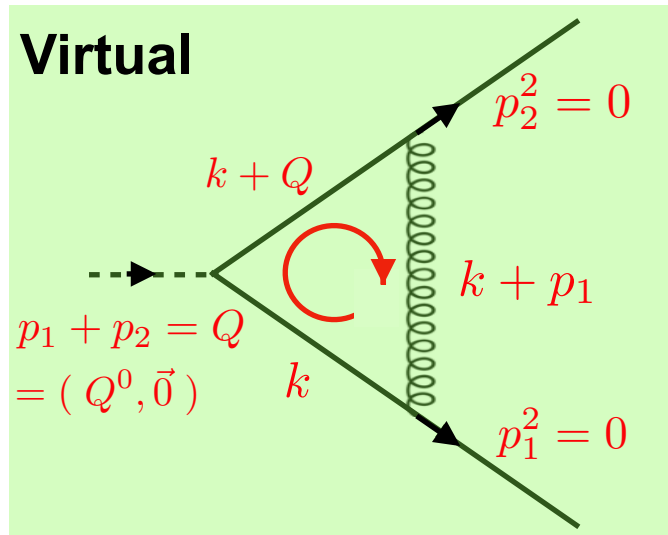
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LOOP INTEGRAND SINGULARITIES

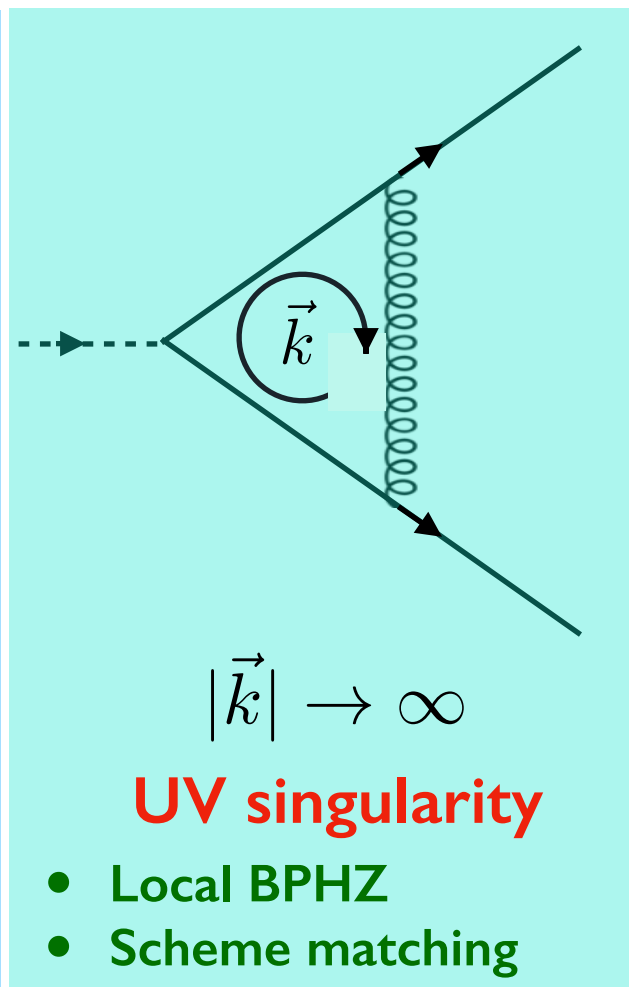
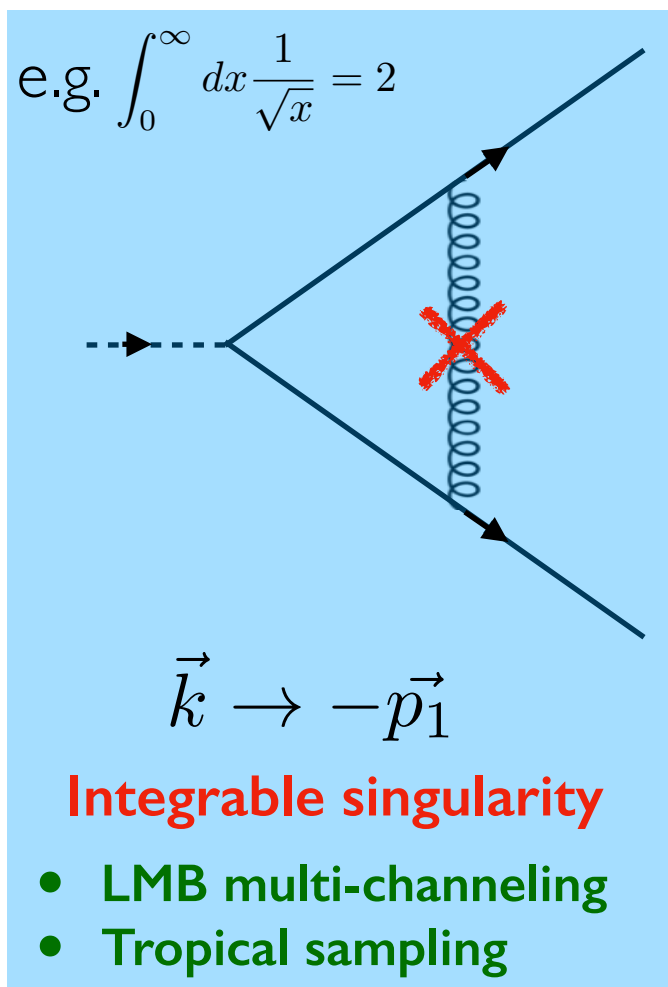


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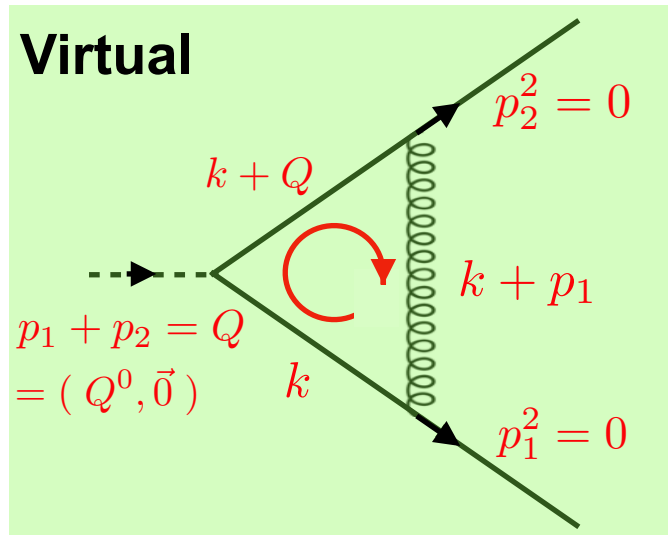
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LOOP INTEGRAND SINGULARITIES



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e.g. $\int_0^\infty dx \frac{1}{\sqrt{x}} = 2$

$\vec{k} \rightarrow -\vec{p}_1$

Integrable singularity

- LMB multi-channeling
- Tropical sampling

$|\vec{k}| \rightarrow \infty$

UV singularity

- Local BPHZ
- Scheme matching

$\delta^-(k^2)$

$\delta^+((k + p_1)^2)$

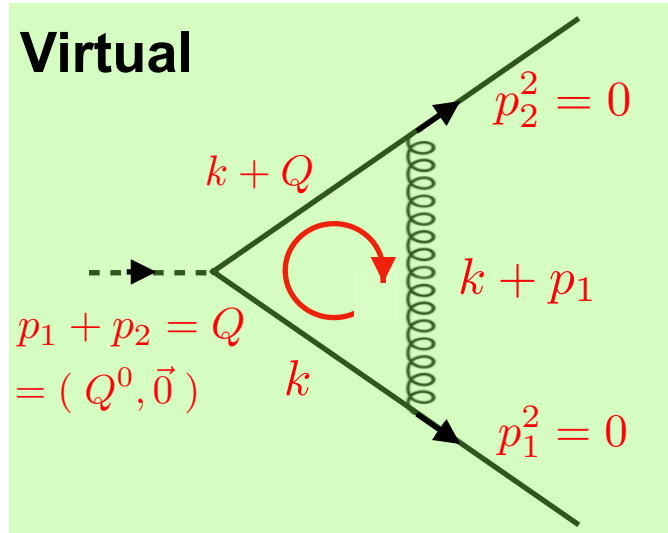
$p_1^2 = 0$

$\vec{k} \rightarrow -x\vec{p}_1 \quad x \in [0, 1]$

IR singularity

- Loop IR counterterms
- Local Unitarity / KLN

LOOP INTEGRAND SINGULARITIES



$$\int dk^0 d^3 \vec{k} \frac{\bar{v}_{p_2} \gamma^\mu \not{k} (\not{k} + \not{Q}) \gamma_\mu u_{p_1}}{k^2 (k + Q)^2 (k + p_1)^2}$$

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cLTD
 $\supset$
CFF

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Integrable singularity

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$|\vec{k}| \rightarrow \infty$

UV singularity

- Local BPHZ
- Scheme matching

$\delta^+((k + p_1)^2)$

$\delta^-(k^2)$

$p_1^2 = 0$

$\vec{k} \rightarrow -x\vec{p}_1 \quad x \in [0, 1]$

IR singularity

- Loop IR counterterms
- Local Unitarity / KLN

$\delta^+((k + Q)^2)$

$\delta^-(k^2)$

$|\vec{k}| \rightarrow Q^0/2$

Threshold singularity

- Contour deformation
- Threshold subtraction

THE THREE-DIMENSIONAL REPS. ZOO

Applying LTD to a two-loop double-triangle: one residue per spanning tree

$$\int d^4k d^4l \text{ (double-triangle diagram) } = \int d^3\vec{k} d^3\vec{l} \left[\begin{array}{cccc} \text{diagram 1} & + & \text{diagram 2} & + & \text{diagram 3} & + & \text{diagram 4} \\ + & \text{diagram 5} & + & \text{diagram 6} & + & \text{diagram 7} & + & \text{diagram 8} \end{array} \right]$$

Interplay of momentum conservation and causal prescription is key to obtain the energy flow

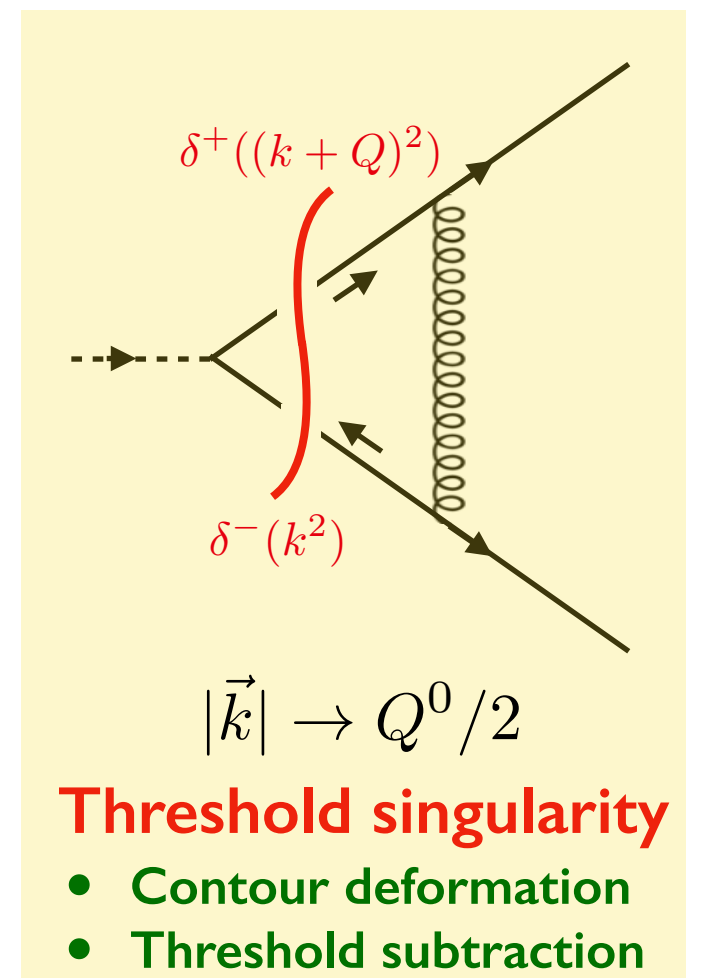
- **Distributional identities:** [Bierenbaum, Catani, Draggiotis, Rodrigo, arxiv: 1007.0194]
- **Averaging procedure:** [Runkel, Scór, Vesga, Weinzierl, arxiv: 1902.02135]
- **Iterative procedure:** [Capatti, VH, Kermanschah, Ruijl, arxiv: 1906.06138]
- **Manifestly causal:** [Capatti, VH, Kermanschah, Pelloni, Ruijl, arxiv: 2009.05509]
- **Cross-Free Family** [Capatti, arxiv: 2211.09653]

Codes : [<https://github.com/apelloni/cLTD>]

[<https://bitbucket.org/wjtorresb/lotty>] [gammaLoop]

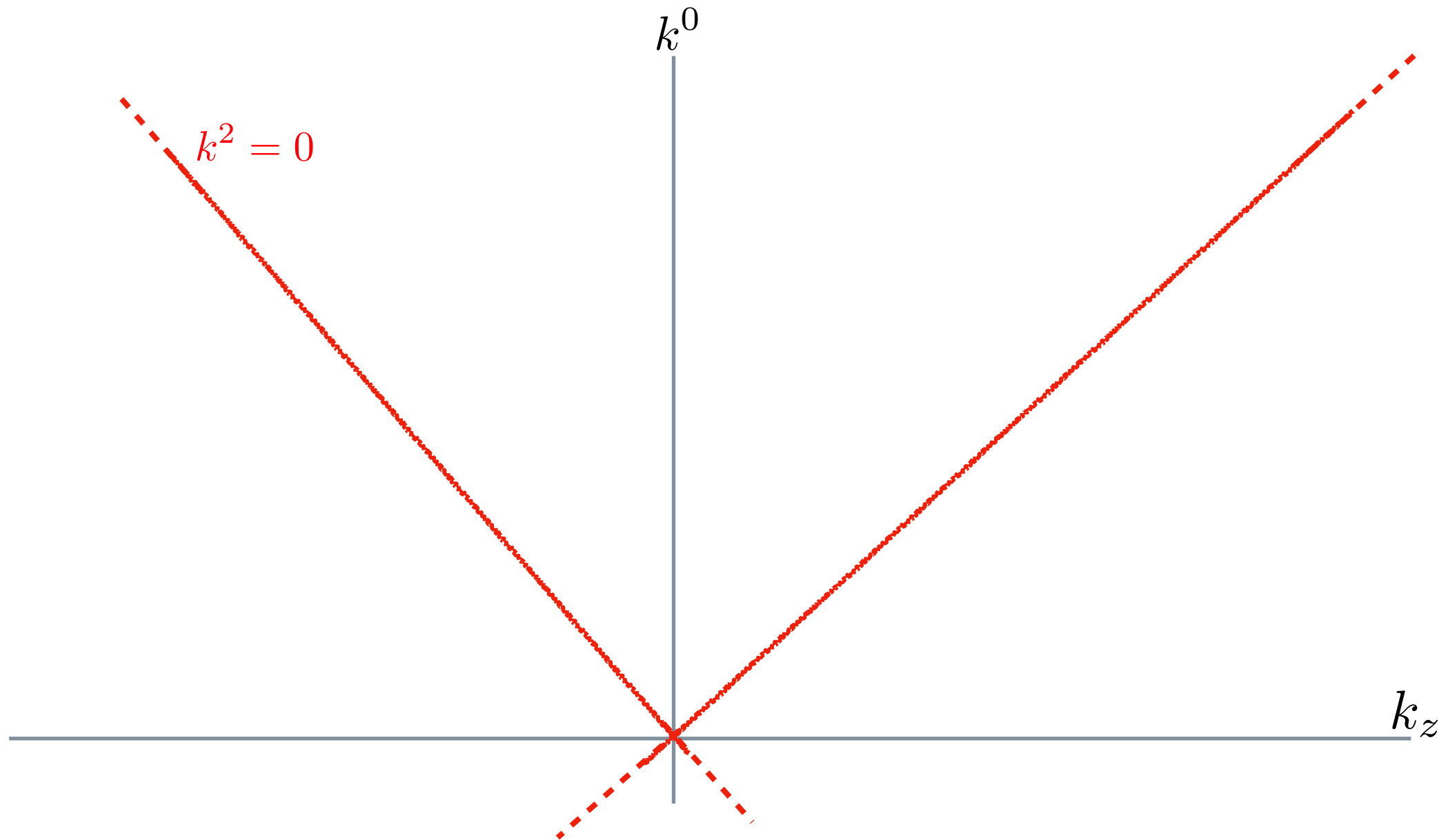
THRESHOLD SINGULARITIES

$$\int d^2\Omega \int_0^\infty |\vec{k}|^2 d|\vec{k}| \frac{a|\vec{k}|^2 + b|\vec{k}| + c}{8|\vec{k}|^2 |\vec{k} + \vec{p}_1|} \frac{1}{|\vec{k}| + |\vec{k} + \vec{p}_1| - |\vec{p}_1|} \frac{1}{|\vec{k}| + |\vec{k} + \vec{Q}| - Q^0 + i\delta}$$



SINGULAR SURFACES IN MINKOWSKI SPACE

$$\int d^4k \frac{1}{\textcolor{red}{k}^2 (\textcolor{green}{k} - p_1)^2 (\textcolor{blue}{k} - p_1 - p_2)^2}$$

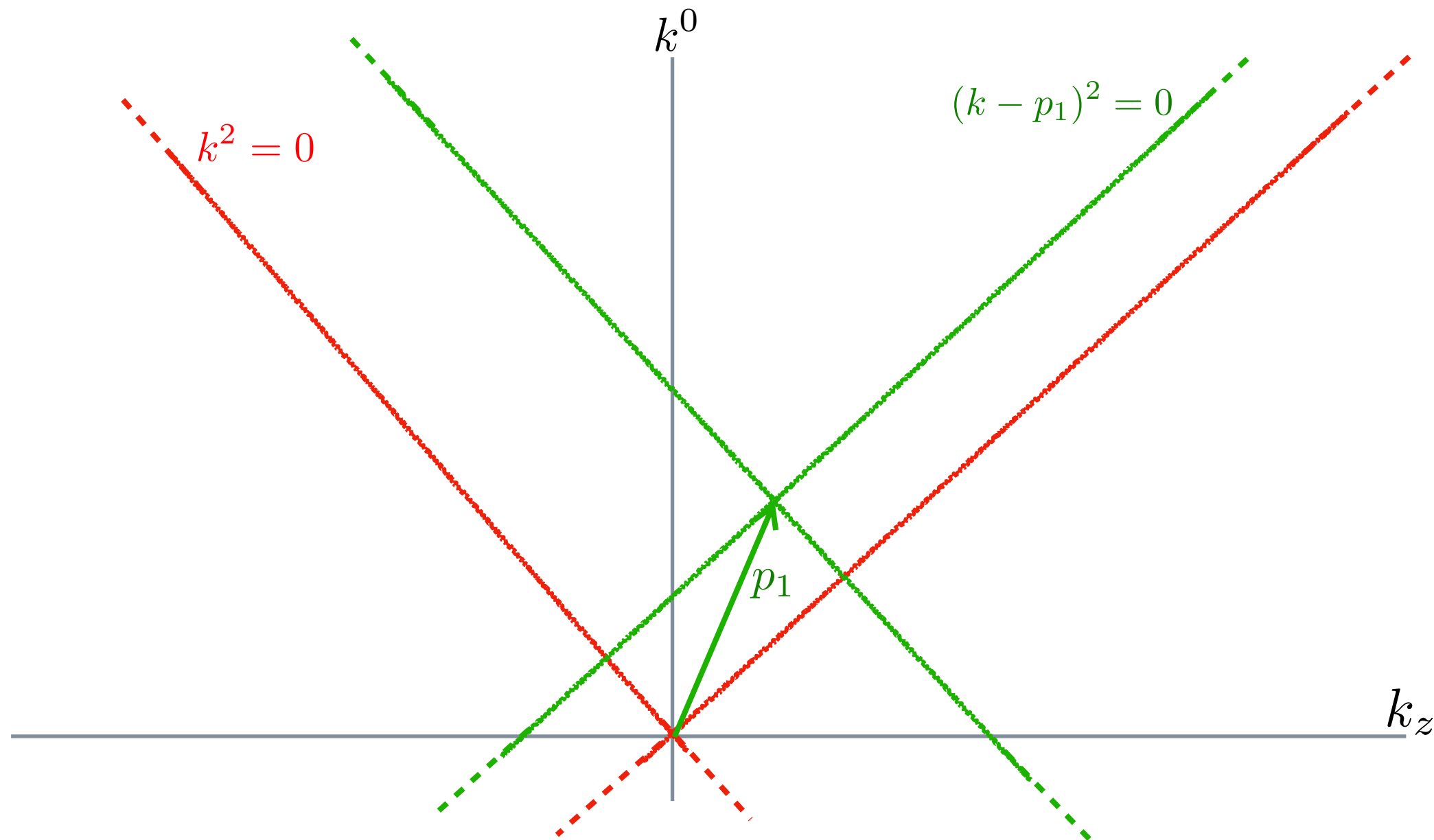


The integrand is singular along each of the coloured surface

Contour deformation in 4D very difficult: [S. Weinzierl & al, arxiv: [1006.4609](#)]
[W. Gong, Z. Nagy, D.Soper, arxiv: [0812.3686](#)]

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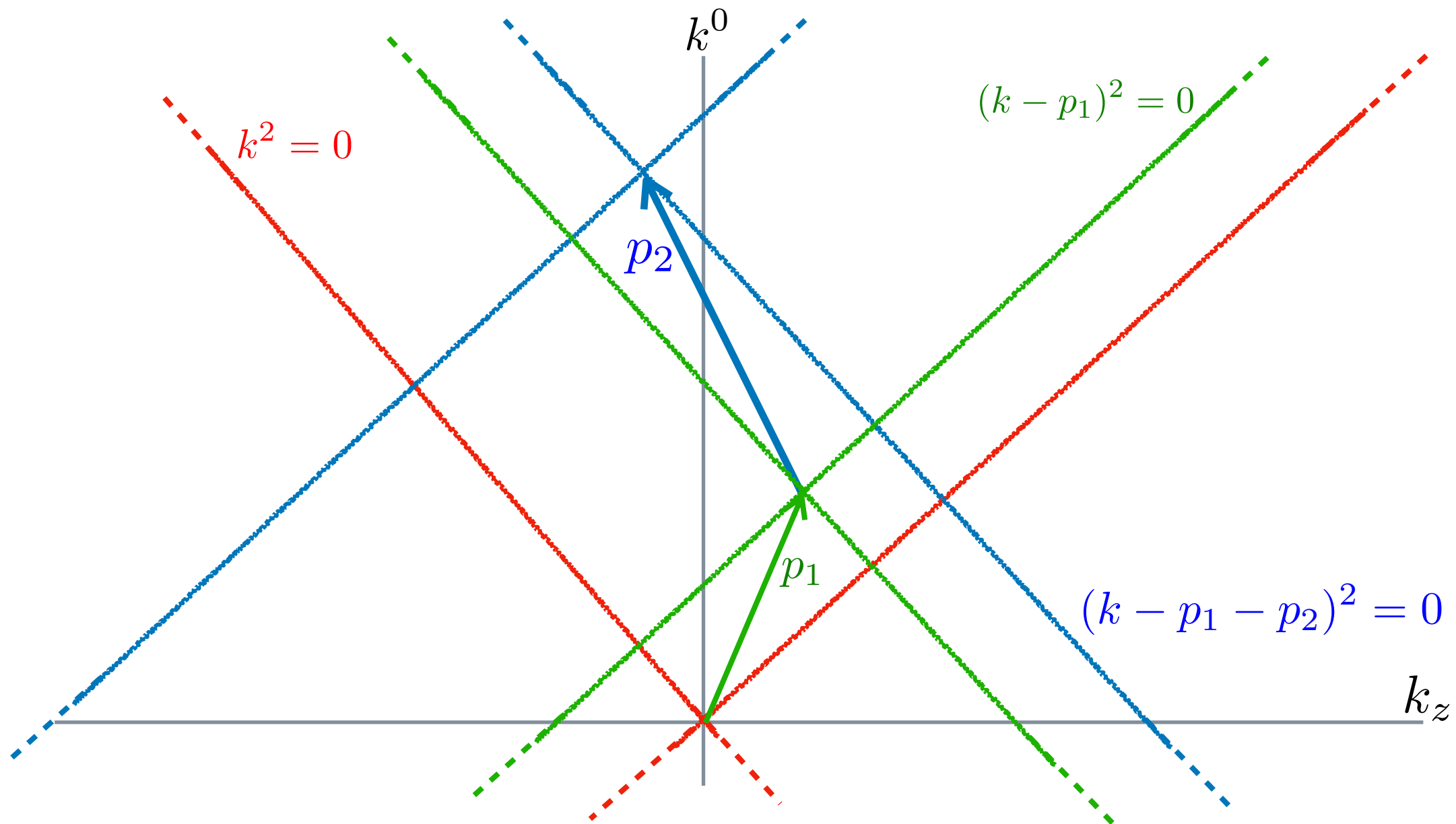


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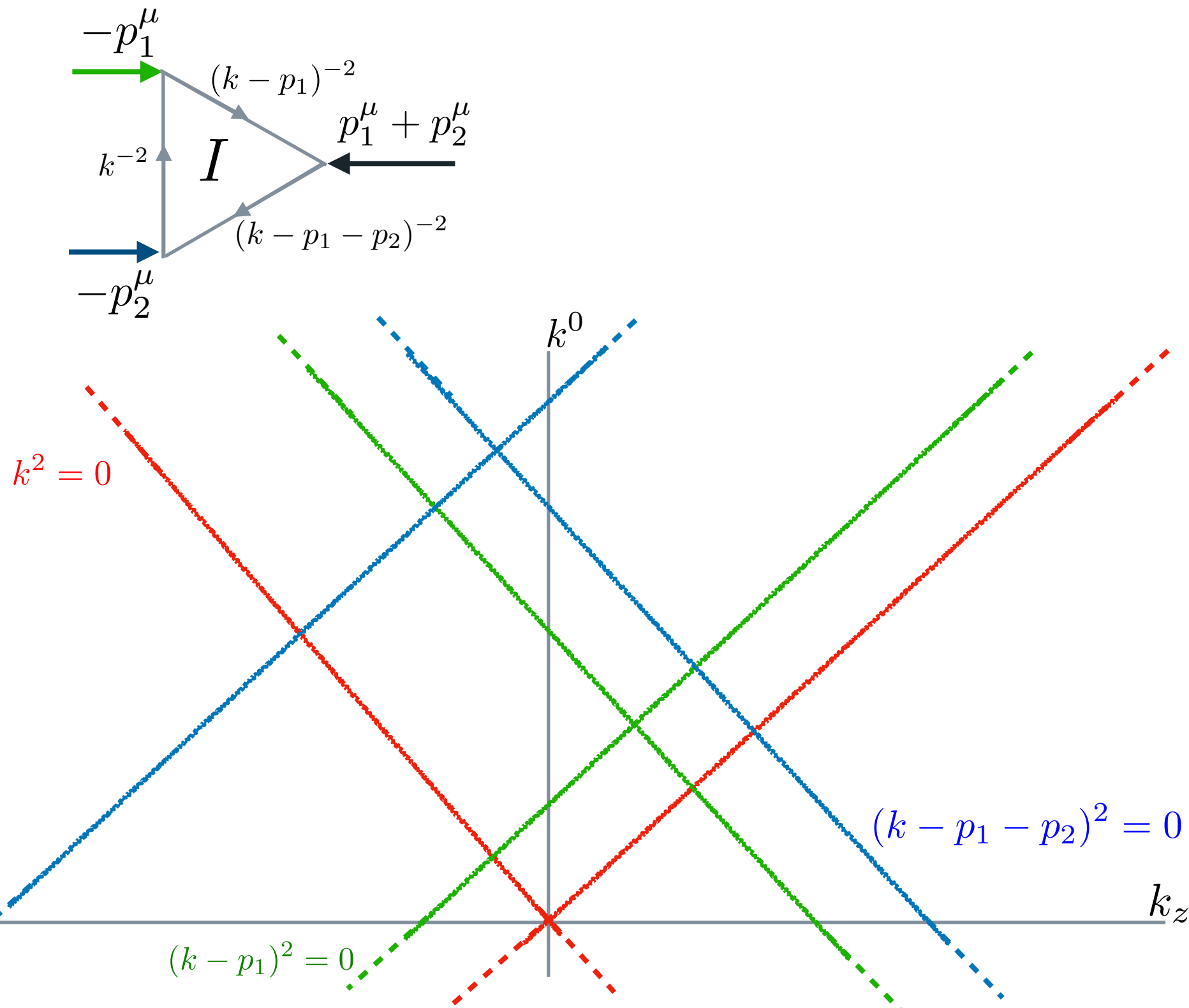


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SINGULAR SURFACES OF THE LTD REPRESENTATION

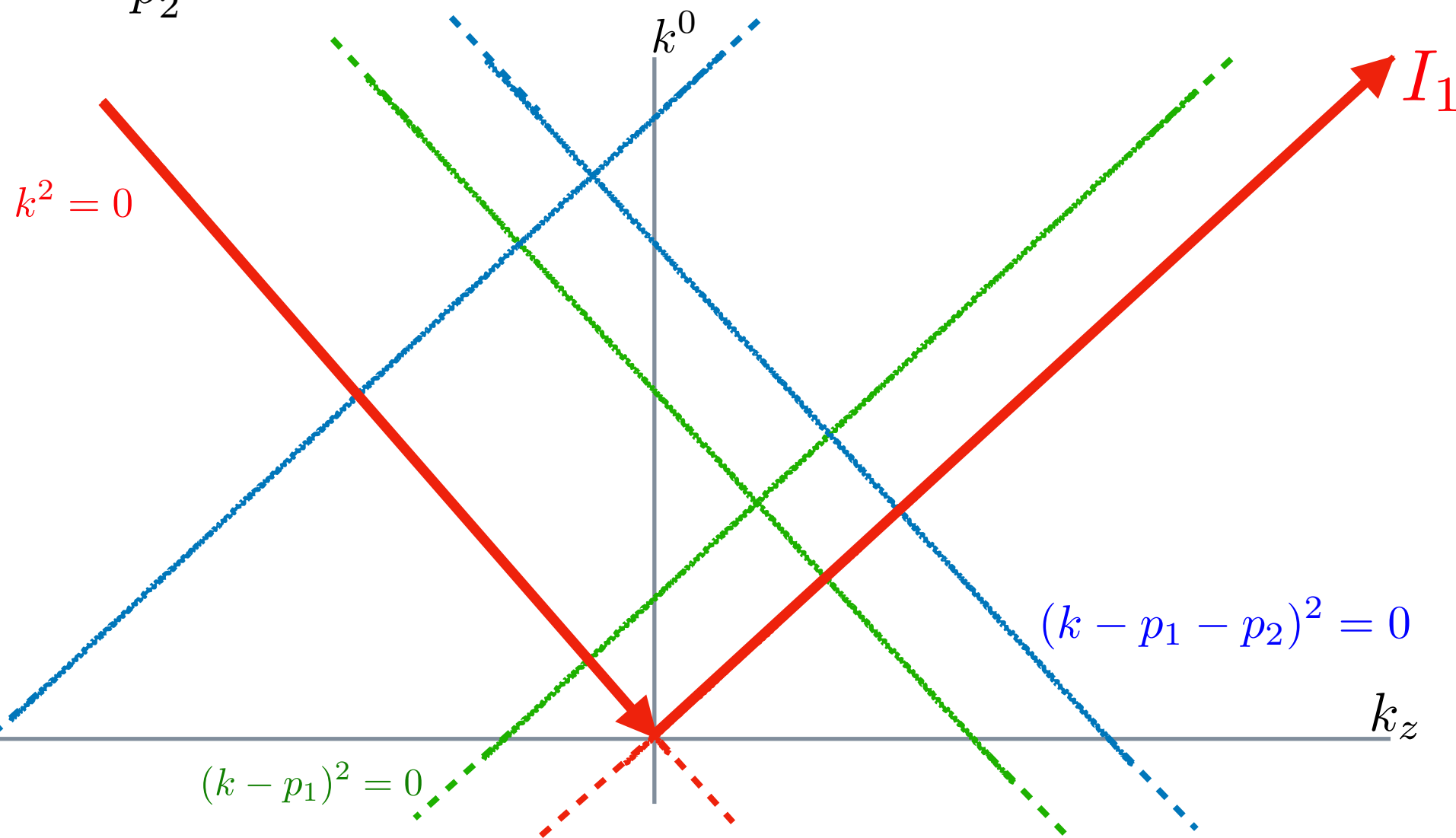
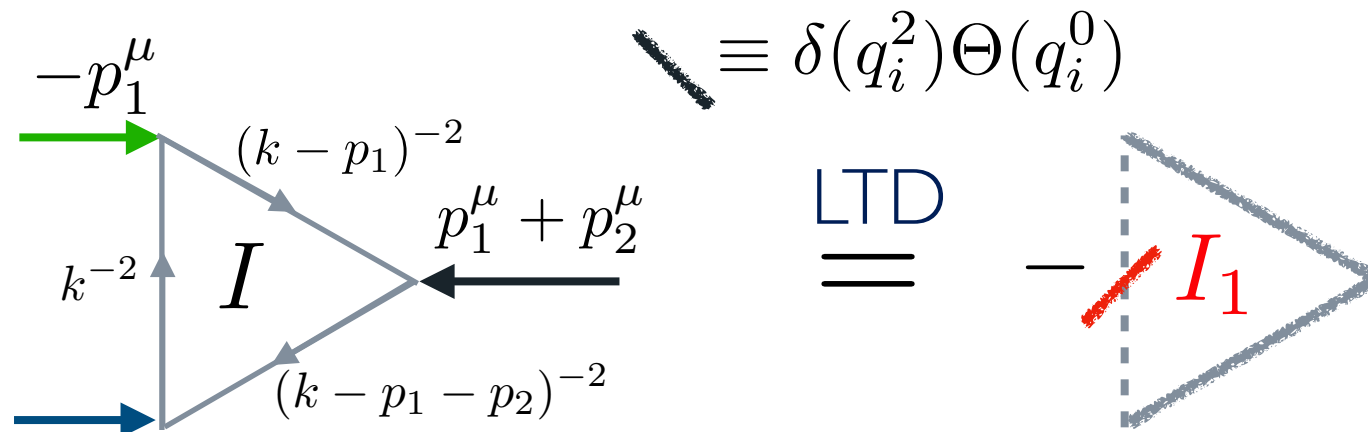
Analytically integrate over the loop energies using Cauchy's theorem (LTD):



$$p_i^2 > 0 \quad \forall i$$

SINGULAR SURFACES OF THE LTD REPRESENTATION

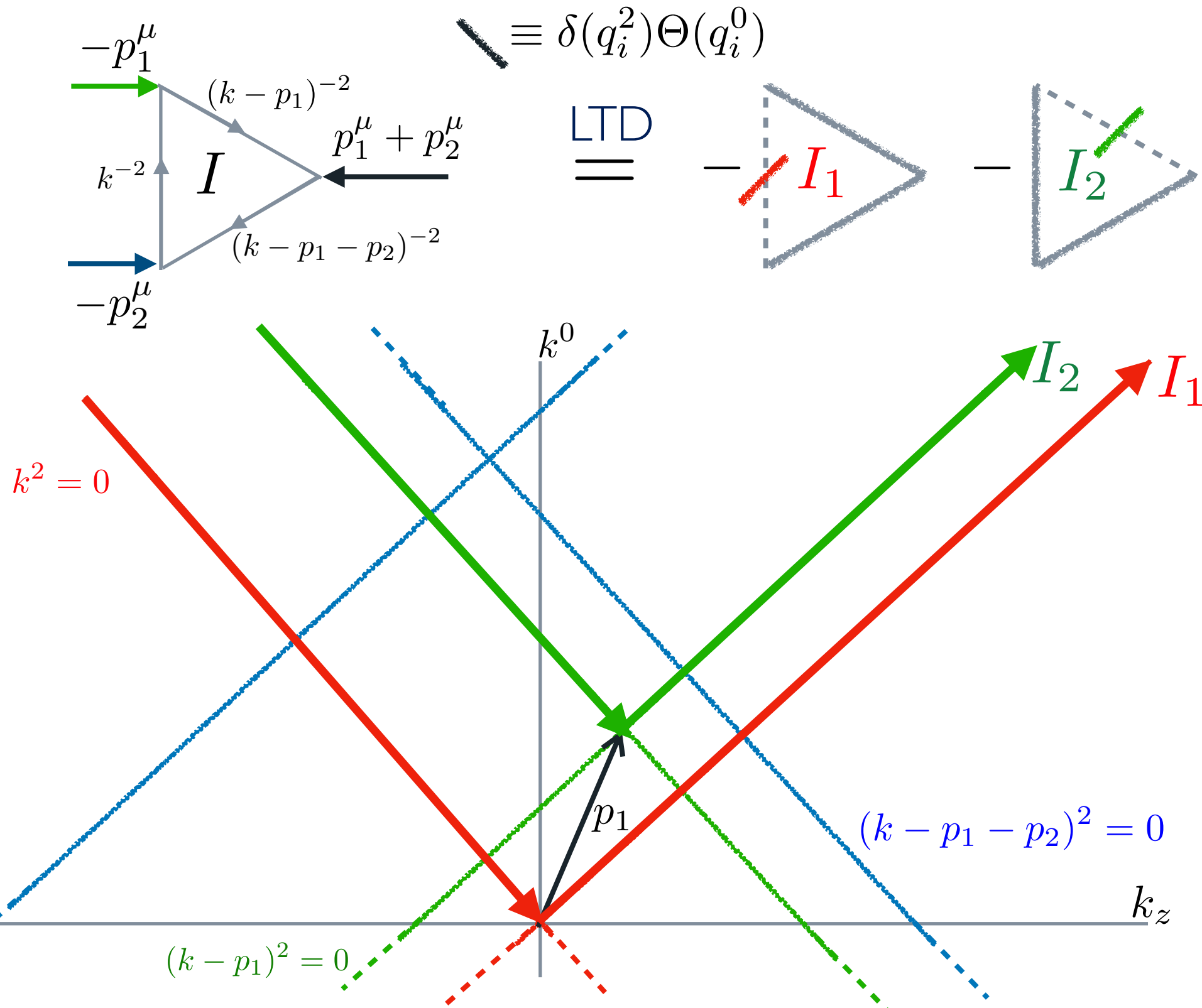
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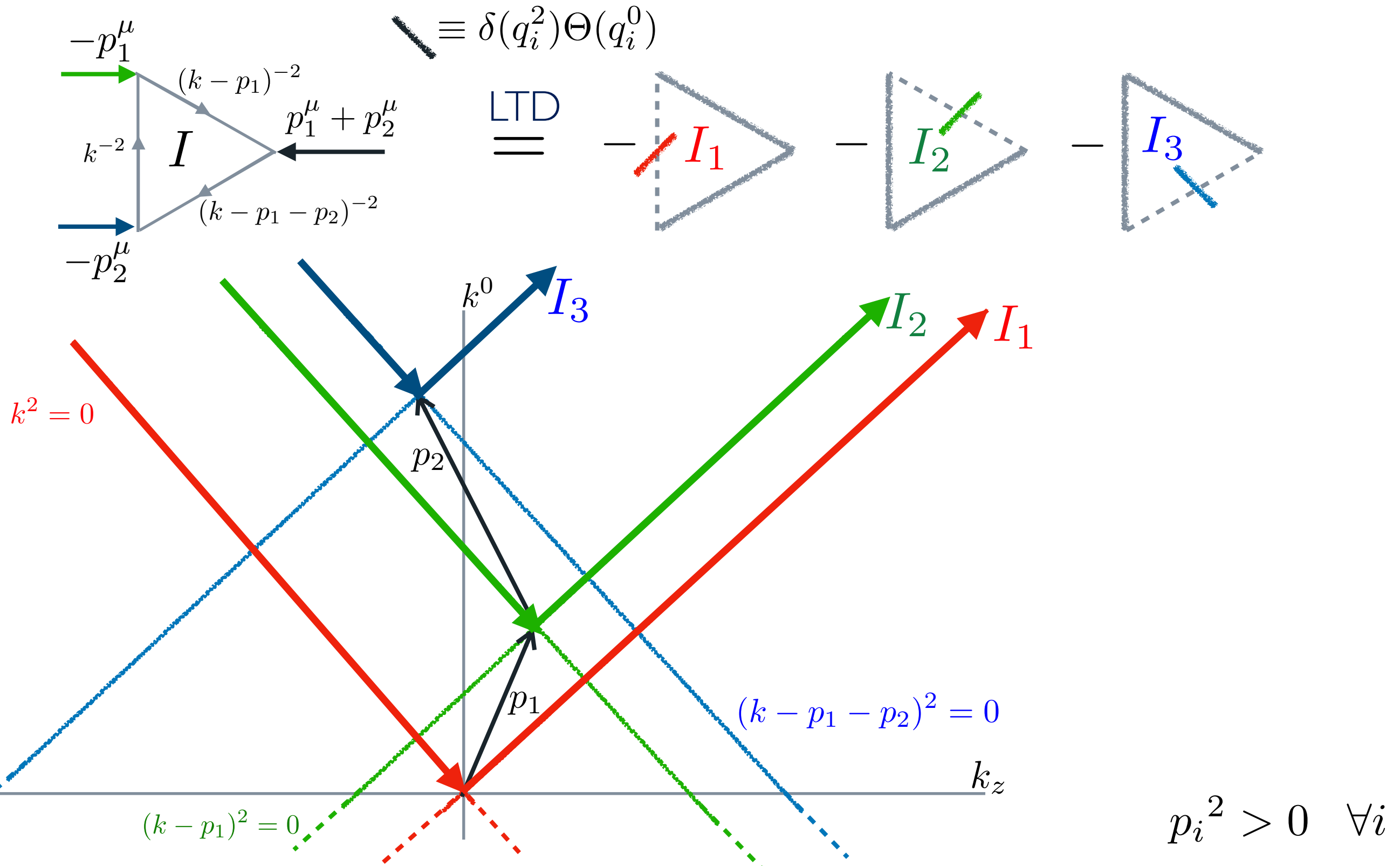
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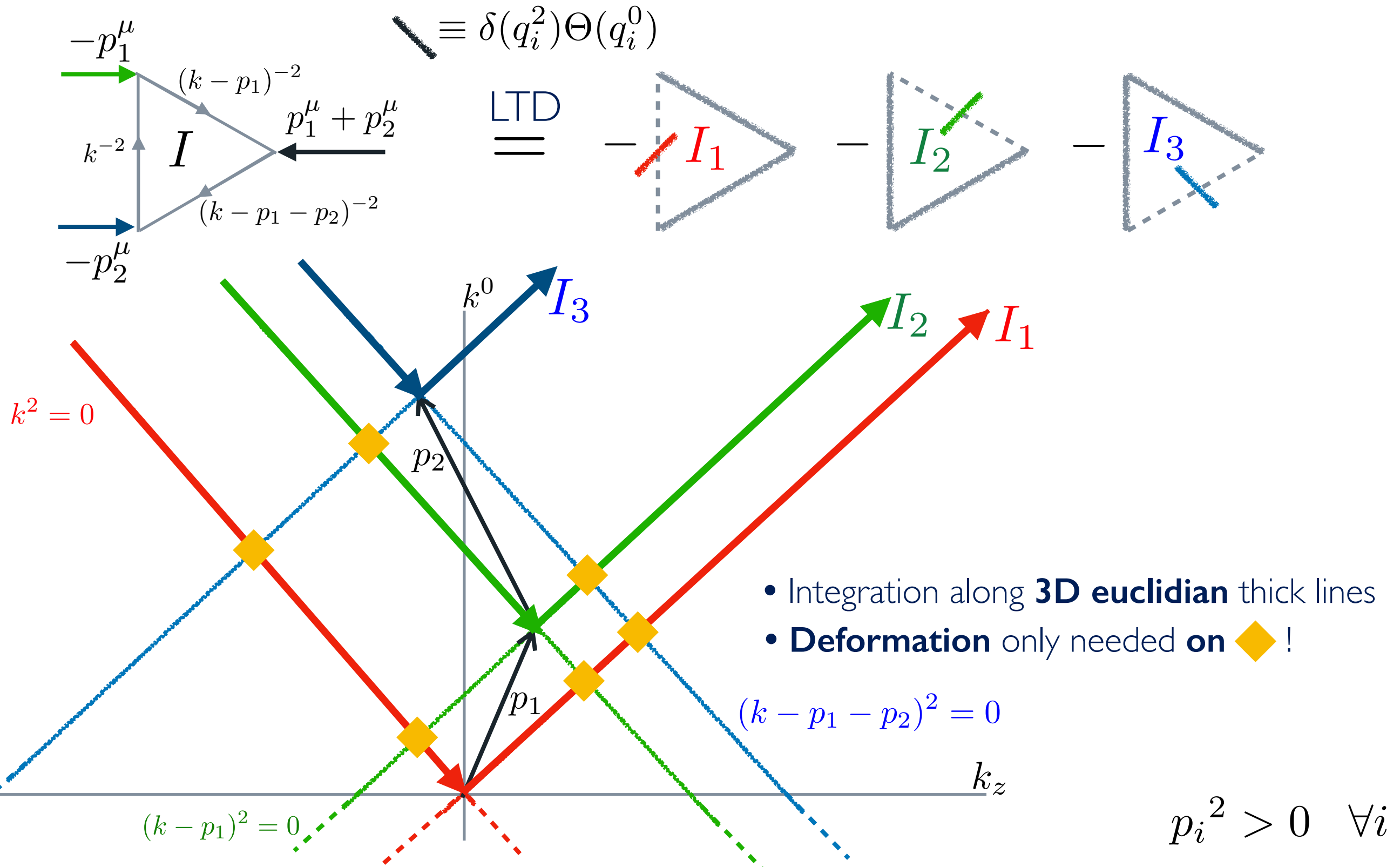
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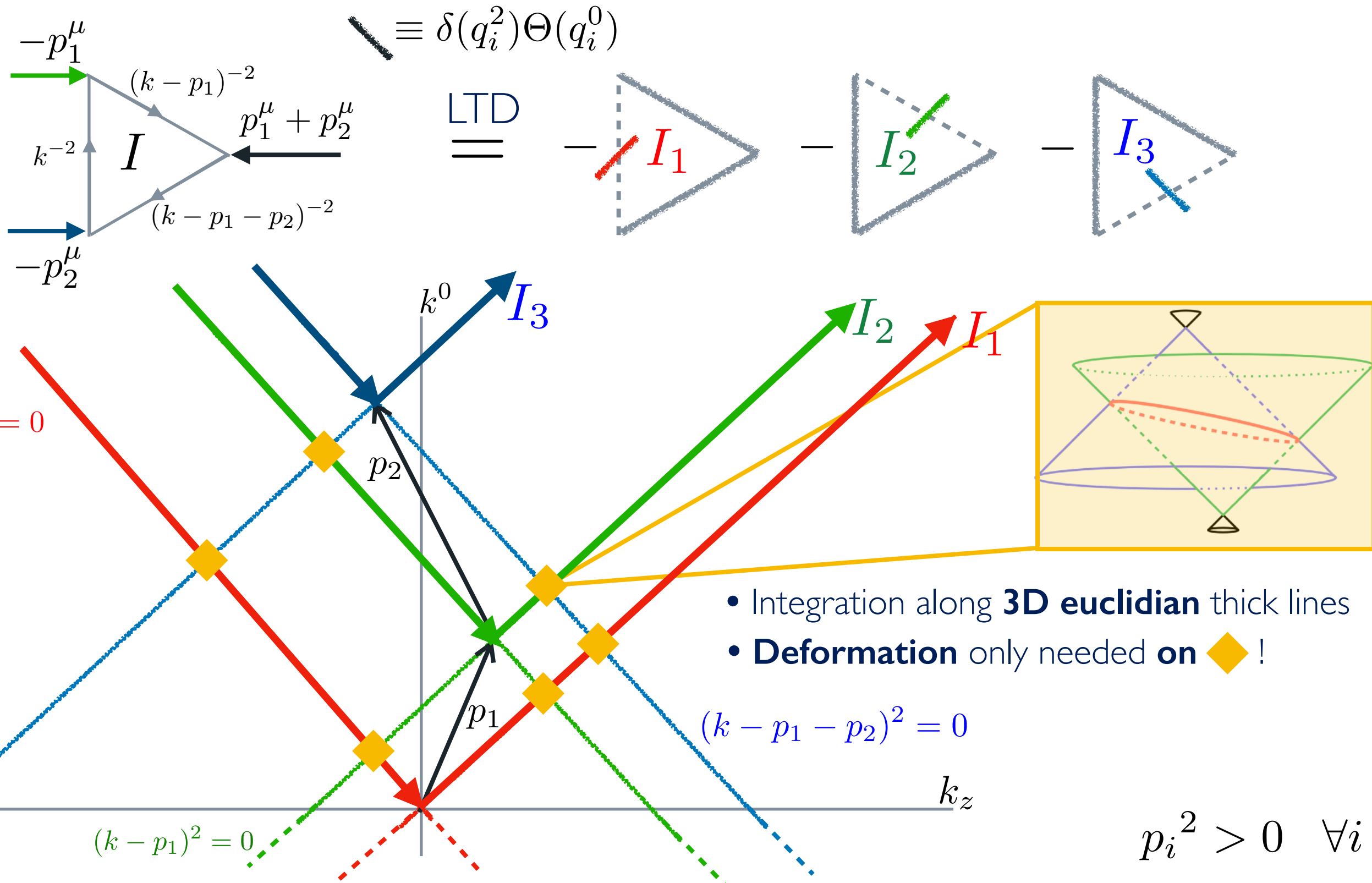
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THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kermanschah, arXiv : [2110.06869](https://arxiv.org/abs/2110.06869)]

$$\frac{1}{E_1 + E_2 - p_1^0} = \frac{1}{|\vec{k}| + |\vec{k} - \vec{p}_1| - p_1^0}$$
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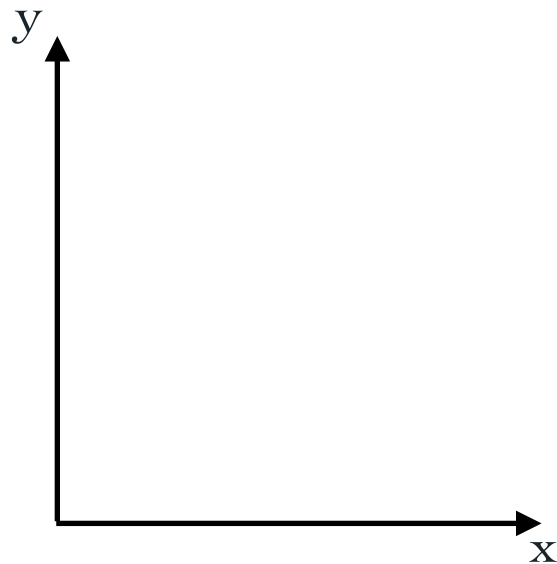
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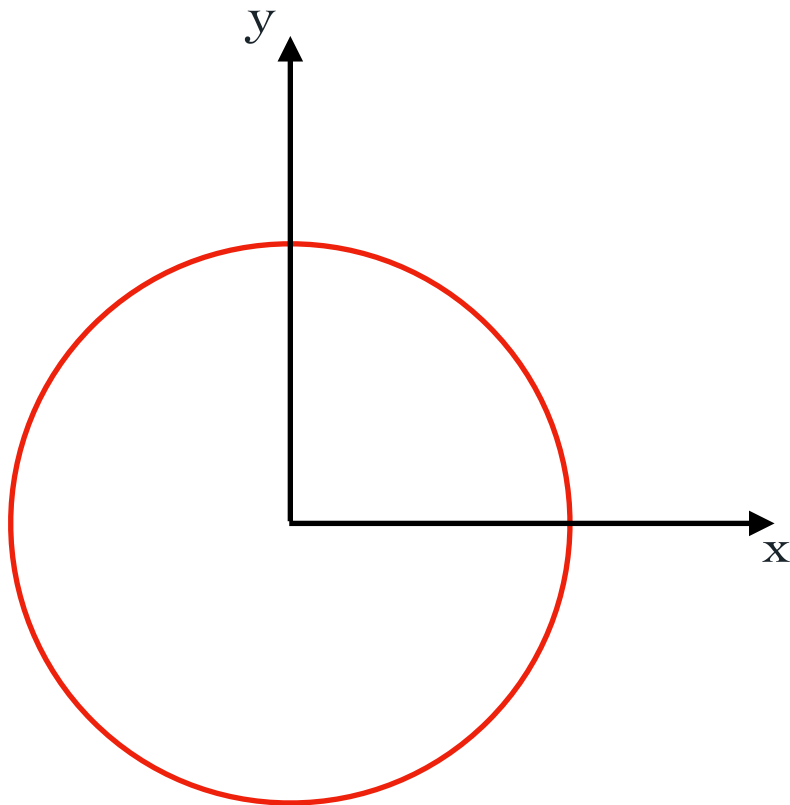
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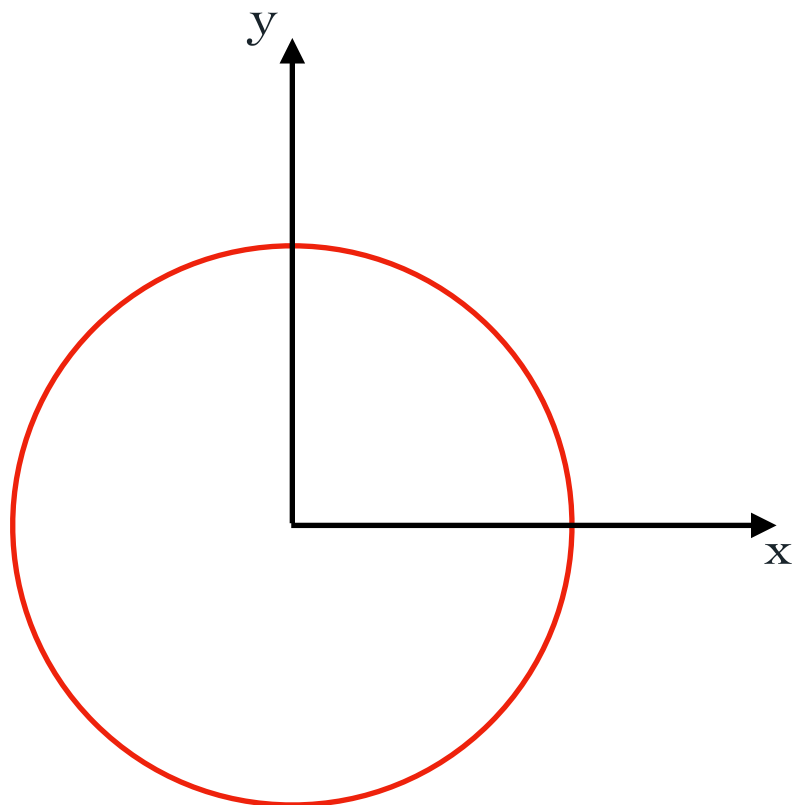
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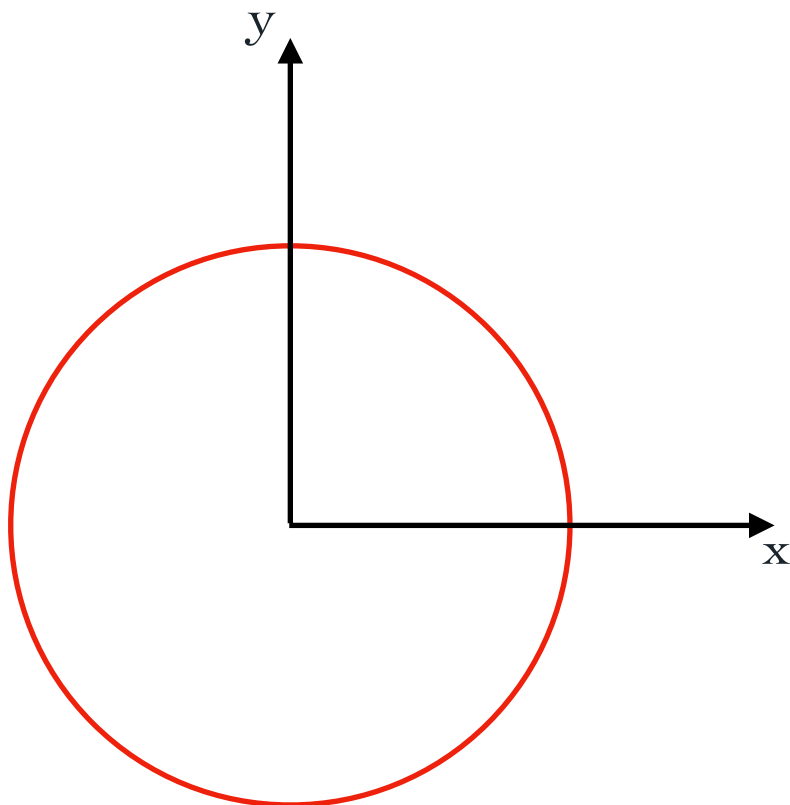
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Local threshold counterterm

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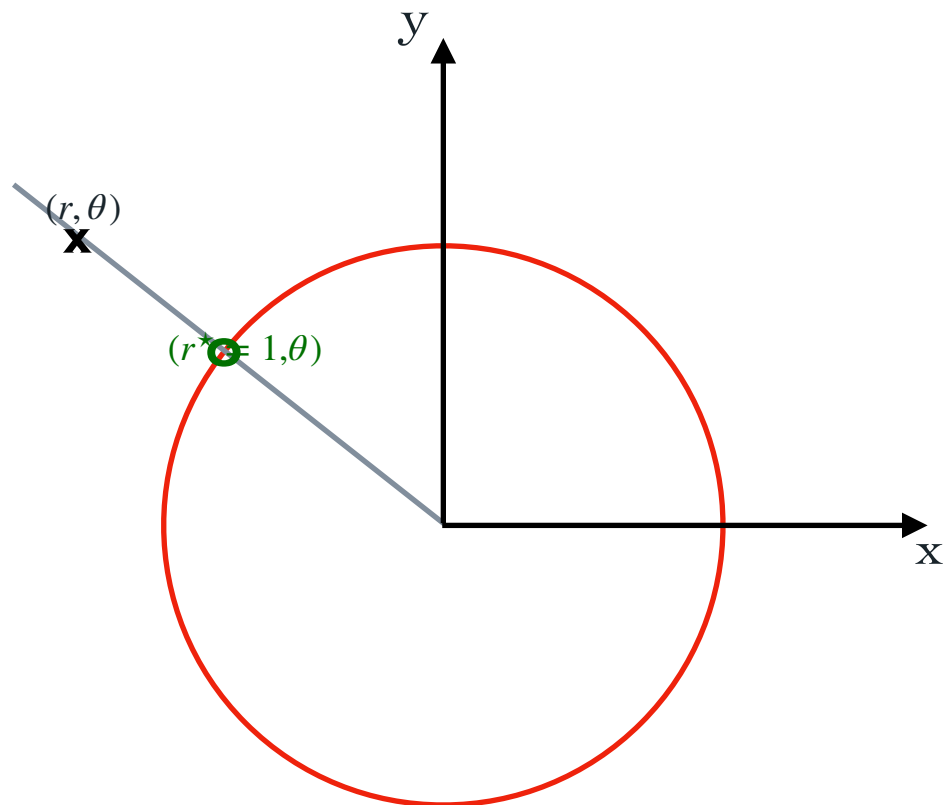
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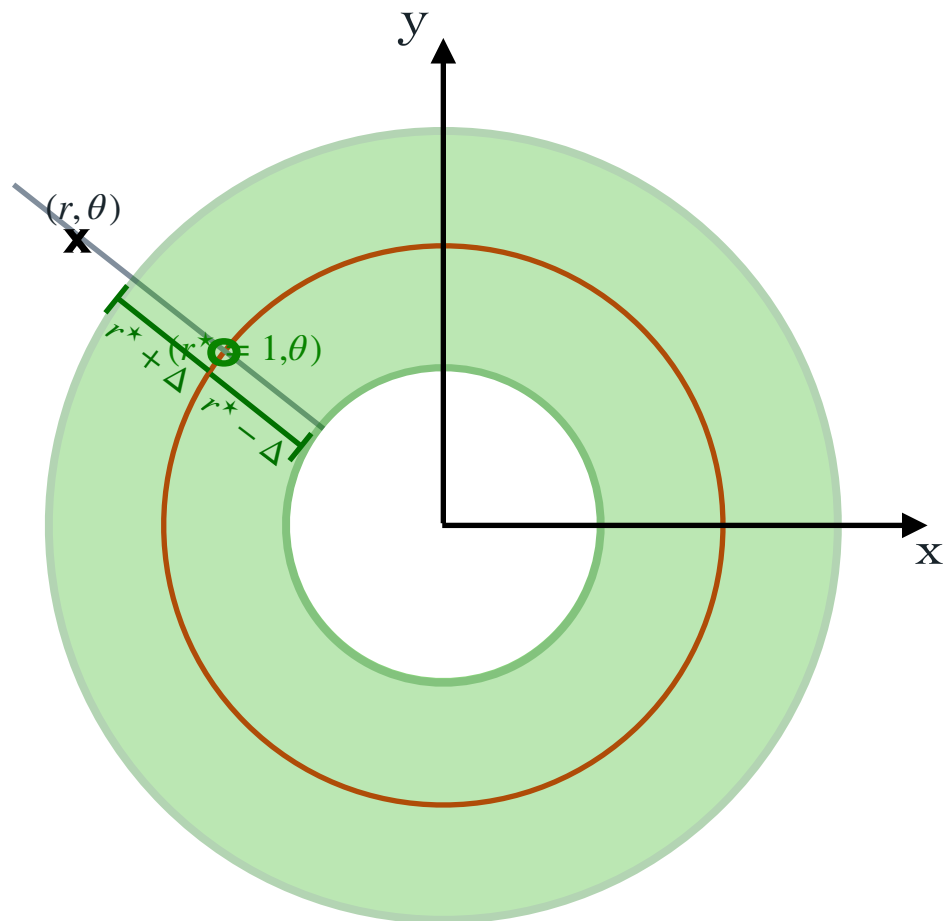
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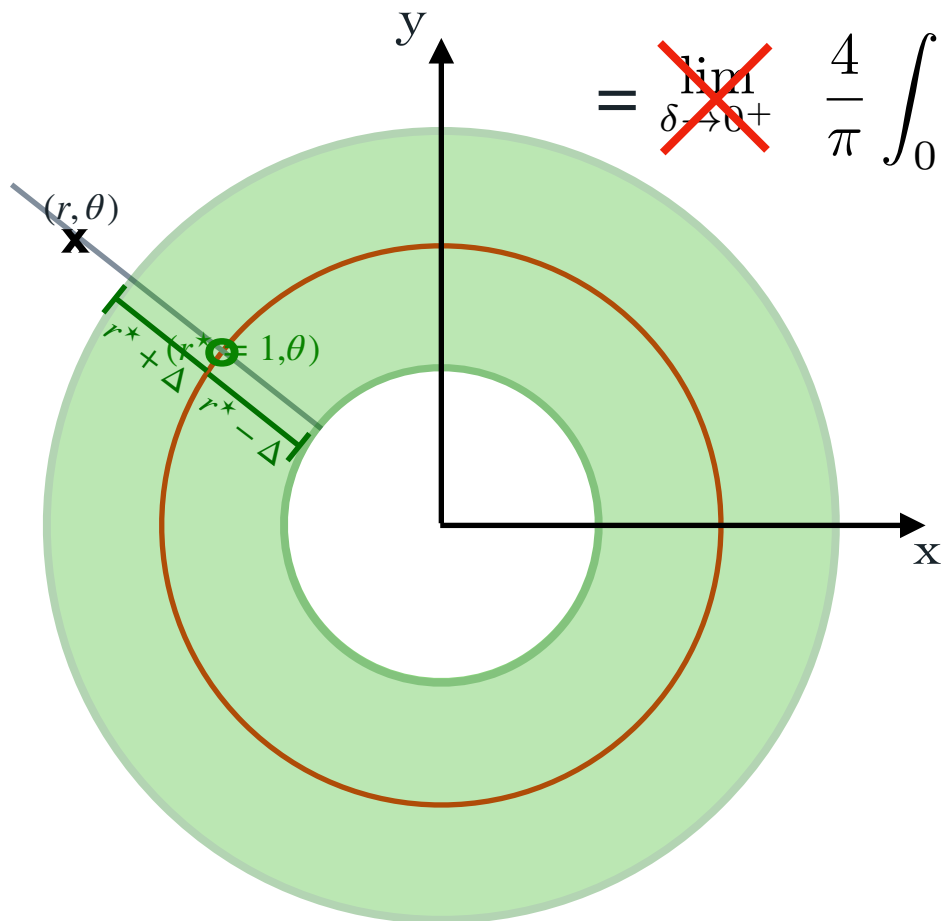
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$\delta = 0$ can be taken safely !



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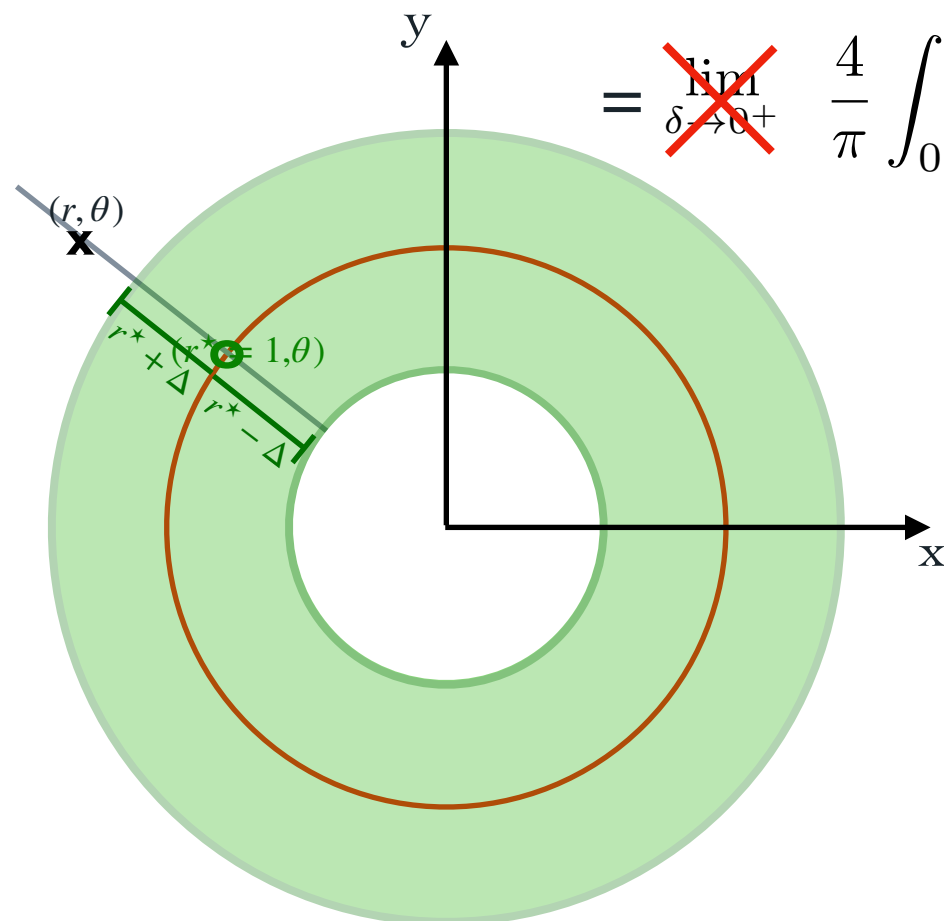
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Local threshold counterterm

Integrated threshold counterterm

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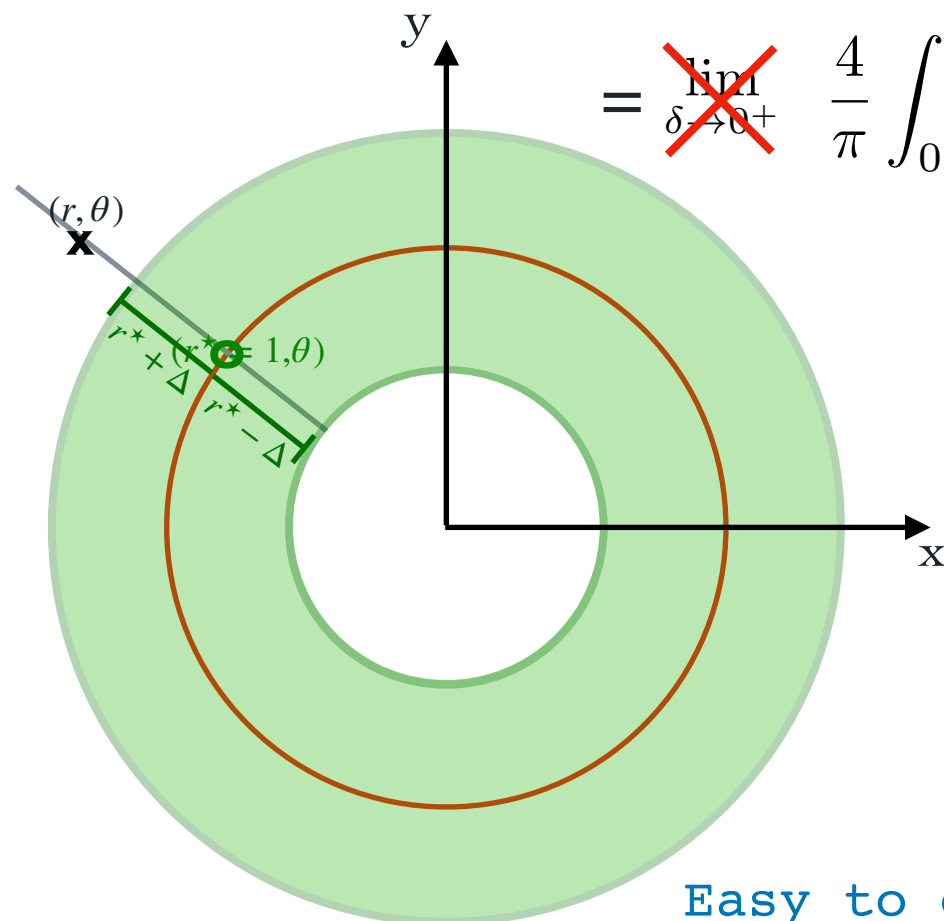
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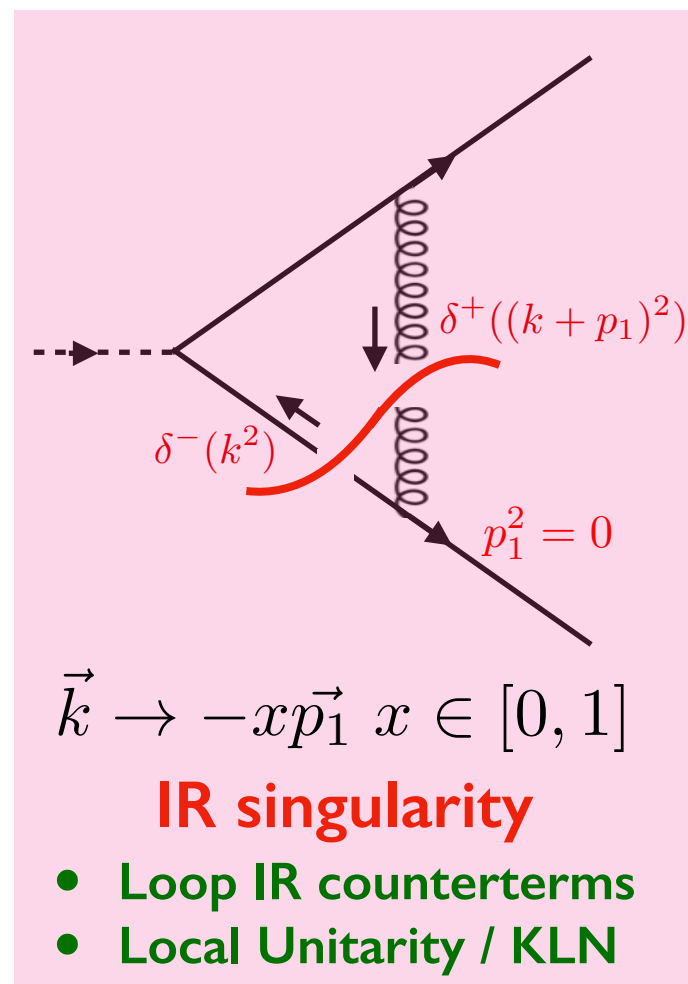
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$$+ \lim_{\delta \rightarrow 0^+} \int_{1-\Delta}^{1+\Delta} dr \frac{4}{\pi} \frac{1}{2} \frac{1}{r - 1 \pm i\delta} \stackrel{\Delta \leq 1}{=} \underbrace{\frac{2}{\pi} \text{PV} \left[\frac{1}{r - 1 \pm i\delta} \right]}_0 \boxed{\mp 2i}$$

Easy to compute since Principal Value is zero by construction !

INFRARED SINGULARITIES

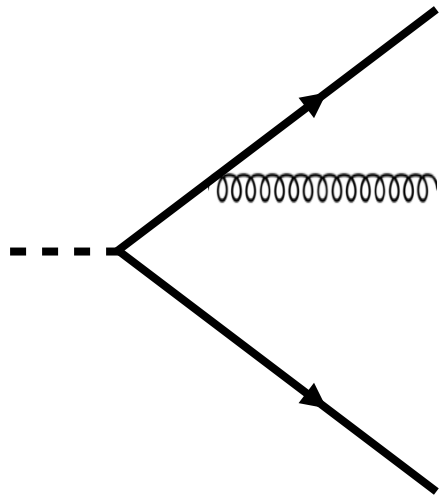
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LOCAL UNITARITY

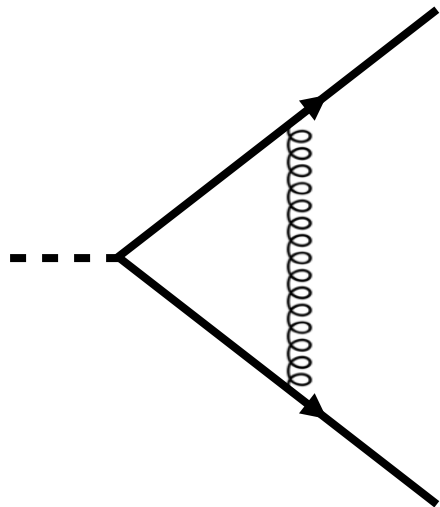
Real-emission (R)

$$\int_0^1 dx \left[\frac{\cos(x)}{x} \right]$$



Virtual (V)

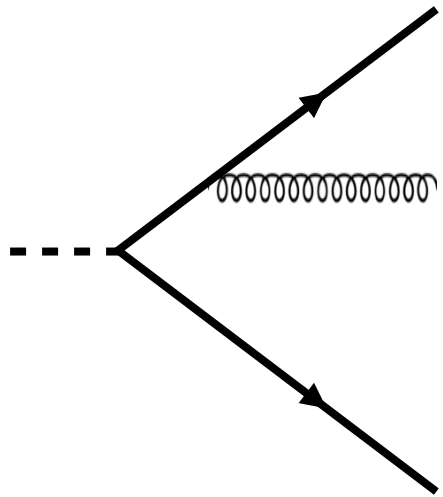
$$\int_0^{10} dy \left[-\frac{e^{-y}}{y} \right]$$



LOCAL UNITARITY

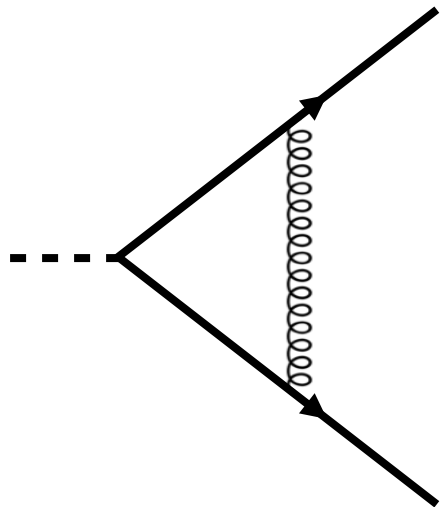
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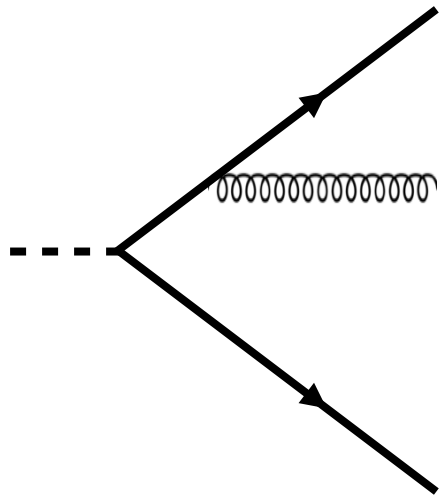
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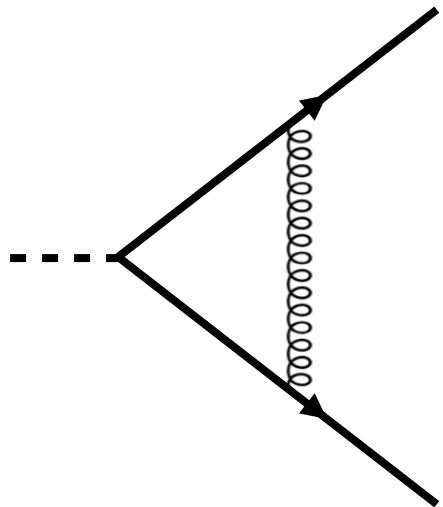
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Finite at $x = 0$!

- Integral performed numerically
- Need to design complicated counterterms

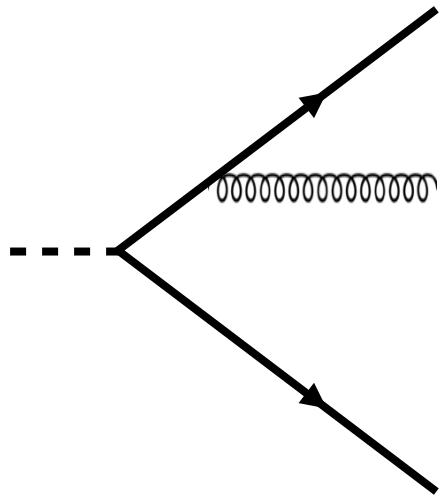
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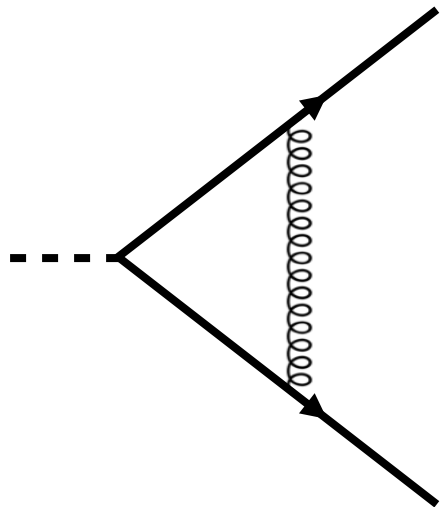
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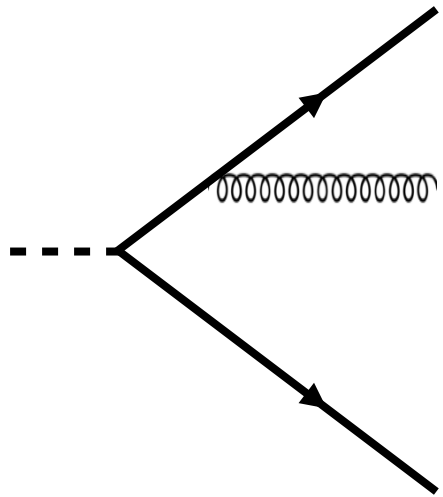
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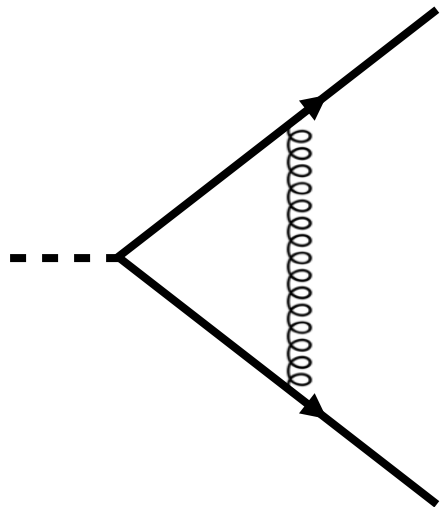


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Virtual (V)



$$\int_0^{10} dy \left[-\frac{e^{-y}}{y} \right] \rightarrow \int_{\cancel{0\epsilon}}^{10} dy \left[-\frac{e^{-y}}{y} \right] + \int_{\cancel{0\epsilon}}^1 dx \left[\frac{1}{x} \right]$$

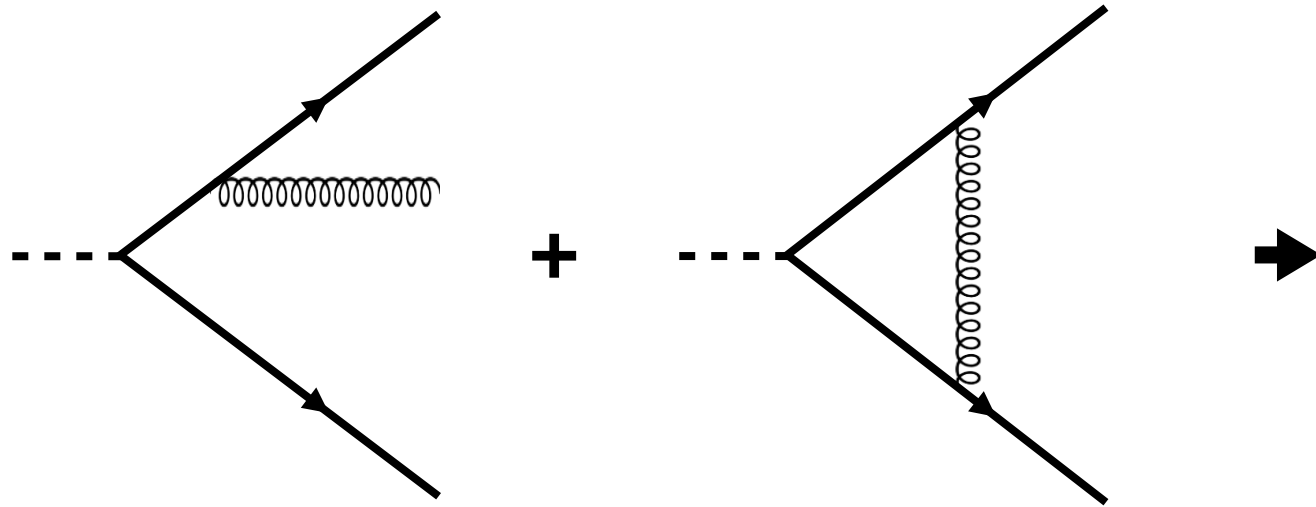
$$\Rightarrow \quad A + \cancel{\log(\epsilon)} \quad + B - \cancel{\log(\epsilon)}$$
$$\qquad\qquad \cancel{+\infty} \qquad\qquad \cancel{-\infty}$$

- Integral performed **analytically**
- No **universal** nor **systematic** solution

LOCAL UNITARITY

Real-emission (R)

Virtual (V)



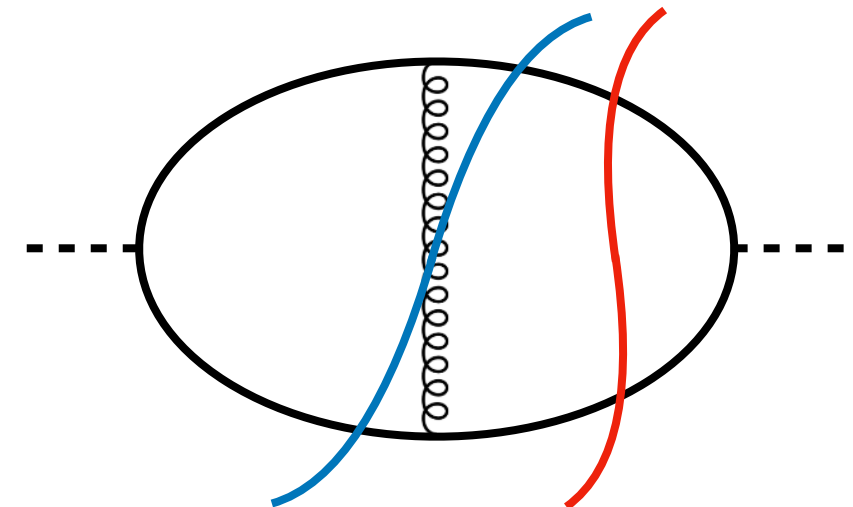
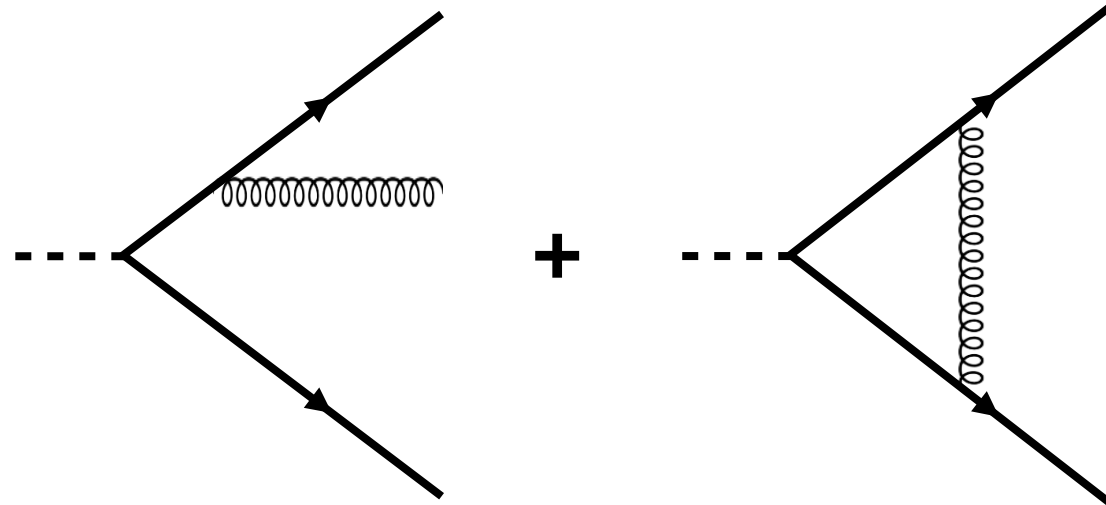
$$\int_0^1 dx \left[\frac{\cos(x)}{x} \right] + \int_0^{10} dy \left[-\frac{e^{-y}}{y} \right] \rightarrow$$

LOCAL UNITARITY

Real-emission (R)

Virtual (V)

Local Unitarity (LU)
“Localised KLN-cancellations”



$$\int_0^1 dx \left[\frac{\cos(x)}{x} \right] + \int_0^{10} dy \left[-\frac{e^{-y}}{y} \right]$$



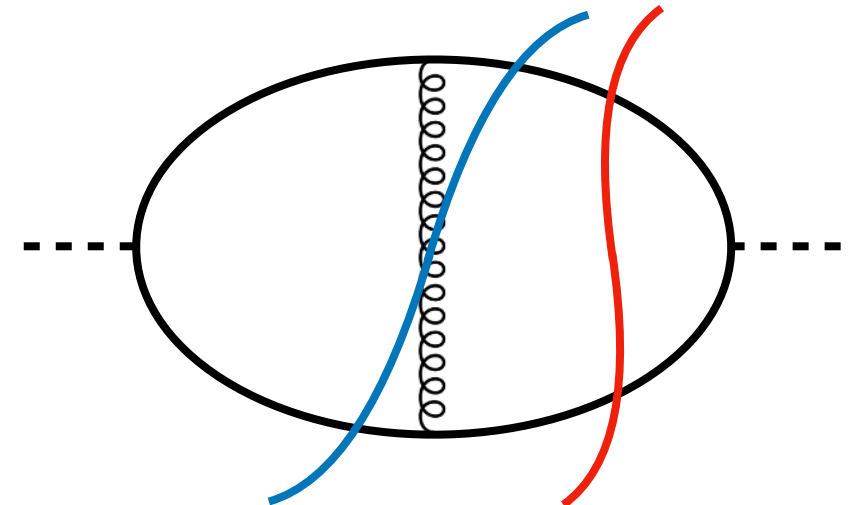
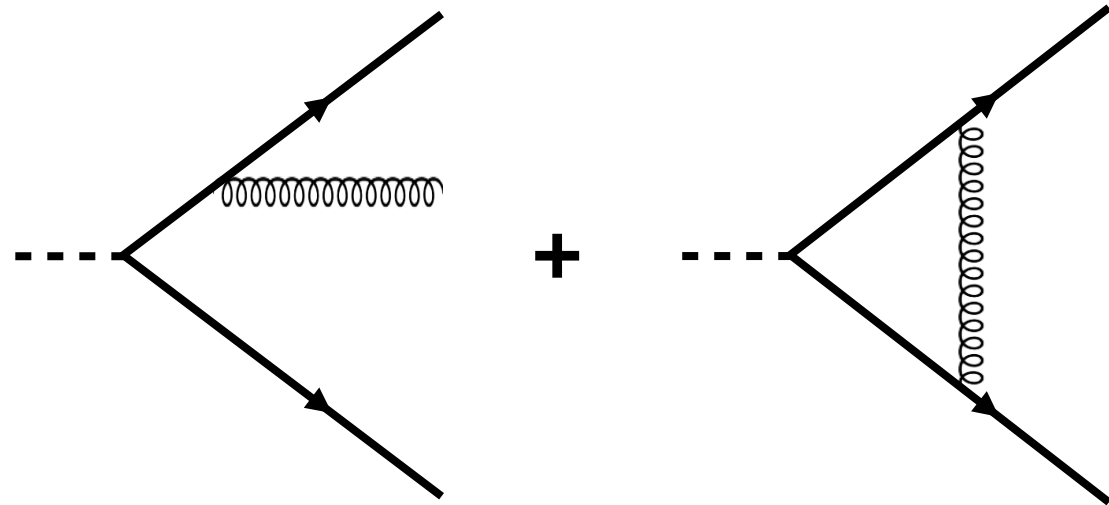
$$\int_0^{10} dx \left[\frac{\cos(x)}{x} \Theta(1-x) - \frac{e^{-x}}{x} \right]$$

LOCAL UNITARITY

Real-emission (R)

Virtual (V)

Local Unitarity (LU)
“Localised KLN-cancellations”



$$\int_0^1 dx \left[\frac{\cos(x)}{x} \right] + \int_0^{10} dy \left[-\frac{e^{-y}}{y} \right] \rightarrow \int_0^{10} dx \left[\frac{\cos(x)}{x} \Theta(1-x) - \frac{e^{-x}}{x} \right]$$

- Direct cancellation of divergences: no regulator

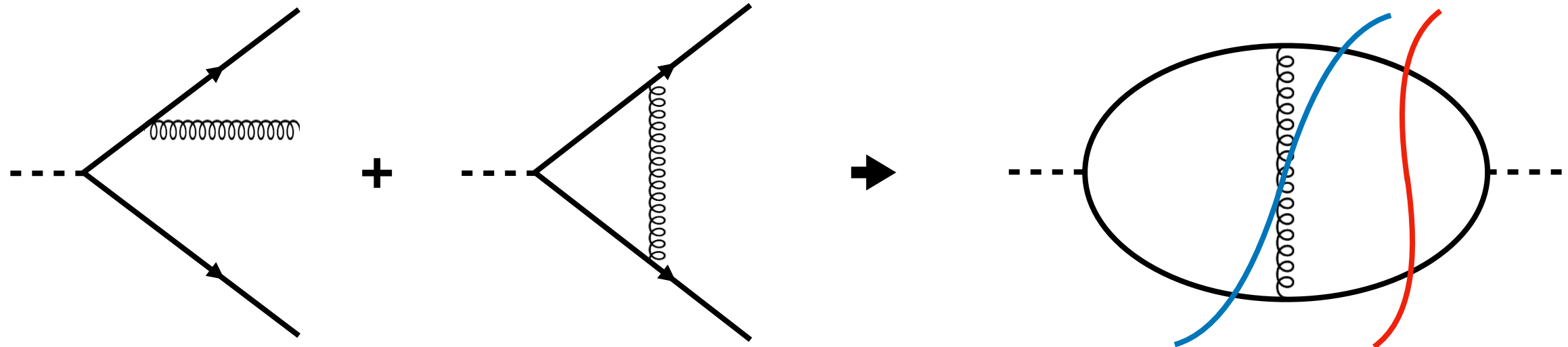
Finite at $x = 0$!

LOCAL UNITARITY

Real-emission (R)

Virtual (V)

Local Unitarity (LU)
“Localised KLN-cancellations”



$$\int_0^1 dx \left[\frac{\cos(x)}{x} \right] + \int_0^{10} dy \left[-\frac{e^{-y}}{y} \right] \rightarrow \int_0^{10} dx \left[\frac{\cos(x)}{x} \Theta(1-x) - \frac{e^{-x}}{x} \right]$$

- Direct cancellation of divergences: no regulator Finite at $x = 0$!
- Very simplified example: applying the idea to QFT is a very challenging

[Capatti, VH, Kermanschah, Ruijl, arxiv: 1906.06138] [Capatti, VH, Kermanschah, Pelloni, Ruijl, arxiv: 1912.09291]

[Capatti, VH, Kermanschah, Pelloni, Ruijl, arxiv: 2009.05509] [Capatti, VH, Pelloni, Ruijl, arxiv: 2010.01068]

[Capatti, VH, Ruijl, arxiv: 2203.11038]

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^*$$

$$+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^*$$

The diagrams represent forward-scattering graphs for the process $\gamma^* \rightarrow d\bar{d}$. Each diagram consists of a fermion loop connected to an incoming photon (wavy line) and an outgoing quark-antiquark pair (curly lines). The diagrams are arranged in two rows, with the first row representing the leading-order contribution and the second row representing the next-to-leading-order contribution. The diagrams are summed and then multiplied by their complex conjugates to give the cross-section.

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} =$$

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^* \\
 &+ \left(\text{Diagram 5} + \text{Diagram 6} \right) \times \left(\text{Diagram 7} + \text{Diagram 8} \right)^* \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 9}
 \end{aligned}$$

The diagrams represent various scattering amplitudes for the process $\gamma^* \rightarrow d\bar{d}$.
 - Diagram 1: A blue square with a wavy line entering from the left and a curved line exiting to the right.
 - Diagram 2: A wavy line entering from the left, a vertical dashed line in the middle, and a curved line exiting to the right.
 - Diagram 3: A green square with a wavy line entering from the left and a curved line exiting to the right.
 - Diagram 4: A wavy line entering from the left, a vertical dashed line in the middle, and a curved line exiting to the right, with a star symbol.
 - Diagram 5: A wavy line entering from the left, a diagonal dashed line in the middle, and a curved line exiting to the right.
 - Diagram 6: A wavy line entering from the left, a diagonal dashed line in the middle, and a curved line exiting to the right.
 - Diagram 7: A wavy line entering from the left, a diagonal dashed line in the middle, and a curved line exiting to the right, with a star symbol.
 - Diagram 8: A wavy line entering from the left, a diagonal dashed line in the middle, and a curved line exiting to the right, with a star symbol.
 - Diagram 9: A blue square and a green square joined by a vertical red line, with a wavy line entering from the left and a wavy line exiting to the right.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^* \\
 &+ \left(\text{Diagram 5} + \text{Diagram 6} \right) \times \left(\text{Diagram 7} + \text{Diagram 8} \right)^* \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 9} + \text{Diagram 10}
 \end{aligned}$$

The diagrams represent various scattering amplitudes for the process $\gamma^* \rightarrow d\bar{d}$.
 - Diagram 1: A wavy line (photon) entering a semi-circular fermion loop.
 - Diagram 2: A wavy line entering a semi-circular fermion loop, with a blue shaded rectangular region on the left and a vertical dashed line on the right.
 - Diagram 3: A wavy line entering a semi-circular fermion loop, with a green shaded rectangular region on the left.
 - Diagram 4: A wavy line entering a semi-circular fermion loop, with a vertical dashed line on the right.
 - Diagram 5: A wavy line entering a semi-circular fermion loop, with a diagonal dashed line on the left.
 - Diagram 6: A wavy line entering a semi-circular fermion loop, with a diagonal dashed line on the right.
 - Diagram 7: A wavy line entering a semi-circular fermion loop, with a diagonal dashed line on the left.
 - Diagram 8: A wavy line entering a semi-circular fermion loop, with a diagonal dashed line on the right.
 - Diagram 9: A wavy line entering a full circular fermion loop, with a red vertical line passing through the center.
 - Diagram 10: A wavy line entering a full circular fermion loop, with a blue shaded rectangular region on the left, a green shaded rectangular region on the right, and a red vertical line passing through the center.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^* \\
 &+ \left(\text{Diagram 5} + \text{Diagram 6} \right) \times \left(\text{Diagram 7} + \text{Diagram 8} \right)^* \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 9} + \text{Diagram 10} + \text{Diagram 11}
 \end{aligned}$$

The diagrams are Feynman-like graphs for the process $\gamma^* \rightarrow d\bar{d}$.
 - Diagram 1: A wavy line (photon) enters from the left and connects to a blue-shaded semi-circular loop.
 - Diagram 2: A wavy line enters from the left and connects to a semi-circular loop with a vertical dashed line inside.
 - Diagram 3: A wavy line enters from the left and connects to an empty semi-circular loop.
 - Diagram 4: A wavy line enters from the left and connects to a green-shaded semi-circular loop with a vertical dashed line inside.
 - Diagram 5: A wavy line enters from the left and connects to a semi-circular loop with a diagonal dashed line inside.
 - Diagram 6: A wavy line enters from the left and connects to a semi-circular loop with a diagonal dashed line inside.
 - Diagram 7: A wavy line enters from the left and connects to a semi-circular loop with a diagonal dashed line inside.
 - Diagram 8: A wavy line enters from the left and connects to a semi-circular loop with a diagonal dashed line inside.
 - Diagram 9: A wavy line enters from the left, goes through a circle with a vertical red line, and exits as a wavy line.
 - Diagram 10: A wavy line enters from the left, goes through a circle with a vertical dashed line and a vertical red line, and exits as a wavy line.
 - Diagram 11: A wavy line enters from the left, goes through a circle with a blue-shaded left half, a vertical dashed line, and a green-shaded right half, with a vertical red line, and exits as a wavy line.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^* \\
 &+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^* \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} \\
 &+ \text{Diagram 8}
 \end{aligned}$$

The diagrams represent various scattering amplitudes for the process $\gamma^* \rightarrow d\bar{d}$.
 - Diagram 1: A wavy line (photon) entering a semi-circular loop from the left, with a vertical dashed line (gluon) inside the loop.
 - Diagram 2: A wavy line entering a semi-circular loop from the left, with a vertical dashed line inside the loop.
 - Diagram 3: A wavy line entering a semi-circular loop from the left, with a diagonal dashed line inside the loop.
 - Diagram 4: A wavy line entering a semi-circular loop from the left, with a diagonal dashed line inside the loop.
 - Diagram 5: A wavy line entering a circular loop from the left, with a vertical dashed line inside the loop.
 - Diagram 6: A wavy line entering a circular loop from the left, with a vertical dashed line inside the loop.
 - Diagram 7: A wavy line entering a circular loop from the left, with a vertical dashed line inside the loop.
 - Diagram 8: A wavy line entering a circular loop from the left, with a diagonal dashed line inside the loop.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^* \\
 &+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^* \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} \\
 &+ \text{Diagram 8} + \text{Diagram 9}
 \end{aligned}$$

The diagrams represent various scattering amplitudes for the process $\gamma^* \rightarrow d\bar{d}$.
 - Diagram 1: A fermion loop with a photon entering from the left and a gluon exiting from the top.
 - Diagram 2: A fermion loop with a photon entering from the left and a gluon exiting from the top, with a vertical gluon line connecting the top and bottom vertices.
 - Diagram 3: A fermion loop with a photon entering from the left and a gluon exiting from the top, with a blue shaded region on the right side.
 - Diagram 4: A fermion loop with a photon entering from the left and a gluon exiting from the top, with a green shaded region on the right side.
 - Diagram 5: A fermion loop with a photon entering from the left and a gluon exiting from the top, with a red vertical line through the center.
 - Diagram 6: A fermion loop with a photon entering from the left and a gluon exiting from the top, with a red vertical line through the center and a vertical gluon line connecting the top and bottom vertices.
 - Diagram 7: A fermion loop with a photon entering from the left and a gluon exiting from the top, with a red vertical line through the center and a vertical gluon line connecting the top and bottom vertices.
 - Diagram 8: A fermion loop with a photon entering from the left and a gluon exiting from the top, with a red vertical line through the center and a diagonal gluon line connecting the top and bottom vertices.
 - Diagram 9: A fermion loop with a photon entering from the left and a gluon exiting from the top, with a red vertical line through the center and a diagonal gluon line connecting the top and bottom vertices, with a blue shaded region on the left and a green shaded region on the right.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^* \\
 &+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^* \\
 \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} \\
 &+ \text{Diagram 8} + \text{Diagram 9} \\
 &+ \text{Diagram 10}
 \end{aligned}$$

The diagrams represent various scattering amplitudes for the process $\gamma^* \rightarrow d\bar{d}$.
 - Diagrams 1 and 2 (in the first row) show a wavy line (photon) interacting with a fermion loop (quark) via a vertical gluon exchange.
 - Diagrams 3 and 4 (in the second row) show a wavy line interacting with a fermion loop via a diagonal gluon exchange. Diagram 3 has a blue shaded region, and Diagram 4 has a green shaded region.
 - Diagrams 5 through 10 (in the third row) show a wavy line interacting with a fermion loop via a vertical gluon exchange, with a red vertical line indicating a cut. Diagrams 6, 7, 8, and 9 have different internal gluon exchange topologies (vertical, diagonal, and diagonal-crossed). Diagram 10 has a blue shaded region on the left and a green shaded region on the right.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^*$$

$$+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^*$$

The diagrams in the first row represent the tree-level forward-scattering graphs for the process $\gamma^* \rightarrow d\bar{d}$. Diagram 1 is a simple vertex. Diagram 2 is a vertex with a vertical gluon loop. Diagram 3 is a vertex with a diagonal gluon loop. Diagram 4 is a vertex with a diagonal gluon loop and a blue shaded region. The second row shows the corresponding complex conjugate terms, with Diagram 4 having a green shaded region.

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10}$$

The diagrams in this equation represent the Local Unitarity (LU) expansion of the forward-scattering graphs. Diagram 5 is a vertex with a red vertical line. Diagram 6 is a vertex with a vertical gluon loop and a red vertical line. Diagram 7 is a vertex with a vertical gluon loop and a red vertical line. Diagram 8 is a vertex with a diagonal gluon loop and a red vertical line. Diagram 9 is a vertex with a diagonal gluon loop and a red vertical line. Diagram 10 is a vertex with a diagonal gluon loop and a red vertical line, with a blue shaded region on the left and a green shaded region on the right.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^*$$

$$+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^*$$

The diagrams in the first row show a wavy line entering a semi-circle from the left. In the first diagram, the semi-circle is empty. In the second, it contains a vertical dashed line. The diagrams in the second row show a wavy line entering a semi-circle from the left, with a diagonal dashed line crossing it. In the first diagram, the diagonal line is solid. In the second, it is dashed.

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10}$$

The diagrams in the second row are circular loops. Each has a wavy line entering from the left and exiting to the right. A red vertical line is drawn through each loop. Diagram 5 is highlighted with a green box. Diagrams 6 and 7 have a vertical dashed line. Diagrams 8 and 9 have a diagonal dashed line. Diagram 10 has a diagonal dashed line and a red vertical line.

 LO

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^*$$

$$+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^*$$

The diagrams in the first row show a fermion line with a self-energy loop (wavy line) and a fermion line with a self-energy loop (wavy line) and a fermion line with a self-energy loop (wavy line). The diagrams in the second row show a fermion line with a self-energy loop (wavy line) and a fermion line with a self-energy loop (wavy line) and a fermion line with a self-energy loop (wavy line).

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10}$$

The diagrams in the first row are grouped into two categories: LO (Low Order) and NLO, Double-Triangle (DT) (Next-to-Leading Order, Double-Triangle). The LO category includes Diagram 5, which is a fermion line with a self-energy loop (wavy line). The NLO, Double-Triangle category includes Diagrams 6, 7, 8, 9, and 10, which are fermion lines with self-energy loops (wavy lines) and a double-triangle loop (red line).

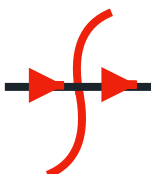
REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{tree} + \text{box} \right) \times \left(\text{tree} + \text{box} \right)^*$$

$$+ \left(\text{triangle} + \text{triangle} \right) \times \left(\text{triangle} + \text{triangle} \right)^*$$

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LO} + \text{NLO, Double-Triangle (DT)} + \text{NLO, Self-Energy (SE)}$$

■ LO
■ NLO, Double-Triangle (DT)
■ NLO, Self-Energy (SE)

 $\equiv \frac{p^2}{2p^0} \delta(p^2) \Theta(p^0)$

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{tree} + \text{box} \right) \times \left(\text{tree} + \text{box} \right)^*$$

$$+ \left(\text{box} + \text{box} \right) \times \left(\text{box} + \text{box} \right)^*$$

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LO} + \text{NLO, DT} + \text{NLO, SE}$$

■ LO
■ NLO, Double-Triangle (DT)
■ NLO, Self-Energy (SE)

$\equiv \frac{p^2}{2p^0} \delta(p^2) \Theta(p^0)$

LOCAL UNITARITY: A CONCEPTUAL SHIFT

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{normal})} = \int \Pi(\text{phase-space}) \left| \text{~}\text{C} + \text{~}\text{C}_{\text{vertical}} + \text{~}\text{C}_{\text{diag1}} + \text{~}\text{C}_{\text{diag2}} \right|^2$$

LOCAL UNITARITY: A CONCEPTUAL SHIFT

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{normal})} = \int \Pi(\text{phase-space}) \left| \text{~} \text{C} + \text{~} \text{C}^{\text{v}} + \text{~} \text{C}^{\text{d}} + \text{~} \text{C}^{\text{u}} \right|^2$$

↓

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{~} \text{C}^{\text{v}} \text{~} \right] + \text{LU} \left[\text{~} \text{C}^{\text{d}} \text{~} \right] + \text{LU} \left[2 \times \text{~} \text{C}^{\text{u}} \text{~} \right]$$

LOCAL UNITARITY: A CONCEPTUAL SHIFT

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{normal})} = \int \Pi^{(\text{phase-space})} \left| \text{~} \text{---} \text{~} + \text{~} \text{---} \text{~} + \text{~} \text{---} \text{~} + \text{~} \text{---} \text{~} \right|^2$$

↓

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{~} \text{---} \text{~} \right] + \text{LU} \left[\text{~} \text{---} \text{~} \right] + \text{LU} \left[2 \times \text{~} \text{---} \text{~} \right]$$

$$\sum_{c \in \{RRR, RRV, RVV, \dots\}} \int \Pi_c^{(\text{phase-space})} \left| \sum_{i_c=1}^{n_{\text{amplitudes}}(c)} \int \Pi_{i_c}^{(\text{loop})} \mathcal{A}_{i_c} \right|_{\text{truncated}}^2$$

IR-subtraction numerical $d = 4$

analytic $d = 4 - 2\epsilon$

LOCAL UNITARITY: A CONCEPTUAL SHIFT

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{normal})} = \int \Pi^{(\text{phase-space})} \left| \text{---} \text{---} + \text{---} \text{---} + \text{---} \text{---} + \text{---} \text{---} \right|^2$$

↓

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{---} \text{---} \right] + \text{LU} \left[\text{---} \text{---} \right] + \text{LU} \left[2 \times \text{---} \text{---} \right]$$

$$\sum_{c \in \{RRR, RRV, RVV, \dots\}} \int \Pi_c^{(\text{phase-space})} \left| \sum_{i_c=1}^{n_{\text{amplitudes}}(c)} \int \Pi_{i_c}^{(\text{loop})} \mathcal{A}_{i_c} \right|_{\text{truncated}}^2$$

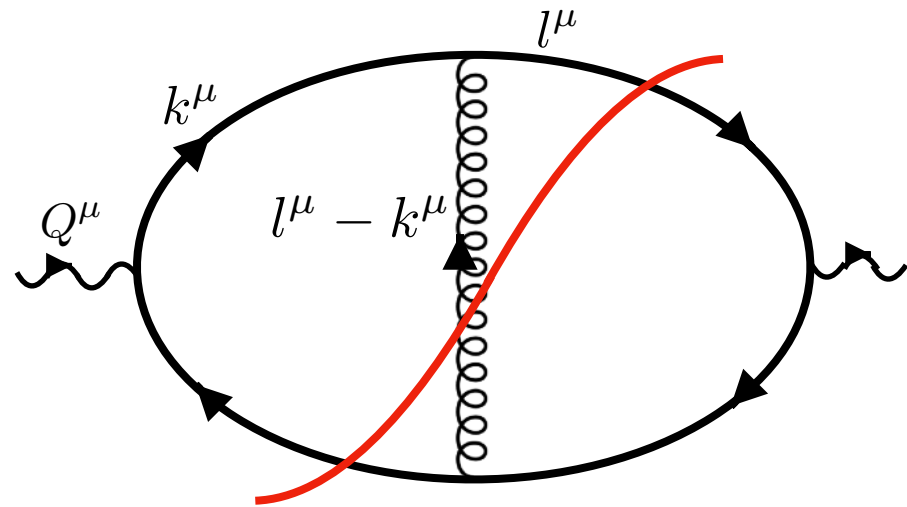
IR-subtraction numerical $d = 4$

analytic $d = 4 - 2\epsilon$

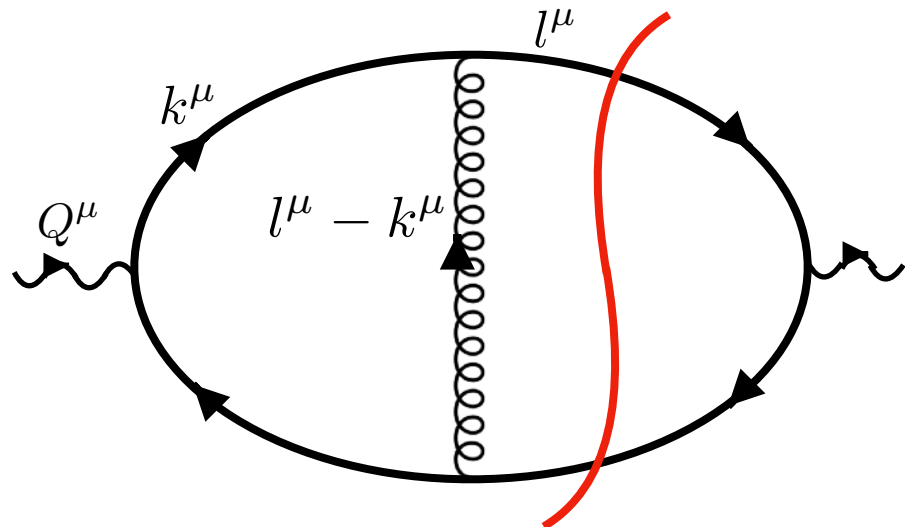
$$\sum_{j=1}^{n_{\text{supergraphs}}} \int \Pi g_j^{(\text{LU})}$$

numerical $d = 4$
NO IR-subtraction

LOCAL UNITARITY: THE ROLE OF LTD



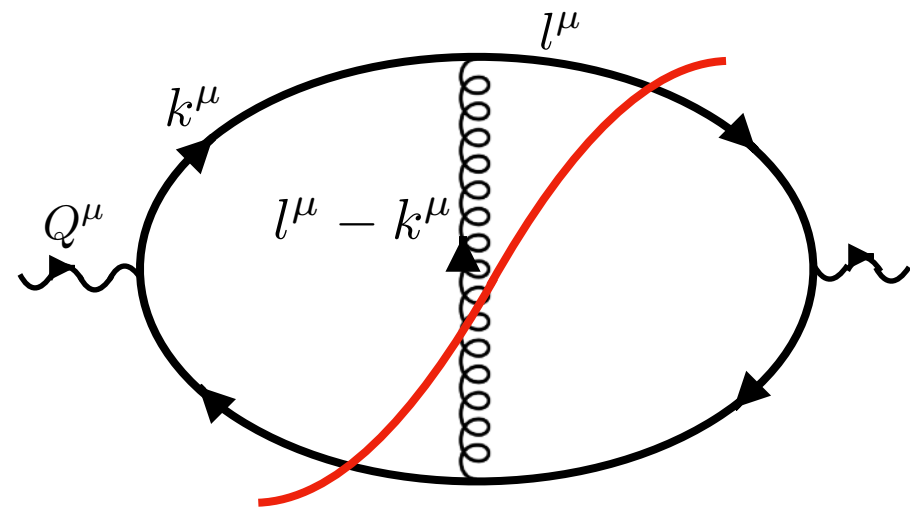
$$\propto \delta^+(l^2) \delta^-((l-k)^2) \delta^-((k-Q)^2)$$



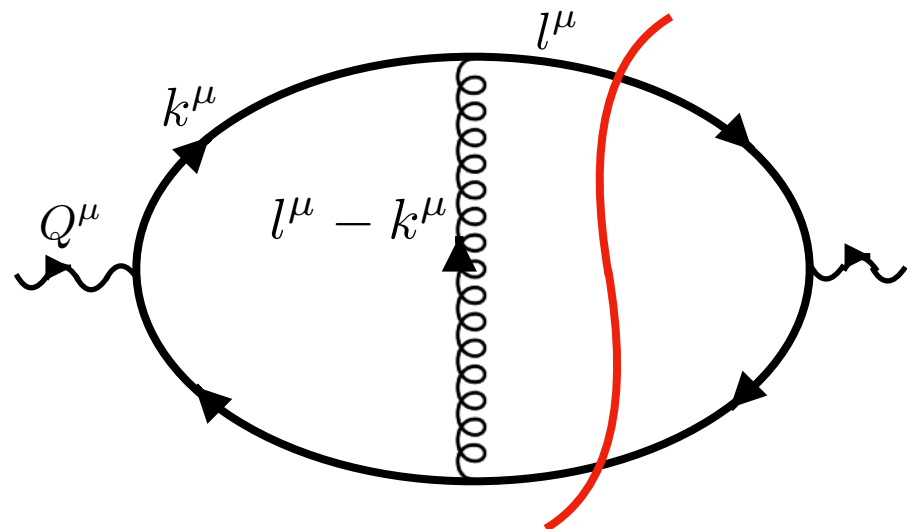
$$\propto \delta^+(l^2) \delta^-((l-Q)^2)$$

Simple and easy-to-follow proceeding show-casing Local Unitarity: [Capatti, arxiv: [2110.15662](https://arxiv.org/abs/2110.15662)]

LOCAL UNITARITY: THE ROLE OF LTD

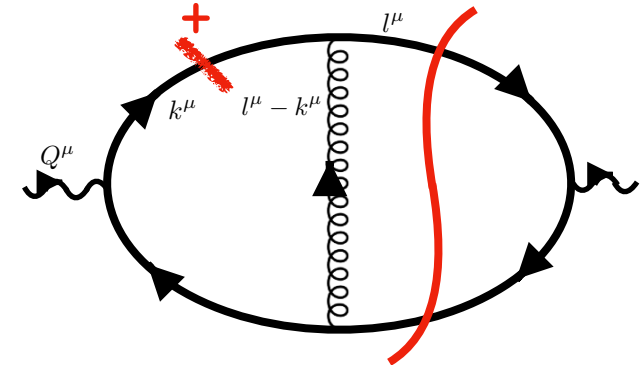


$$\propto \delta^+(l^2) \delta^-((l-k)^2) \delta^-((k-Q)^2)$$



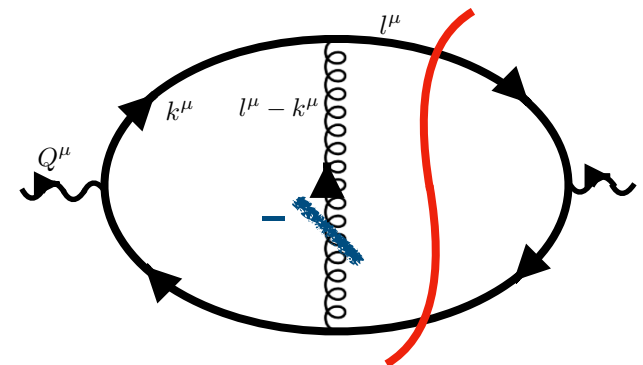
$$\propto \delta^+(l^2) \delta^-((l-Q)^2)$$

LTD
=



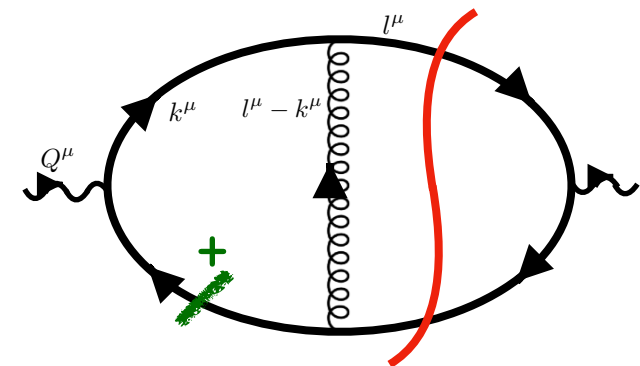
$$\propto \delta^+(l^2) \delta^-((l-Q)^2) \delta^+(k^2)$$

+



$$\propto \delta^+(l^2) \delta^-((l-Q)^2) \delta^+((l-k)^2)$$

+



$$\propto \delta^+(l^2) \delta^-((l-Q)^2) \delta^+((k-Q)^2)$$

Simple and easy-to-follow proceeding show-casing Local Unitarity: [Capatti, arxiv: [2110.15662](https://arxiv.org/abs/2110.15662)]

LOCALITY UNITARITY

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :

— = Cutkosky cut — = threshold singularity

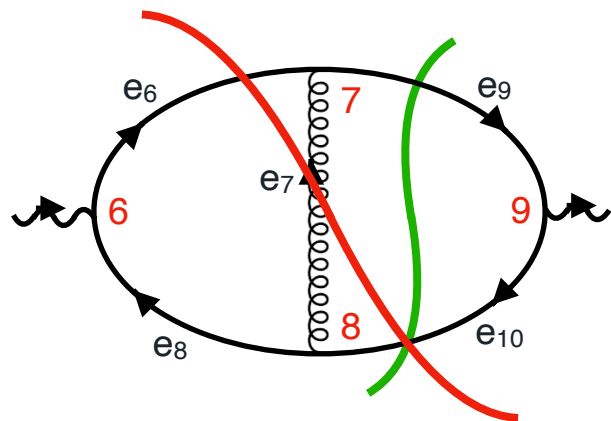


LOCALITY UNITARITY

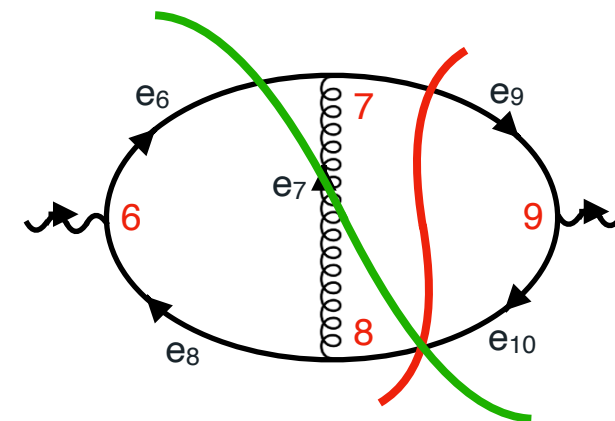
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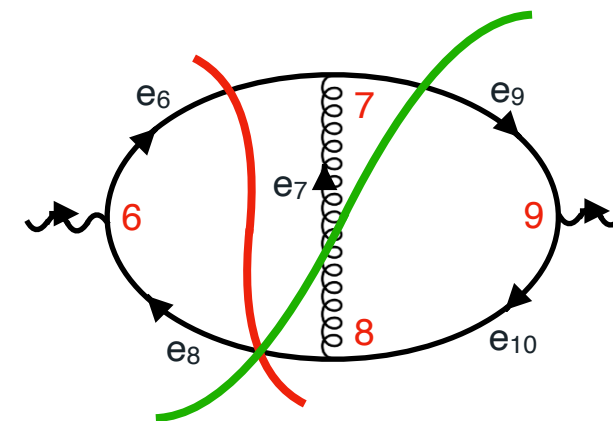
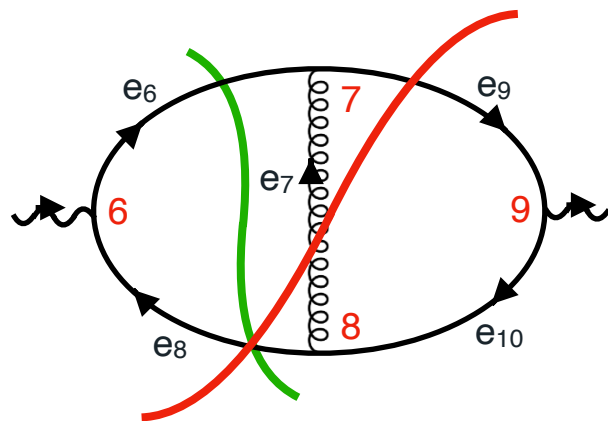
— = Cutkosky cut — = threshold singularity



cancel



cancel

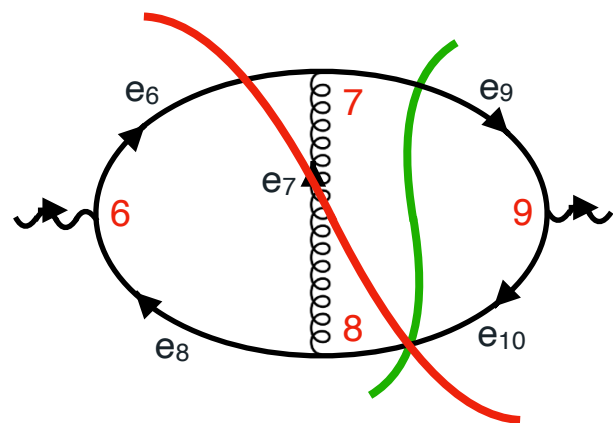


LOCALITY UNITARITY

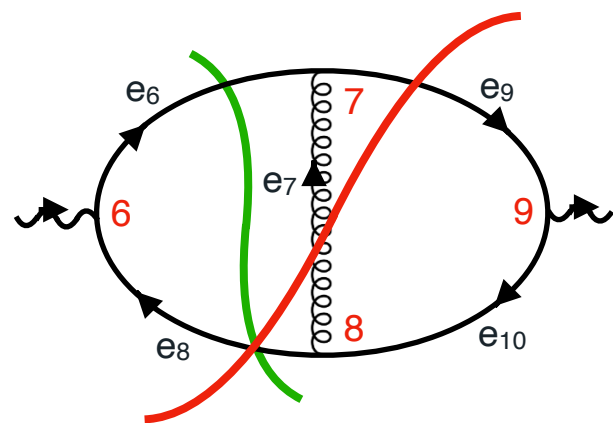
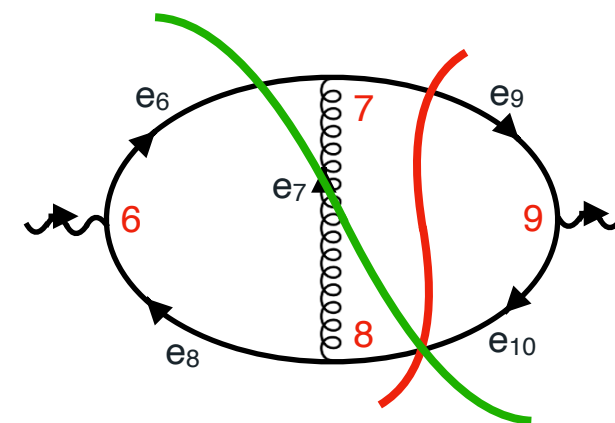
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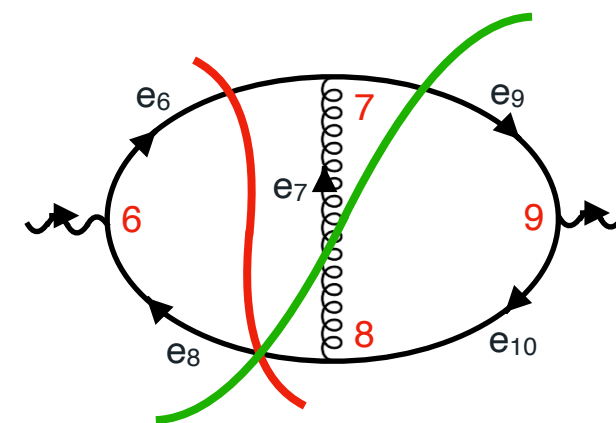
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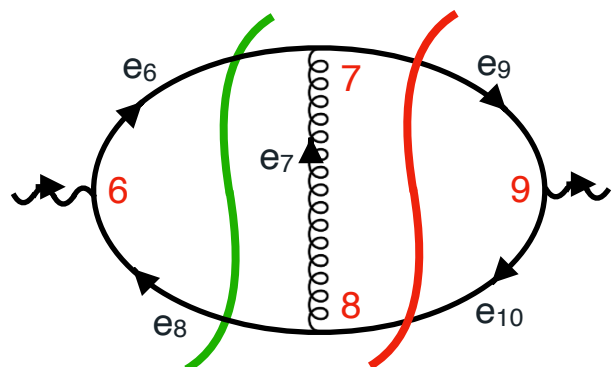
cancels



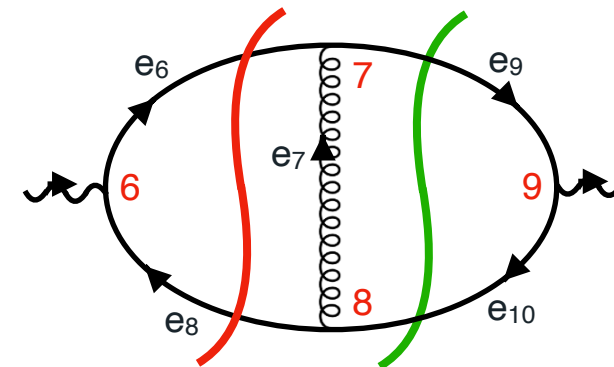
cancels



Even for **non-pinched singular threshold** ! (when $\mathcal{O}_s \equiv 1$) :



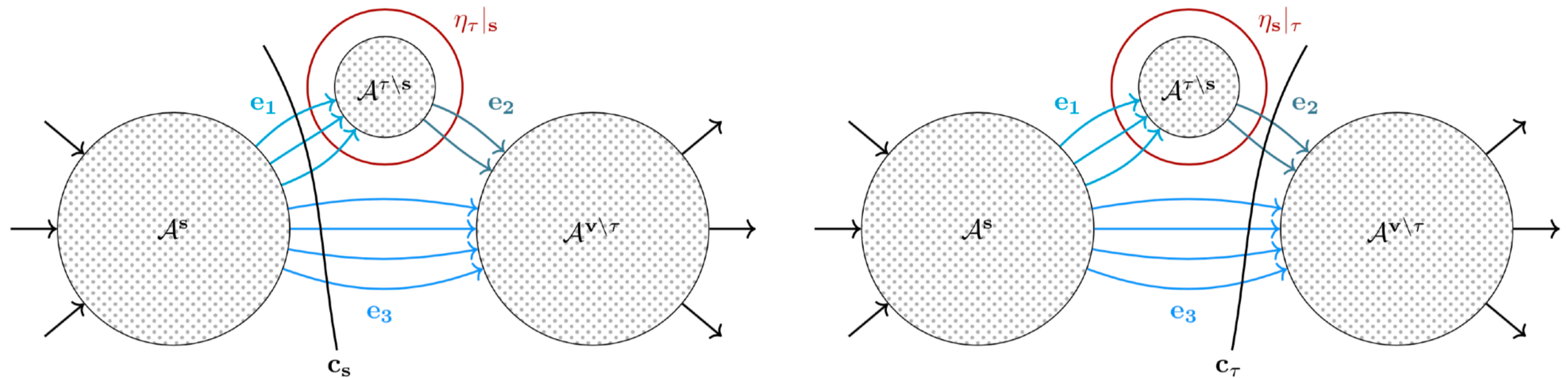
cancels



LOCALITY UNITARITY

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

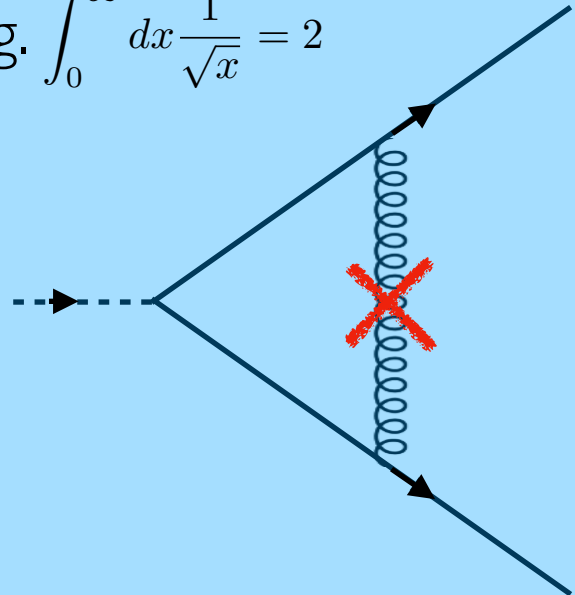
This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :



INTEGRABLE SINGULARITIES

$$\int d^2\Omega \int_0^\infty |\vec{k}|^2 d|\vec{k}| \frac{a|\vec{k}|^2 + b|\vec{k}| + c}{8|\vec{k}|^2 |\vec{k} + \vec{p}_1|} \frac{1}{|\vec{k}| + |\vec{k} + \vec{p}_1| - |\vec{p}_1|} \frac{1}{|\vec{k}| + |\vec{k} + \vec{Q}| - Q^0 + i\delta}$$

e.g. $\int_0^\infty dx \frac{1}{\sqrt{x}} = 2$



$\vec{k} \rightarrow -\vec{p}_1$

Integrable singularity

- LMB multi-channeling
- Tropical sampling

TROPICAL SAMPLING IN MOMENTUM SPACE

FOR TAMING INTEGRABLE SINGULARITIES

$$I^{(\text{eucl.})}[f] = \int d^D L \mathbf{k} \frac{f(\mathbf{k})}{\prod_e D_e(\mathbf{k})^{\nu_e}}$$

TROPICAL SAMPLING IN MOMENTUM SPACE

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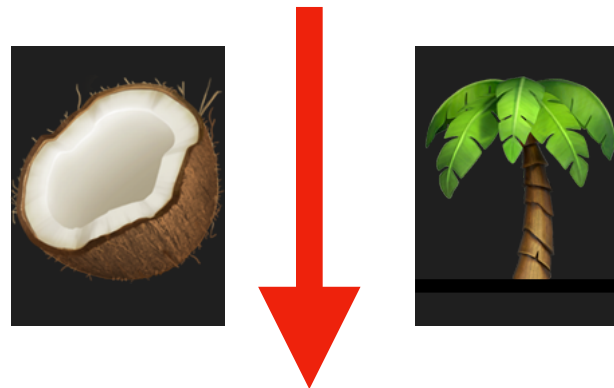
$$D_e(\mathbf{k}) = (\mathbf{M} \cdot \mathbf{k} + \mathbf{Q})^2 + m_e^2$$

TROPICAL SAMPLING IN MOMENTUM SPACE

FOR TAMING INTEGRABLE SINGULARITIES

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TROPICAL SAMPLING OF EUCLIDEAN FEYNMAN INTEGRALS

$$I^{\text{eucl.}}[f = 1]$$

E	$\ell(G)$	σ_I/I	samples per second	preprocessing time	RAM
6	3	0.9	$1.1 \cdot 10^6 / s$	$3.0 \cdot 10^{-5} s$	1 KB
8	4	1.1	$7.5 \cdot 10^5 / s$	$1.3 \cdot 10^{-4} s$	4 KB
10	5	1.3	$5.1 \cdot 10^5 / s$	$6.0 \cdot 10^{-4} s$	16 KB
12	6	1.6	$4.1 \cdot 10^5 / s$	$2.7 \cdot 10^{-3} s$	64 KB
14	7	1.8	$3.2 \cdot 10^5 / s$	$1.2 \cdot 10^{-2} s$	256 KB
16	8	2.1	$2.6 \cdot 10^5 / s$	$5.3 \cdot 10^{-2} s$	1 MB
18	9	2.5	$2.1 \cdot 10^5 / s$	$2.3 \cdot 10^{-1} s$	4 MB
20	10	2.8	$1.4 \cdot 10^5 / s$	$1.1 \cdot 10^0 s$	16 MB
22	11	3.2	$1.0 \cdot 10^5 / s$	$4.7 \cdot 10^0 s$	64 MB
24	12	3.7	$8.6 \cdot 10^4 / s$	$2.1 \cdot 10^1 s$	256 MB
26	13	4.2	$6.9 \cdot 10^4 / s$	$9.5 \cdot 10^1 s$	1 GB
28	14	4.8	$5.9 \cdot 10^4 / s$	$4.4 \cdot 10^2 s$	4 GB
30	15	5.3	$5.1 \cdot 10^4 / s$	$1.9 \cdot 10^3 s$	16 GB
32	16	6.3	$4.3 \cdot 10^4 / s$	$8.7 \cdot 10^3 s$	64 GB
34	17	7.2	$3.6 \cdot 10^4 / s$	$3.9 \cdot 10^4 s$	256 GB

Table 1: Benchmark of Feynman integral evaluations with different numbers of edges.

[M. Borinsky, arXiv : [2008.12310](https://arxiv.org/abs/2008.12310)]

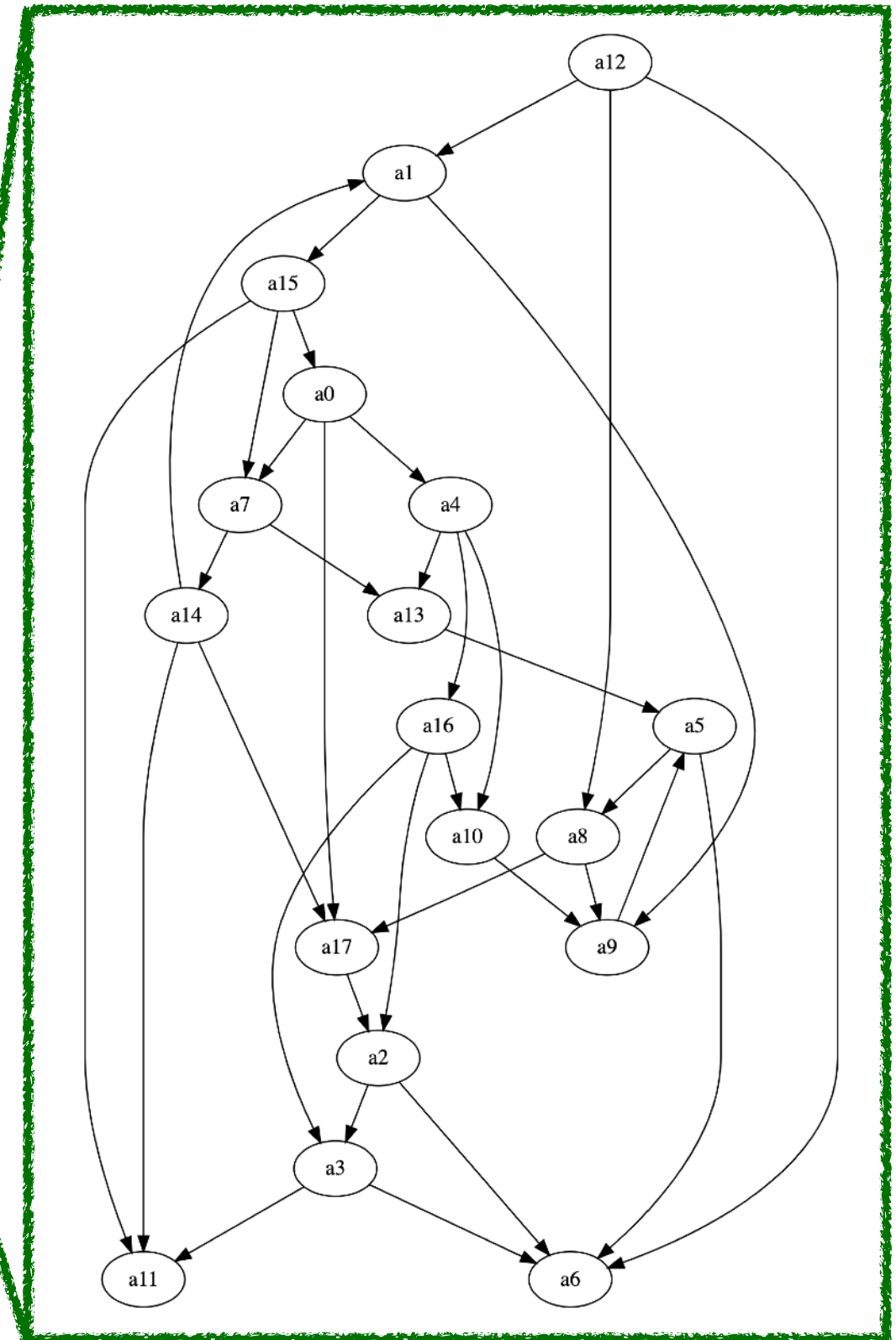
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TROPICAL SAMPLING OF LOOP-TREE DUALITY INTEGRANDS

[Inspired from and in collaboration with M. Borinsky, arXiv : [2008.12310](https://arxiv.org/abs/2008.12310)]

[Investigation from **Mathijs Fraije**]

$$I^{(\text{LTD})} \supset \int d^3 \vec{k} \frac{1}{\boxed{|\vec{k}|} \boxed{|\vec{k} + \vec{p}_1|} \boxed{|\vec{k} - \vec{p}_2|}} \frac{\mathcal{N}(\vec{k})}{((E_1 + E_2 - p_1^0)(E_1 + E_2 + p_1^0))}$$

TROPICAL SAMPLING OF LOOP-TREE DUALITY INTEGRANDS

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$$I^{(\text{LTD})} \supset \int d^3 \vec{k} \frac{1}{\underbrace{|\vec{k}|}_{\text{green}} \underbrace{|\vec{k} + \vec{p}_1|}_{\text{blue}} \underbrace{|\vec{k} - \vec{p}_2|}_{\text{yellow}}} \frac{\mathcal{N}(\vec{k})}{((E_1 + E_2 - p_1^0)(E_1 + E_2 + p_1^0))}$$

Multi-channeling is the canonical approach to flatten these three integrable singularities.

But it would be far better to build a single parameterisation whose Jacobian vanishes

simultaneously at *all three points* $\vec{k} = \{ \vec{0}, -\vec{p}_1, \vec{p}_2 \}$:

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$$= \int d^3 \vec{k} \frac{1}{D_1^{\frac{1}{2}} D_2^{\frac{1}{2}} D_3^{\frac{1}{2}}} f(\vec{k})$$

with **euclidean** propagators :

$$\begin{aligned} D_1 &= \vec{k} \cdot \vec{k} \\ D_2 &= (\vec{k} + \vec{p}_1) \cdot (\vec{k} + \vec{p}_1) \\ D_3 &= (\vec{k} - \vec{p}_2) \cdot (\vec{k} - \vec{p}_2) \end{aligned}$$

Ready for being tropical-sampled !

$$= \int_{[0,1]^{(DL+2|E|+1)}} d\mathbf{z} \underbrace{K(\mathbf{z})}_{\text{bounded}} f(\mathbf{k}(\mathbf{z}))$$

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$$\omega = \sum_e \nu_e - DL/2$$

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Easy to sample momenta from this PDF **exactly**

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- Tropical sampling idea: let $p(x, y) = 12x^2 - 7y^3 + 5xy$ for which $p^{\text{tr}}(x, y) = \max \{x^2, y^3, xy\}$

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- Moreover, a systematic algorithm exists to sample according to p^{tr} (note: sampling dimensionality increases)
- One can show that the ratio $K = \frac{p^{\text{tr}}}{p}$ is bounded from below and above when $p \equiv \mathcal{U}$ or $\mathcal{V}$

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$$I^{\text{eucl.}}[f] = \int d^{DL} \mathbf{k} \frac{f(\mathbf{k})}{\prod_e D_e(\mathbf{k})^{\nu_e}} \propto \int d\mathbf{k} \underbrace{\int_{\mathbb{P}_+^E} \prod_e x_e^{\nu_e}}_{\text{Feynman params.}} \frac{f(\mathbf{k})}{(\sum_e x_e D_e(\mathbf{k}))^{\sum_e \nu_e}}$$

$$\omega = \sum_e \nu_e - DL/2$$

$$\mathcal{V} = \mathcal{F}/\mathcal{U}$$

Quite a bit of algebra, leading to $\sim$ Schwinger param.

$$\propto \underbrace{\int_{\mathbb{P}_+^E} \prod_e x_e^{\nu_e} \frac{1}{\mathcal{U}(\mathbf{x})^{D/2} \mathcal{V}(\mathbf{x})^\omega}}_{\text{Tropical sampling [M. Borinsky 2008.12310]}} \int_0^\infty d\lambda \lambda^{\omega-1} e^{-\lambda} \int \prod_l d^D q_l e^{-q_l^2/2} f(\mathbf{k}(\mathbf{x}, \mathbf{q}, \lambda))$$

Easy to sample momenta from this PDF **exactly**

$$= \int_{[0,1]^{(DL+2|E|+1)}} d\mathbf{z} \underbrace{K(\mathbf{z})}_{\text{bounded}} f(\mathbf{k}(\mathbf{z}))$$

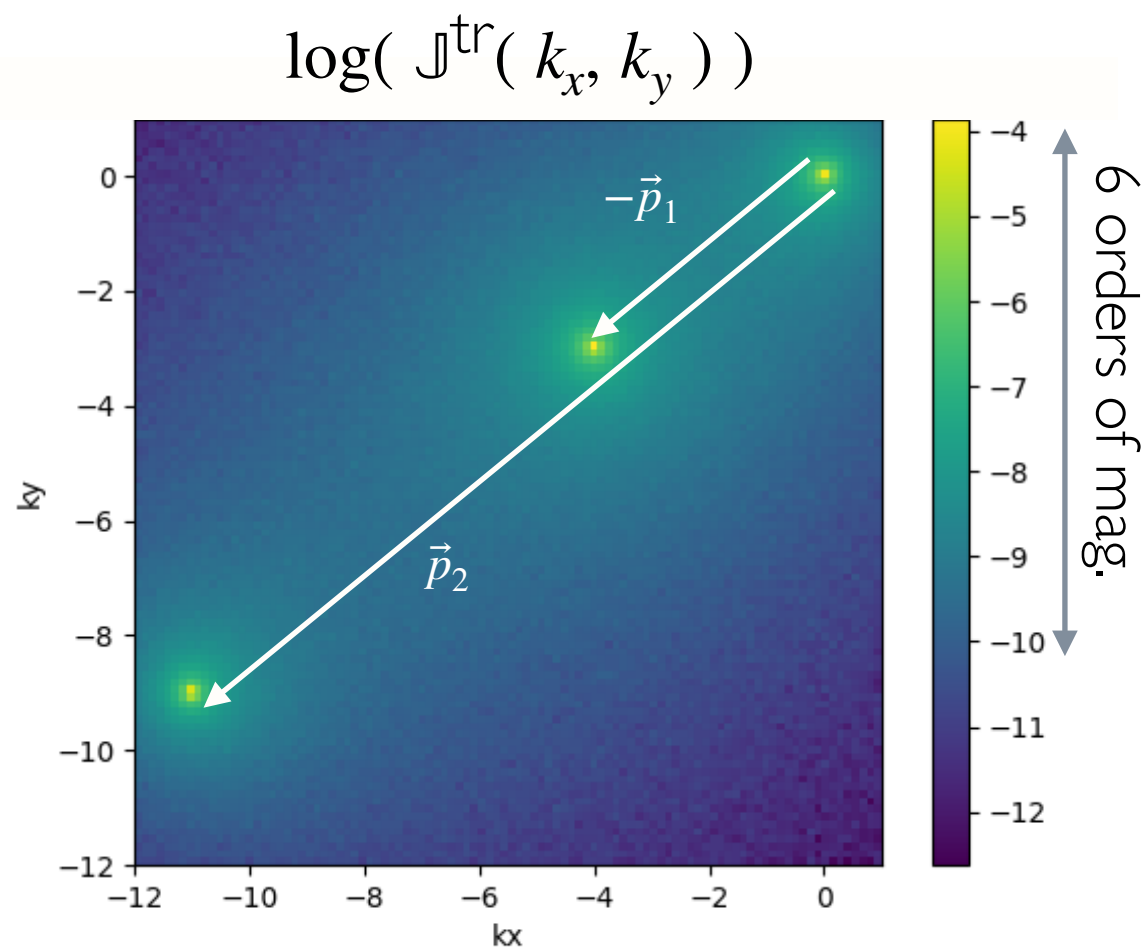
$$K(\mathbf{z}) \propto \left(\frac{\mathcal{U}^{\text{tr}}(\mathbf{x})}{\mathcal{U}(\mathbf{x})} \right)^{D/2} \left(\frac{\mathcal{V}^{\text{tr}}(\mathbf{x})}{\mathcal{V}(\mathbf{x})} \right)^\omega$$

- Tropical sampling idea: let $p(x, y) = 12x^2 - 7y^3 + 5xy$ for which $p^{\text{tr}}(x, y) = \max\{x^2, y^3, xy\}$
- Moreover, a systematic algorithm exists to sample according to p^{tr} (note: sampling dimensionality increases)
- One can show that the ratio $K = \frac{p^{\text{tr}}}{p}$ is bounded from below and above when $p \equiv \mathcal{U}$ or $\mathcal{V}$
- M. Fraije and M. Borinsky will soon publish a **Rust** implementation of the map: $\phi^{\text{tr}}[\Gamma](\mathbf{z}) \rightarrow (K(\mathbf{z}), \mathbf{k}(\mathbf{z}))$

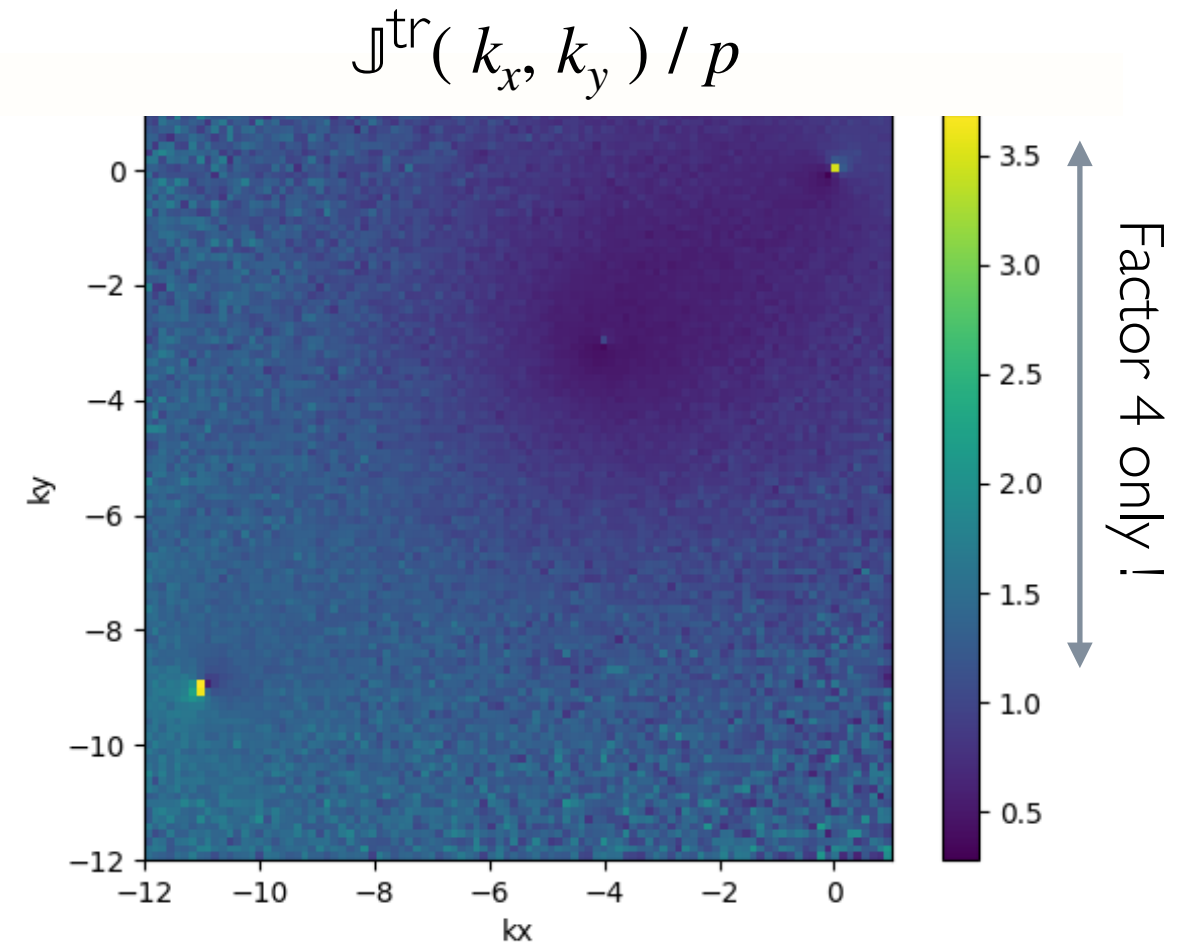
TROPICAL SAMPLING OF THE LTD TRIANGLE INTEGRAL

Let's apply this approach to our example.

In 2D with tropical sampling for removing blow ups from $p := \frac{1}{|\vec{k}||\vec{k} + \vec{p}_1||\vec{k} - \vec{p}_2|}$:



Log of tropical ~sampling density

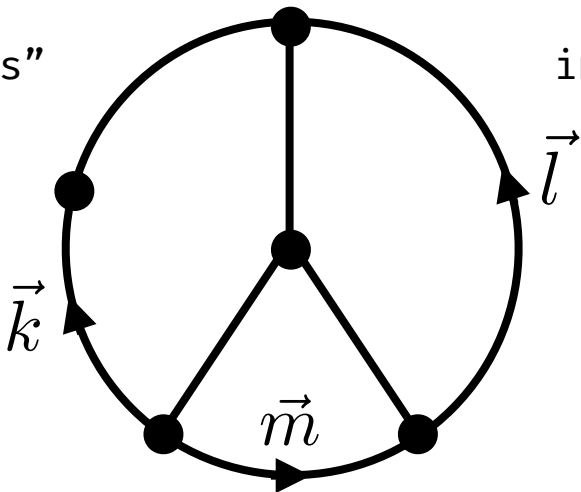


Integrable singularities conquered !

- Many details of this approach omitted here. In our implementation: **arbitrary numerators** supported!

TROPICAL SAMPLING RESULTS

“scalar vacuum mercedes”



internal mass: $m = 1$

SAMPLING STRATEGY	# DIM.	IM. CENTRAL VALUE	PULL
Analytic result	–	$-3.0525714828514\text{e-}7$	ref.
Prod. of spherical params [†]	9	$-2.990(50)\text{e-}7$	-1.2 σ

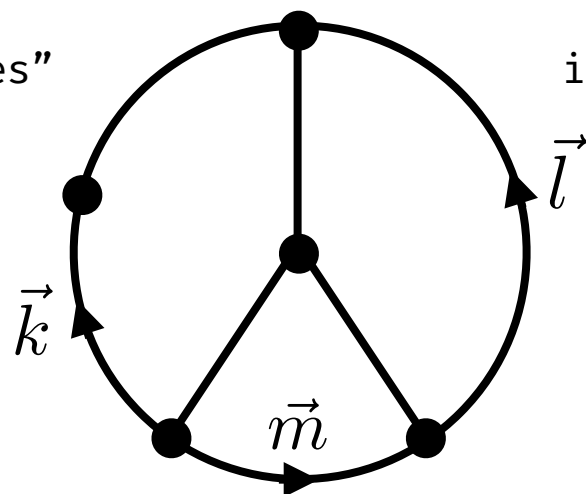
$$\int \left[d\Omega_k |\vec{k}|^2 d|\vec{k}| \right] \times \left[d\Omega_l |\vec{l}|^2 d|\vec{l}| \right] \times \left[d\Omega_m |\vec{m}|^2 d|\vec{m}| \right]$$

[†]: 5 x 2M samples, 10 μs /sample

Credits: **Mathijs Fraaije**

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SAMPLING STRATEGY	# DIM.	IM. CENTRAL VALUE	PULL
Analytic result	–	$-3.0525714828514\text{e-}7$	ref.
Prod. of spherical params [†]	9	$-2.990(50)\text{e-}7$	-1.2 σ
multi-channeling [†]	9	$-3.0484(51)\text{e-}7$	-0.8 σ

$$\int \left[d\Omega_k |\vec{k}|^2 d|\vec{k}| \right] \times \left[d\Omega_l |\vec{l}|^2 d|\vec{l}| \right] \times \left[d\Omega_m |\vec{m}|^2 d|\vec{m}| \right]$$

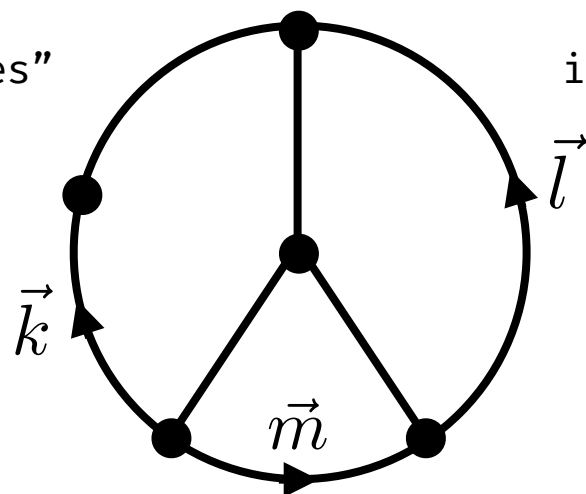
$$1 = \sum_{\text{lmb}} \left[\frac{\prod_{i \in \text{lmb}} E_i^{-\alpha}}{\left(\sum_{\text{lmb}}, \prod_{j \in \text{lmb}} E_j^{-\alpha} \right)} \right]$$

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TROPICAL SAMPLING RESULTS

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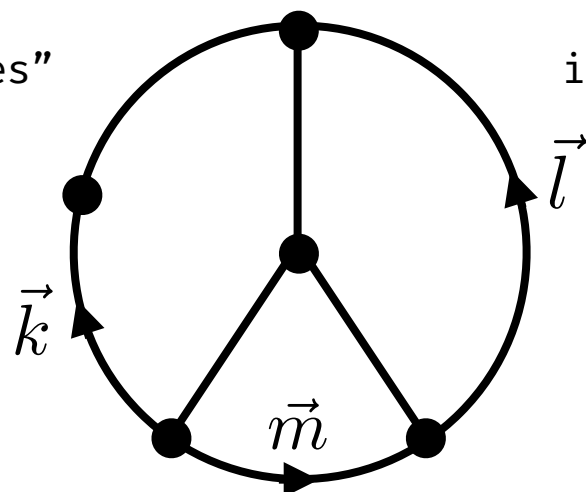
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Tropical sampling [†]	23	–3.05349(43)e–7	+2.1 σ

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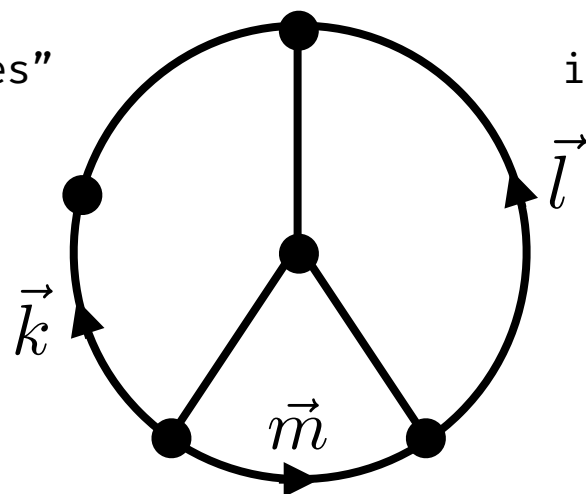
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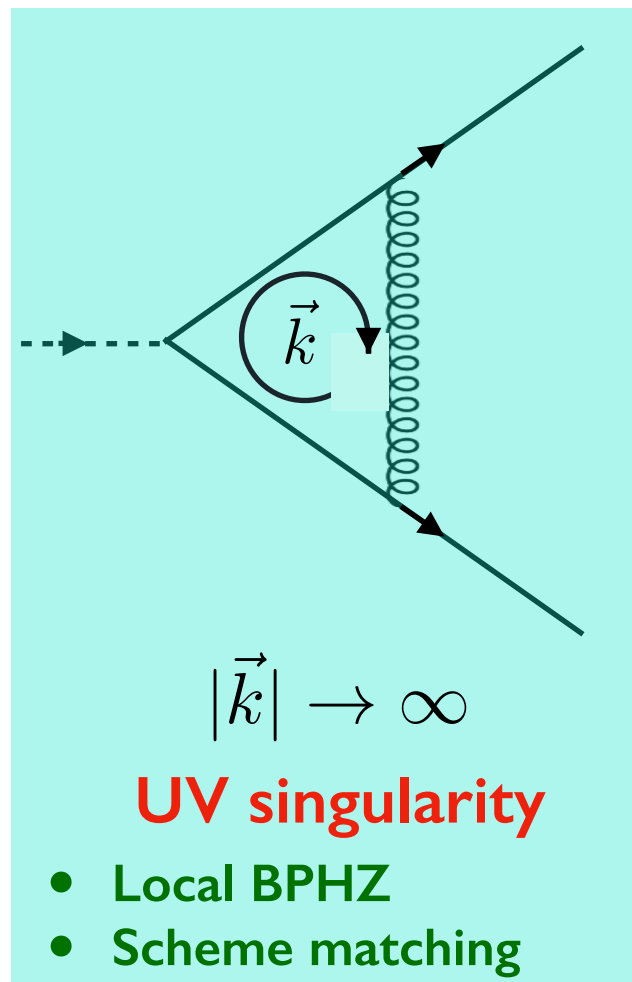
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Neural Importance Sampling	dep. on sampling	Theo Heimel prelim: <u>2x+ var. reduction</u> !	\o/

[†]: 5 x 2M samples, 10 μs /sample

Credits: **Mathijs Fraaije**

ULTRAVIOLET SINGULARITIES

$$\int d^2\Omega \int_0^\infty |\vec{k}|^2 d|\vec{k}| \frac{a|\vec{k}|^2 + b|\vec{k}| + c}{8|\vec{k}|^2 |\vec{k} + \vec{p}_1|} \frac{1}{|\vec{k}| + |\vec{k} + \vec{p}_1| - |\vec{p}_1|} \frac{1}{|\vec{k}| + |\vec{k} + \vec{Q}| - Q^0 + i\delta}$$



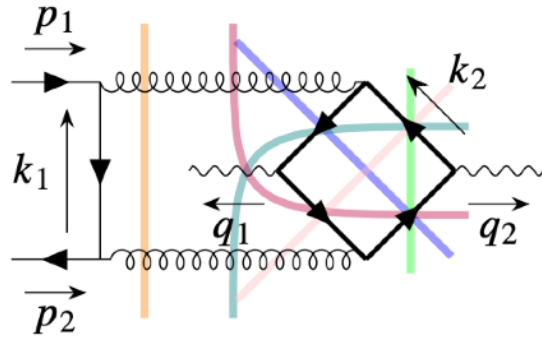
➔ *Easy!* Lucien's talk

APPLICATIONS

GENERIC COMPUTATIONAL FRAMEWORK

To accommodate the many applications, on top of collider simulations

Collider:



$pp \rightarrow VV(V) @ \text{NNLO}$

[D.Kermanschah, M. Vicini,
C. Anastasiou, & al.]

$pp \rightarrow t\bar{t} @ \text{NLO}$

[Z. Capatti, L. Huber, & al.]

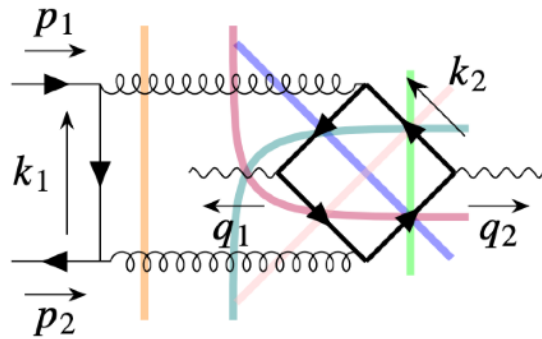
$pp \rightarrow [\text{high-mult}] @ \text{NLO}$

[S. Pozzorini, G. Bertolotti, & al.]

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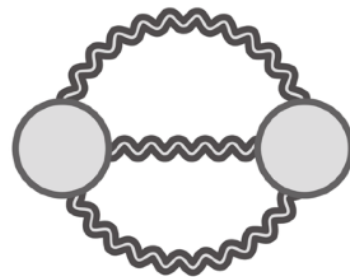
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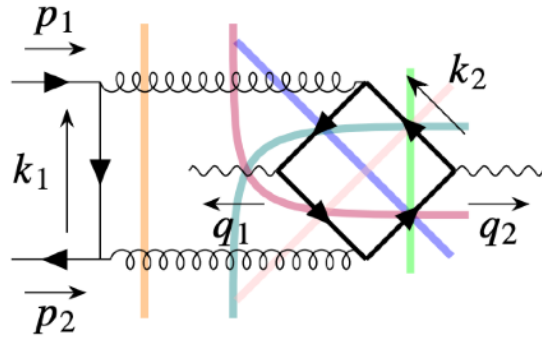
Neutron stars EoS

[Kärkkäinen & al.: 2501.17921]

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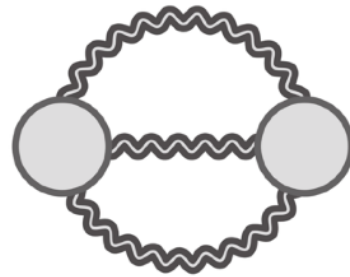
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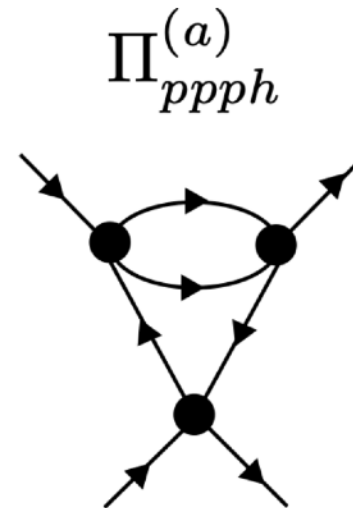
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Neutron stars EoS

[Kärkkäinen & al.: 2501.17921] [R. Beane & al.: 2407.20168]

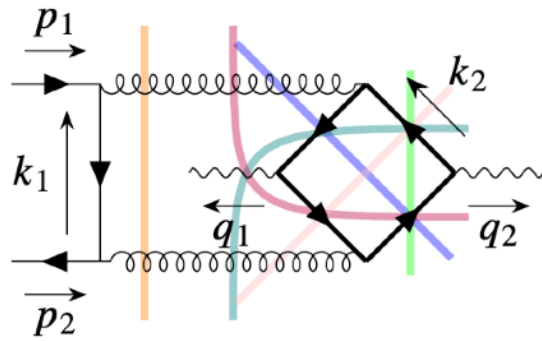


Fermi gas

GENERIC COMPUTATIONAL FRAMEWORK

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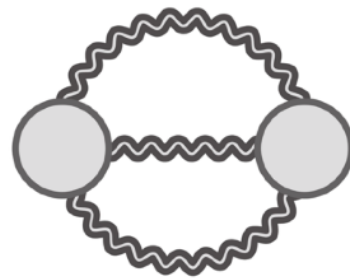
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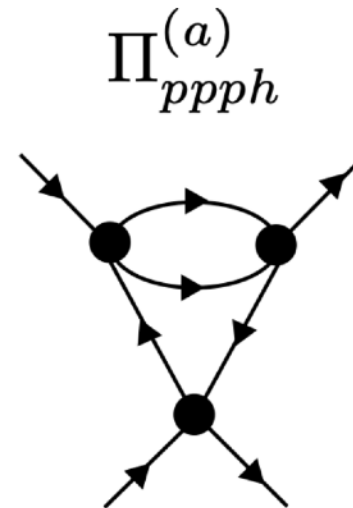
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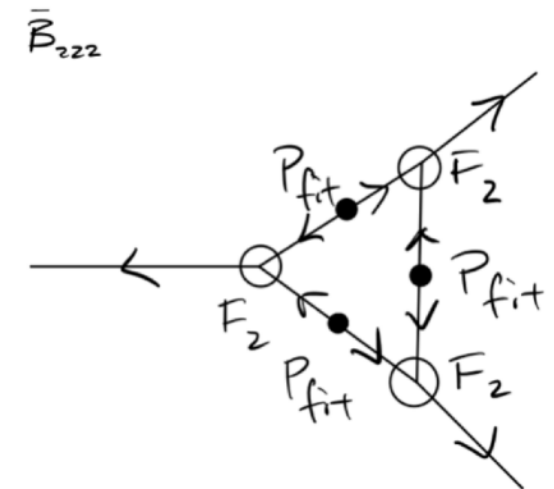


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Fermi gas



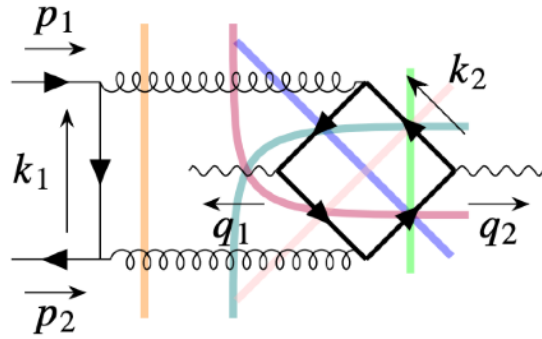
EFT for Large Scale Structure

[Anastasiou & al.: 2212.07421]

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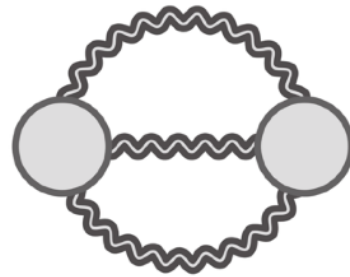
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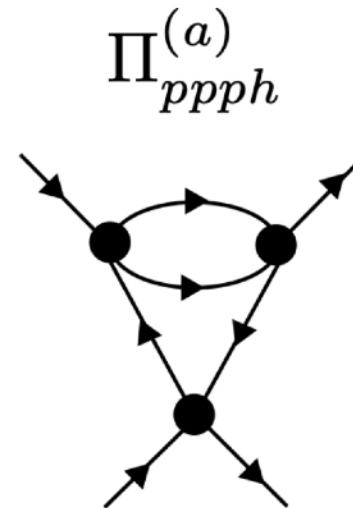
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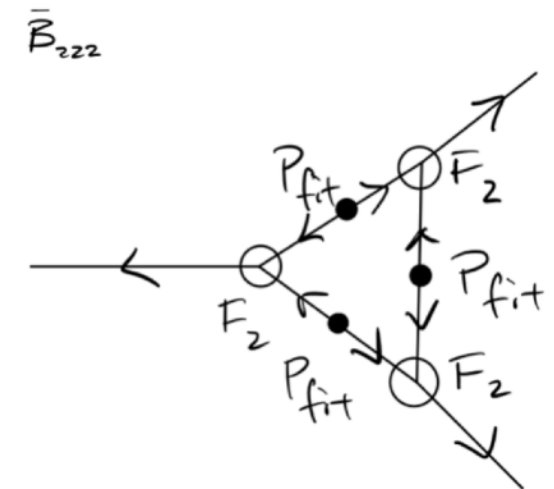


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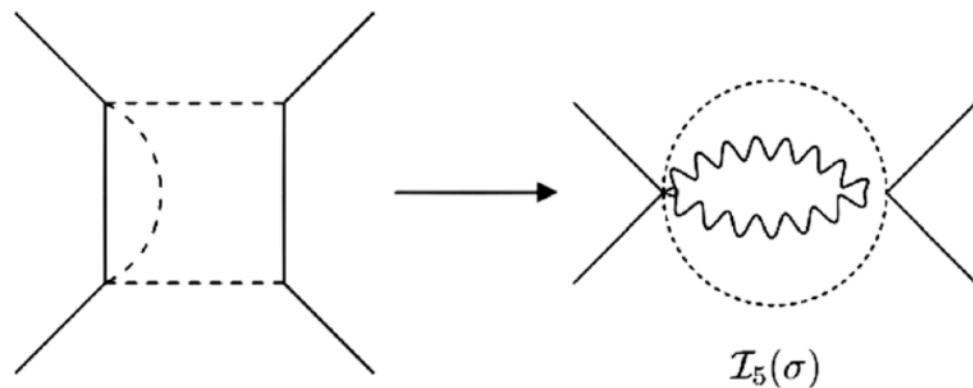


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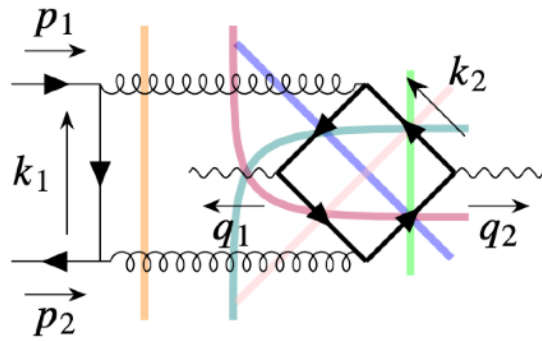
Gravitational waves from classical limit of loops

[Emil & al.: 2104.04510]

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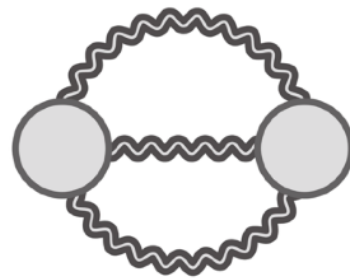
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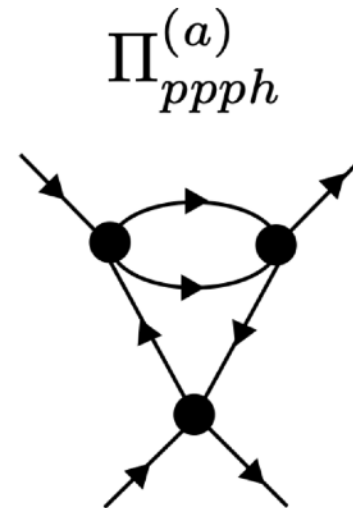
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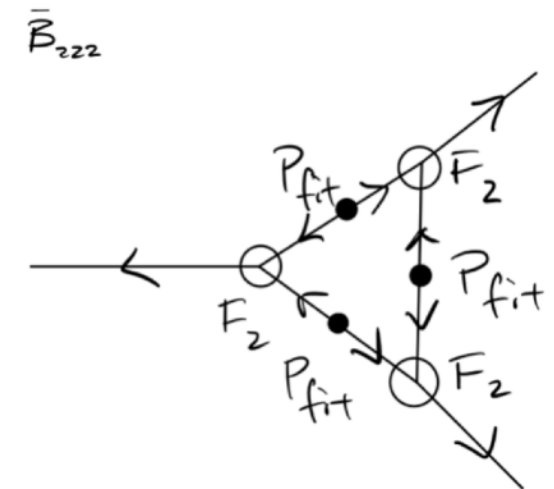


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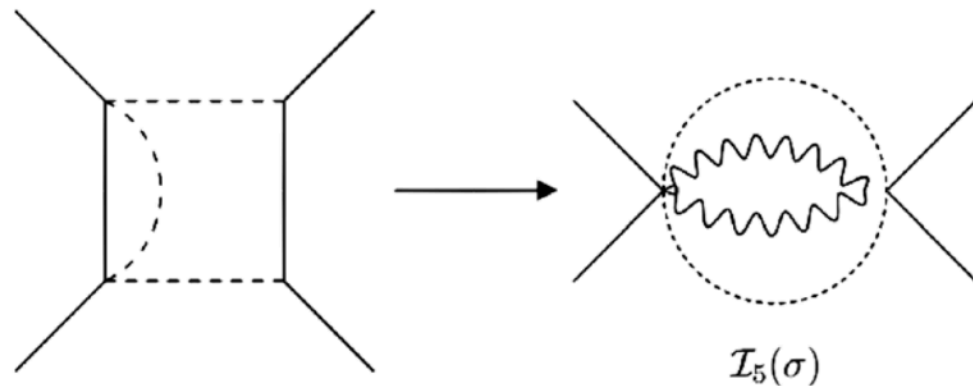


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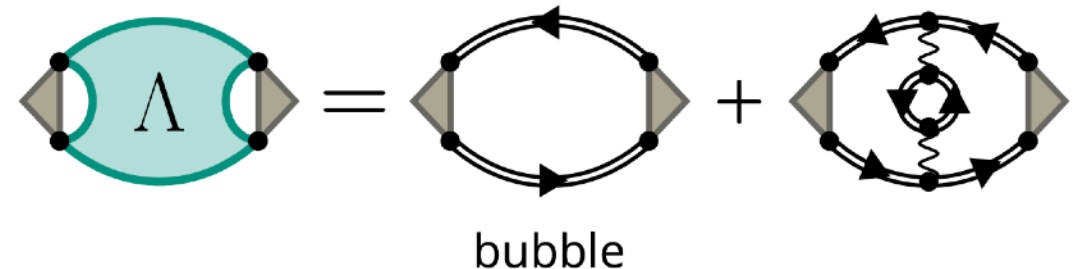
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“RFDiagMC” for high- T_c superconductivity

[Ferrero & al.: 2501.19118]

...

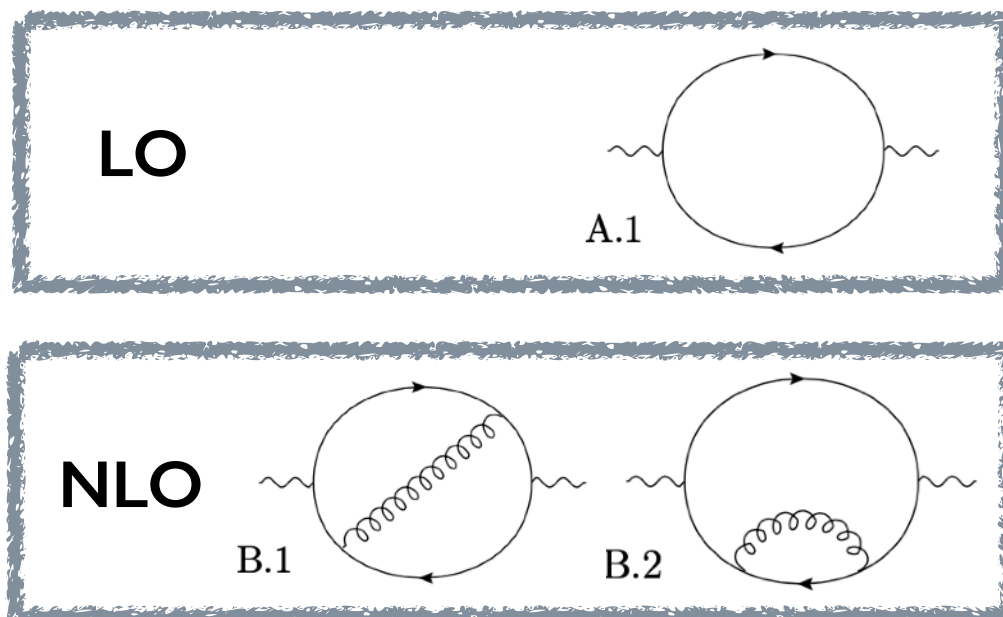
NUMERICAL RESULTS

[Z. Capatti, VH, B. Ruijl, arxiv: 2203.11038]

We have computed the following fixed-order processes with **Local Unitarity**:

NNLO	$e^+e^- \rightarrow \gamma \rightarrow jj$	$p_t(j_1)$ distribution	NNLO	$\gamma^* \rightarrow jj$	inclusive
	$e^+e^- \rightarrow \gamma \rightarrow jjj$	semi-inclusive		$\gamma^* \rightarrow t\bar{t}$	inclusive
	$e^+e^- \rightarrow \gamma \rightarrow t\bar{t}h$	(semi-)inclusive			

First **NNLO** cross-sections computed **fully numerically** in momentum space.



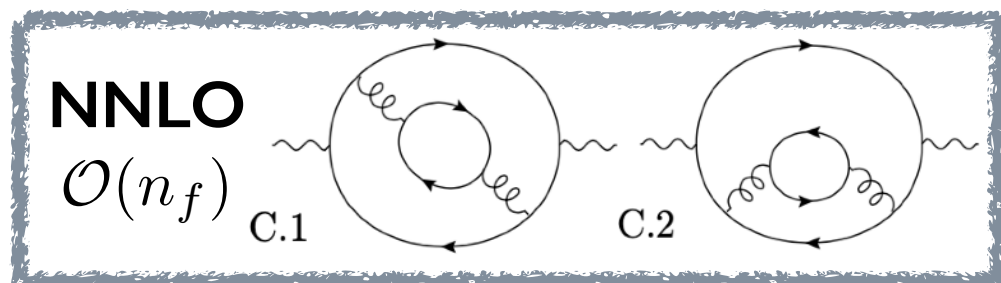
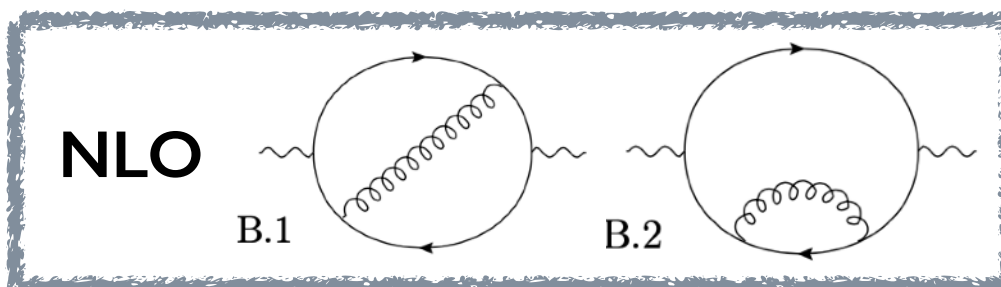
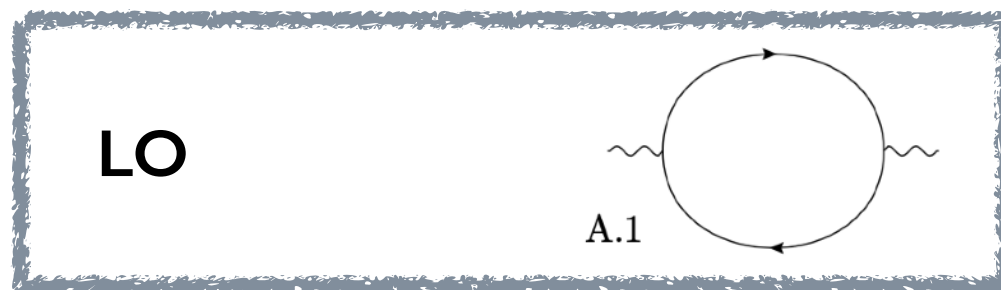
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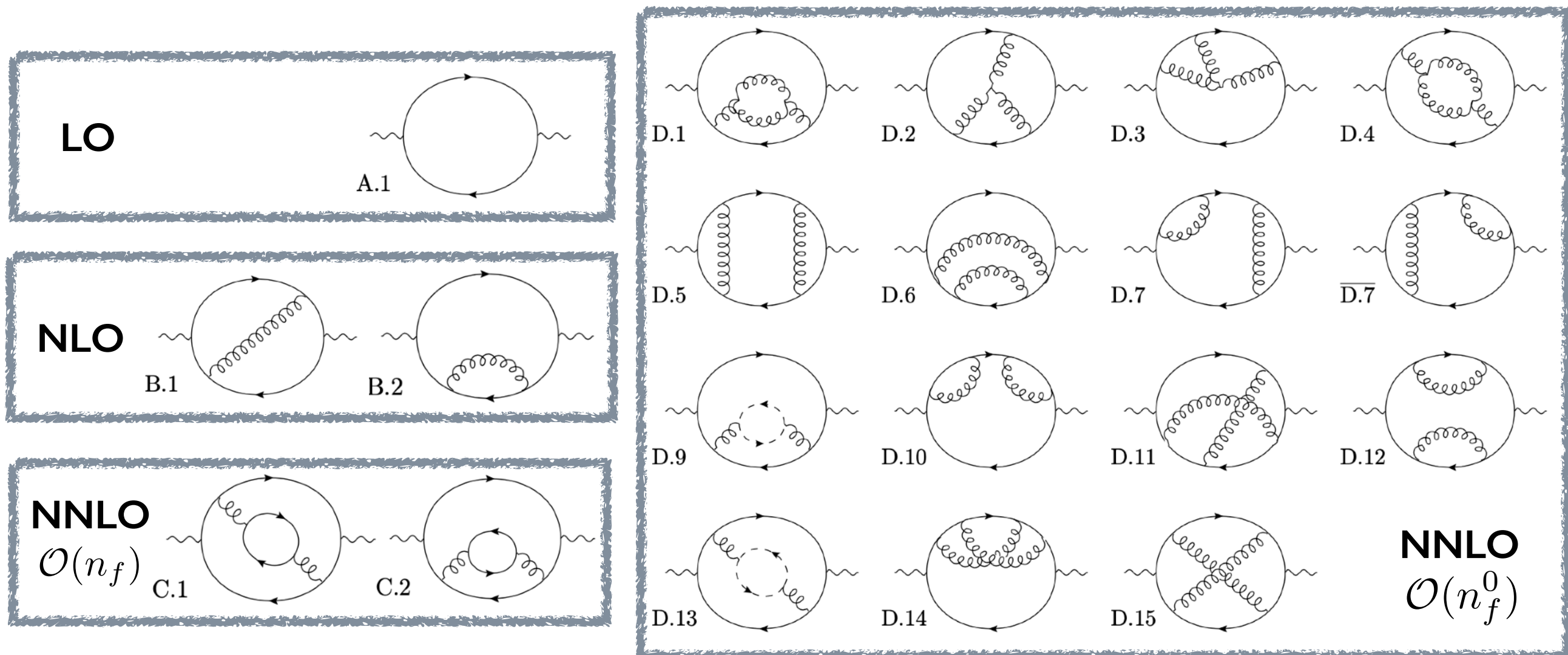
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First NNLO cross-sections computed **fully numerically** in momentum space.



IN ULTRA PERIPHERAL COLLISIONS (UPC) AT HADRON COLLIDERS

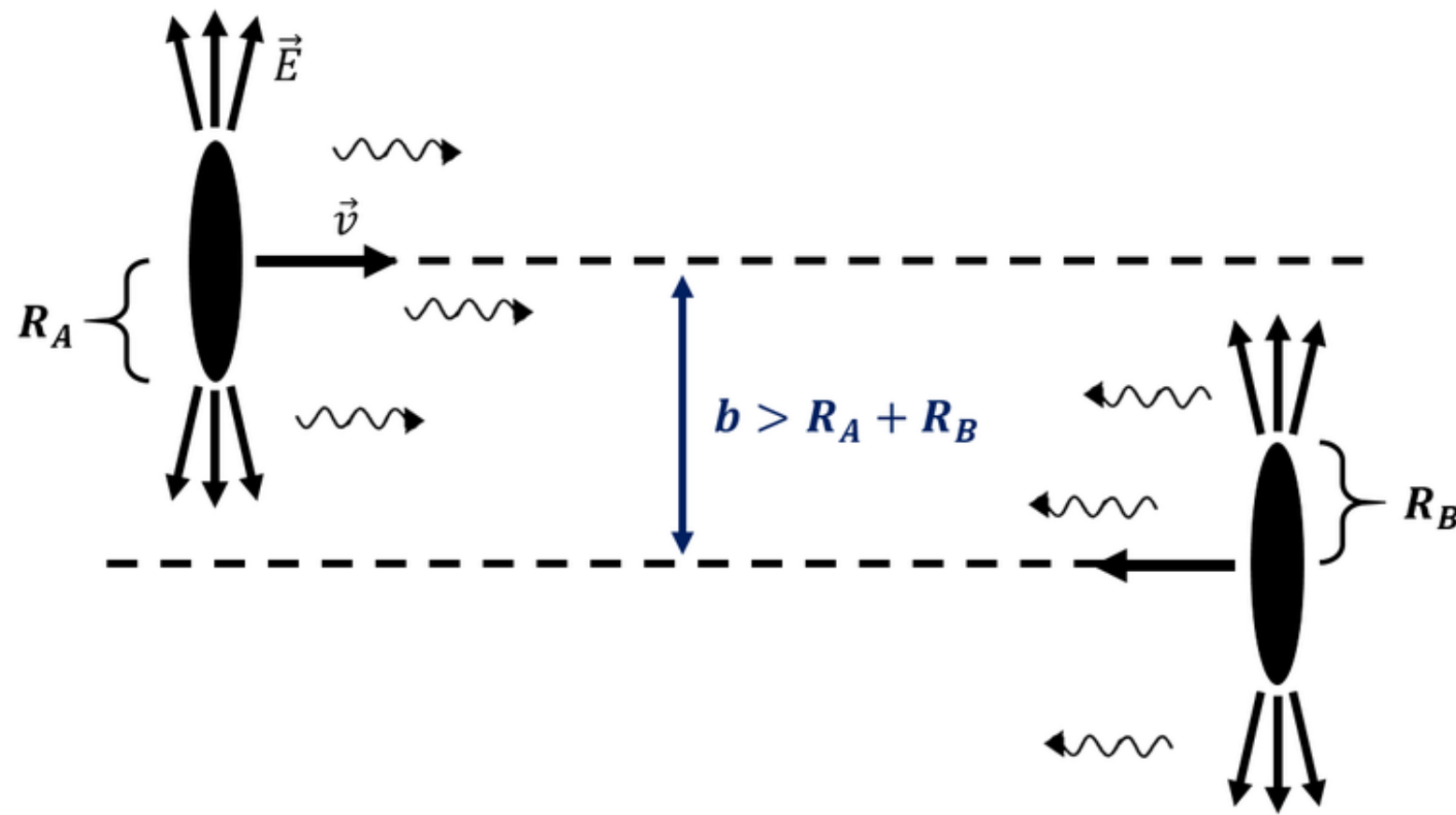
$$\gamma\gamma \rightarrow \gamma\gamma$$

AND

$$\gamma\gamma \rightarrow Q\bar{Q}$$

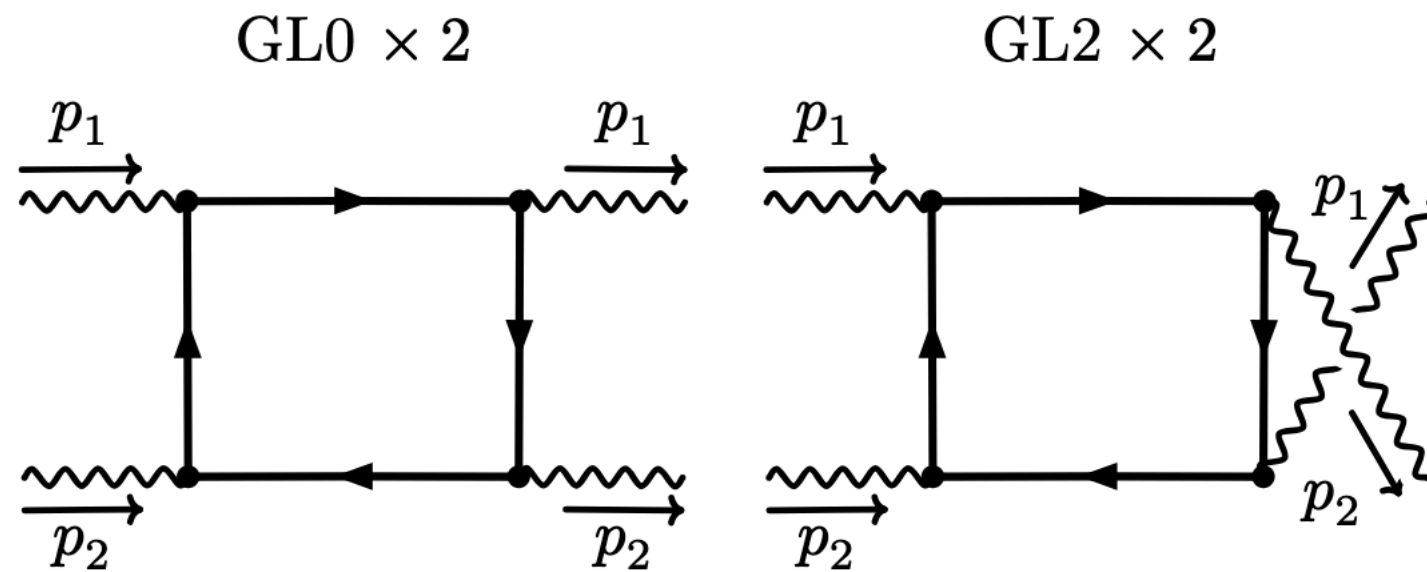
[Ajjath, Ekta, Fraaije, VH, H.S.Shao arxiv: [2312.16956](#)]

[Capatti, Fraaije, VH, Huber, Ruijl, H.S.Shao arxiv: [2509.13439](#)]



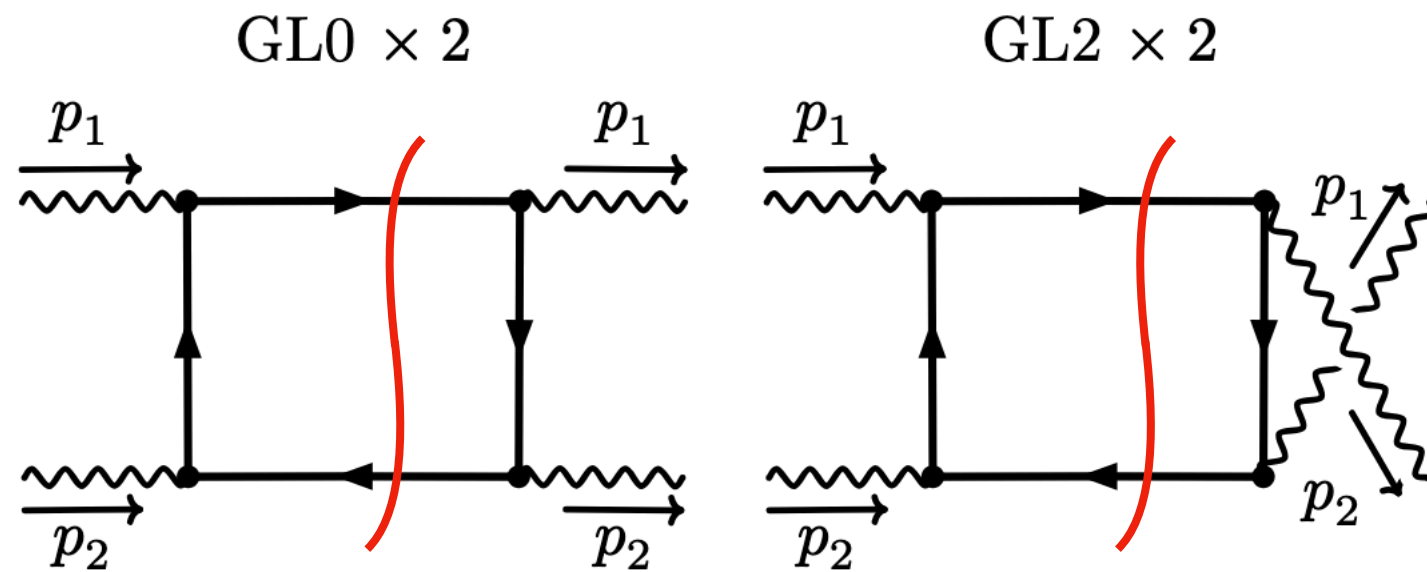
NNLO QCD WITH LOCAL UNITARITY

LO: 2 distinct FS graphs



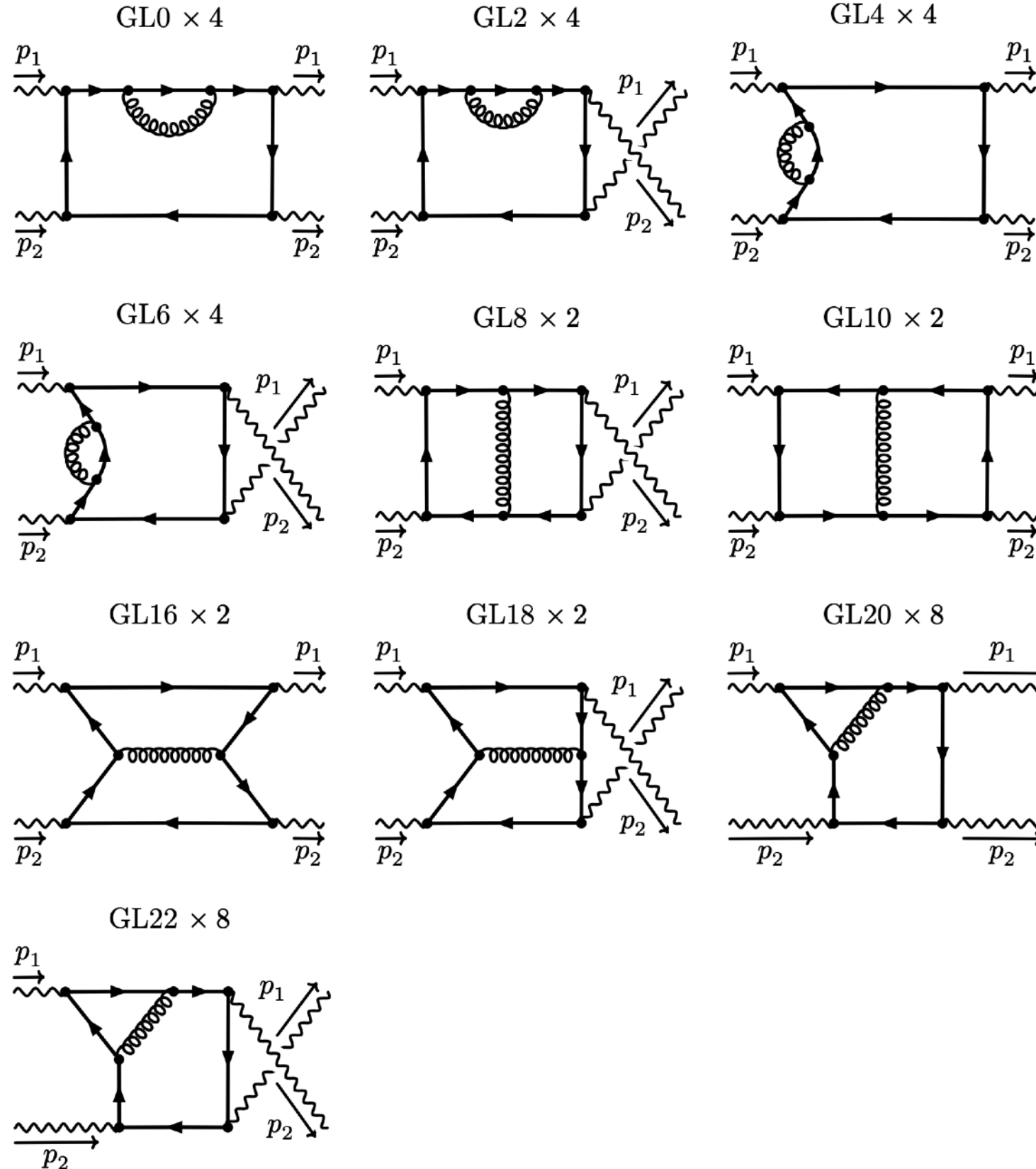
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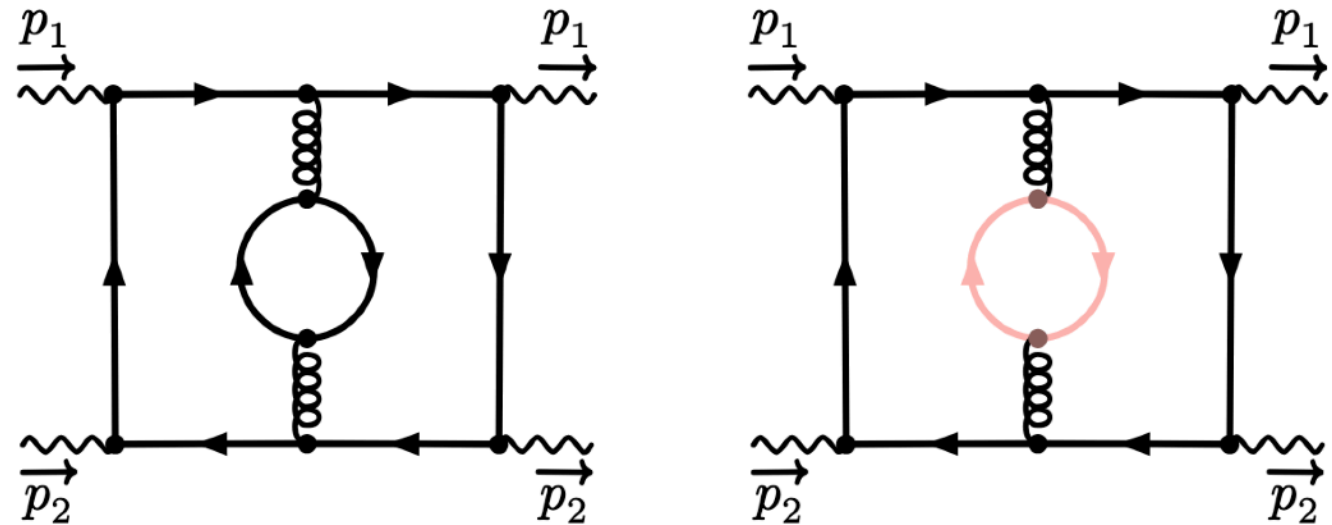
NNLO QCD WITH LOCAL UNITARITY

NLO:
10 distinct FS graphs

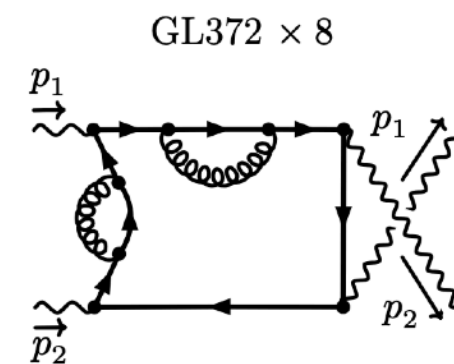
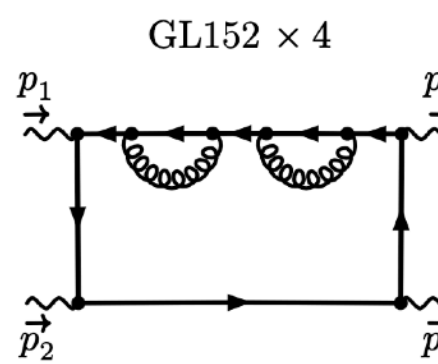
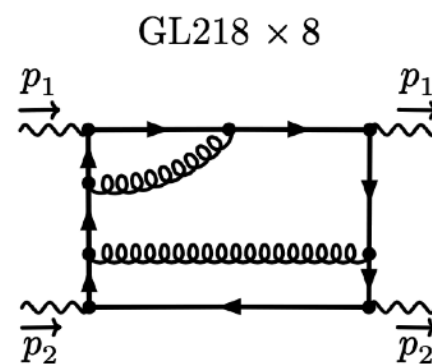
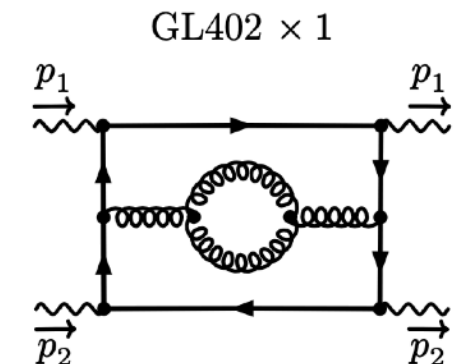
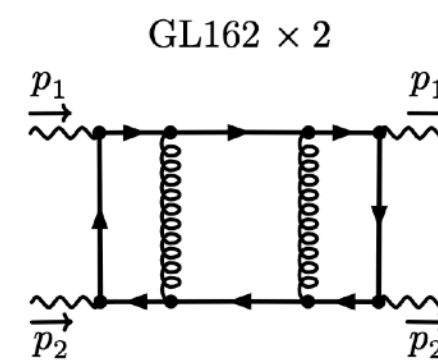
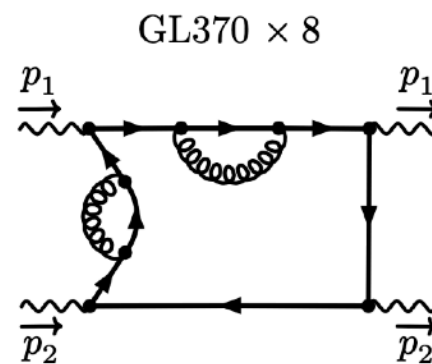


NNLO QCD WITH LOCAL UNITARITY

NNLO : nh, nf
2 x 10 distinct FS graphs

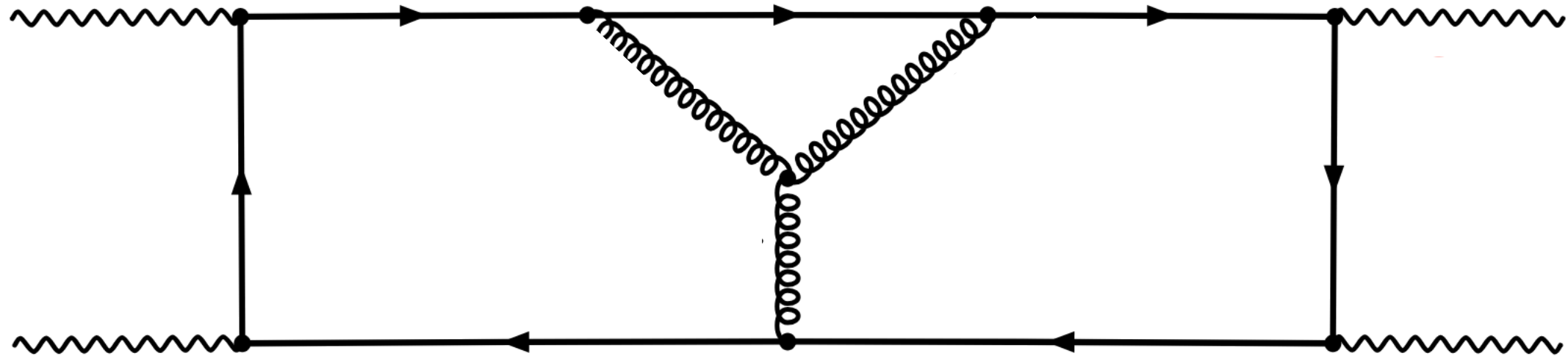


NNLO : non-singlet
114 distinct FS graphs



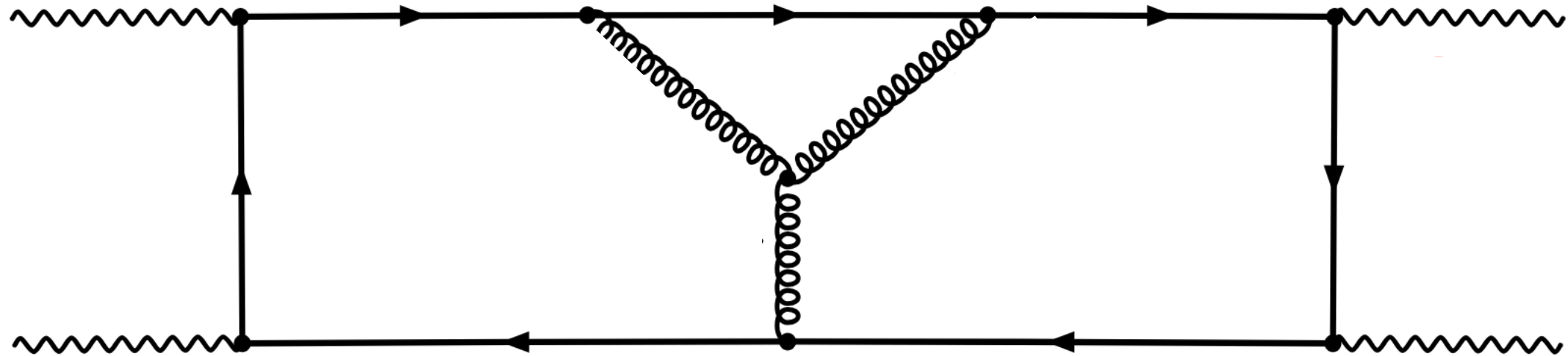
NNLO QCD WITH LOCAL UNITARITY

Reminder: when you see:

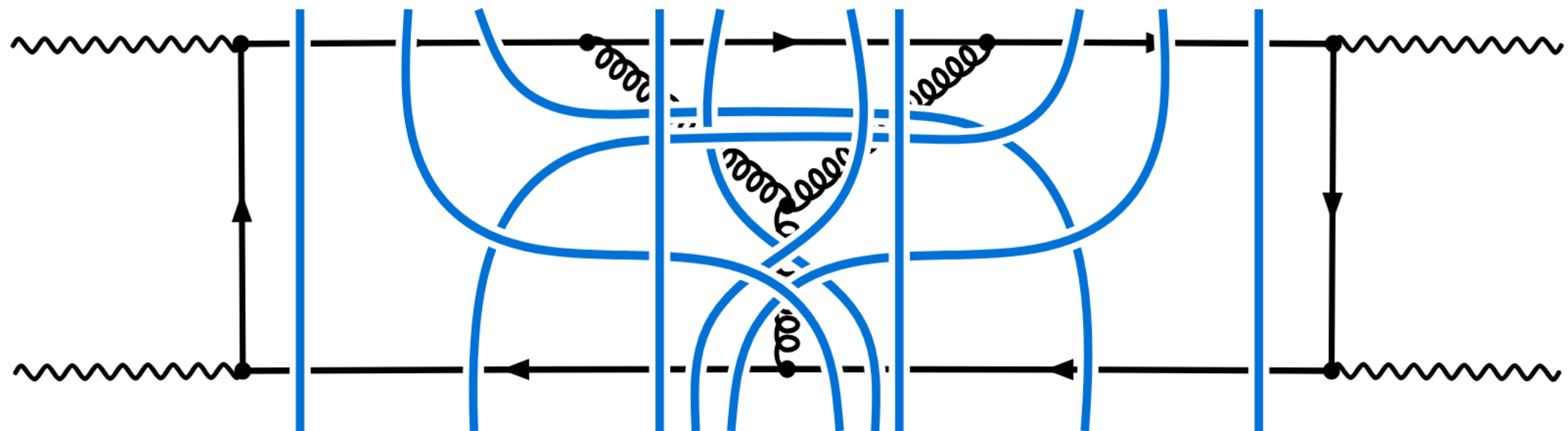


NNLO QCD WITH LOCAL UNITARITY

Reminder: when you see:



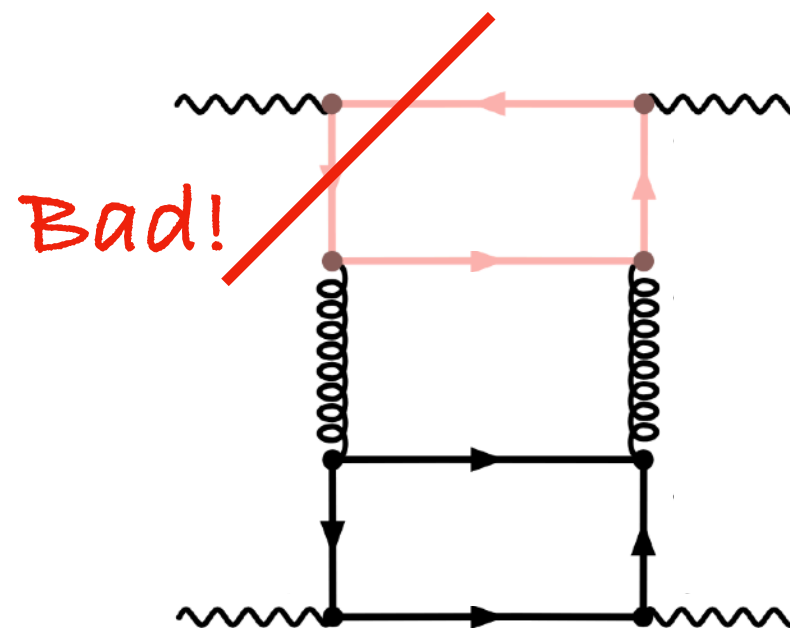
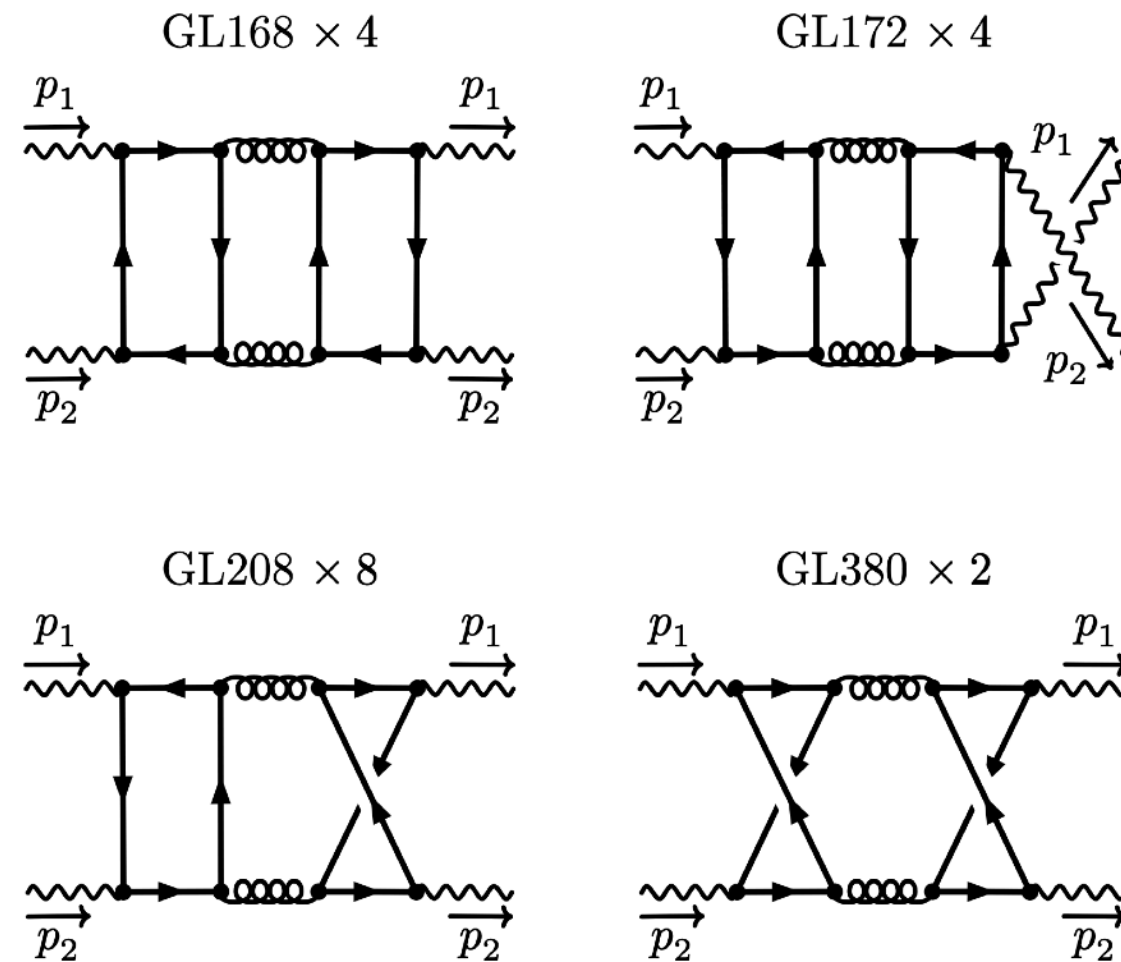
Think:



All cuts included !

NNLO QCD WITH LOCAL UNITARITY

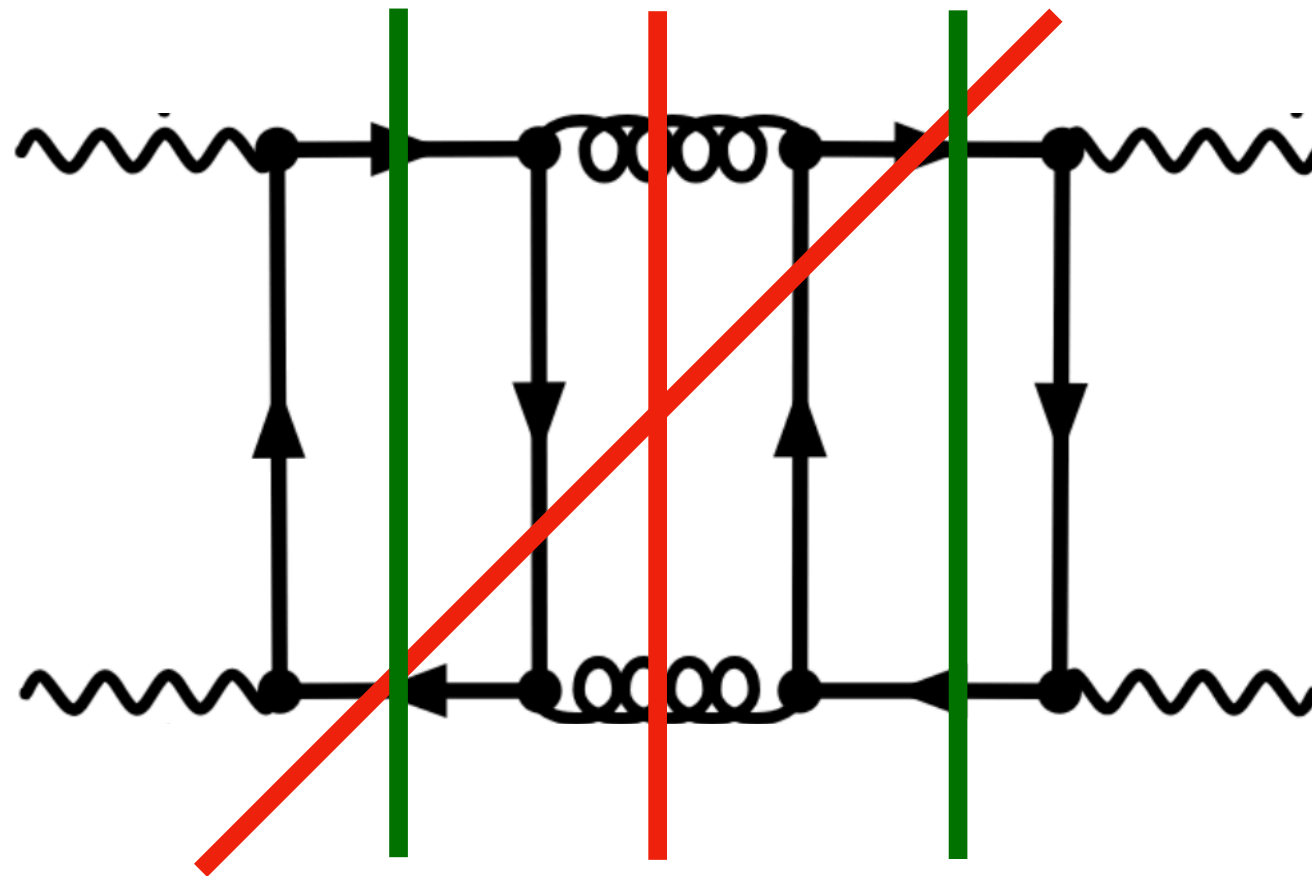
NNLO: singlets
4 distinct FS graphs



NNLO : singlets with ISR excluded

NNLO QCD WITH LOCAL UNITARITY

What about the singlet contributions ?



Corresponds to: $\gamma\gamma \rightarrow gg$

Keep to avoid threshold regularisation!

And subtract it back, computed as with MadGraph!

Corresponds to: $\gamma\gamma \rightarrow t\bar{t}t\bar{t}$

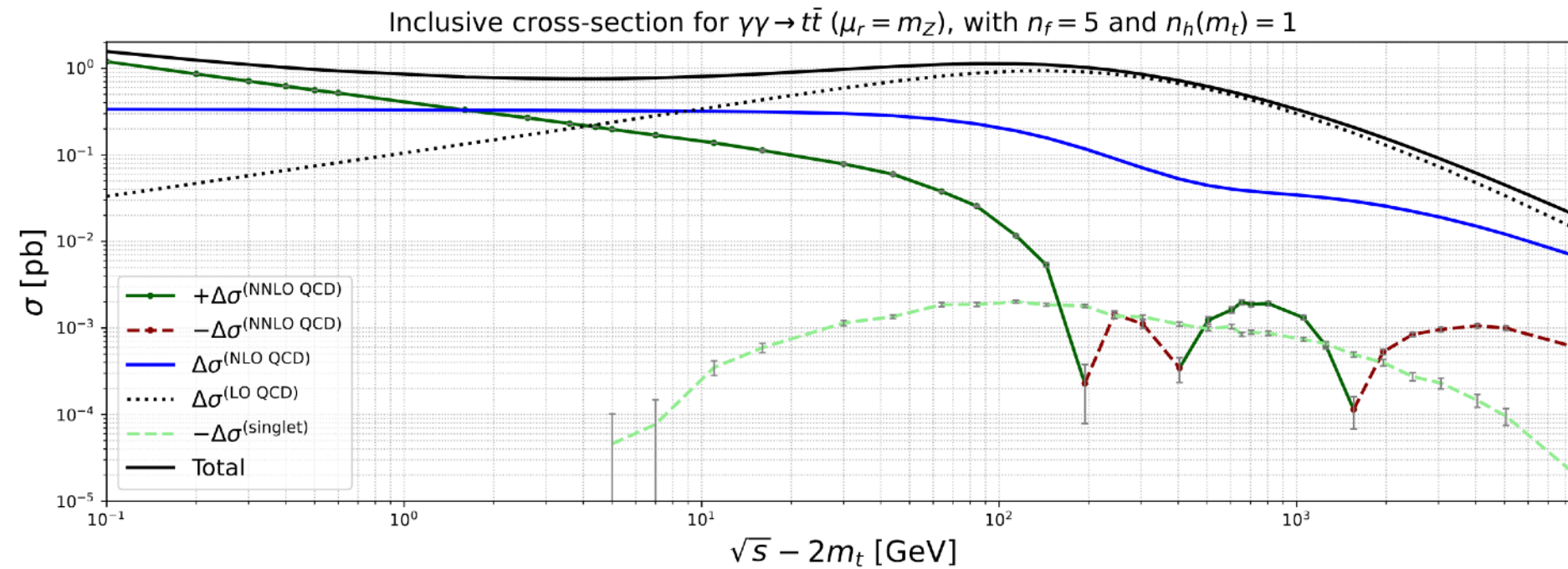
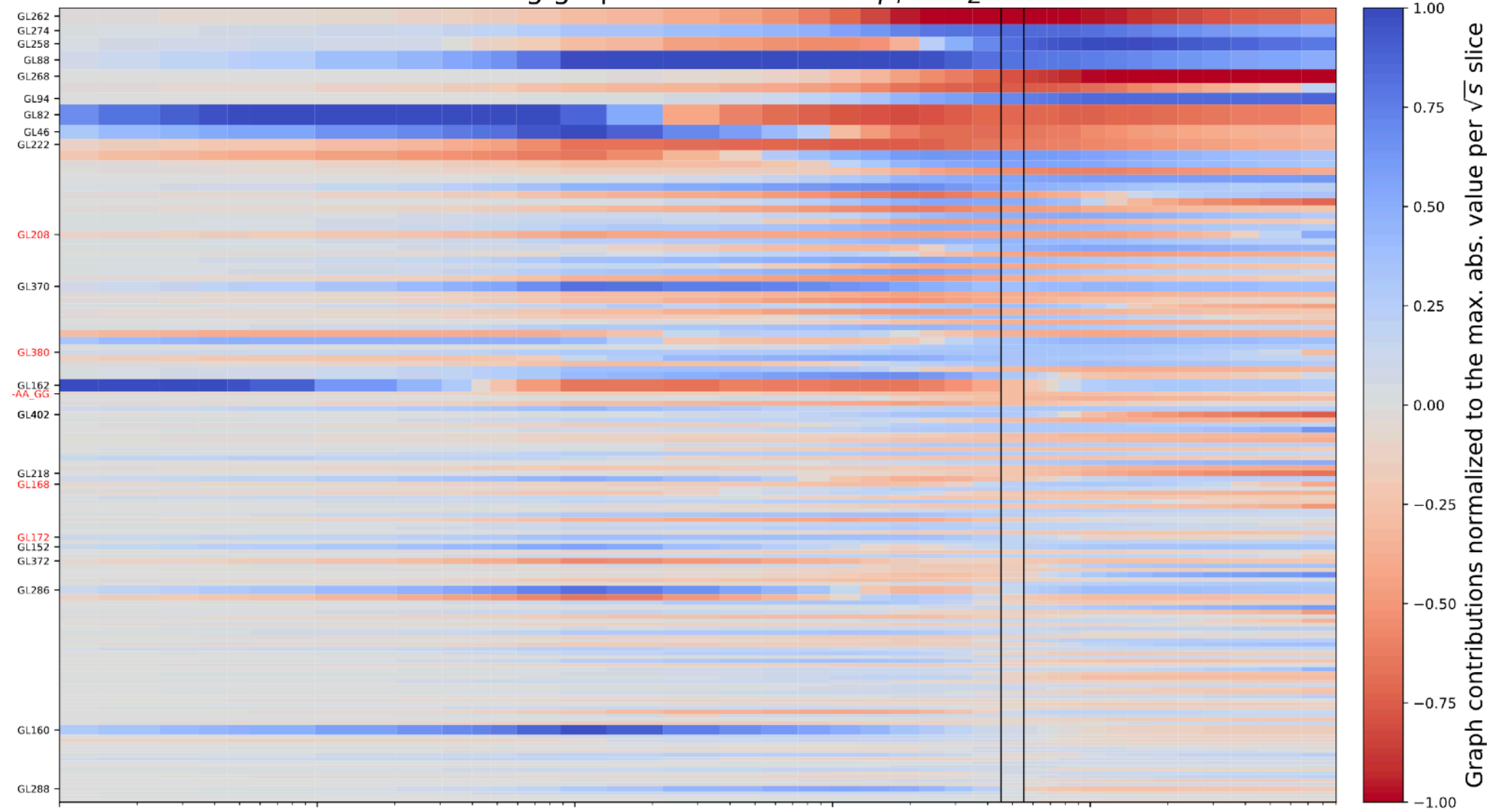
Safe to remove!

It cannot intersect with $\gamma\gamma \rightarrow t\bar{t}$ cuts!

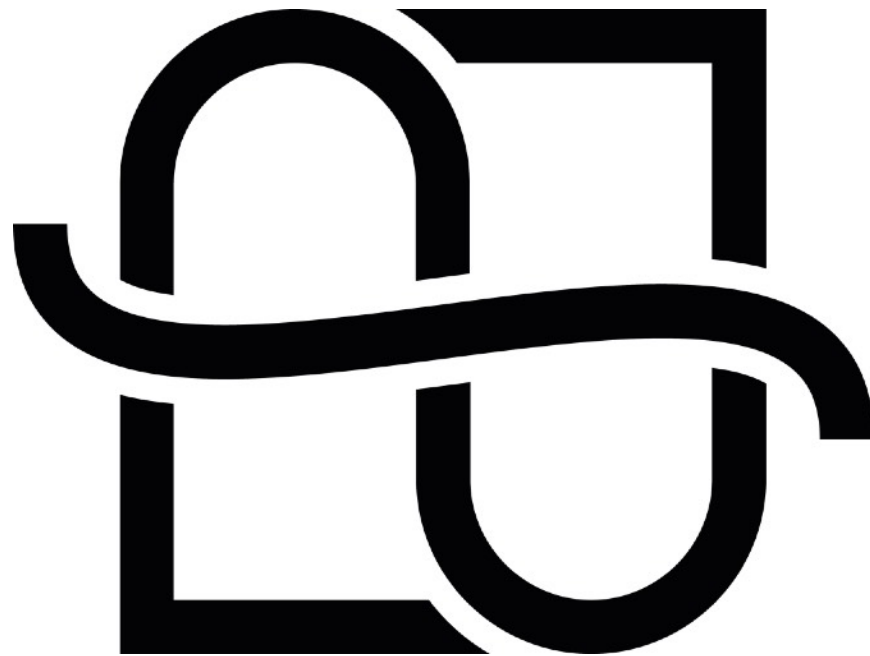
$$\sum_{G \in \text{class C}} \sigma_G(\gamma\gamma \rightarrow Q\bar{Q}, Q\bar{Q}g) = \sum_{G \in \text{class C}} \underbrace{\sigma_G(\gamma\gamma \rightarrow Q\bar{Q}, Q\bar{Q}g, gg)}_{\text{LU}} - \underbrace{\sum_{G \in \text{class C}} \sigma_G(\gamma\gamma \rightarrow gg)}_{\text{MG5_AMC}} .$$

RESULTS:

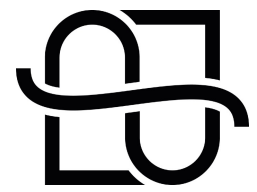
Individual forward scattering graph contributions at $\mu_r = m_Z$ as a function of $\sqrt{s}$



γ Loop : DEMO



MODULAR CODES FOR MAXIMUM FLEXIBILITY



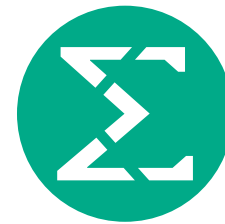
γ Loop



User-friendly



Low-level



Symbolica

~ “Open-source CAS / Compiler”

Components often distributed independently as sub-crates on:

<https://github.com/alpha100p/gammaloop/tree/main/crates/XXX>

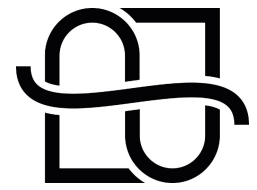
- Fast and flexible numerator algebra



spenso

[spenso](#) / [idenso](#)

MODULAR CODES FOR MAXIMUM FLEXIBILITY



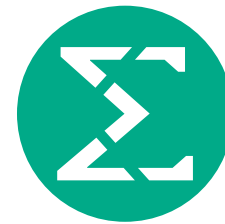
γ Loop



User-friendly



Low-level



Symbolica

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Components often distributed independently as sub-crates on:

<https://github.com/alpha100p/gammaloop/tree/main/crates/XXX>

- Fast and flexible numerator algebra
- Diagram generation and manipulations



spenso

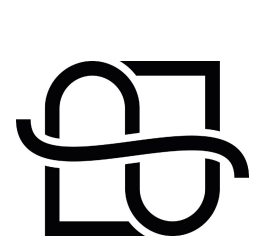
[spenso](#) / [idenso](#)



feyngen

[through gammaloop CLI only](#) / [linnet](#)

MODULAR CODES FOR MAXIMUM FLEXIBILITY



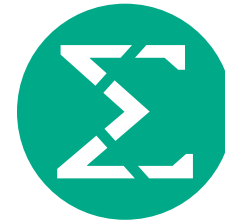
γ Loop



User-friendly



Low-level



Symbolica

~ “Open-source CAS / Compiler”

Components often distributed independently as sub-crates on:

<https://github.com/alpha100p/gammaloop/tree/main/crates/XXX>

- Fast and flexible numerator algebra
- Diagram generation and manipulations
- Analytical vacuum integrals



spenso

[spenso](#) / [idenso](#)



feyngen

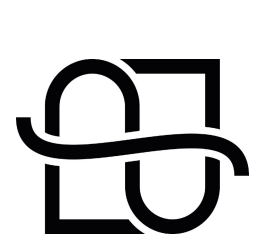
[through gammaloop CLI only](#) / [linnet](#)



vakint

[single-scale vacuum integrals](#)

MODULAR CODES FOR MAXIMUM FLEXIBILITY



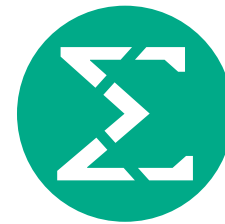
γ Loop



User-friendly



Low-level



Symbolica

~ “Open-source CAS / Compiler”

Components often distributed independently as sub-crates on:

<https://github.com/alpha100p/gammaploop/tree/main/crates/XXX>

- Fast and flexible numerator algebra
- Diagram generation and manipulations
- Analytical vacuum integrals
- Topology-informed momentum sampling



spenso

[spenso](#) / [idenso](#)



feyngen

[through gammaploop CLI only](#) / [linnet](#)



vakint

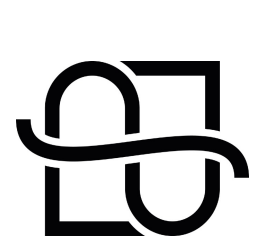
[single-scale vacuum integrals](#)



momtrop

[tropical sampling with momentum space integrals](#)

MODULAR CODES FOR MAXIMUM FLEXIBILITY



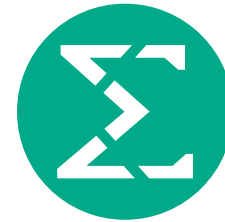
γ Loop



User-friendly



Low-level



Symbolica

~ “Open-source CAS / Compiler”

Components often distributed independently as sub-crates on:

<https://github.com/alpha100p/gammaploop/tree/main/crates/XXX>

- Fast and flexible numerator algebra
- Diagram generation and manipulations
- Analytical vacuum integrals
- Topology-informed momentum sampling
- Neural multi-channel importance sampling



spenso

[spenso](#) / [idenso](#)



feyngen

[through gammaploop CLI only](#) / [linnet](#)



vakint

[single-scale vacuum integrals](#)



momtrop

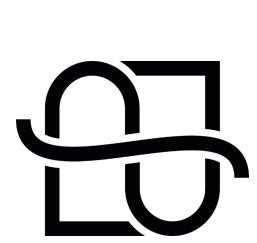
[tropical sampling with momentum space integrals](#)



MadNIS

[T. Heimerl & al tool tailored to gammaBoard](#)

MODULAR CODES FOR MAXIMUM FLEXIBILITY



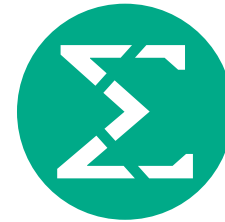
γ Loop



User-friendly



Low-level



Symbolica

~ “Open-source CAS / Compiler”

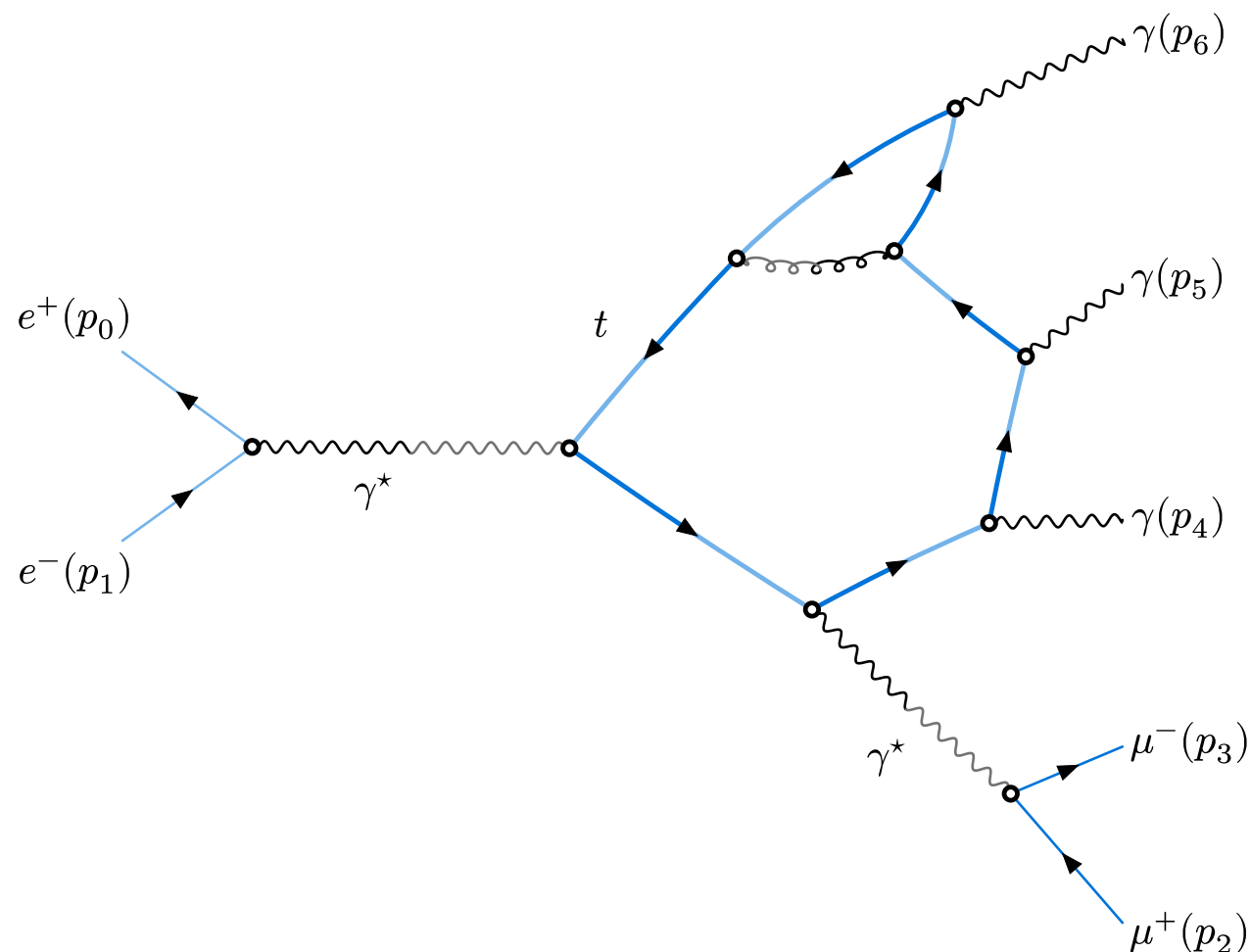
Components often distributed independently as sub-crates on:

<https://github.com/alpha100p/gammaploop/tree/main/crates/XXX>

- Fast and flexible numerator algebra → **spenso**
[spenso](#) / [idenso](#)
- Diagram generation and manipulations → **feyngen**
[through gammaploop CLI only](#) / [linnet](#)
- Analytical vacuum integrals → **vakint**
[single-scale vacuum integrals](#)
- Topology-informed momentum sampling → **momtrop**
[tropical sampling with momentum space integrals](#)
- Neural multi-channel importance sampling → **MadNIS**
[T. Heimerl & al tool tailored to gammaBoard](#)
- Monte-Carlo integration harness / dashboard → **GammaBoard**
[Full generic: use it!](#)

THE LOOPFEST 2026 CHALLENGE

Two-loop 5-point challenge diagram



Two-loop 5-point challenge diagram

```
digraph BNL {
  num = "1";
  overall_factor = "1";
  overall_factor_evaluated = "1";
  projector = "vbar(0,spenso::bis(4,hedge(0)))
    * u(1,spenso::bis(4,hedge(1)))
    * v(2,spenso::bis(4,hedge(2)))
    * ubar(3,spenso::bis(4,hedge(3)))
    * ebar(4,spenso::mink(4,hedge(4)))
    * ebar(5,spenso::mink(4,hedge(5)))
    * ebar(6,spenso::mink(4,hedge(6)))";

  ext [style=invis]
  ext -> 0:0 [id=0 particle="e+"];
  ext -> 0:1 [id=1 particle="e-"];

  1:2 -> ext [id=2 particle="mu+"];
  1:3 -> ext [id=3 particle="mu-"];
  2:4 -> ext [id=4 particle="a"];
  3:5 -> ext [id=5 particle="a"];
  4:6 -> ext [id=6 particle="a"];

  0 -> 5 [id=7 particle="a"];
  6 -> 1 [id=8 particle="a"];

  5 -> 6 [id=9 particle="t" lmb_id=0];
  6 -> 2 [id=10 particle="t"];
  2 -> 3 [id=11 particle="t"];
  3 -> 7 [id=12 particle="t"];
  7 -> 4 [id=13 particle="t"];
  4 -> 8 [id=14 particle="t"];
  8 -> 5 [id=15 particle="t"];
  7 -> 8 [id=16 particle="g" lmb_id=1];
}
```

["dot" format documentation](#)

THE LOOPFEST 2026 DEMO

<https://gammaloop.hirschi.lu>

Welcome to the LoopFest XXIV demo

Use the live GammaLoop demo tools below to monitor distributed runs, generate graph data, or continue the challenge workflow.



Gammaboard

Monitor and control distributed Monte Carlo integration runs.

Open board

[GammaLoop source on GitHub](#)
[Gammaboard source on GitHub](#)

Feynmangraph (WIP)

Generate and inspect Feynman graph data with the local GammaLoop backend.

Open graph tool

[FeynmanGraph source on GitHub](#)
(credits: Elijah Cavan)

GammaLoop Challenge

Enter the LoopFest challenge ratios and check the generated result.

Open challenge

THE LOOPFEST 2026 DEMO

<https://gammaloop.hirschi.lu/board>



Real-time Monte Carlo simulation monitoring

Operator mode LOG OUT

[RUNS](#) [MANAGEMENT](#) [PERFORMANCE](#) [LOGS](#) [SETTINGS](#)

Connected to local

[COPY RUN TOML](#) [NEW RUN](#)

Select Run Show child runs
gammaloop_bnl_r4_demo paused | completed samples 1,100,000

all [ASSIGN](#) [UNASSIGN](#) [PAUSE RUN](#) [DELETE RUN](#)

Task Queue

[CLONE RUN](#) [ADD TASK](#) [DELETE TASK](#)

Name	State	Kind	Failure	Goal	Completed
accumulator	completed	set_accumulator	-	-	0
havana-train	completed	sample	-	100,000	100,000
sample	completed	sample	-	1,000,000	1,000,000

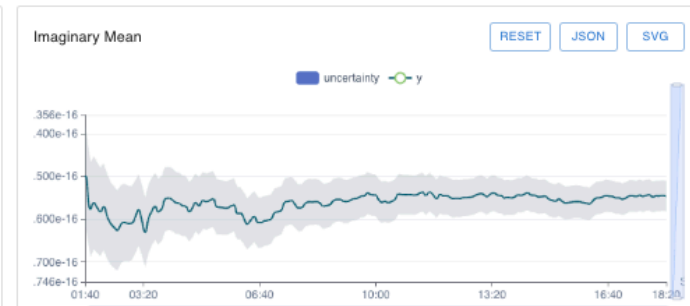
Selected Run TOML
gammaloop_bnl_r4_demo

Selected Task TOML
sample

Selected Task Output

Sample Progress
1,000,000 / 1,000,000

Estimate Summary			
Count	1,100,000	RSD	1.5488
Real Mean	$(7.661 \pm 0.034) \times 10^{-16}$ (0.443%) Full precision (f64)	Imag Mean	$(-2.544 \pm 0.036) \times 10^{-16}$ (1.399%) Full precision (f64)
Abs Mean	$(3.2700 \pm 0.0039) \times 10^{-15}$ (0.119%) Full precision (f64)		



THE LOOPFEST 2026 DEMO

<https://gamma.loop.hirschi.lu/challenge>

The LoopFest XXIV Challenge

[Challenge Rules](#)



0x926fAdD8CD8c63f4565a610d203a
9b9D97A0fb5F

[See it on etherscan](#)

Enter the four complex-valued ratios with real and imaginary parts separated. Each entry must use a signed two-significant-digit form such as -0.31 , $+0.20$, or 2.3 .

R1/R0	R2/R0	R3/R0	R4/R0
Real -4.8	Real +2.2	Real -1.2	Real -0.17
Imaginary +0.20	Imaginary +5.5	Imaginary +3.4	Imaginary +0.34

Allowed forms: -0.31 , $+0.31$, 0.31 , -2.3 , or 2.3 .

[Test!](#)

INCORRECT

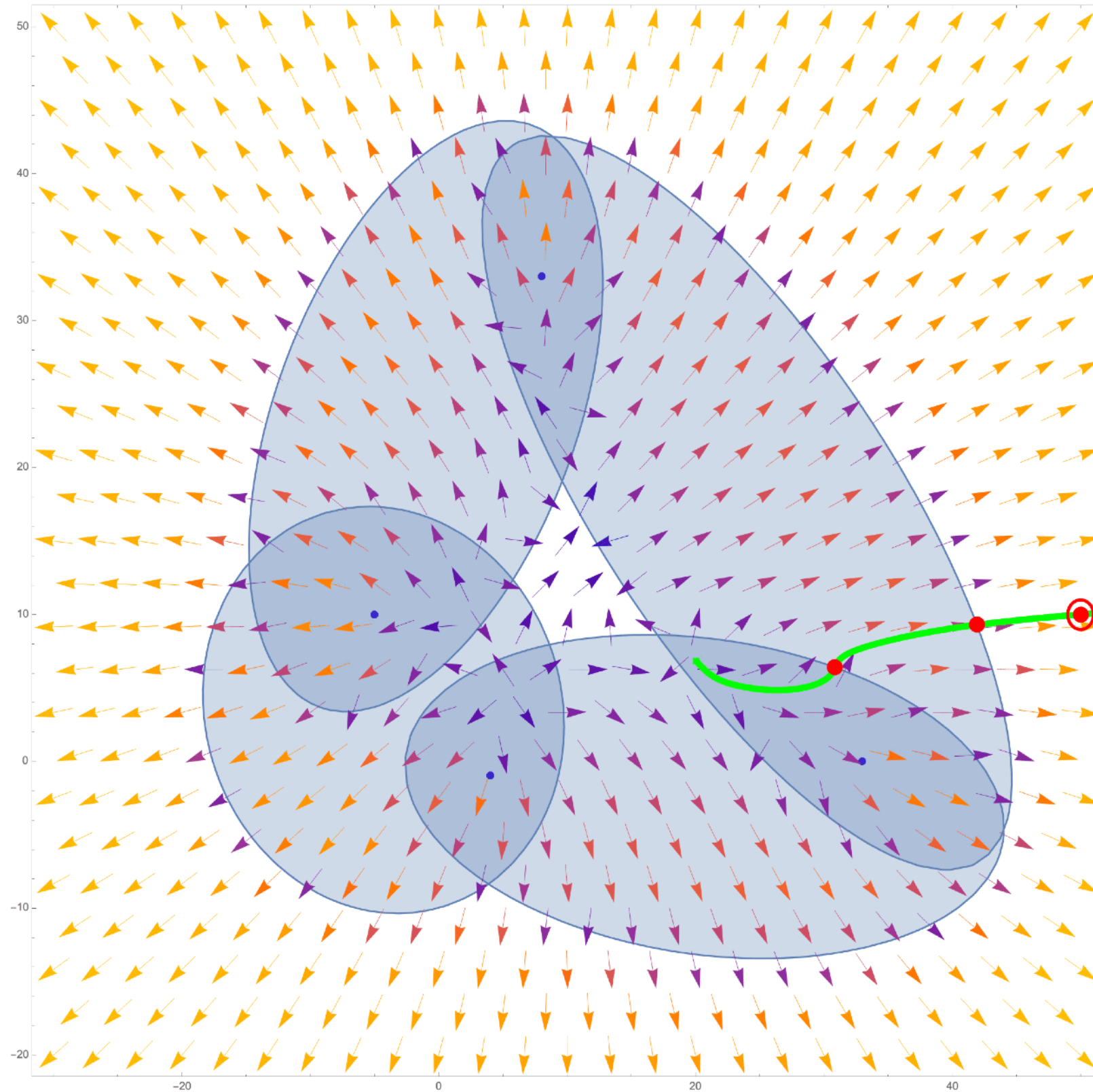
PASSPHRASE

PUBLIC KEY

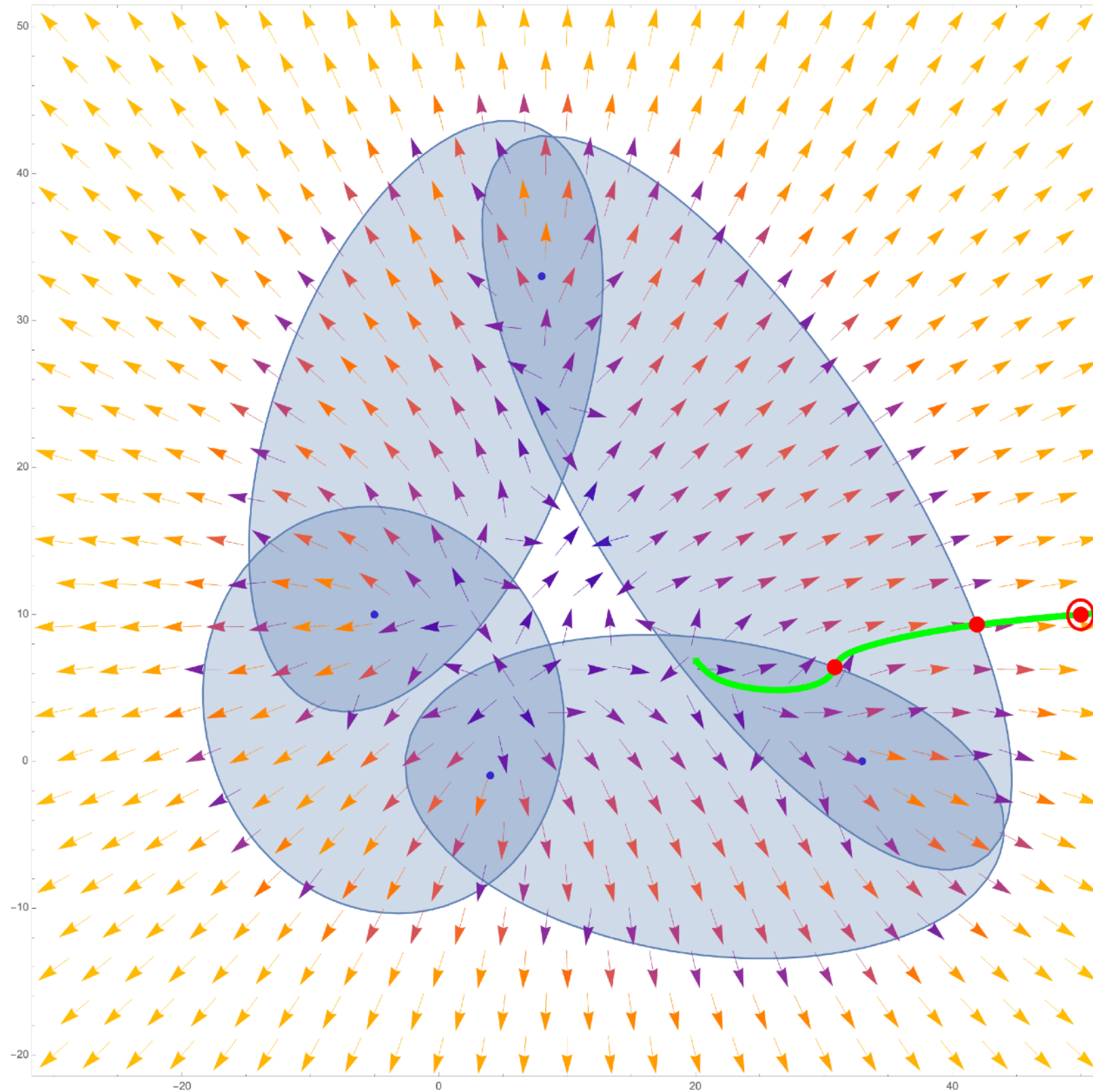
PRIVATE KEY

ETHEREUM ADDRESS

THANK YOU FOR YOUR ATTENTION



THANK YOU FOR YOUR ATTENTION



BACKUP SLIDES

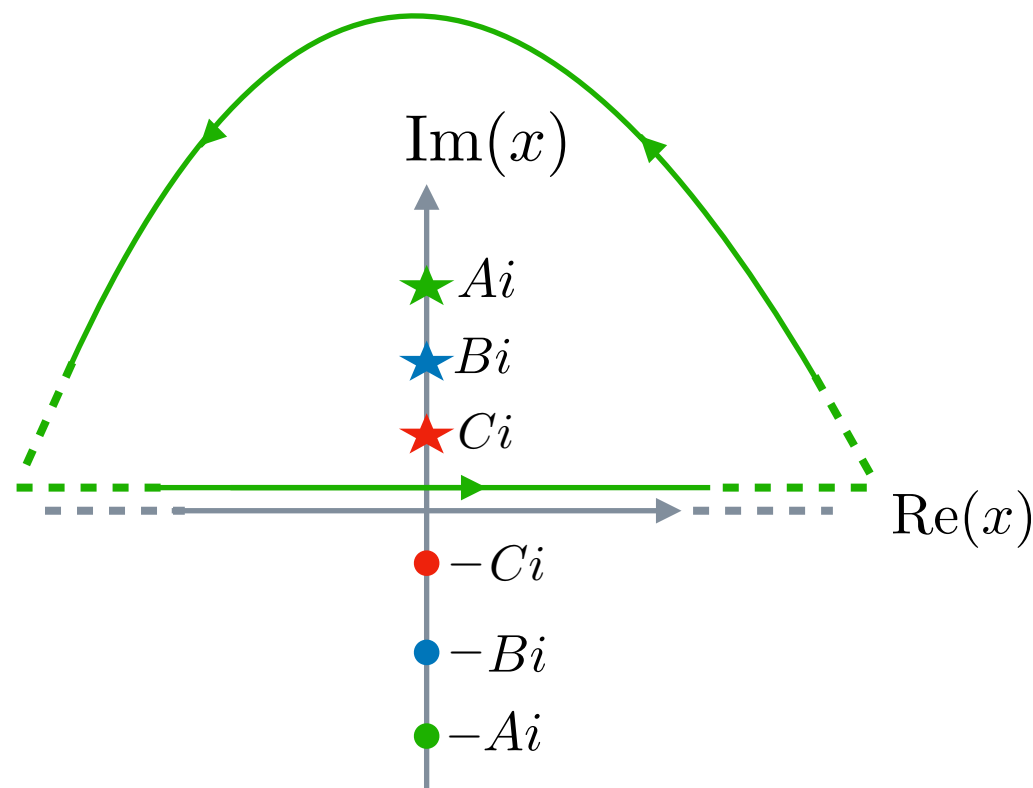
LOOP-TREE DUALITY

(ONE)-LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx F(x) \quad F(x) = \frac{1}{x^2 + A^2} \frac{1}{x^2 + B^2} \frac{1}{x^2 + C^2}$$

$$F(x) = \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(x - Bi)(x + Bi)} \frac{1}{(x - Ci)(x + Ci)}$$

(Assumptions $\rightarrow \{A > 0, B > 0, C > 0\}$)



Cauchy: $(R(x^*) \equiv \text{Res}(F, x = x^*))$

$$I = (-2\pi i) [R(Ai) + R(Bi) + R(Ci)]$$

What does it correspond to for a one-loop integral?

(ONE-)LOOP TREE DUALITY

$$\begin{aligned} \frac{1}{k^2 - M^2 + i\delta} &= \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} \\ &= \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right) \left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)} \end{aligned}$$

(ONE-)LOOP TREE DUALITY

$$\begin{aligned} \frac{1}{k^2 - M^2 + i\delta} &= \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} \\ &= \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right) \left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)} \end{aligned}$$

Pole selected for each propagator

(ONE-)LOOP TREE DUALITY

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta}$$

$$= \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right) \left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

Pole selected for each propagator

Analogous 1-loop triangle case:

$$\text{triangle diagram} \equiv \delta(q_i^2) \Theta(q_i^0)$$

$$\stackrel{\text{LTD}}{=} -I_1 - I_2 - I_3$$

(ONE-)LOOP TREE DUALITY

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right) \left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

Pole selected for each propagator

Analogous 1-loop triangle case:

$$\text{Diagram} \equiv \delta(q_i^2) \Theta(q_i^0)$$

Diagram description: A triangle loop with external momenta $-p_1^\mu$ (top left), $-p_2^\mu$ (bottom left), and $p_1^\mu + p_2^\mu$ (right). Internal momenta are k (left), $k - p_1$ (top), and $k - p_1 - p_2$ (bottom). The diagram is labeled I .

Equation: $I \stackrel{\text{LTD}}{=} -I_1 - I_2 - I_3$

Diagram description: Three triangle diagrams labeled I_1 (red), I_2 (green), and I_3 (blue). Each diagram has a dashed line and a solid line, with a colored slash indicating the pole selection.

General 1-loop case:

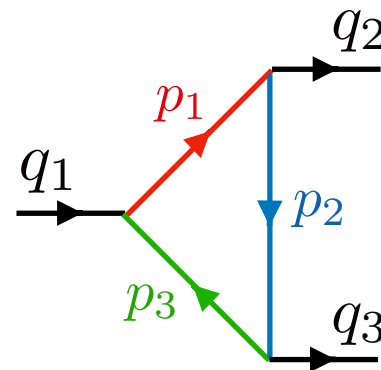
[Catani & al., arxiv:0804.3170]

$$I = - \int_q \sum_{i=1}^N \tilde{\delta}(q_i) \prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{q_j^2 - i0 \eta(q_j - q_i)}$$

MULTI-LOOP TREE DUALITY: CLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

Using partial fractioning, H-surfaces can be algebraically canceled:



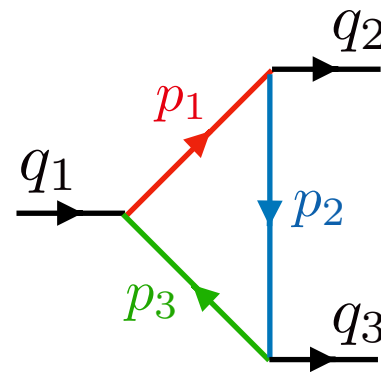
The diagram shows a triangle loop. An incoming horizontal line from the left is labeled q_1 . It splits into a red diagonal line labeled p_1 (pointing up-right) and a green diagonal line labeled p_3 (pointing down-right). The red line p_1 and a blue vertical line labeled p_2 (pointing down) meet at a vertex. From this vertex, an outgoing horizontal line to the right is labeled q_2 . The blue line p_2 and the green line p_3 meet at another vertex. From this vertex, an outgoing horizontal line to the right is labeled q_3 .

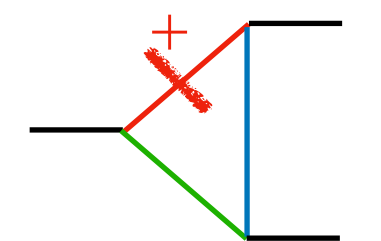
$$= \frac{1}{2\pi i} \int_{\mathbb{R}} dk^0 \frac{1}{(k^0 + p_1^0)^2 - E_1^2} \frac{1}{(k^0 + p_2^0)^2 - E_2^2} \frac{1}{(k^0 + p_3^0)^2 - E_3^2} \stackrel{\text{LTD}}{=}$$

MULTI-LOOP TREE DUALITY: CLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

Using partial fractioning, H-surfaces can be **algebraically canceled**:



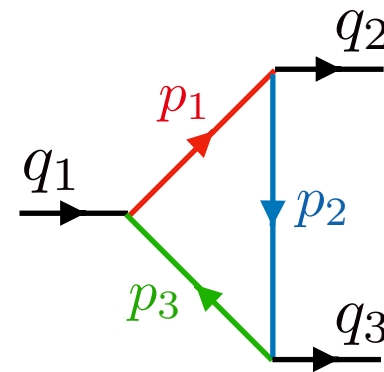
$$= \frac{1}{2\pi i} \int_{\mathbb{R}} dk^0 \frac{1}{(k^0 + p_1^0)^2 - E_1^2} \frac{1}{(k^0 + p_2^0)^2 - E_2^2} \frac{1}{(k^0 + p_3^0)^2 - E_3^2} \stackrel{\text{LTD}}{=}$$


$$\frac{1}{2E_1} \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

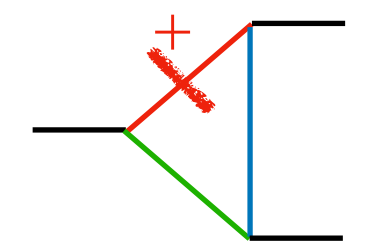
MULTI-LOOP TREE DUALITY: CLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

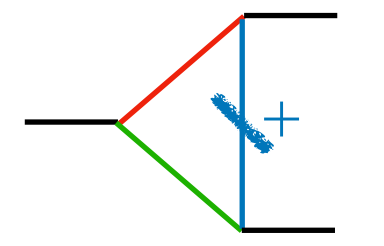
Using partial fractioning, H-surfaces can be algebraically canceled:



$$= \frac{1}{2\pi i} \int_{\mathbb{R}} dk^0 \frac{1}{(k^0 + p_1^0)^2 - E_1^2} \frac{1}{(k^0 + p_2^0)^2 - E_2^2} \frac{1}{(k^0 + p_3^0)^2 - E_3^2} \stackrel{\text{LTD}}{=} \\$$



$$\frac{1}{2E_1} \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$



$$+ \frac{1}{(E_2 - p_2^0 + p_1^0)^2 - E_1^2} \frac{1}{2E_2} \frac{1}{(E_2 - p_2^0 + p_3^0)^2 - E_3^2}$$

MULTI-LOOP TREE DUALITY: CLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

Using partial fractioning, H-surfaces can be **algebraically canceled**:

$$\begin{aligned}
 & \text{Diagram 1: } q_1 \text{ (in), } p_1 \text{ (red, up), } p_3 \text{ (green, down), } p_2 \text{ (blue, down), } q_2 \text{ (out), } q_3 \text{ (out)} \\
 & = \frac{1}{2\pi i} \int_{\mathbb{R}} dk^0 \frac{1}{(k^0 + p_1^0)^2 - E_1^2} \frac{1}{(k^0 + p_2^0)^2 - E_2^2} \frac{1}{(k^0 + p_3^0)^2 - E_3^2} \stackrel{\text{LTD}}{=} \\
 & \begin{aligned}
 & \text{Diagram 2: } \frac{1}{2E_1} \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2} \\
 & + \frac{1}{(E_2 - p_2^0 + p_1^0)^2 - E_1^2} \frac{1}{2E_2} \frac{1}{(E_2 - p_2^0 + p_3^0)^2 - E_3^2} \\
 & + \frac{1}{(E_3 - p_3^0 + p_1^0)^2 - E_1^2} \frac{1}{(E_3 - p_3^0 + p_2^0)^2 - E_2^2} \frac{1}{2E_3}
 \end{aligned}
 \end{aligned}$$

MULTI-LOOP TREE DUALITY: CLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

Using partial fractioning, H-surfaces can be **algebraically canceled**:

$$\begin{aligned}
 & \text{Diagram 1: A tree diagram with three external lines } q_1, q_2, q_3 \text{ and three internal lines } p_1, p_2, p_3. \\
 & = \frac{1}{2\pi i} \int_{\mathbb{R}} dk^0 \frac{1}{(k^0 + p_1^0)^2 - E_1^2} \frac{1}{(k^0 + p_2^0)^2 - E_2^2} \frac{1}{(k^0 + p_3^0)^2 - E_3^2} \stackrel{\text{LTD}}{=} \\
 & \begin{aligned}
 & \text{Diagram 2: A tree diagram with a red line and a blue line, and a green line.} \\
 & \frac{1}{2E_1} \boxed{\frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2}} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2} \\
 & + \frac{1}{(E_2 - p_2^0 + p_1^0)^2 - E_1^2} \frac{1}{2E_2} \frac{1}{(E_2 - p_2^0 + p_3^0)^2 - E_3^2} \\
 & + \frac{1}{(E_3 - p_3^0 + p_1^0)^2 - E_1^2} \frac{1}{(E_3 - p_3^0 + p_2^0)^2 - E_2^2} \frac{1}{2E_3}
 \end{aligned}
 \end{aligned}$$

MULTI-LOOP TREE DUALITY: CLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

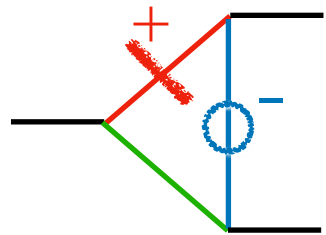
Using partial fractioning, H-surfaces can be **algebraically canceled**:

$$\begin{aligned}
 & \text{Diagram 1: A vertex with three external lines } q_1, q_2, q_3 \text{ and three internal lines } p_1, p_2, p_3. \\
 & = \frac{1}{2\pi i} \int_{\mathbb{R}} dk^0 \frac{1}{(k^0 + p_1^0)^2 - E_1^2} \frac{1}{(k^0 + p_2^0)^2 - E_2^2} \frac{1}{(k^0 + p_3^0)^2 - E_3^2} \stackrel{\text{LTD}}{=} \\
 & \begin{aligned}
 & \text{Diagram 2: } \frac{1}{2E_1} \boxed{\frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2}} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2} \\
 & + \boxed{\frac{1}{(E_2 - p_2^0 + p_1^0)^2 - E_1^2}} \frac{1}{2E_2} \frac{1}{(E_2 - p_2^0 + p_3^0)^2 - E_3^2} \\
 & + \frac{1}{(E_3 - p_3^0 + p_1^0)^2 - E_1^2} \frac{1}{(E_3 - p_3^0 + p_2^0)^2 - E_2^2} \frac{1}{2E_3}
 \end{aligned}
 \end{aligned}$$

MULTI-LOOP TREE DUALITY: CLTD

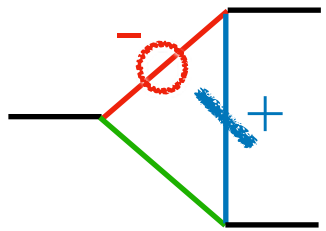
[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

Using partial fractioning , H-surfaces can be algebraically canceled: $\frac{1}{x^2 - y^2} = \frac{1}{2y} \left(\frac{1}{x - y} - \frac{1}{x + y} \right)$



$$\frac{1}{2E_1}$$

$$\frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$



$$+ \frac{1}{(E_2 - p_2^0 + p_1^0)^2 - E_1^2}$$

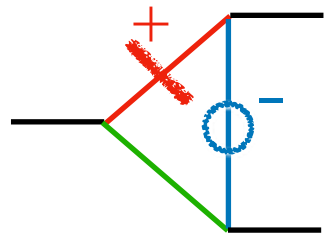
$$\frac{1}{2E_2}$$

$$\frac{1}{(E_2 - p_2^0 + p_3^0)^2 - E_3^2}$$

MULTI-LOOP TREE DUALITY: CLTD

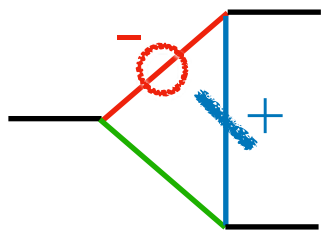
[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

Using partial fractioning , H-surfaces can be algebraically canceled: $\frac{1}{x^2 - y^2} = \frac{1}{2y} \left(\frac{1}{x - y} - \frac{1}{x + y} \right)$



$$\frac{1}{2E_1} \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_2} \left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2} \right)$$

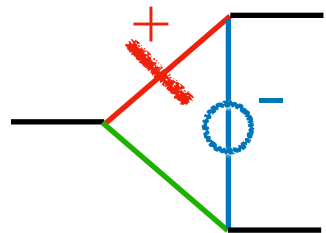


$$+ \frac{1}{(E_2 - p_2^0 + p_1^0)^2 - E_1^2} \frac{1}{2E_2} \frac{1}{(E_2 - p_2^0 + p_3^0)^2 - E_3^2}$$

MULTI-LOOP TREE DUALITY: CLTD

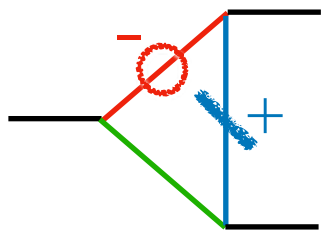
[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

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$$\frac{1}{2E_1} \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_2} \left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2} \right)$$



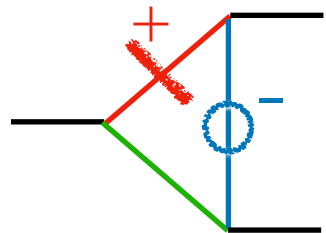
$$+ \frac{1}{(E_2 - p_2^0 + p_1^0)^2 - E_1^2} \frac{1}{2E_2} \frac{1}{(E_2 - p_2^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_1} \left(\frac{1}{(E_2 - p_2^0 + p_1^0) - E_1} - \frac{1}{(E_2 - p_2^0 + p_1^0) + E_1} \right)$$

MULTI-LOOP TREE DUALITY: CLTD

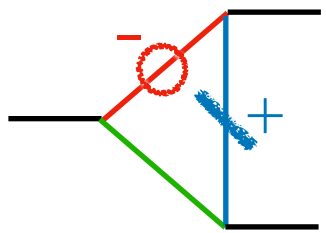
[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

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$$\frac{1}{2E_1} \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_2} \left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2} \right)$$



$$+ \frac{1}{(E_2 - p_2^0 + p_1^0)^2 - E_1^2} \frac{1}{2E_2} \frac{1}{(E_2 - p_2^0 + p_3^0)^2 - E_3^2}$$

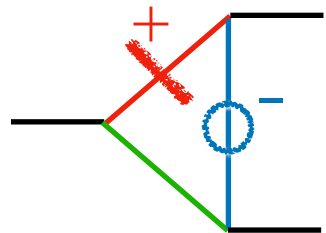
$$= \frac{1}{2E_1} \left(\frac{1}{(E_2 - p_2^0 + p_1^0) - E_1} - \frac{1}{(E_2 - p_2^0 + p_1^0) + E_1} \right)$$

$$\supset \frac{1}{2E_1} \frac{1}{2E_2} \left(\frac{1}{E_1 - p_1^0 + p_2^0 - E_2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2} + \right.$$

MULTI-LOOP TREE DUALITY: CLTD

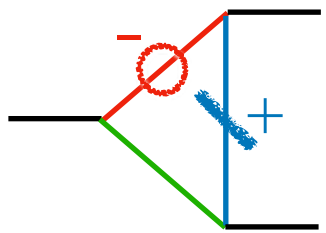
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$$\frac{1}{2E_1} \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_2} \left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2} \right)$$



$$+ \frac{1}{(E_2 - p_2^0 + p_1^0)^2 - E_1^2} \frac{1}{2E_2} \frac{1}{(E_2 - p_2^0 + p_3^0)^2 - E_3^2}$$

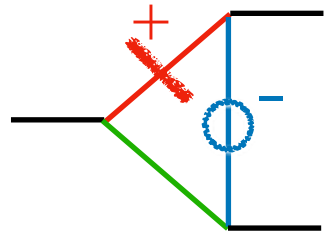
$$= \frac{1}{2E_1} \left(\frac{1}{(E_2 - p_2^0 + p_1^0) - E_1} - \frac{1}{(E_2 - p_2^0 + p_1^0) + E_1} \right)$$

$$\supset \frac{1}{2E_1} \frac{1}{2E_2} \left(\frac{1}{E_1 - p_1^0 + p_2^0 - E_2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2} + \frac{1}{E_2 - p_2^0 + p_1^0 - E_1} \frac{1}{(E_2 - p_2^0 + p_3^0)^2 - E_3^2} \right)$$

MULTI-LOOP TREE DUALITY: CLTD

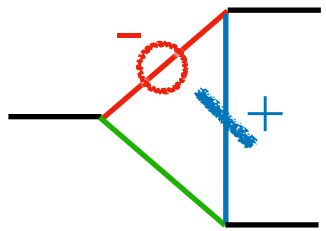
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$$\frac{1}{2E_1} \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_2} \left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2} \right)$$



$$+ \frac{1}{(E_2 - p_2^0 + p_1^0)^2 - E_1^2} \frac{1}{2E_2} \frac{1}{(E_2 - p_2^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_1} \left(\frac{1}{(E_2 - p_2^0 + p_1^0) - E_1} - \frac{1}{(E_2 - p_2^0 + p_1^0) + E_1} \right)$$

$$\supset \frac{1}{2E_1} \frac{1}{2E_2} \left(\frac{1}{E_1 - p_1^0 + p_2^0 - E_2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2} + \frac{1}{E_2 - p_2^0 + p_1^0 - E_1} \frac{1}{(E_2 - p_2^0 + p_3^0)^2 - E_3^2} \right)$$

$$= \frac{1}{2E_1} \frac{1}{2E_2} \frac{E_1 + E_2 - p_1 - p_2 + 2p_3}{((E_1 - p_1^0 + p_3^0)^2 - E_3^2) ((E_2 - p_2^0 + p_3^0)^2 - E_3^2)}$$

Manifestly **divergence-free** at :

$$E_2(\vec{k}) = E_1(\vec{k}) + p_2^0 - p_1^0$$

(TWO-)LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \left[\right.$

$\left. \right]$

(TWO-)LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \left[\right.$

$$\frac{1}{2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} \Theta(A)$$

]

(TWO-)LOOP TREE DUALITY MOCK-UP

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$$\frac{1}{2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} \Theta(A)$$

$$+ \frac{1}{-2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} \Theta(-A)$$

]

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$$+ \frac{1}{-2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} \Theta(-A)$$

$$+ \frac{1}{(Ci - y - Ai)(Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{2Ci} \Theta(\Im[Ci - y])$$

]

(TWO-)LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

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$$+ \frac{1}{(Ci - y - Ai)(Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{2Ci} \Theta(\Im[Ci - y])$$

$$+ \frac{1}{(-Ci - y - Ai)(-Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{-2Ci} \Theta(\Im[-Ci - y]) \quad \left. \right]$$

(TWO-)LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \left[\right.$

$$\frac{1}{2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} \overset{\equiv 1}{\Theta(A)}$$

$$+ \frac{1}{-2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} \overset{\equiv 0}{\Theta(-A)}$$

$$+ \frac{1}{(Ci - y - Ai)(Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{2Ci} \Theta(\Im[Ci - y])$$

$$+ \frac{1}{(-Ci - y - Ai)(-Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{-2Ci} \Theta(\Im[-Ci - y]) \quad \left. \right]$$

(TWO-)LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

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$$\frac{1}{2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} \overset{\equiv 1}{\Theta(A)}$$

$$+ \frac{1}{-2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} \overset{\equiv 0}{\Theta(-A)}$$

$$+ \frac{1}{(Ci - y - Ai)(Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{2Ci} \overset{\equiv 1}{\Theta(\Im[Ci - y])}$$

$$+ \frac{1}{(-Ci - y - Ai)(-Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{-2Ci} \overset{\equiv 0}{\Theta(\Im[-Ci - y])} \left. \right]$$

(TWO-)LOOP TREE DUALITY MOCK-UP

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]

(TWO-)LOOP TREE DUALITY MOCK-UP

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(**+1**, **0**, **0**)

+ $\frac{1}{(Ci - y - Ai)(Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{2Ci}$

(**0**, **0**, **+1**)

$\left. \right]$

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

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$$(+1, 0, 0)$$

$$(0, 0, +1)$$

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$$(+1, 0, 0)$$

$$(0, 0, +1)$$

Integrate y then: $I = (-2\pi i)^2 \left[\right.$

$\left. \right]$

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \left[\right.$

$$+ \frac{1}{2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} + \frac{1}{(Ci - y - Ai)(Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{2Ci} \left. \right]$$

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Integrate y then: $I = (-2\pi i)^2 \left[\right.$

$$\frac{1}{2Ai} \frac{1}{2Bi} \frac{1}{(Ai + Bi - Ci)(Ai + Bi + Ci)}$$

$$(+1, +1, 0)$$

$\left. \right]$

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \left[\right.$

$$+ \frac{1}{2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} + \frac{1}{(Ci - y - Ai)(Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{2Ci} \left. \right]$$

$$(+1, 0, 0)$$

$$(0, 0, +1)$$

Integrate y then: $I = (-2\pi i)^2 \left[\right.$

$$+ \frac{1}{2Ai} \frac{1}{2Bi} \frac{1}{(Ai + Bi - Ci)(Ai + Bi + Ci)} + \frac{1}{2Ai} \frac{1}{(Ci - Ai - Bi)(Ci - Ai + Bi)} \frac{1}{2Ci} \Theta(C - A)$$

$$(+1, +1, 0)$$

$$(+1, 0, +1)$$

$\left. \right]$

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \left[\right.$

$$+ \frac{1}{2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} + \frac{1}{(Ci - y - Ai)(Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{2Ci} \left. \right]$$

$$(+1, 0, 0)$$

$$(0, 0, +1)$$

Integrate y then: $I = (-2\pi i)^2 \left[\right.$

$$+ \frac{1}{2Ai} \frac{1}{2Bi} \frac{1}{(Ai + Bi - Ci)(Ai + Bi + Ci)} + \frac{1}{2Ai} \frac{1}{(Ci - Ai - Bi)(Ci - Ai + Bi)} \frac{1}{2Ci} \Theta(C - A)$$

$$(+1, +1, 0)$$

$$(+1, 0, +1)$$

$$\sim \lim_{\delta \rightarrow 0} \Theta(\delta |\vec{k}| - \delta |\vec{l}|) = ???$$

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \left[\right.$

$$+ \frac{1}{2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} + \frac{1}{(Ci - y - Ai)(Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{2Ci} \left. \right]$$

$$(+1, 0, 0)$$

$$(0, 0, +1)$$

Integrate y then: $I = (-2\pi i)^2 \left[\right.$

$$+ \frac{1}{2Ai} \frac{1}{2Bi} \frac{1}{(Ai + Bi - Ci)(Ai + Bi + Ci)} + \frac{1}{2Ai} \frac{1}{(Ci - Ai - Bi)(Ci - Ai + Bi)} \frac{1}{2Ci} \Theta(C - A) - \frac{1}{2Ai} \frac{1}{(Ci - Ai - Bi)(Ci - Ai + Bi)} \frac{1}{2Ci} \Theta(C - A)$$

$$(+1, +1, 0)$$

$$(+1, 0, +1)$$

$$(+1, 0, +1)$$

$$\left. \right]$$

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

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$$(+1, 0, 0)$$

$$(0, 0, +1)$$

Integrate **y** then: $I = (-2\pi i)^2 \left[\right.$

$$+ \frac{1}{2Ai} \frac{1}{2Bi} \frac{1}{(Ai + Bi - Ci)(Ai + Bi + Ci)} + \frac{1}{2Ai} \frac{1}{(Ci - Ai - Bi)(Ci - Ai + Bi)} \frac{1}{2Ci} \Theta(C - A) - \frac{1}{2Ai} \frac{1}{(Ci - Ai - Bi)(Ci - Ai + Bi)} \frac{1}{2Ci} \Theta(C - A) - \frac{1}{-2Ai} \frac{1}{(Ai + Ci - Bi)(Ai + Ci + Bi)} \frac{1}{2Ci} \left. \right]$$

$$(+1, +1, 0)$$

$$(+1, 0, +1)$$

$$(+1, 0, +1)$$

$$(-1, 0, +1)$$

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \left[\right.$

$$+ \frac{1}{2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} + \frac{1}{(Ci - y - Ai)(Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{2Ci} \left. \right]$$

$$(+1, 0, 0)$$

$$(0, 0, +1)$$

Integrate y then: $I = (-2\pi i)^2 \left[\right.$

$$+ \frac{1}{2Ai} \frac{1}{2Bi} \frac{1}{(Ai + Bi - Ci)(Ai + Bi + Ci)} + \frac{1}{2Ai} \frac{1}{(Ci - Ai - Bi)(Ci - Ai + Bi)} \frac{1}{2Ci} \Theta(C - A) - \frac{1}{2Ai} \frac{1}{(Ci - Ai - Bi)(Ci - Ai + Bi)} \frac{1}{2Ci} \Theta(C - A) - \frac{1}{-2Ai} \frac{1}{(Ai + Ci - Bi)(Ai + Ci + Bi)} \frac{1}{2Ci} + \frac{1}{(Ci - Bi - Ai)(Ci - Bi + Ai)} \frac{1}{2Bi} \frac{1}{2Ci} \left. \right]$$

$$(+1, +1, 0)$$

$$(+1, 0, +1)$$

$$(+1, 0, +1)$$

$$(-1, 0, +1)$$

$$(0, +1, +1)$$

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \left[\right.$

$$\begin{aligned} & \frac{1}{2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} \\ & + \frac{1}{(Ci - y - Ai)(Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{2Ci} \end{aligned} \left. \right]$$

$(+1, 0, 0)$

$(0, 0, +1)$

Integrate y then: $I = (-2\pi i)^2 \left[\right.$

$$\begin{aligned} & \frac{1}{2Ai} \frac{1}{2Bi} \frac{1}{(Ai + Bi - Ci)(Ai + Bi + Ci)} \\ & + \frac{1}{2Ai} \frac{1}{(Ci - Ai - Bi)(Ci - Ai + Bi)} \frac{1}{2Ci} \quad \text{O}(C - A) \\ & - \frac{1}{2Ai} \frac{1}{(Ci - Ai - Bi)(Ci - Ai + Bi)} \frac{1}{2Ci} \quad \text{O}(C - A) \\ & - \frac{1}{-2Ai} \frac{1}{(Ai + Ci - Bi)(Ai + Ci + Bi)} \frac{1}{2Ci} \\ & + \frac{1}{(Ci - Bi - Ai)(Ci - Bi + Ai)} \frac{1}{2Bi} \frac{1}{2Ci} \end{aligned} \left. \right]$$

$(+1, +1, 0)$

$(+1, 0, +1)$

$(+1, 0, +1)$

$(-1, 0, +1)$

$(0, +1, +1)$

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$$+ \frac{1}{2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} + \frac{1}{(Ci - y - Ai)(Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{2Ci} \left. \right]$$

(+1, 0, 0)

(0, 0, +1)

Integrate **y** then: $I = (-2\pi i)^2 \left[\right.$

$$+ \frac{1}{2Ai} \frac{1}{2Bi} \frac{1}{(Ai + Bi - Ci)(Ai + Bi + Ci)} + \frac{1}{2Ai} \frac{1}{(Ci - Ai - Bi)(Ci - Ai + Bi)} \frac{1}{2Ci} - \frac{1}{2Ai} \frac{1}{(Ci - Ai - Bi)(Ci - Ai + Bi)} \frac{1}{2Ci} - \frac{1}{-2Ai} \frac{1}{(Ai + Ci - Bi)(Ai + Ci + Bi)} \frac{1}{2Ci} + \frac{1}{(Ci - Bi - Ai)(Ci - Bi + Ai)} \frac{1}{2Bi} \frac{1}{2Ci} \left. \right]$$

(+1, +1, 0)

(+1, 0, +1)

(+1, 0, +1)

(-1, 0, +1)

(0, +1, +1)

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \left[\right.$

$$\begin{aligned} & \frac{1}{2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} \\ & + \frac{1}{(Ci - y - Ai)(Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{2Ci} \end{aligned} \left. \right]$$

(+1, 0, 0)

(0, 0, +1)

Integrate y then: $I = (-2\pi i)^2 \left[\right.$

$$\begin{aligned} & \frac{1}{2Ai} \frac{1}{2Bi} \frac{1}{(Ai + Bi - Ci)(Ai + Bi + Ci)} \\ & + \frac{1}{2Ai} \frac{1}{(Ai + Ci - Bi)(Ai + Ci + Bi)} \frac{1}{2Ci} \\ & + \frac{1}{(Ci - Bi - Ai)(Ci - Bi + Ai)} \frac{1}{2Bi} \frac{1}{2Ci} \end{aligned} \left. \right]$$

(+1, +1, 0)

(-1, 0, +1)

(0, +1, +1)

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(x + y - Ci)(x + y + Ci)}$$

Integrate x first: $I = (-2\pi i) \int_{-\infty}^{\infty} dy \left[\frac{1}{2Ai} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{(Ai + y - Ci)(Ai + y + Ci)} + \frac{1}{(Ci - y - Ai)(Ci - y + Ai)} \frac{1}{(y - Bi)(y + Bi)} \frac{1}{2Ci} \right]$

$$(+1, 0, 0)$$

$$(0, 0, +1)$$

Integrate y then: $I = (-2\pi i)^2 \left[\frac{1}{2Ai} \frac{1}{2Bi} \frac{1}{(Ai + Bi - Ci)(Ai + Bi + Ci)} - \frac{1}{-2Ai} \frac{1}{(Ai + Ci - Bi)(Ai + Ci + Bi)} \frac{1}{2Ci} + \frac{1}{(Ci - Bi - Ai)(Ci - Bi + Ai)} \frac{1}{2Bi} \frac{1}{2Ci} \right]$

$$(+1, +1, 0)$$

$$(-1, 0, +1)$$

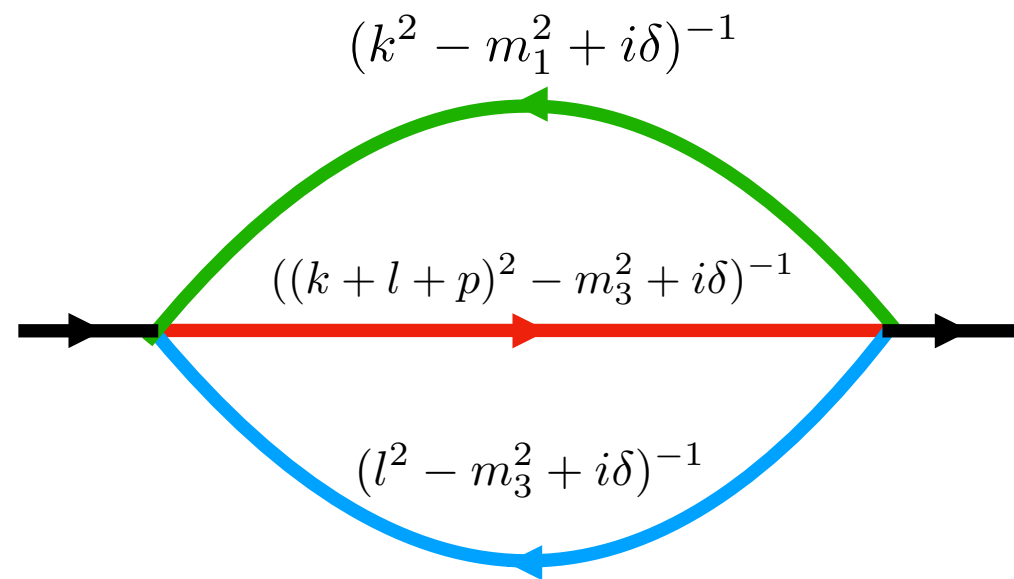
$$(0, +1, +1)$$

The **signs** of the residues selected are **non-trivial** !

MULTI-LOOP TREE DUALITY

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

Analogous sunrise case:



$$I = (2\pi i)^2 \left[\begin{array}{l} \mathbf{R}(+1, +1, 0) \\ + \mathbf{R}(-1, 0, +1) \\ + \mathbf{R}(0, +1, +1) \end{array} \right]$$

Now, what about **beyond two-loop** and **arbitrary topologies** ?

- **Distributional identities:** [Bierenbaum, Catani, Draggiotis, Rodrigo, arxiv: 1007.0194]
- **Averaging procedure:** [Runkel, Scór, Vesga, Weinzierl, arxiv: 1902.02135]
- **Iterative procedure:** [Capatti, VH, Kermanschah, Ruijl, arxiv: 1906.06138]

MULTI-LOOP TREE DUALITY

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

We iteratively apply the residue theorem:

$$I = (-i)^n \int \prod_{j=1}^n \frac{d^3 \vec{k}_j}{(2\pi)^3} \sum_{\mathbf{b} \in \mathcal{B}} \text{Res}_{\mathbf{b}}[f], \quad \text{Res}_{\mathbf{b}}[f] = \frac{1}{\prod_{i \in \mathbf{b}} 2E_i} \frac{N}{\prod_{i \in \mathbf{e} \setminus \mathbf{b}} D_i} \Big|_{\{q_j^0 = \sigma_j^{\mathbf{b}} E_j\}_{j \in \mathbf{b}}}$$

MULTI-LOOP TREE DUALITY

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

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Key characteristics:

- A single residue per spanning-tree of the graph, with a common sign.

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- No surviving Heaviside factors, all identically simplify to zero or one.

MULTI-LOOP TREE DUALITY

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

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Key characteristics:

- A single residue per spanning-tree of the graph, with a common sign.
- No surviving Heaviside factors, all identically simplify to zero or one.
- ✓ Verified explicitly up to six loops.
- ✓ Also valid for complex momenta[†] and masses.

Algorithmic determination of the cut-structure $\sigma_j^{\mathbf{b}}$:

- Given a:
- loop momentum basis
 - choice of complex contour closure (upper or lower)
 - ordering in the integration of loop momenta energies

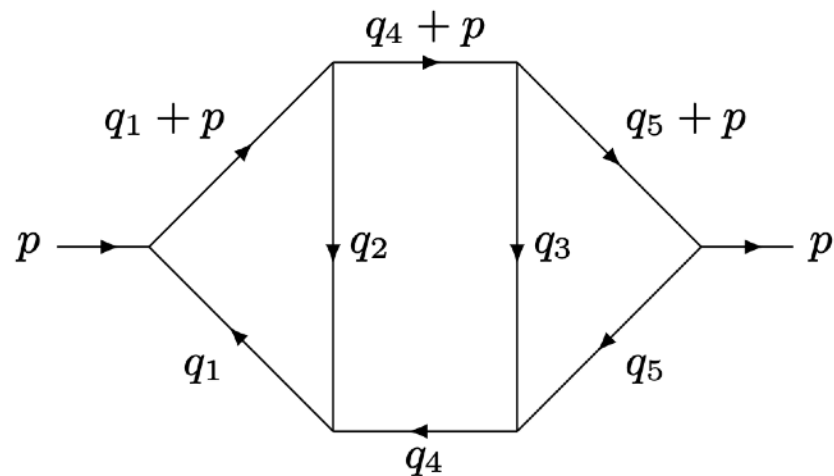
we can compute the cut structure signs for **any** multi-loop topology.

[†] Credits to S. Weinzierl

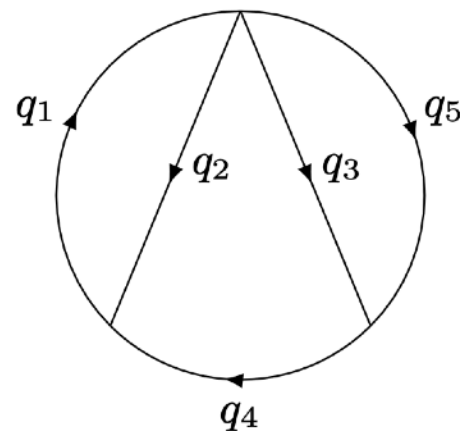
MULTI-LOOP TREE DUALITY

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

Three-loop example (showcasing the irrelevance of external momenta).



(a) Original diagram.



(b) Reduced diagram.

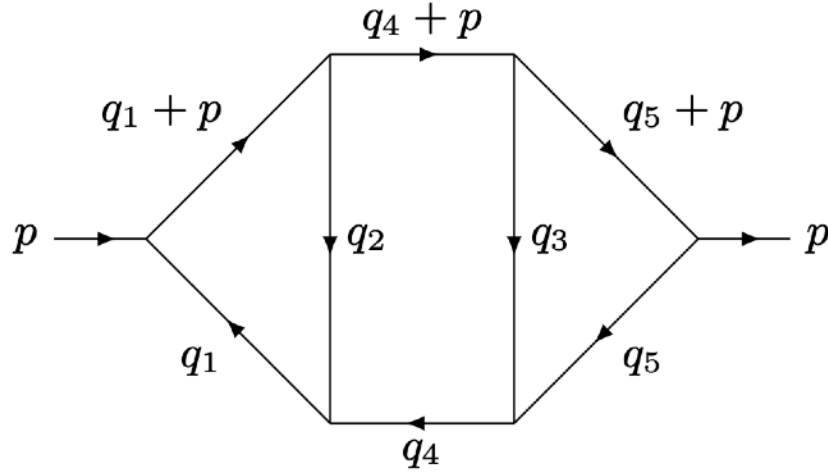
$q_1 = k_1$	$\mathbf{s}_1 = (1, 0, 0)$
$q_2 = k_2$	$\mathbf{s}_2 = (0, 1, 0)$
$q_3 = k_3$	$\mathbf{s}_3 = (0, 0, 1)$
$q_4 = k_1 - k_2$	$\mathbf{s}_4 = (1, -1, 0)$
$q_5 = k_1 - k_2 - k_3$	$\mathbf{s}_5 = (1, -1, -1)$

(c) Momenta and signatures.

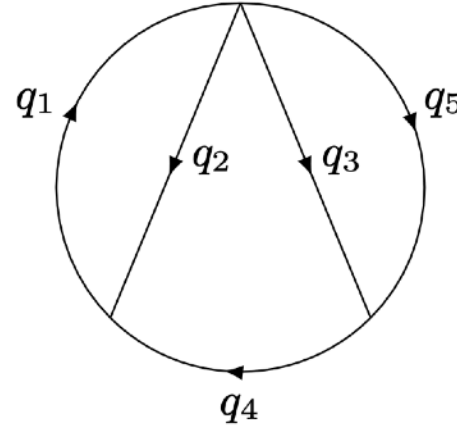
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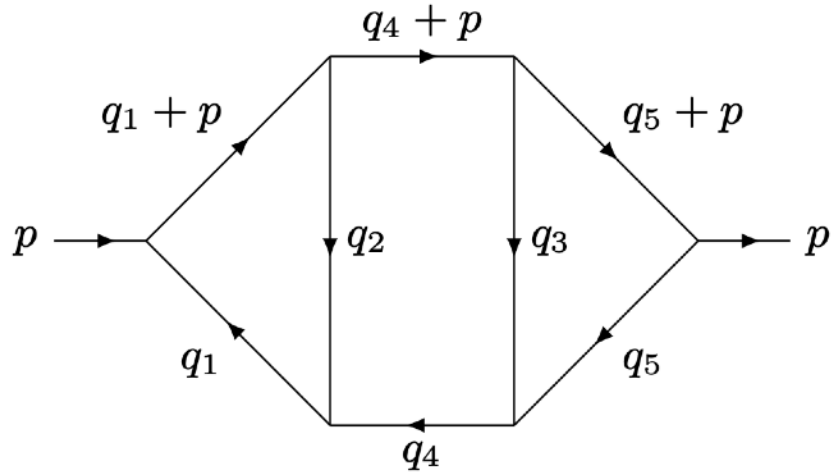
(c) Momenta and signatures.

$$\sigma_j^{\mathbf{b}} \equiv \begin{bmatrix} \Sigma_1 = (1, 1, 1, 0, 0), \\ \Sigma_2 = (1, -1, 0, 0, -1), \\ \Sigma_3 = (1, 0, 1, -1, 0), \\ \Sigma_4 = (1, 0, 1, 0, -1), \\ \Sigma_5 = (1, 0, 0, 1, -1), \\ \Sigma_6 = (0, 1, 1, 1, 0), \\ \Sigma_7 = (0, 1, 1, 0, 1), \\ \Sigma_8 = (0, 1, 0, 1, -1) \end{bmatrix}$$

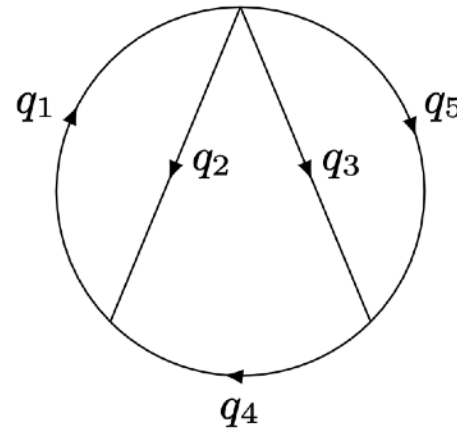
MULTI-LOOP TREE DUALITY

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

Three-loop example (showcasing the irrelevance of external momenta).



(a) Original diagram.

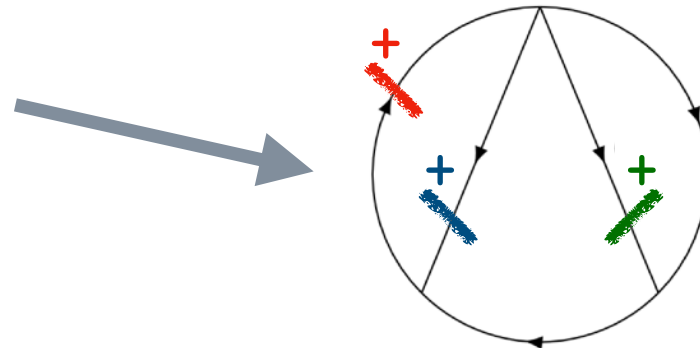


(b) Reduced diagram.

$q_1 = k_1$	$\mathbf{s}_1 = (1, 0, 0)$
$q_2 = k_2$	$\mathbf{s}_2 = (0, 1, 0)$
$q_3 = k_3$	$\mathbf{s}_3 = (0, 0, 1)$
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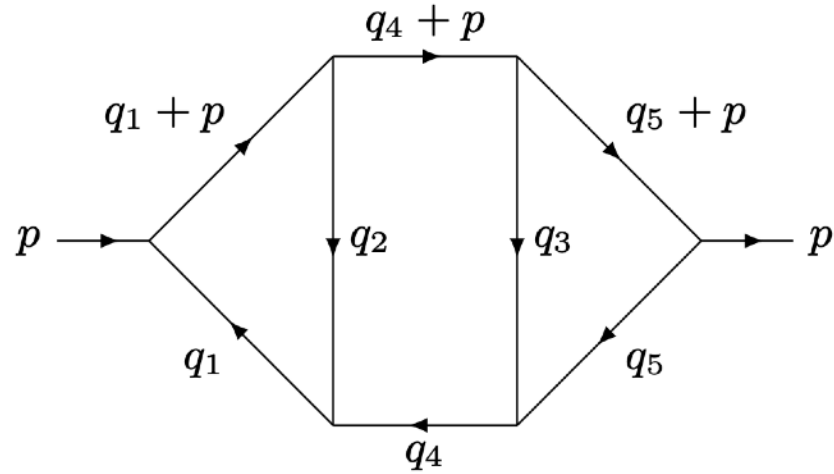
$$\sigma_j^{\mathbf{b}} \equiv \left[\begin{array}{l} \Sigma_1 = (1, 1, 1, 0, 0), \\ \Sigma_2 = (1, -1, 0, 0, -1), \\ \Sigma_3 = (1, 0, 1, -1, 0), \\ \Sigma_4 = (1, 0, 1, 0, -1), \\ \Sigma_5 = (1, 0, 0, 1, -1), \\ \Sigma_6 = (0, 1, 1, 1, 0), \\ \Sigma_7 = (0, 1, 1, 0, 1), \\ \Sigma_8 = (0, 1, 0, 1, -1) \end{array} \right]$$



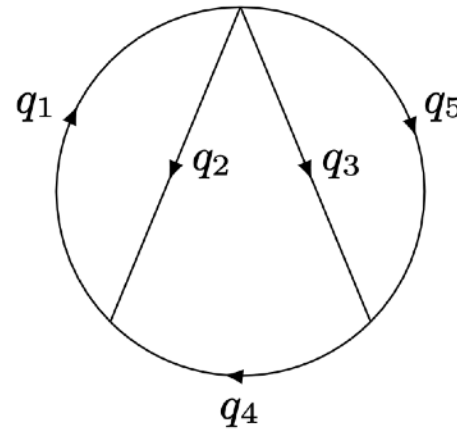
MULTI-LOOP TREE DUALITY

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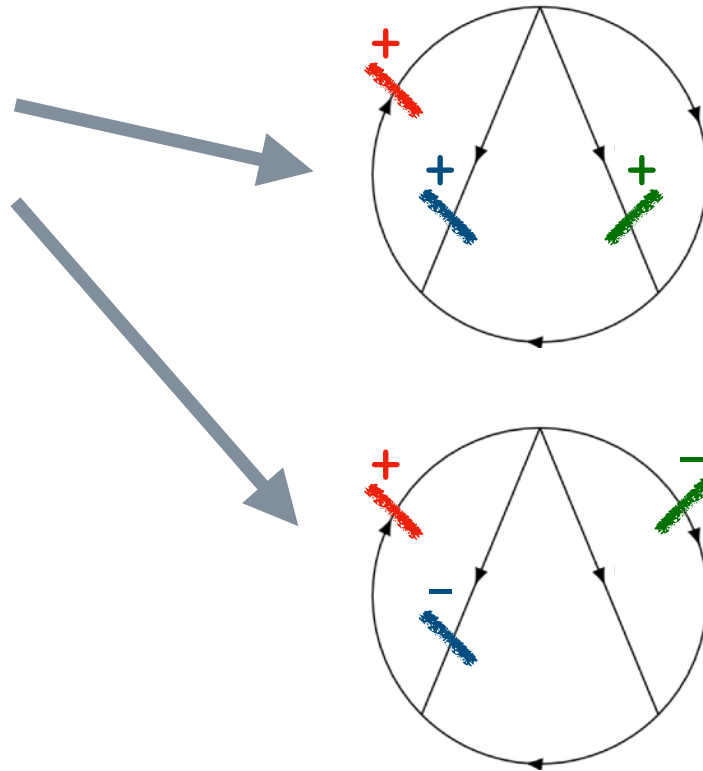


(b) Reduced diagram.

$$\begin{aligned} q_1 &= k_1 & \mathbf{s}_1 &= (1, 0, 0) \\ q_2 &= k_2 & \mathbf{s}_2 &= (0, 1, 0) \\ q_3 &= k_3 & \mathbf{s}_3 &= (0, 0, 1) \\ q_4 &= k_1 - k_2 & \mathbf{s}_4 &= (1, -1, 0) \\ q_5 &= k_1 - k_2 - k_3 & \mathbf{s}_5 &= (1, -1, -1) \end{aligned}$$

(c) Momenta and signatures.

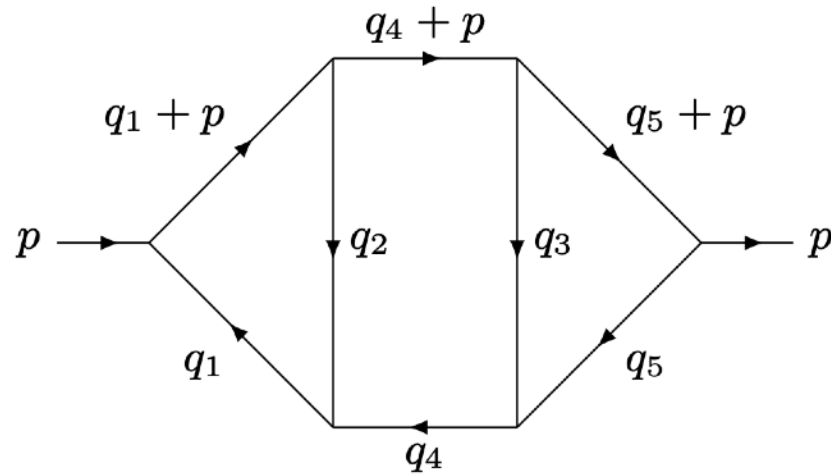
$$\sigma_j^b \equiv \begin{bmatrix} \Sigma_1 = (1, 1, 1, 0, 0), \\ \Sigma_2 = (1, -1, 0, 0, -1), \\ \Sigma_3 = (1, 0, 1, -1, 0), \\ \Sigma_4 = (1, 0, 1, 0, -1), \\ \Sigma_5 = (1, 0, 0, 1, -1), \\ \Sigma_6 = (0, 1, 1, 1, 0), \\ \Sigma_7 = (0, 1, 1, 0, 1), \\ \Sigma_8 = (0, 1, 0, 1, -1) \end{bmatrix}$$



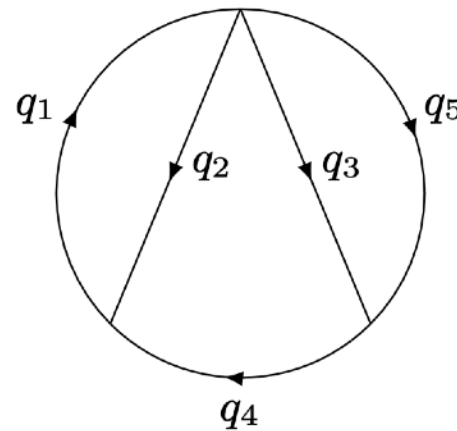
MULTI-LOOP TREE DUALITY

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

Three-loop example (showcasing the irrelevance of external momenta).



(a) Original diagram.

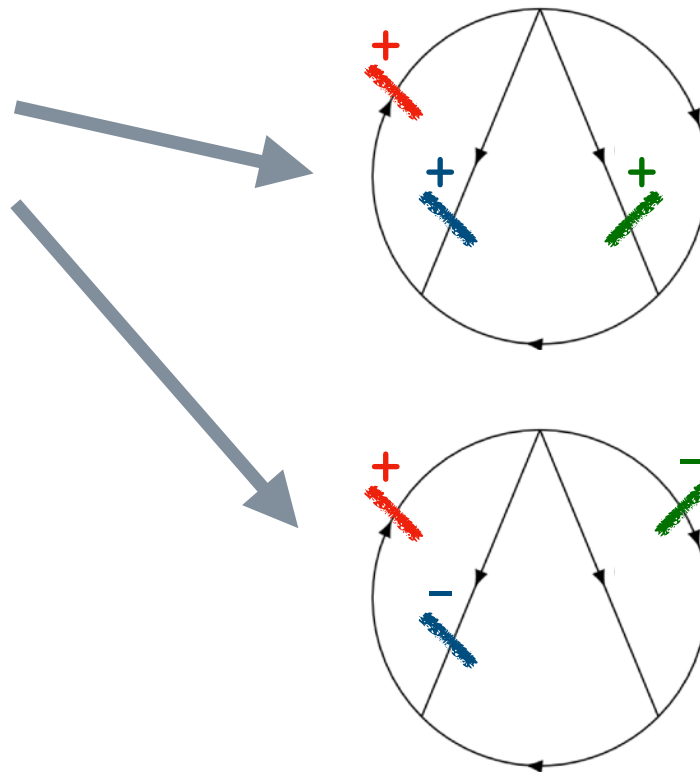


(b) Reduced diagram.

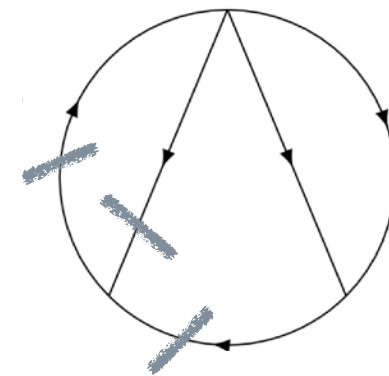
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$$\sigma_j^b \equiv \begin{bmatrix} \Sigma_1 = (1, 1, 1, 0, 0), \\ \Sigma_2 = (1, -1, 0, 0, -1), \\ \Sigma_3 = (1, 0, 1, -1, 0), \\ \Sigma_4 = (1, 0, 1, 0, -1), \\ \Sigma_5 = (1, 0, 0, 1, -1), \\ \Sigma_6 = (0, 1, 1, 1, 0), \\ \Sigma_7 = (0, 1, 1, 0, 1), \\ \Sigma_8 = (0, 1, 0, 1, -1) \end{bmatrix}$$

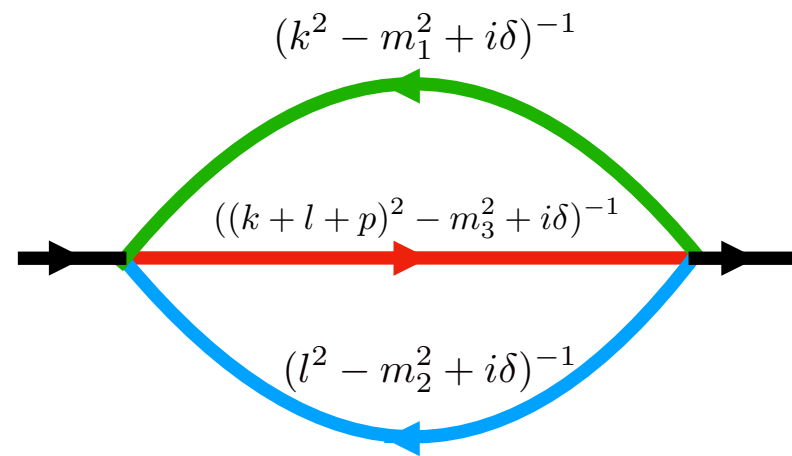


Ex. of an invalid spanning tree:



MULTI-LOOP TREE DUALITY: SINGULARITIES

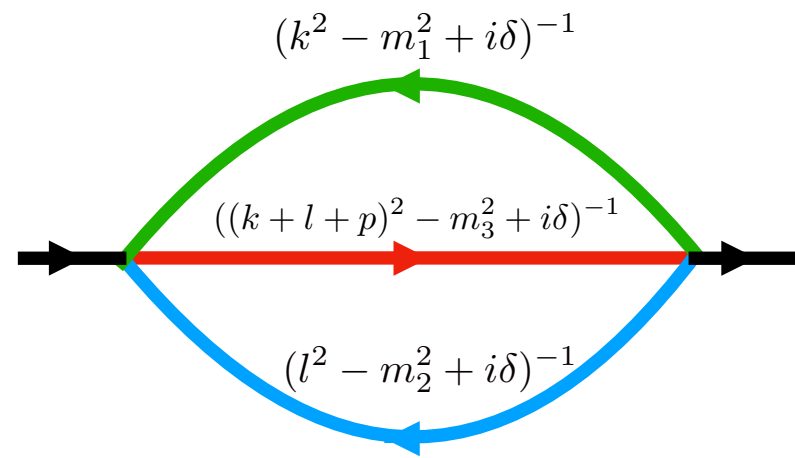
The sunrise again, with more details:



$$I = (2\pi i)^2 \left[\begin{array}{l} \text{R}(+1, +1, 0) \\ + \text{R}(-1, 0, +1) \\ + \text{R}(0, +1, +1) \end{array} \right]$$

MULTI-LOOP TREE DUALITY: SINGULARITIES

The sunrise again, with more details:

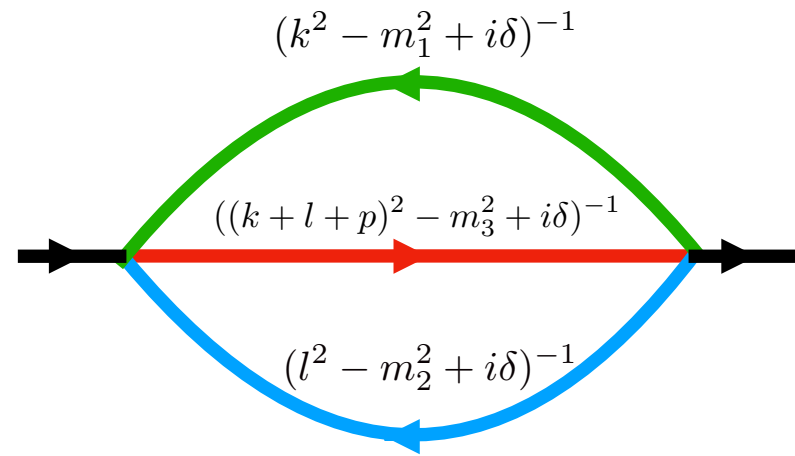


$$I = (2\pi i)^2 \left[\begin{array}{l} \mathbf{R}(+1, +1, 0) \\ + \mathbf{R}(-1, 0, +1) \\ + \mathbf{R}(0, +1, +1) \end{array} \right]$$

$$\mathbf{R}(+1, +1, 0) \stackrel{k^0 = \sqrt{\vec{k}^2 + m_1^2 - i\delta}}{=} \stackrel{l^0 = \sqrt{\vec{l}^2 + m_1^2 - i\delta}}{=} (-i)^2 \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{d^3 \vec{l}}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2 + m_1^2 - i\delta}} \frac{1}{2\sqrt{\vec{l}^2 + m_2^2 - i\delta}} \frac{1}{\Delta_3^+ \Delta_3^-}$$

MULTI-LOOP TREE DUALITY: SINGULARITIES

The sunrise again, with more details:



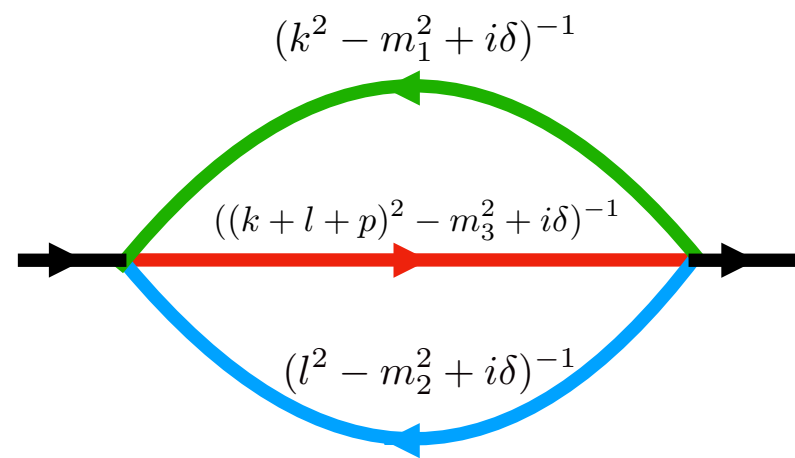
$$I = (2\pi i)^2 \left[\begin{array}{l} \mathbf{R}(+1, +1, 0) \\ + \mathbf{R}(-1, 0, +1) \\ + \mathbf{R}(0, +1, +1) \end{array} \right]$$

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$$\begin{aligned} \Delta_3^\pm(\vec{k}, \vec{l}) &= +k^0 + l^0 + p^0 \pm \sqrt{(\vec{k} + \vec{l} + \vec{p})^2 + m_3^2 - i\delta} \\ &= +\sqrt{\vec{k}^2 + m_1^2 - i\delta} + \sqrt{\vec{l}^2 + m_2^2 - i\delta} + p^0 \pm \sqrt{(\vec{k} + \vec{l} + \vec{p})^2 + m_3^2 - i\delta} \end{aligned}$$

MULTI-LOOP TREE DUALITY: SINGULARITIES

The sunrise again, with more details:



$$I = (2\pi i)^2 \left[\begin{array}{l} \mathbf{R}(+1, +1, 0) \\ + \mathbf{R}(-1, 0, +1) \\ + \mathbf{R}(0, +1, +1) \end{array} \right]$$

$$\mathbf{R}(+1, +1, 0) \stackrel{k^0 = \sqrt{\vec{k}^2 + m_1^2 - i\delta}}{=} \stackrel{l^0 = \sqrt{\vec{l}^2 + m_2^2 - i\delta}}{=} (-i)^2 \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{d^3 \vec{l}}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2 + m_1^2 - i\delta}} \frac{1}{2\sqrt{\vec{l}^2 + m_2^2 - i\delta}} \frac{1}{\Delta_3^+ \Delta_3^-}$$

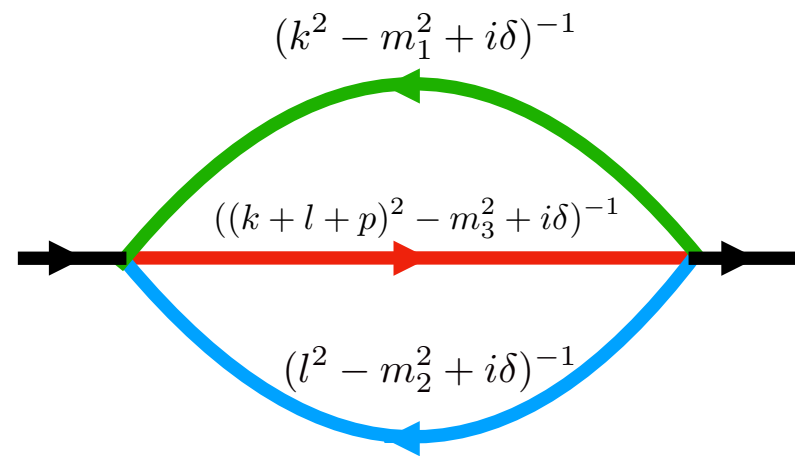
$$\Delta_3^\pm(\vec{k}, \vec{l}) = +k^0 + l^0 + p^0 \pm \sqrt{(\vec{k} + \vec{l} + \vec{p})^2 + m_3^2 - i\delta}$$

$$= \textcircled{+} \sqrt{\vec{k}^2 + m_2^2 - i\delta} \textcircled{+} \sqrt{\vec{l}^2 + m_2^2 - i\delta} + p^0 \textcircled{\pm} \sqrt{(\vec{k} + \vec{l} + \vec{p})^2 + m_3^2 - i\delta}$$

identical signs : **bounded** ellipsoid-like surface **requiring deformation**: **E-surfaces**
otherwise : **unbounded** hyperboloid-like surface **dual-cancelling**: **H-surfaces**

MULTI-LOOP TREE DUALITY: SINGULARITIES

The sunrise again, with more details:



$$I = (2\pi i)^2 \left[\begin{array}{l} \mathbf{R}(+1, +1, 0) \\ + \mathbf{R}(-1, 0, +1) \\ + \mathbf{R}(0, +1, +1) \end{array} \right]$$

$$\mathbf{R}(+1, +1, 0) \stackrel{k^0 = \sqrt{\vec{k}^2 + m_1^2 - i\delta}, l^0 = \sqrt{\vec{l}^2 + m_2^2 - i\delta}}{=} (-i)^2 \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{d^3 \vec{l}}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2 + m_1^2 - i\delta}} \frac{1}{2\sqrt{\vec{l}^2 + m_2^2 - i\delta}} \frac{1}{\Delta_3^+ \Delta_3^-}$$

$$\Delta_3^\pm(\vec{k}, \vec{l}) = +k^0 + l^0 + p^0 \pm \sqrt{(\vec{k} + \vec{l} + \vec{p})^2 + m_3^2 - i\delta}$$

$$= \textcircled{+} \sqrt{\vec{k}^2 + m_1^2 - i\delta} \textcircled{+} \sqrt{\vec{l}^2 + m_2^2 - i\delta} + p^0 \textcircled{\pm} \sqrt{(\vec{k} + \vec{l} + \vec{p})^2 + m_3^2 - i\delta}$$

identical signs : **bounded** ellipsoid-like surface **requiring deformation**: **E-surfaces**
otherwise : **unbounded** hyperboloid-like surface **dual-cancelling**: **H-surfaces**

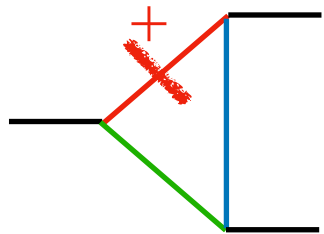
For **Euclidean** kinematics ($p_i^2 < 0$) : (on-shell energies: $E_i(\vec{k}_i) = \sqrt{\vec{k}_i^2 + m_i^2 - i\delta}$)

E-surface has **no solution** $\rightarrow$ **no** non-integrable **singularity** $\rightarrow$ **no deformation** needed

MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

The **cLTD** representation is closely related to **Time Ordered Perturbation Theory**



$$\frac{1}{2E_1}$$

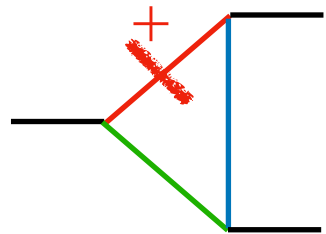
$$\frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2}$$

$$\frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

MULTI-LOOP TREE DUALITY: cLTD

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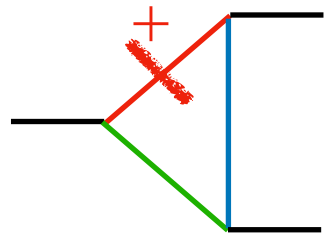


$$\begin{aligned}
 & \frac{1}{2E_1} \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2} \\
 &= \frac{1}{2E_2} \left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2} \right)
 \end{aligned}$$

MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

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$$\frac{1}{2E_1}$$

$$= \frac{1}{2E_3} \left(\frac{1}{E_1 - p_1^0 + p_3^0 - E_3} - \frac{1}{E_1 - p_1^0 + p_3^0 + E_3} \right)$$

$$\frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2}$$

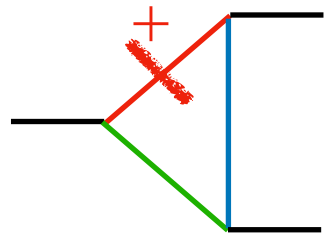
$$\frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_2} \left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2} \right)$$

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$$= \frac{1}{2E_3} \left(\frac{1}{E_1 - p_1^0 + p_3^0 - E_3} - \frac{1}{E_1 - p_1^0 + p_3^0 + E_3} \right)$$

$$\frac{1}{2E_1} \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

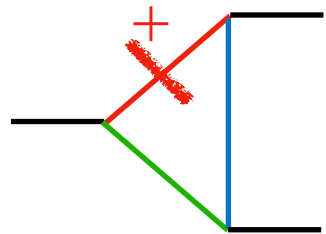
$$= \frac{1}{2E_2} \left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2} \right)$$

$$\supset \frac{1}{2E_1}$$

MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

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$$= \frac{1}{2E_3} \left(\frac{1}{E_1 - p_1^0 + p_3^0 - E_3} - \frac{1}{E_1 - p_1^0 + p_3^0 + E_3} \right)$$

$$\frac{1}{2E_1} \left(\frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} - \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2} \right)$$

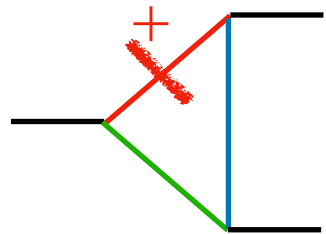
$$= \frac{1}{2E_2} \left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2} \right)$$

$$\supset \frac{1}{2E_1} \frac{1}{2E_2} \frac{-1}{E_1 - p_1^0 + p_2^0 + E_2}$$

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$$= \frac{1}{2E_3} \left(\frac{1}{E_1 - p_1^0 + p_3^0 - E_3} - \frac{1}{E_1 - p_1^0 + p_3^0 + E_3} \right)$$

$$\frac{1}{2E_1} \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

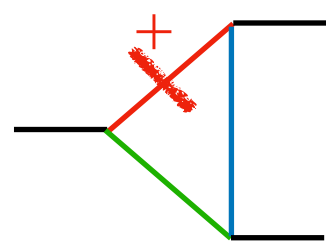
$$= \frac{1}{2E_2} \left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2} \right)$$

$$\supset \frac{1}{2E_1} \frac{1}{2E_2} \frac{1}{2E_3} \frac{-1}{E_1 - p_1^0 + p_2^0 + E_2} \frac{-1}{E_1 - p_1^0 + p_3^0 + E_3}$$

MULTI-LOOP TREE DUALITY: cLTD

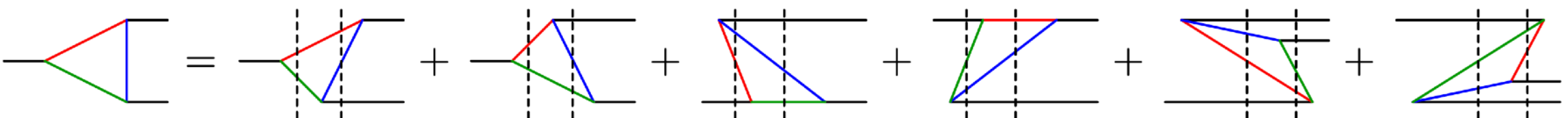
[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

The **cLTD** representation is closely related to **Time Ordered Perturbation Theory**



$$\begin{aligned}
 &= \frac{1}{2E_3} \left(\frac{1}{E_1 - p_1^0 + p_3^0 - E_3} - \frac{1}{E_1 - p_1^0 + p_3^0 + E_3} \right) \\
 &\quad \frac{1}{2E_1} \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2} \\
 &= \frac{1}{2E_2} \left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2} \right) \\
 &\supset \frac{1}{2E_1} \frac{1}{2E_2} \frac{1}{2E_3} \frac{-1}{E_1 - p_1^0 + p_2^0 + E_2} \frac{-1}{E_1 - p_1^0 + p_3^0 + E_3}
 \end{aligned}$$

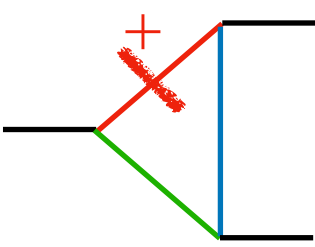
TOPT :



MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

The **cLTD** representation is closely related to **Time Ordered Perturbation Theory**



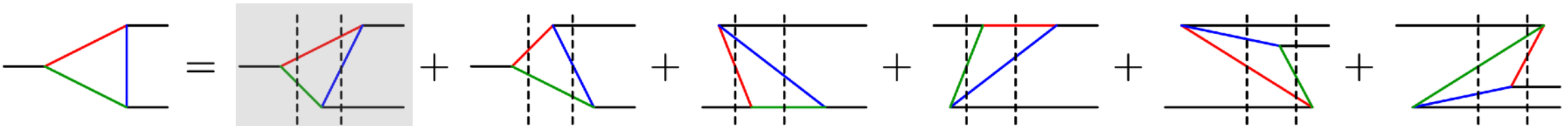
$$= \frac{1}{2E_3} \left(\frac{1}{E_1 - p_1^0 + p_3^0 - E_3} - \frac{1}{E_1 - p_1^0 + p_3^0 + E_3} \right)$$

$$\frac{1}{2E_1} \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_2} \left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2} \right)$$

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TOPT :



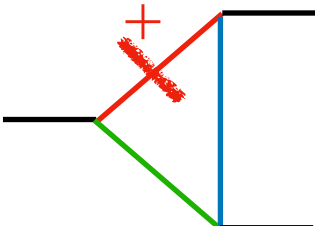
$$= \frac{1}{2E_1} \frac{1}{2E_2} \frac{1}{2E_3} \frac{-1}{-q_1^0 + q_3^0 + (E_1 + E_2)} \frac{-1}{-q_1^0 + (E_1 + E_3)}$$

Identical !

MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

The **cLTD** representation is closely related to **Time Ordered Perturbation Theory**



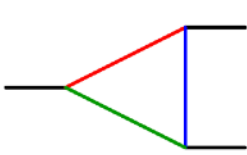
$$= \frac{1}{2E_3} \left(\frac{1}{E_1 - p_1^0 + p_3^0 - E_3} - \frac{1}{E_1 - p_1^0 + p_3^0 + E_3} \right)$$

$$\frac{1}{2E_1} \frac{1}{(E_1 - p_1^0 + p_2^0)^2 - E_2^2} \frac{1}{(E_1 - p_1^0 + p_3^0)^2 - E_3^2}$$

$$= \frac{1}{2E_2} \left(\frac{1}{(E_1 - p_1^0 + p_2^0) - E_2} - \frac{1}{(E_1 - p_1^0 + p_2^0) + E_2} \right)$$

$$\supset \frac{1}{2E_1} \frac{1}{2E_2} \frac{1}{2E_3} \frac{-1}{E_1 - p_1^0 + p_2^0 + E_2} \frac{-1}{E_1 - p_1^0 + p_3^0 + E_3}$$

TOPT :



$$= \frac{1}{2E_1} \frac{1}{2E_2} \frac{1}{2E_3} \frac{-1}{-q_1^0 + q_3^0 + (E_1 + E_2)} \frac{-1}{-q_1^0 + (E_1 + E_3)}$$

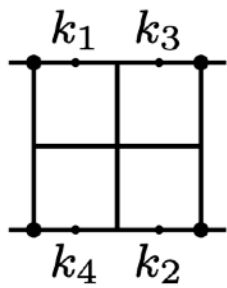
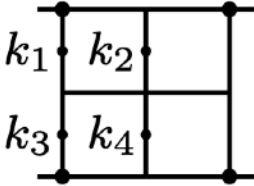
Identical !

In general however, **no one-to-one** correspondence. **cLTD** has fewer terms.

MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

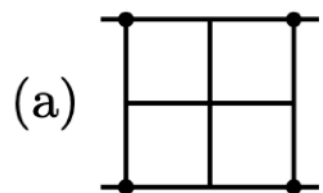
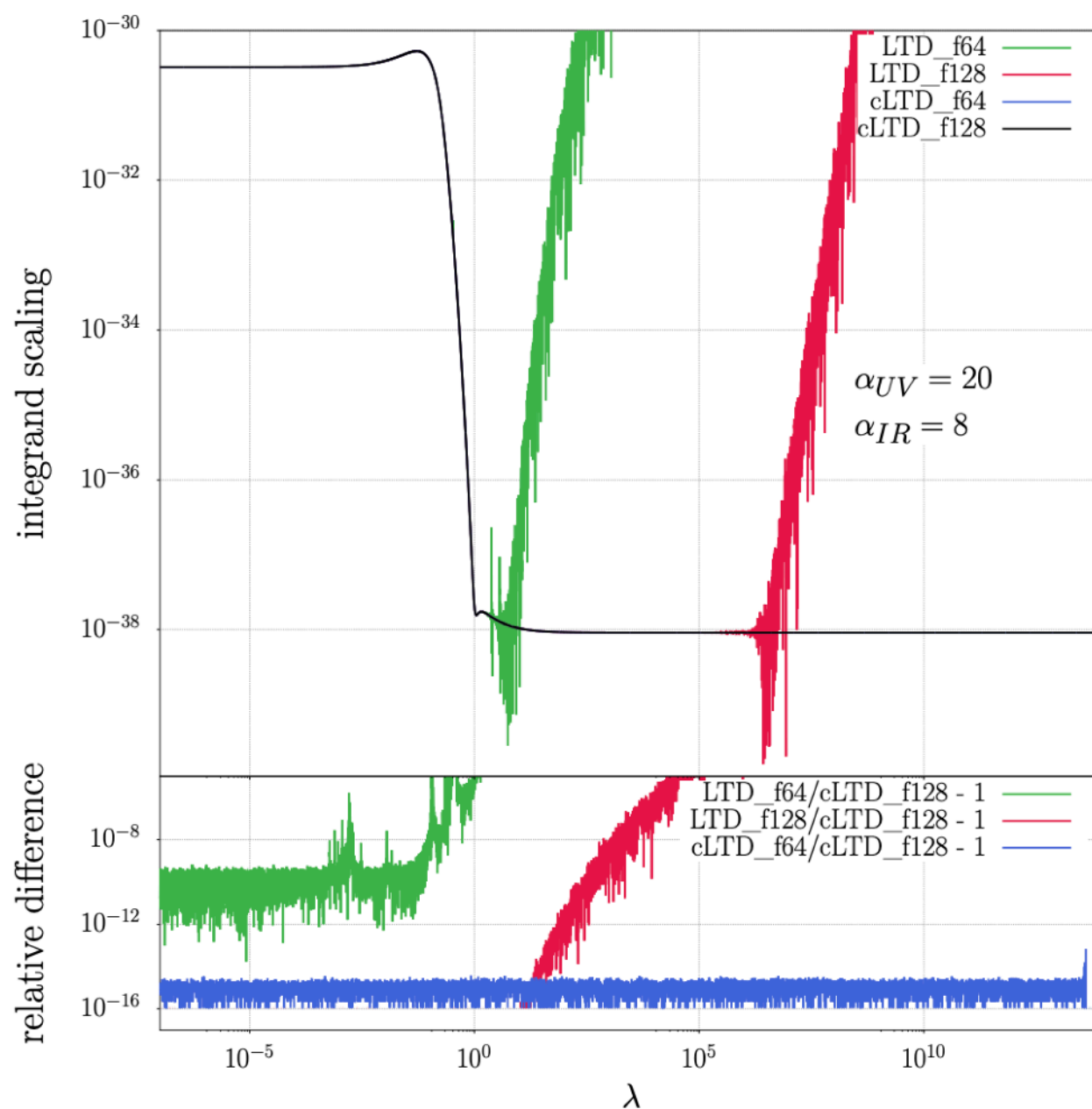
We give a completely general and **algorithmic** construction of **cLTD**:

Loop momentum basis	Input		$n_{\text{opt}}=0$		$n_{\text{opt}}=1000$			
	signatures σ	n_{prop}	N_+	$N_\times$	$t_{[\text{min}]}^{(\text{gen})}$	N_+	$N_\times$	$t_{[\mu\text{s}]}^{(\text{run})}$
	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$	22'851	159'964	10.2	3'515	3'520	7.5
	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$	85'535	598'752	154.7	21'351	20'944	87

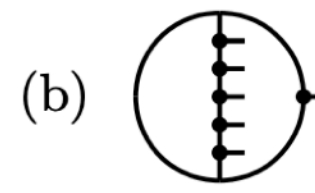
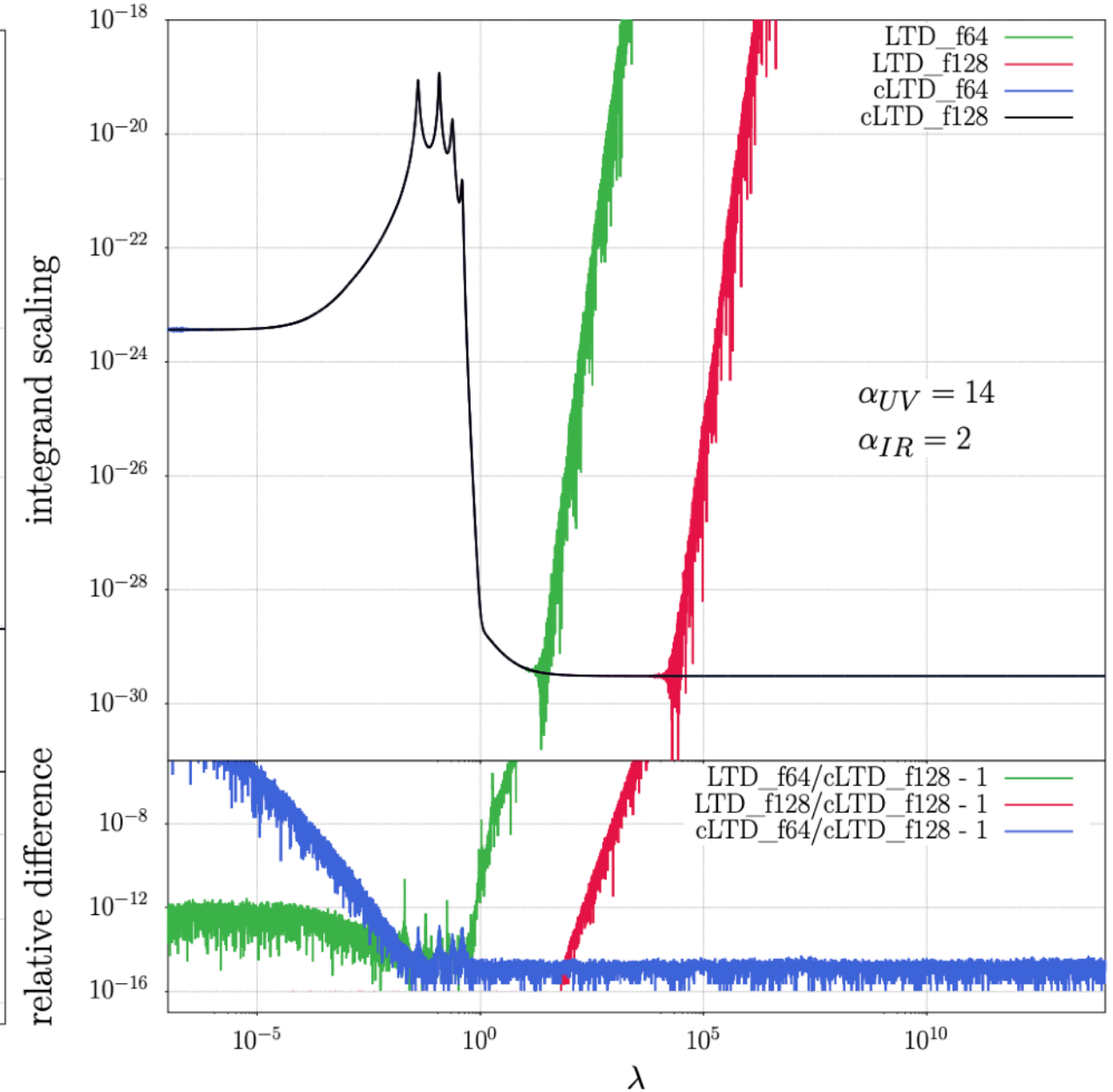
MULTI-LOOP TREE DUALITY: cLTD

[Capatti, VH, Pelloni, Kermanschah, Ruijl, arxiv:2009.05509]

cLTD completely cures **numerical stability** issue in the **UV** region:



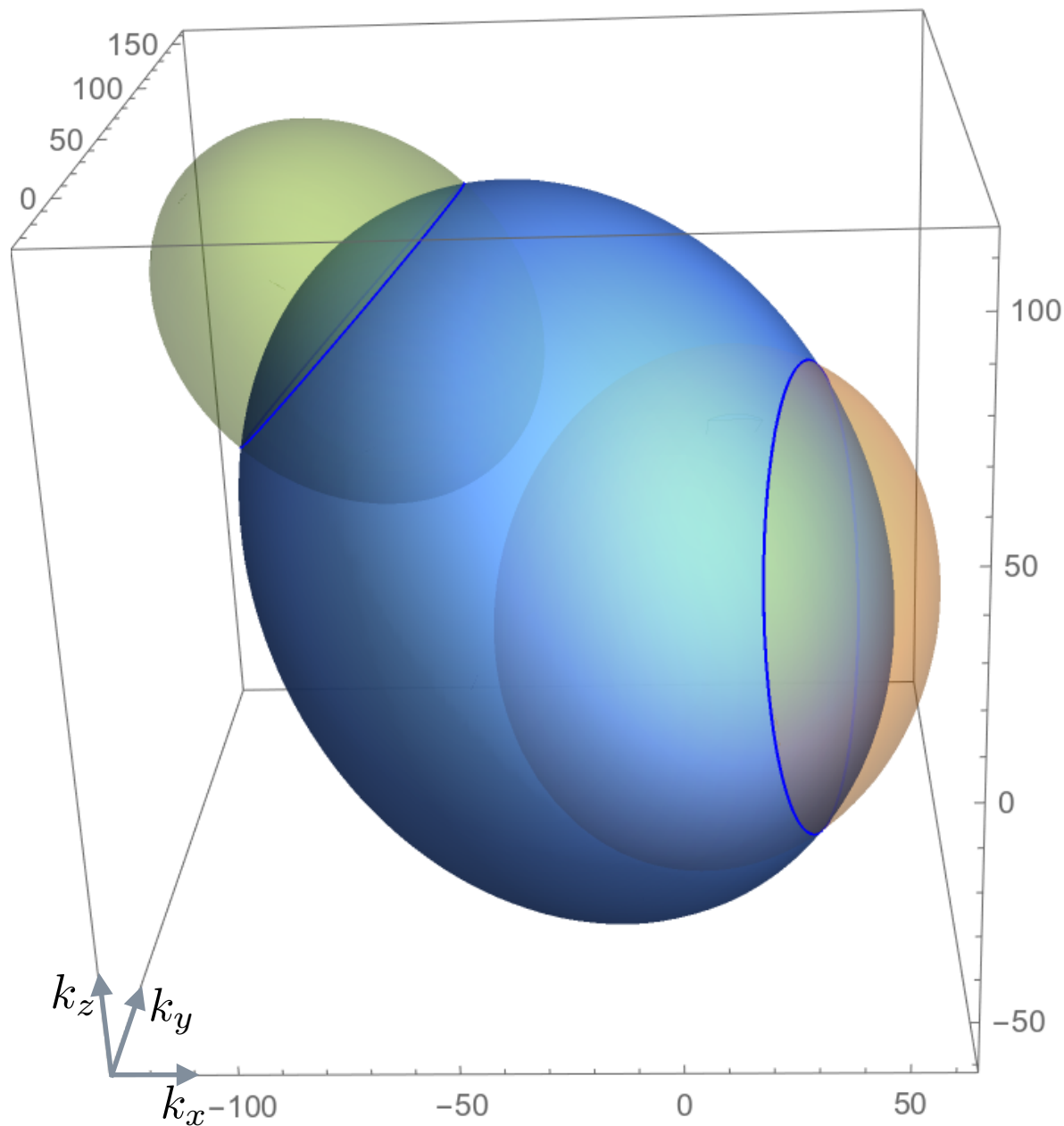
Stability of the 2x2 fishnet 4-loop integral with a constant numerator set to 1.



Stability of a 2-loop 6-point integral with the rank-2 numerator $(k_1 + k_2) \cdot p_1 + k_1 \cdot k_2$.

THRESHOLD REGULARISATION

SINGULAR SURFACES - 2D ELLIPSOIDS



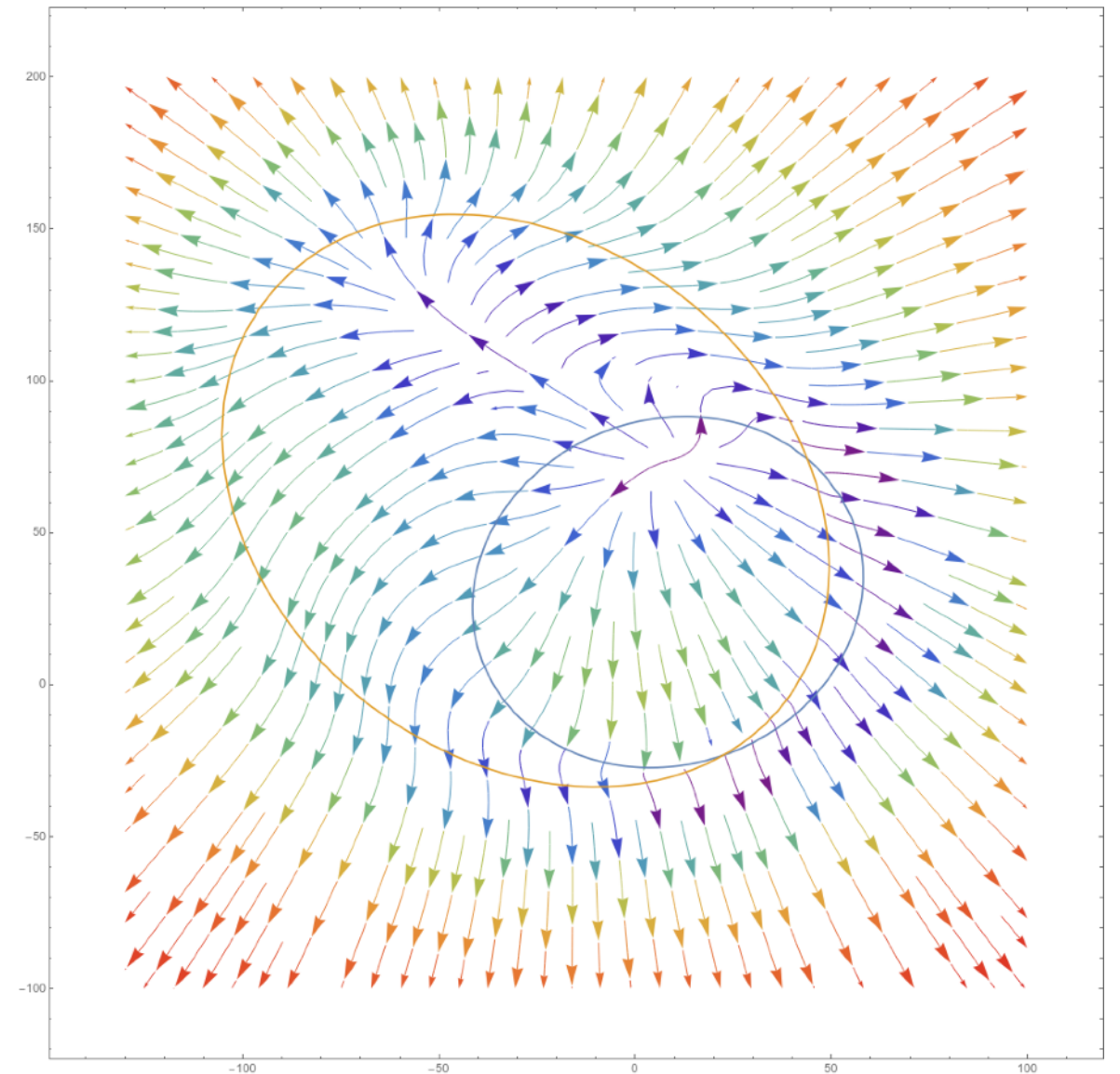
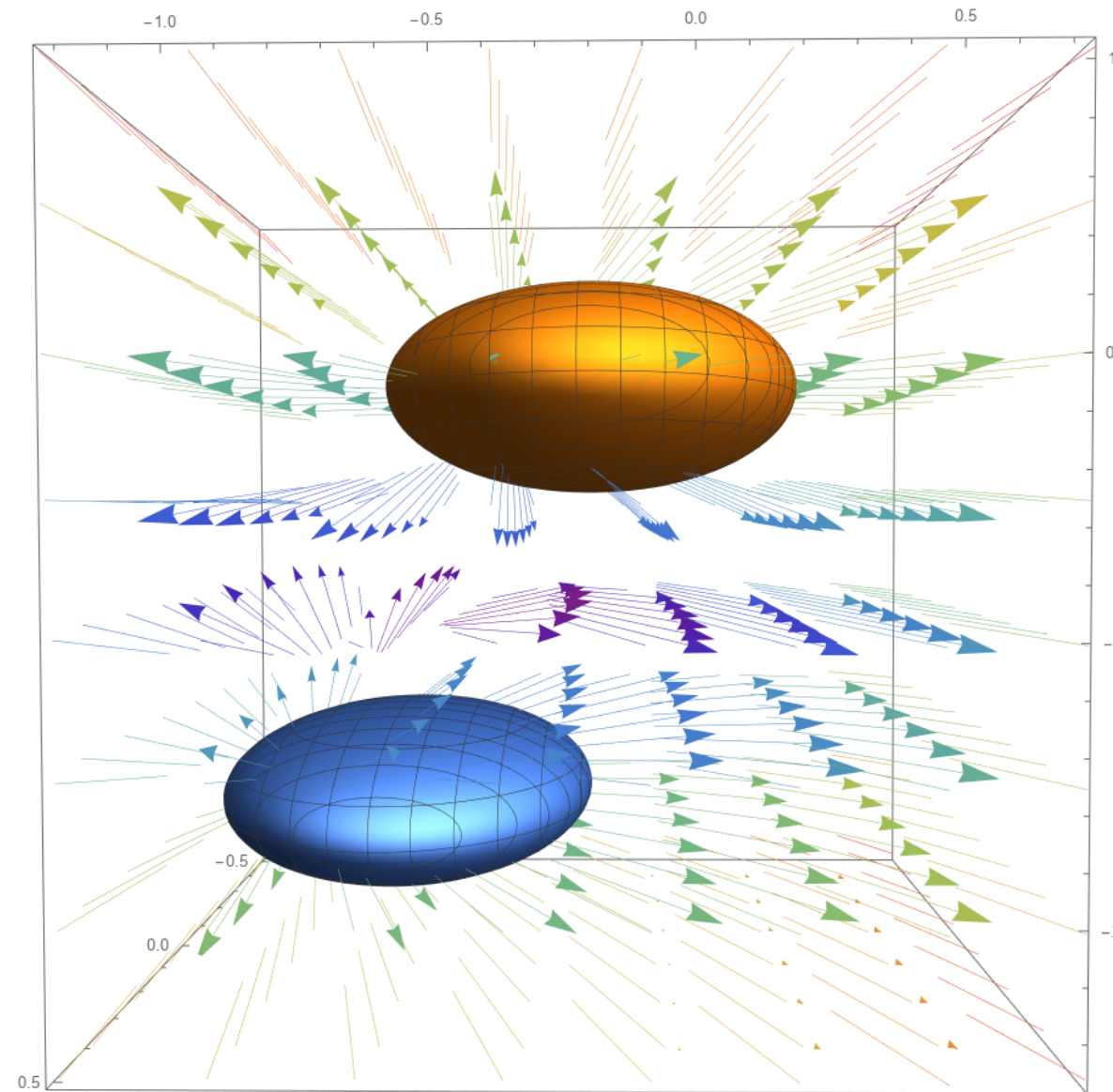
► General **one-loop** ellipsoid **expression**:

$$E_{ij}(\vec{k}) = \sqrt{\left(\vec{k} + \vec{p}_i\right)^2 + m_i^2 - i\delta} + \sqrt{\left(\vec{k} + \vec{p}_j\right)^2 + m_j^2 - i\delta} - p_i^0 + p_j^0$$

DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

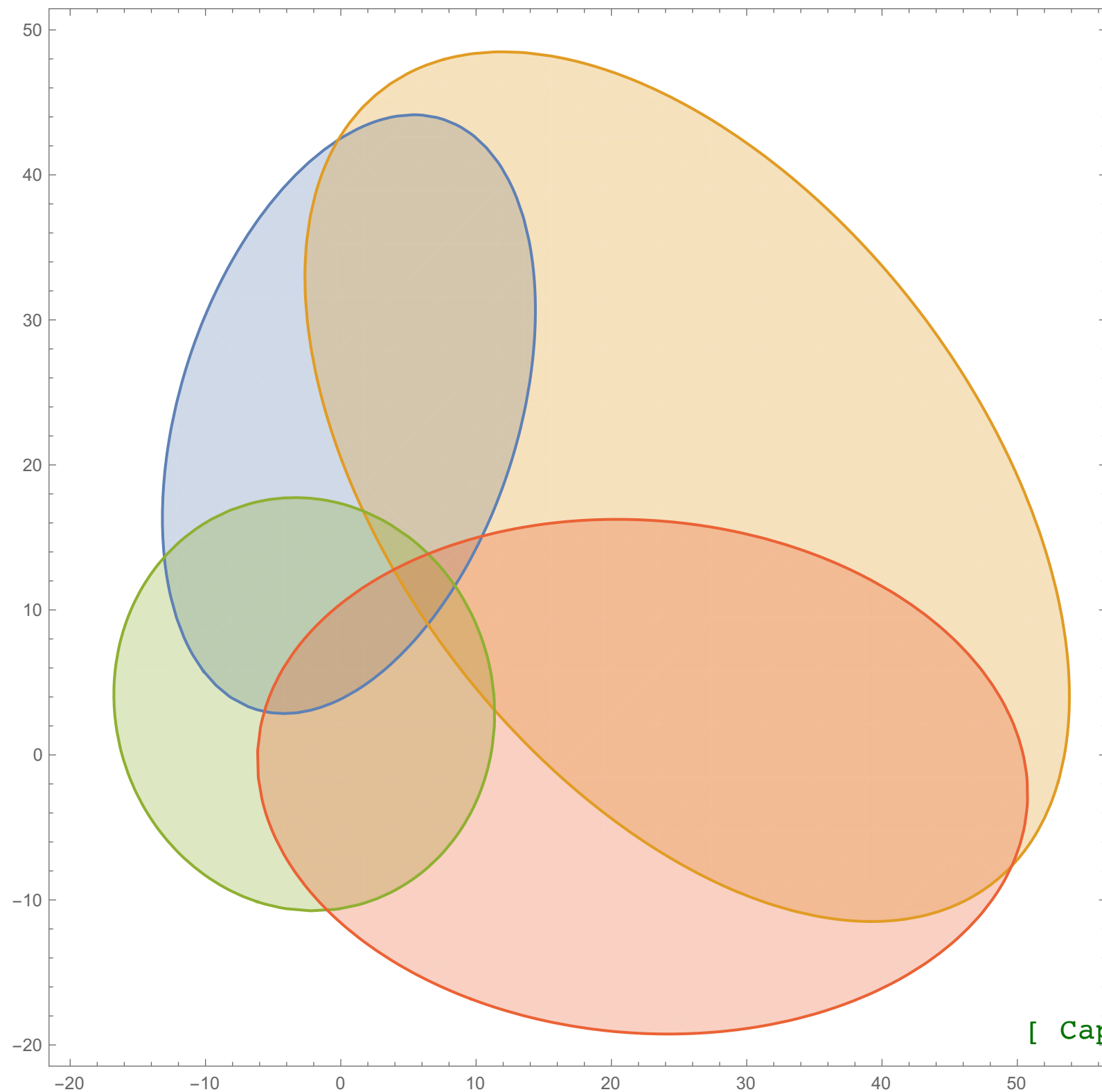
Deformation: $\vec{k} \rightarrow \vec{k} - i\vec{\kappa}$

Causal prescription imposes: $\vec{\kappa} \cdot \vec{n}_{E_{ij}} > 0$



DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

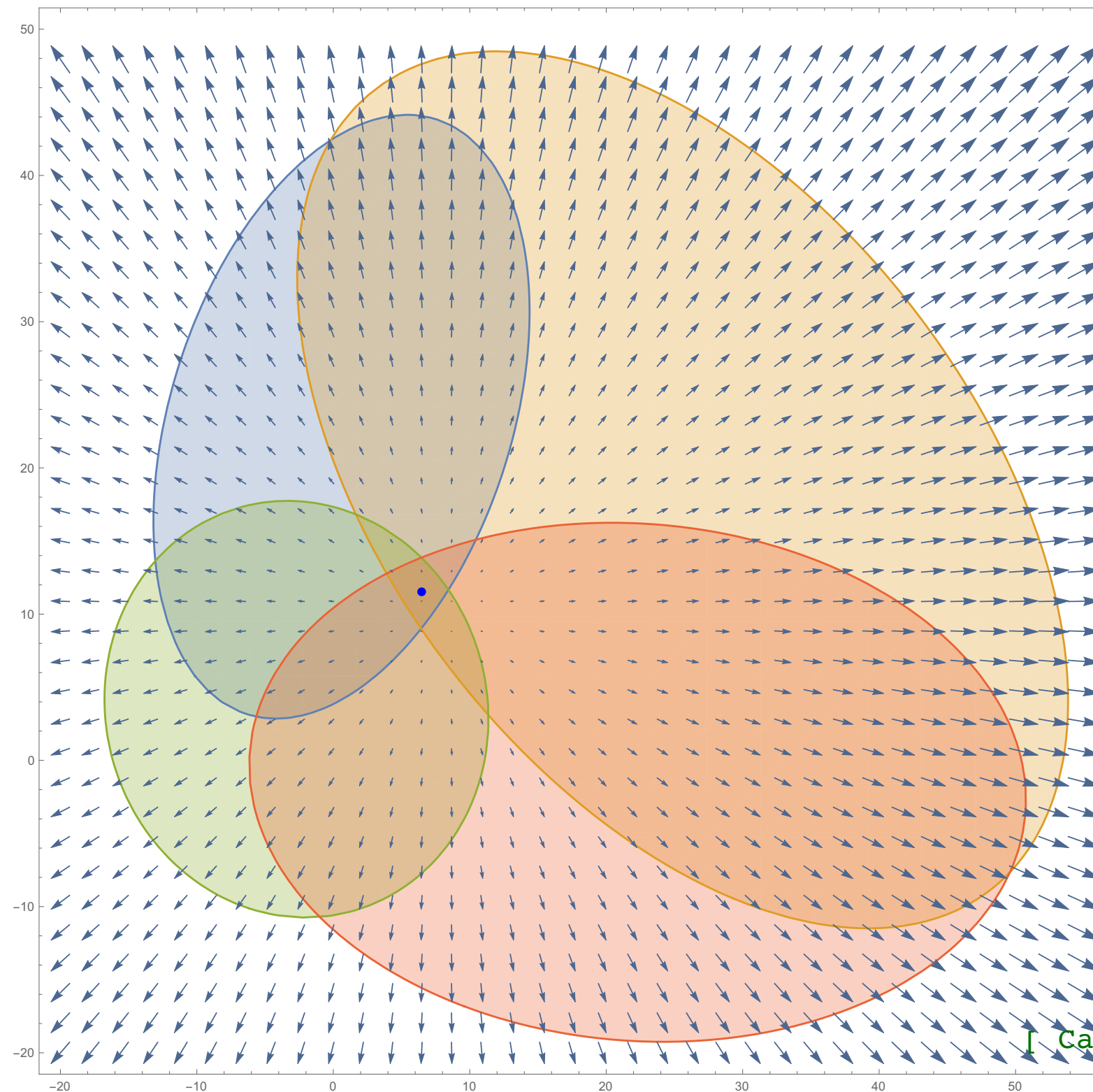
How to construct such a field? For example for this case:



[Capatti, VH, Kermanschah, Pelloni, Ru
[arxiv:1906.06138]

DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

How to construct such a field? For example for this case:

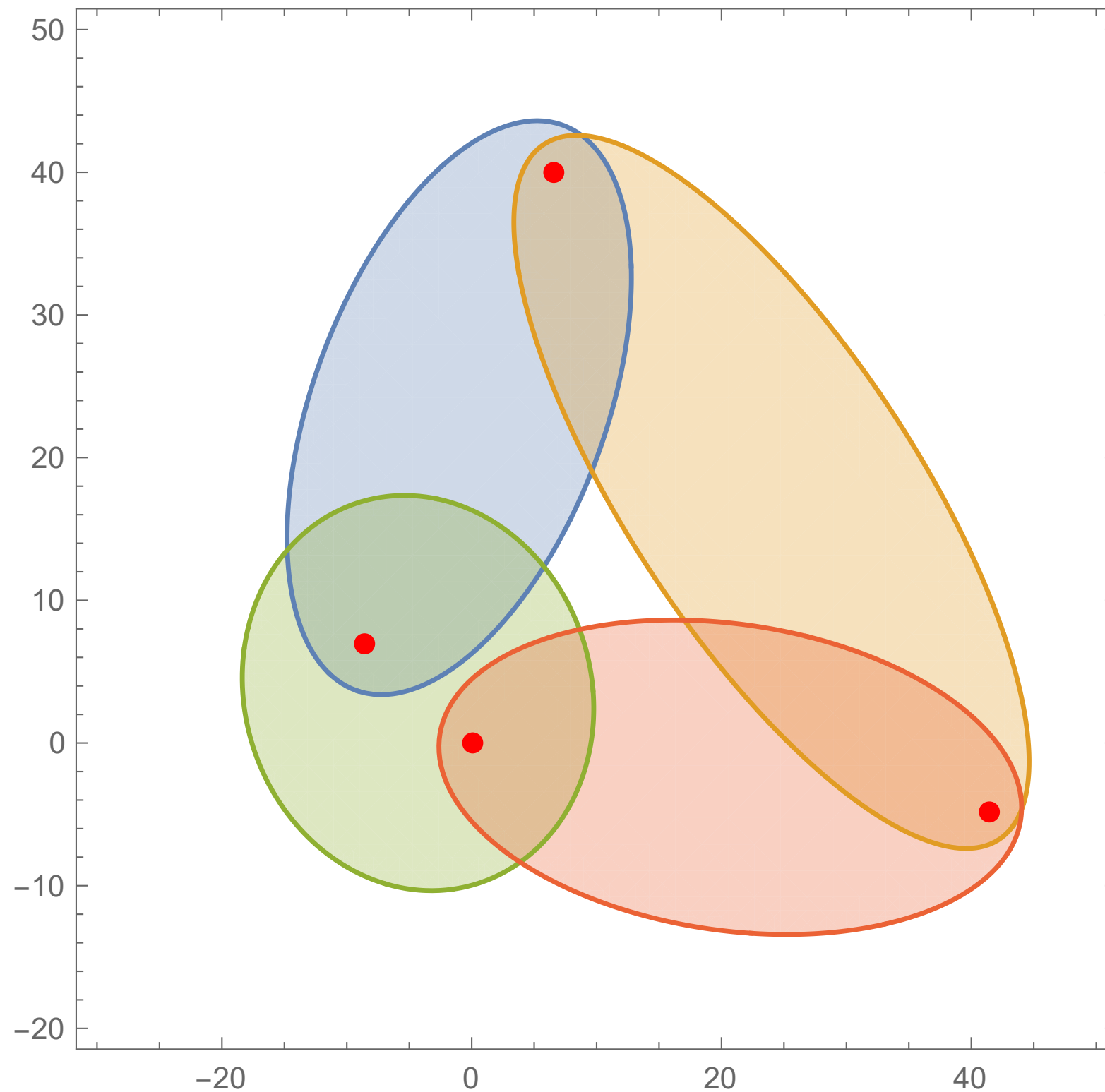


A **radial field** centered in
the inside of all ellipsoids!

[Capatti, VH, Kermanschah, Pelloni, Ru
[[arxiv:1906.06138](https://arxiv.org/abs/1906.06138)]

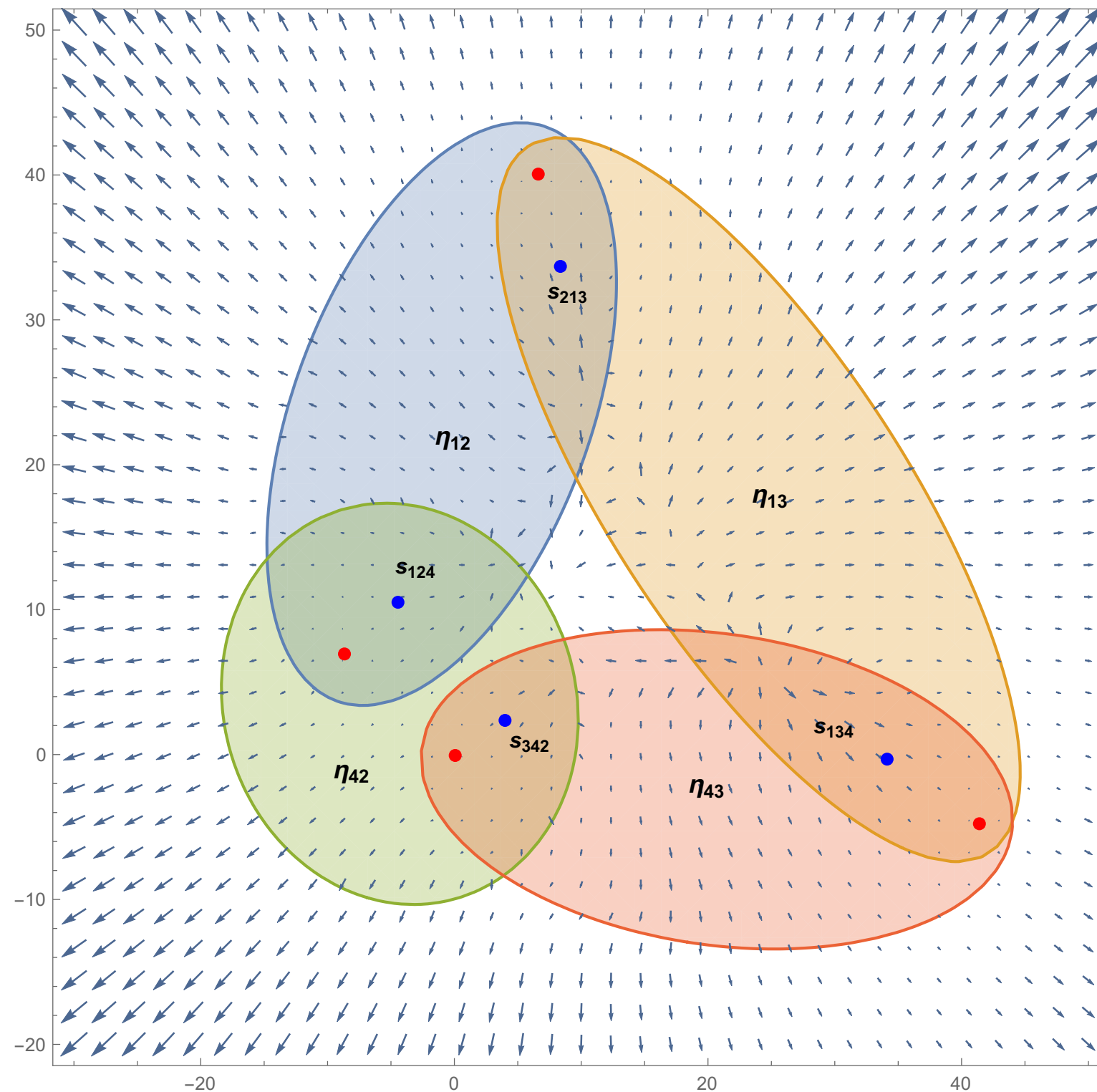
DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

But then what if there is no point in the inside of **all ellipsoids** (Box4E example) ?



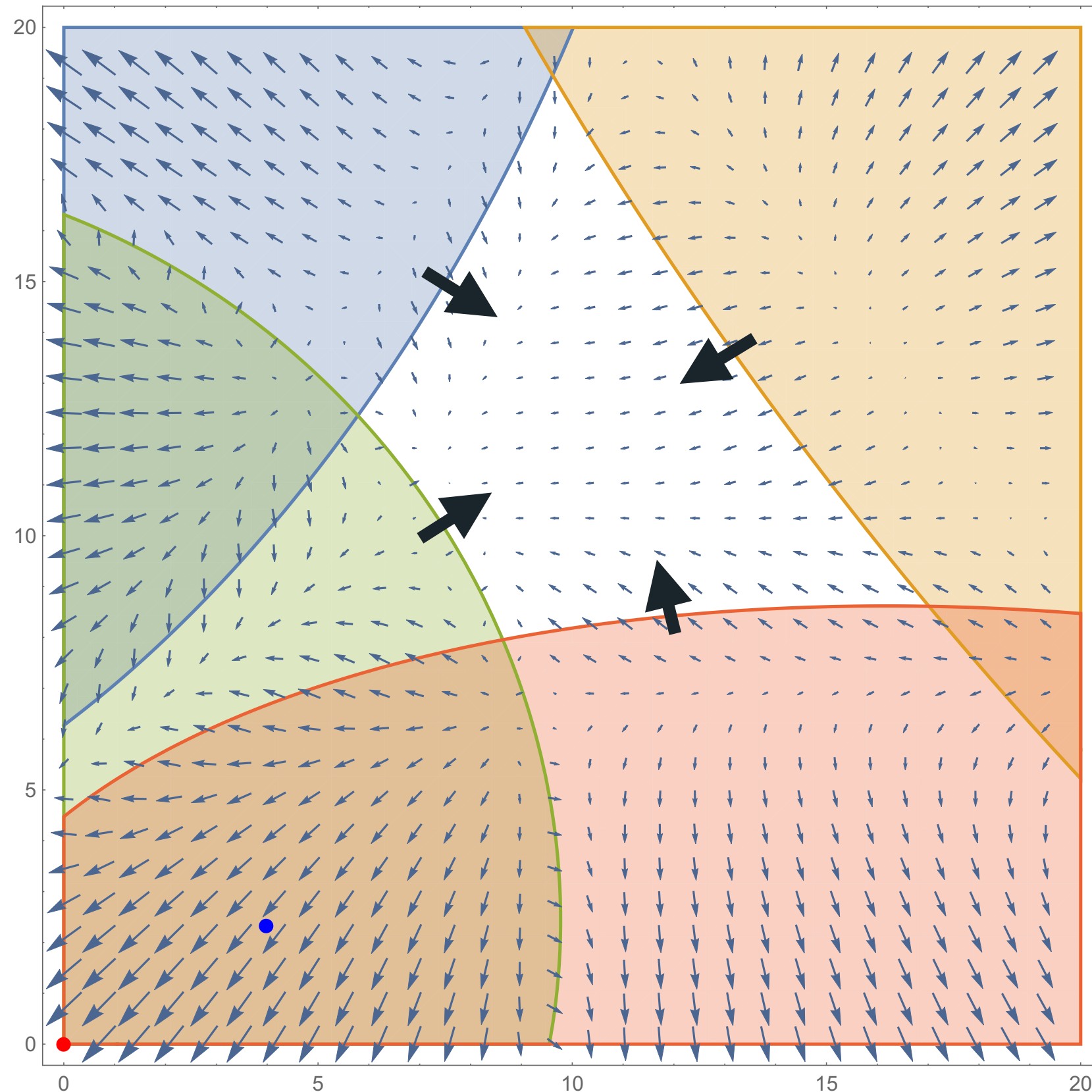
DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

But then what if there is no point in the inside of **all ellipsoids** (Box4E example) ?



DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

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THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kermanschah, arXiv : [2110.06869](https://arxiv.org/abs/2110.06869)]

$$\frac{1}{E_1 + E_2 - p_1^0} = \frac{1}{|\vec{k}| + |\vec{k} - \vec{p}_1| - p_1^0}$$
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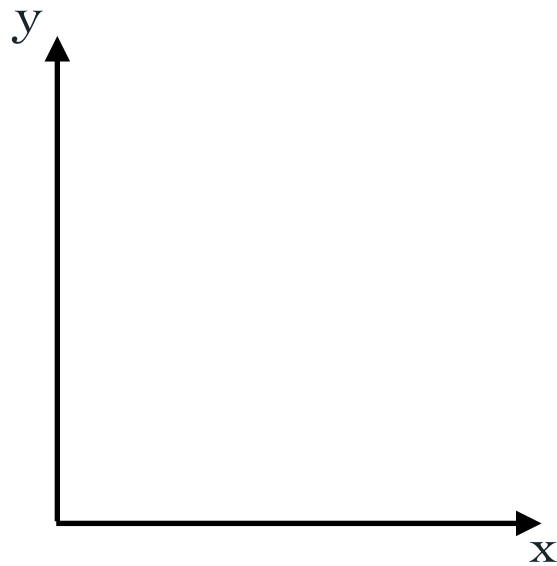
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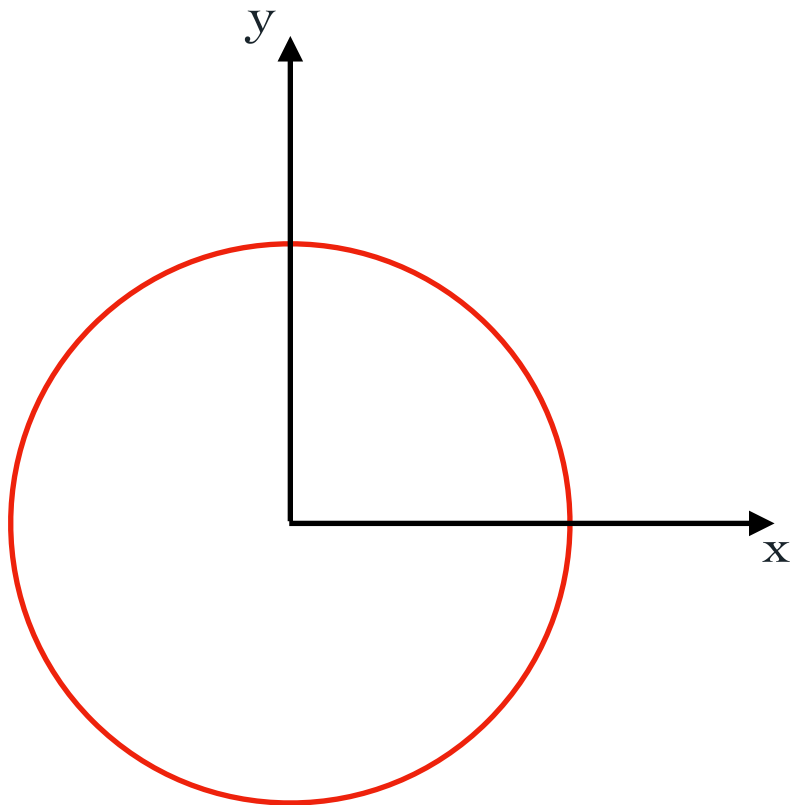
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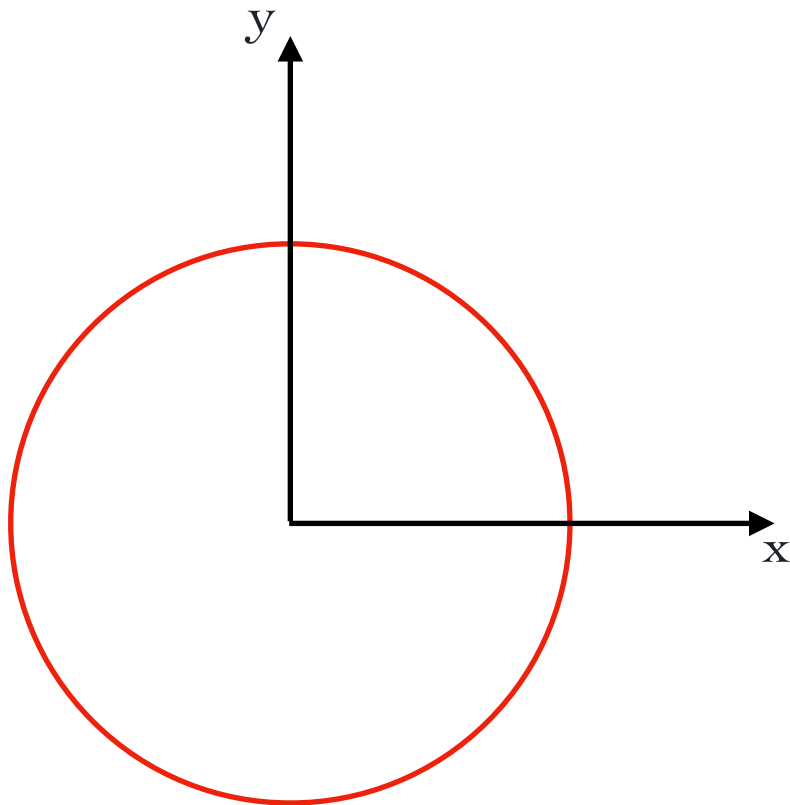
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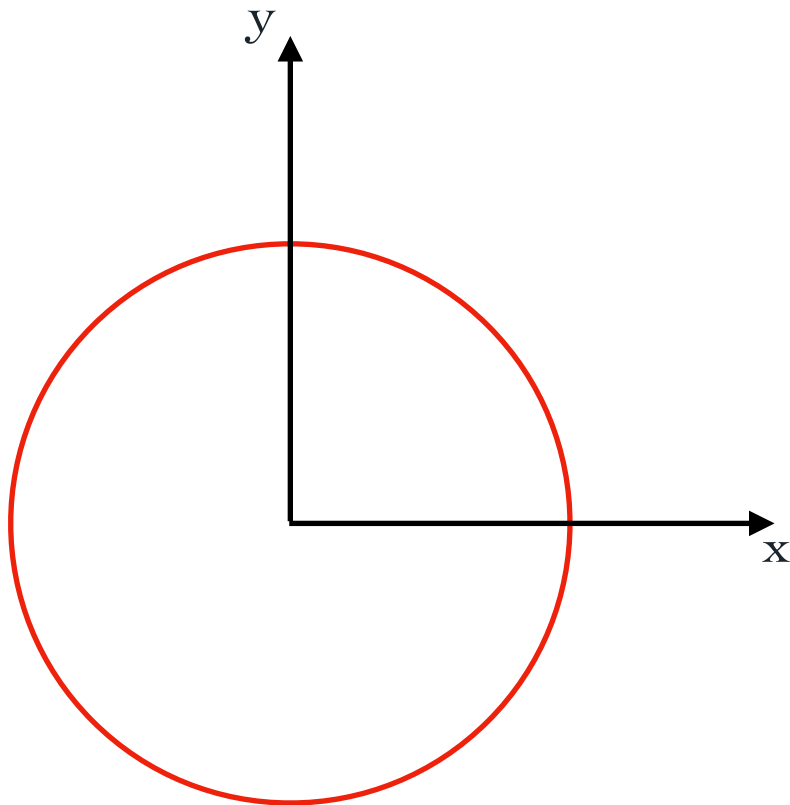
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Local threshold counterterm

$$= \lim_{\delta \rightarrow 0^+} \frac{4}{\pi} \left[\int_0^\infty dr \left(\frac{1}{r^2 + 1} \right) \frac{1}{1 - r \pm i\delta} - \int_{1-\Delta}^{1+\Delta} dr \left(\frac{1}{(1)^2 + 1} \right) \frac{1}{1 - r \pm i\delta} \right]$$



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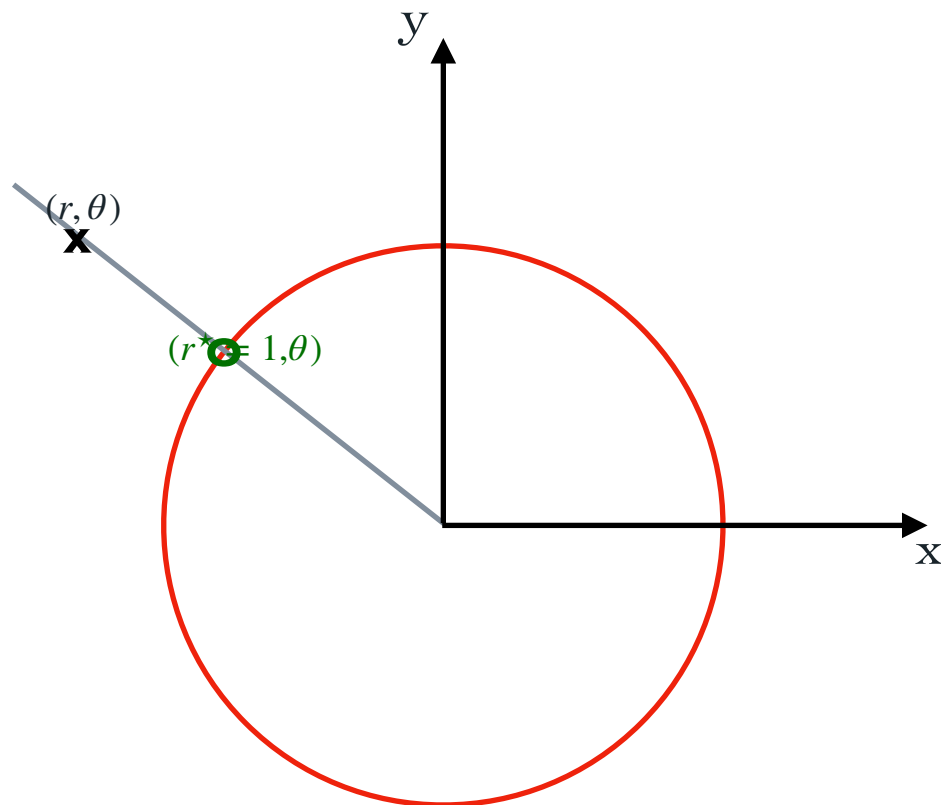
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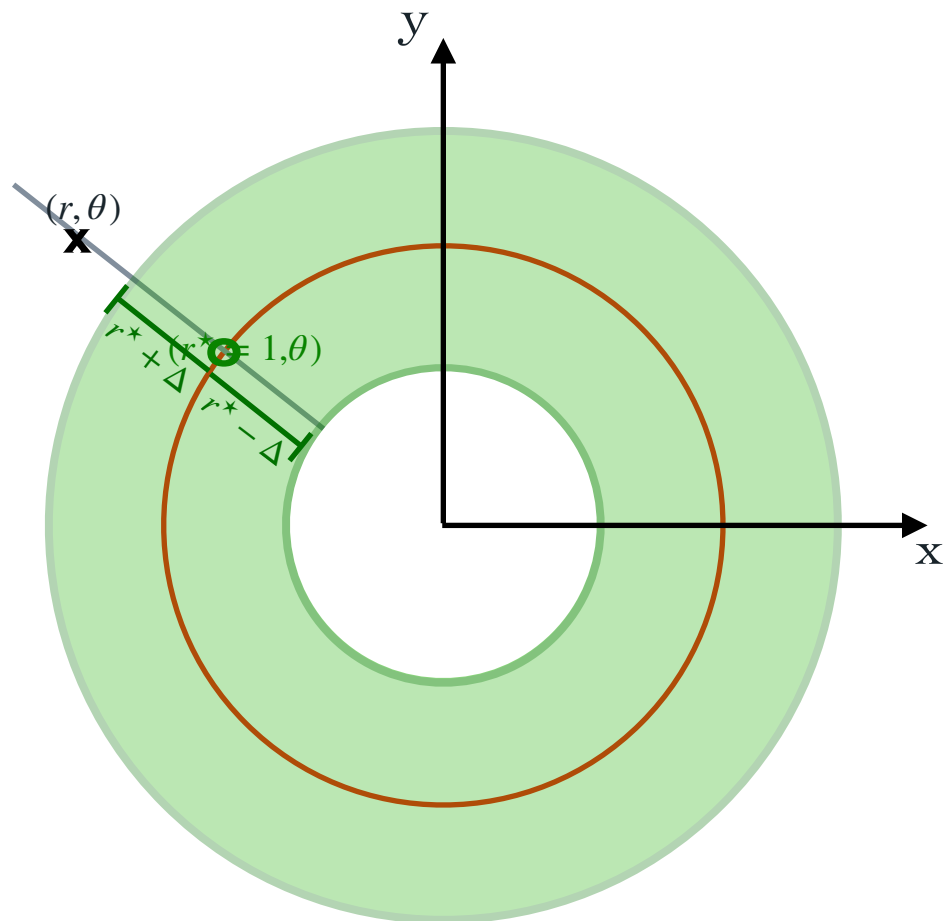
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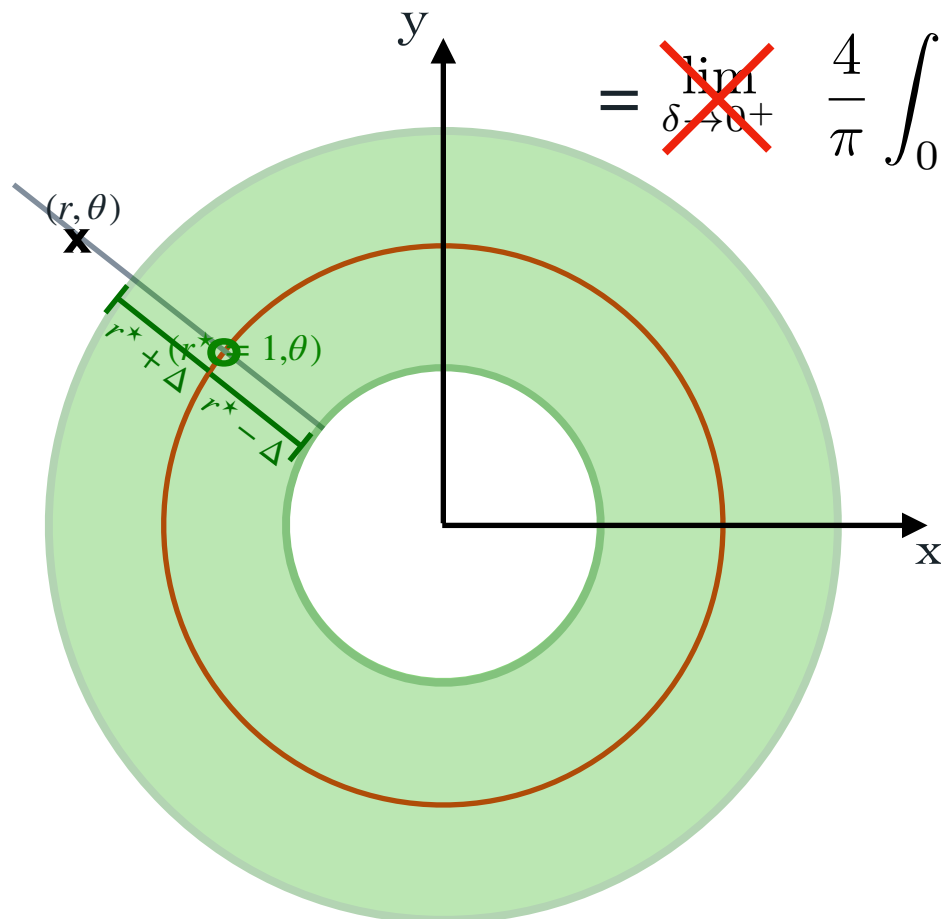
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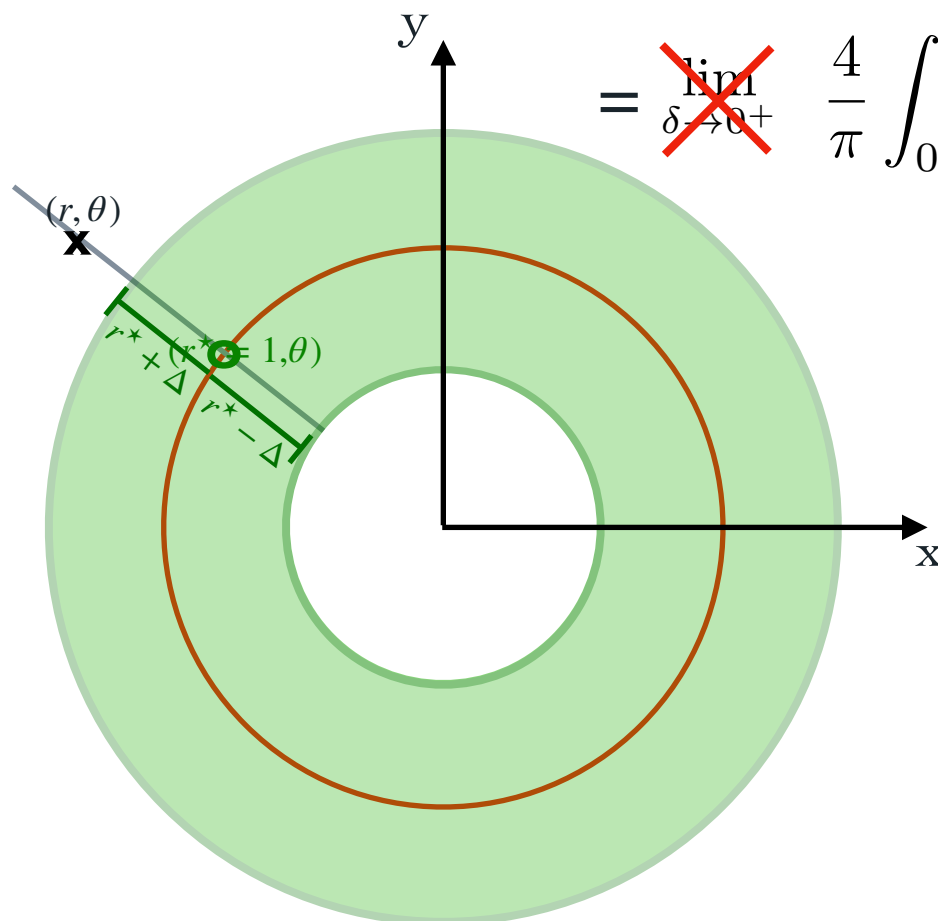
Local threshold counterterm

Integrated threshold counterterm

$$= \lim_{\delta \rightarrow 0^+} \frac{4}{\pi} \left[\int_0^\infty dr \left(\frac{1}{r^2 + 1} \right) \frac{1}{1 - r \pm i\delta} - \int_{\boxed{1-\Delta}}^{\boxed{1+\Delta}} dr \left(\frac{1}{(1)^2 + 1} \right) \frac{1}{1 - r \pm i\delta} \right] + \frac{4}{\pi} \int_{1-\Delta}^{1+\Delta} dr \left(\frac{1}{(1)^2 + 1} \right) \frac{1}{1 - r \pm i\delta}$$

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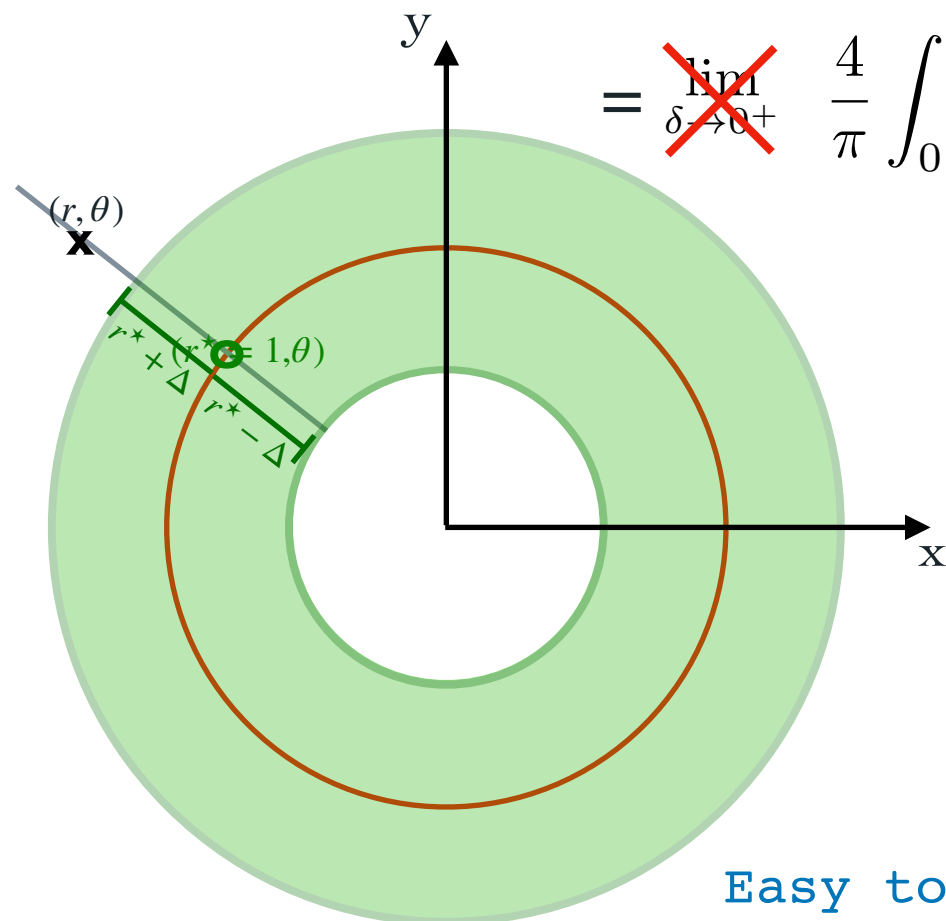
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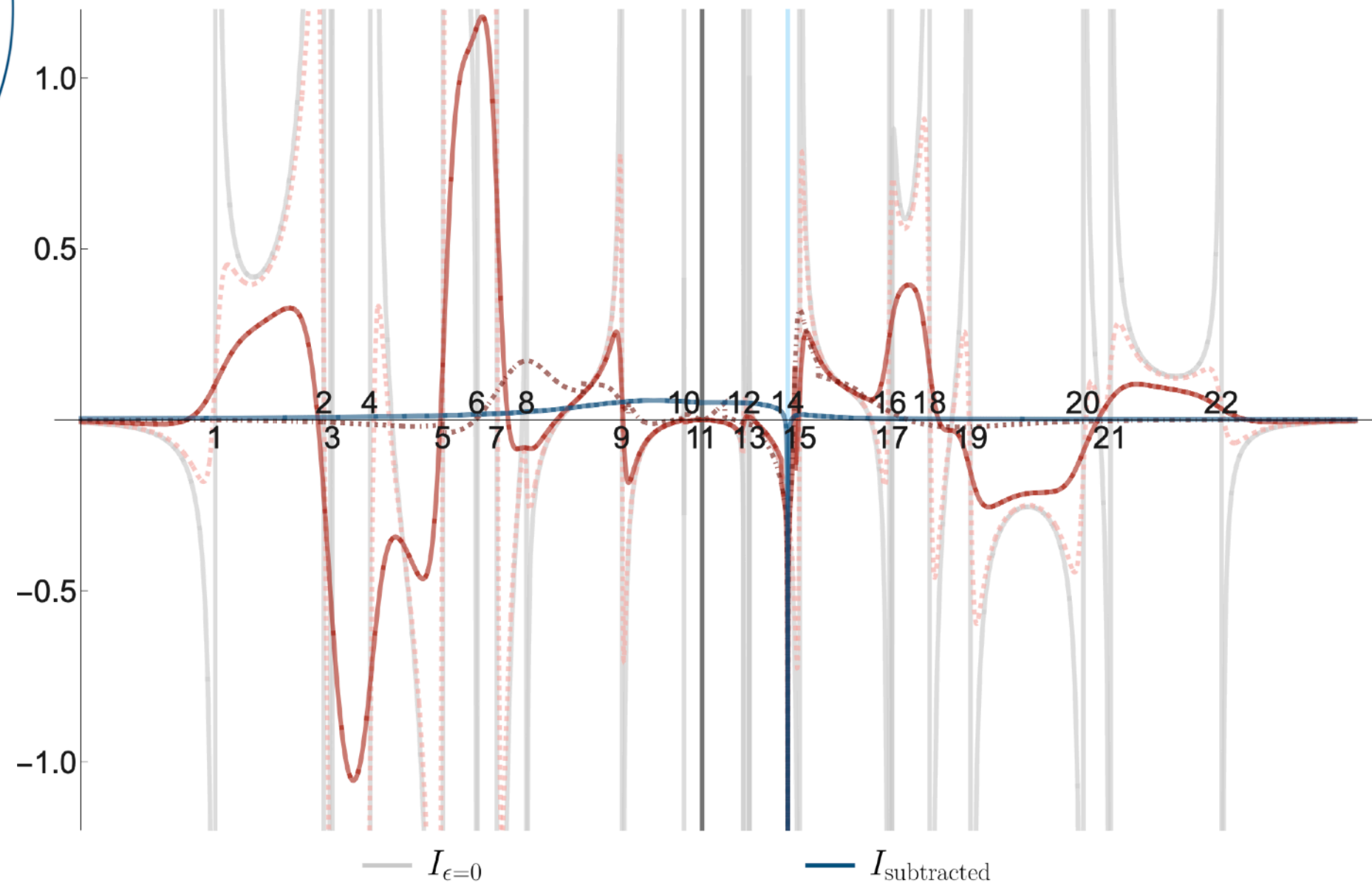
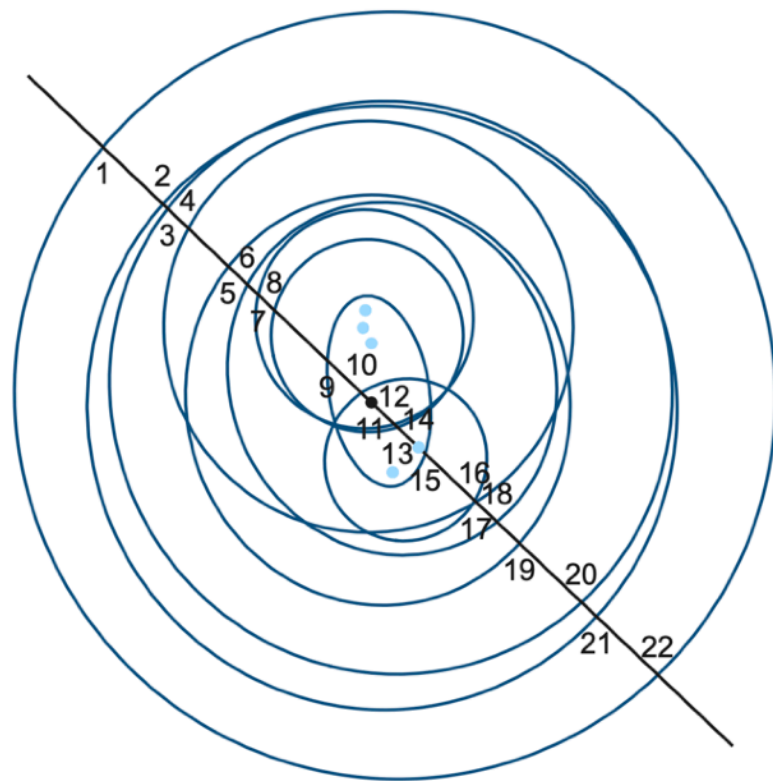
$\delta = 0$ can be taken safely !

$$+ \lim_{\delta \rightarrow 0^+} \int_{1-\Delta}^{1+\Delta} dr \frac{4}{\pi} \frac{1}{2} \frac{1}{r - 1 \pm i\delta} \stackrel{\Delta \leq 1}{=} \underbrace{\frac{2}{\pi} \text{PV} \left[\frac{1}{r - 1 \pm i\delta} \right]}_0 \boxed{\mp 2i}$$

Easy to compute since Principal Value is zero by construction

THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

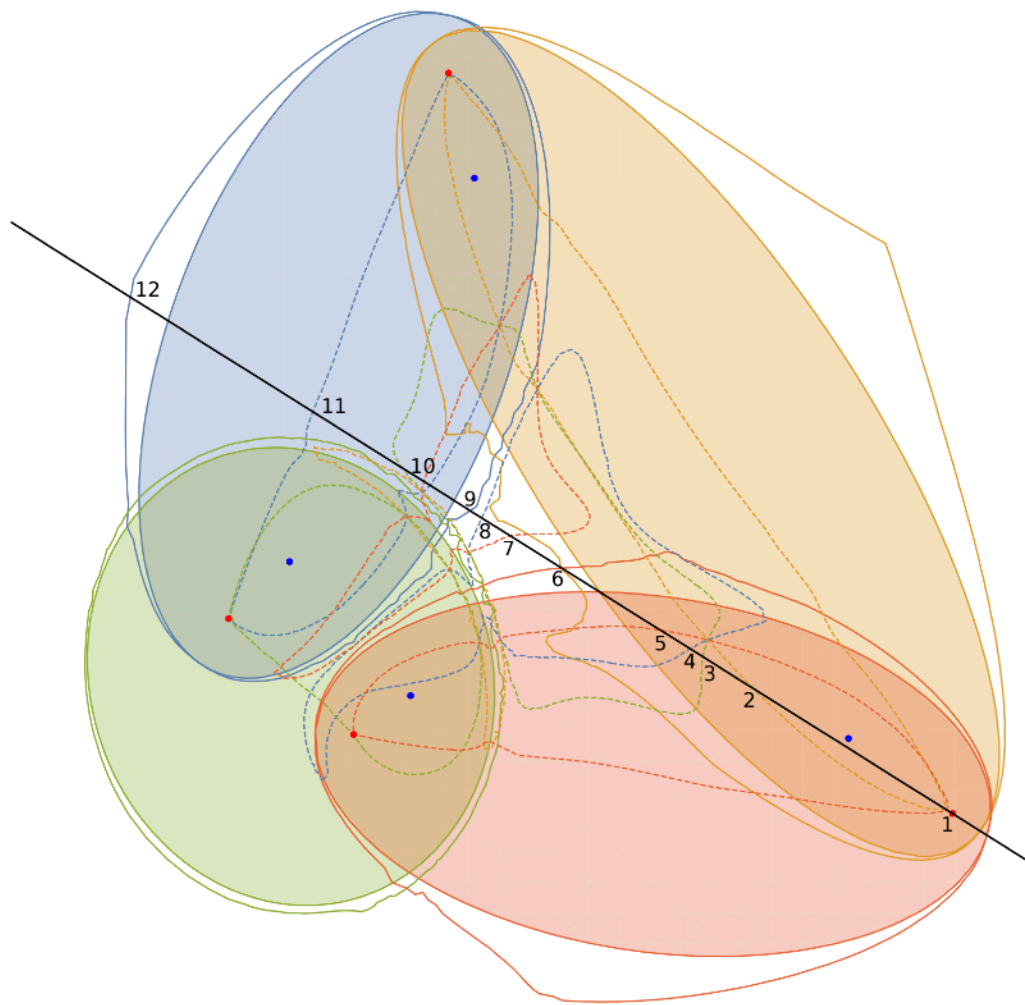
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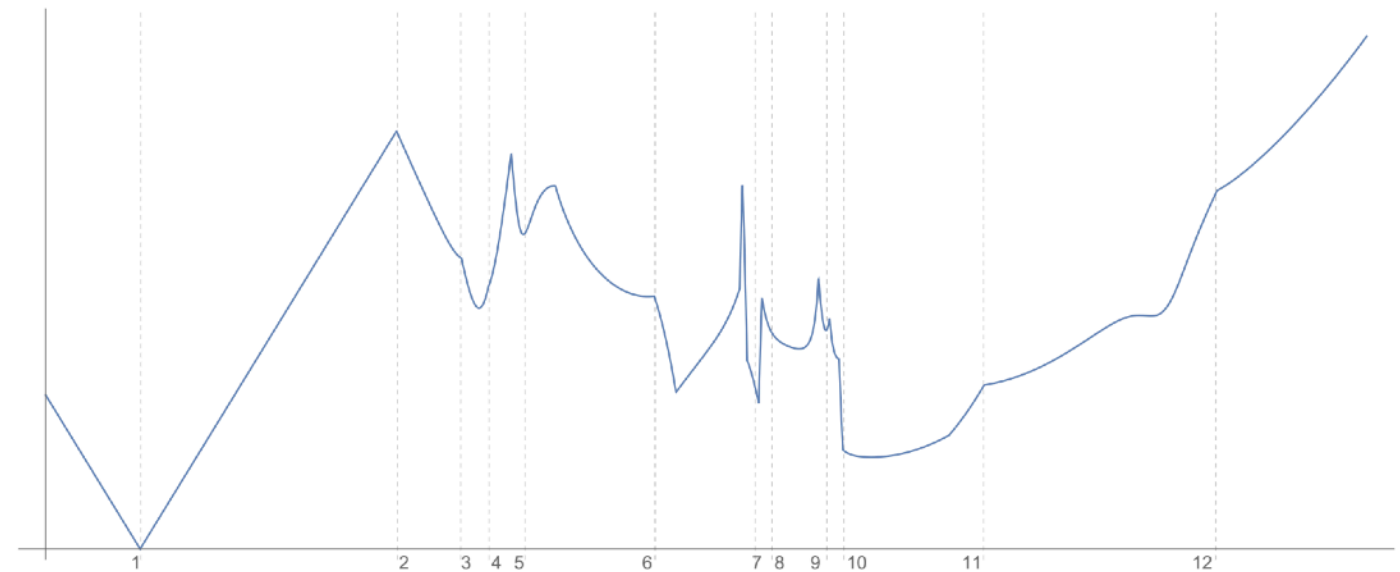
$\cdots \text{Re } I_{\text{deformed}}(\lambda_{\text{max}} = 3)$
 $\text{— Re } I_{\text{deformed}}(\lambda_{\text{max}} = 10)$
 $\cdots \text{Re } I_{\text{deformed}}(\lambda_{\text{max}} = 300)$

CONTOUR DEFORMATION MAGNITUDE

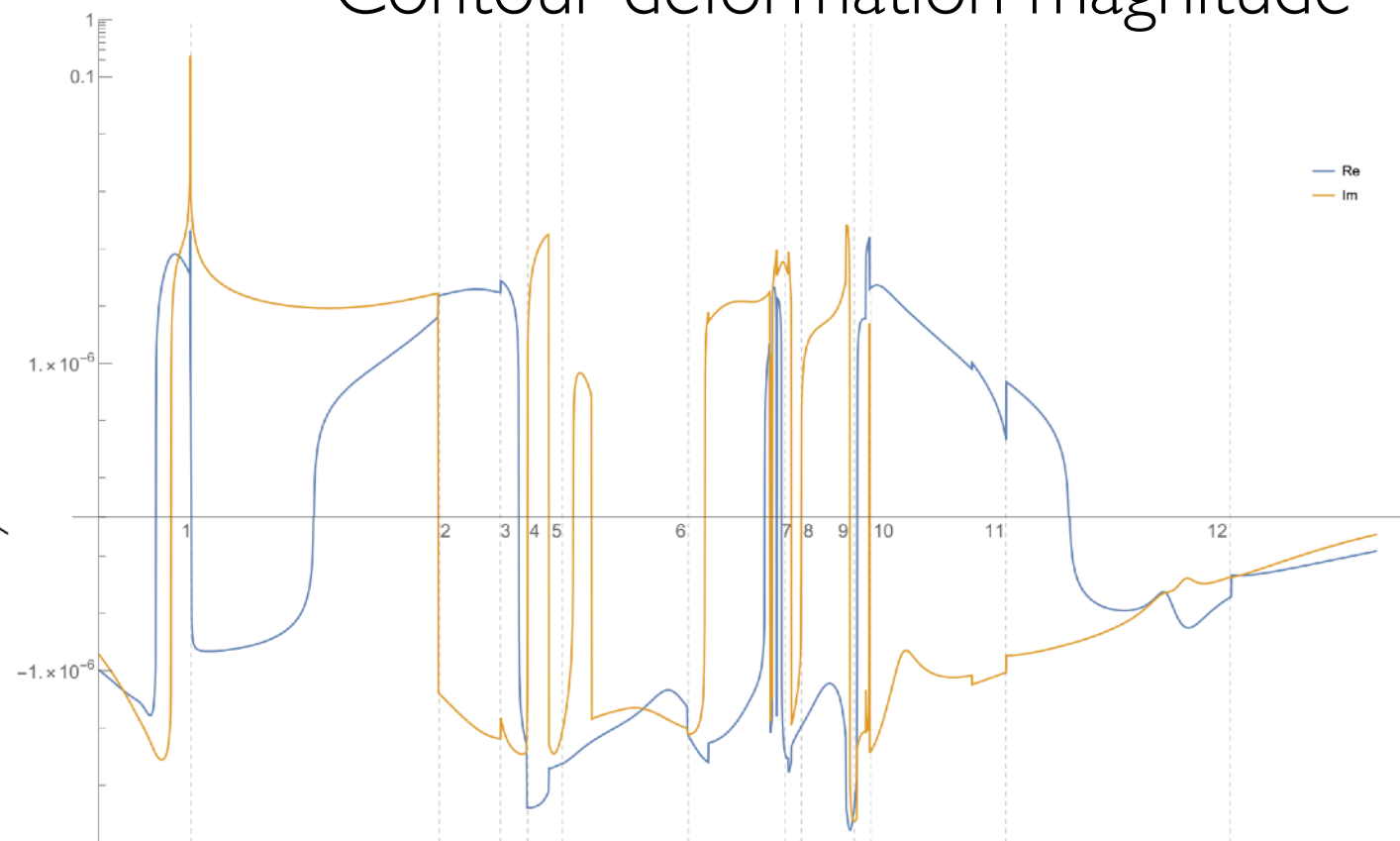
[Capatti, VH, Kermanschah. Ruiil arXiv : 1912.09291 1



Threshold configuration



Contour deformation magnitude



integrand x jacobian weight

UV SUBTRACTION

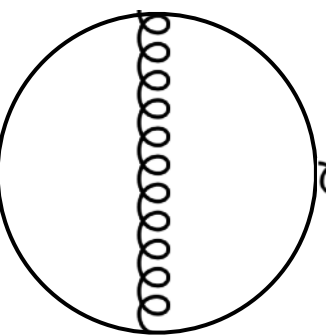
LOCALISED RENORMALISATION

[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ [refs.](#)]

$$R(\Gamma) = \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K \left(\sum_{S \in W(\gamma) \setminus \gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma') \right)$$

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$$R \left(\Gamma = \text{diagram} \right) = \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K \left(\sum_{S \in W(\gamma) \setminus \gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma') \right)$$


LOCALISED RENORMALISATION

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$$R \left(\Gamma = \text{diagram of a circle with a vertical line of nodes through it} \right) = \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K \left(\sum_{S \in W(\gamma) \setminus \gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma') \right)$$

UV subgraphs :

$$\text{dod}(\gamma_{\{1,2,3\}}) = 0$$

$$\text{dod}(\gamma_4) = 2$$

$$\gamma_1 = \text{[diagram of a loop with a vertical dashed line inside]}$$

$$\gamma_2 = \text{Diagram: a semi-circular loop with a vertical dashed line through its center and a wavy line extending from the right side.}$$

$$\gamma_3 = \text{diagram of a circle with four external legs}$$

$$\gamma_4 = \text{Diagram: a circle with a vertical wavy line through the center, and two external wavy lines on the left and right sides.}$$

LOCALISED RENORMALISATION

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UV subgraphs :

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$$\text{dod}(\gamma_4) = 2$$

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$$\gamma_2 = \text{Diagram: a semi-circle with a vertical dashed line through its center and a wavy line extending from its right side.}$$

$$\gamma_3 = \text{diagram of a circle with four external legs}$$

$$\gamma_4 = \text{Diagram showing a circle with two external wavy lines on opposite sides, representing a loop diagram contribution to the four-point function.}$$

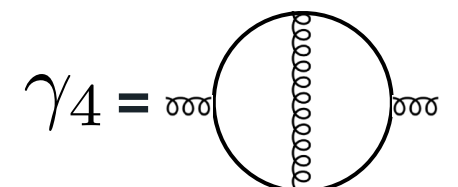
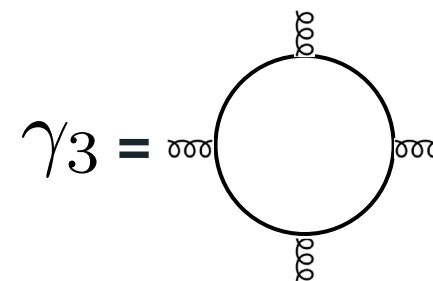
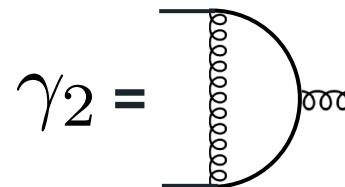
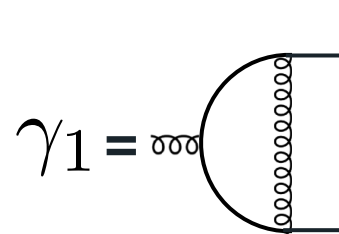
$$\begin{aligned} R(\Gamma) = & \Gamma - K(\gamma_1) * \Gamma \setminus \gamma_1 - K(\gamma_2) * \Gamma \setminus \gamma_2 - K(\gamma_3) * \Gamma \setminus \gamma_3 - K(\gamma_4) * \Gamma \setminus \gamma_4 \\ & + K(K(\gamma_1) * \Gamma \setminus \gamma_1) + K(K(\gamma_2) * \Gamma \setminus \gamma_2) + K(K(\gamma_3) * \Gamma \setminus \gamma_3) \end{aligned}$$

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UV subgraphs :
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$$R(\Gamma) = \Gamma - K(\gamma_1) * \Gamma \setminus \gamma_1 - K(\gamma_2) * \Gamma \setminus \gamma_2 - K(\gamma_3) * \Gamma \setminus \gamma_3 - K(\gamma_4) * \Gamma \setminus \gamma_4 \\ + K(K(\gamma_1) * \Gamma \setminus \gamma_1) + K(K(\gamma_2) * \Gamma \setminus \gamma_2) + K(K(\gamma_3) * \Gamma \setminus \gamma_3)$$

What is the operator $K(\gamma)$? Anything we want ! so long as it:

- Locally cancels UV divergences of γ , even in the presence of nestings
- Yields results immediately renormalised in the chosen scheme ($\overline{\text{MS}}$ + OS)
- Minimal analytics: at most single-scale all-massive vacuum integrals

LOCAL RENORMALISATION OPERATOR K

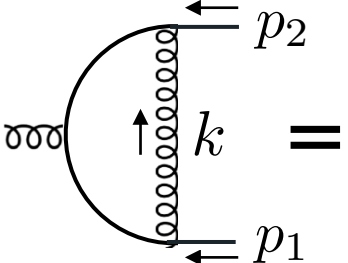
Our solution: $K(\gamma) := T(\gamma)$

$T(\gamma) := \text{Local CT}$: Taylor expansion around the “UV point” up to $\text{dod}(\gamma)$

LOCAL RENORMALISATION OPERATOR K

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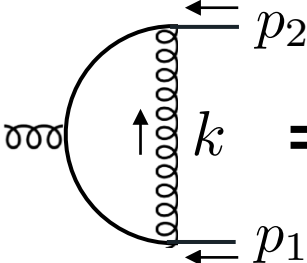
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$$\gamma_1 = \text{diagram} = \frac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{((k - p_1)^2 - m^2)(k^2)((k + p_2)^2 - m^2)}$$


LOCAL RENORMALISATION OPERATOR K

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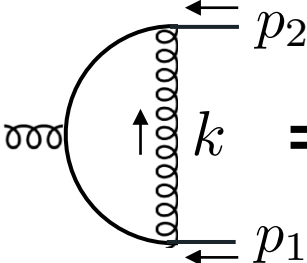
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$$\gamma_1^\lambda := \frac{\mathcal{N}_{\gamma_1}(k, \lambda p_1, \lambda p_2, \lambda m)}{(k - \lambda p_1)^2 - m_{UV}^2 - \lambda^2(m^2 - m_{UV}^2))(k^2 - m_{UV}^2)((k + \lambda p_2)^2 - m_{UV}^2 - \lambda^2(m^2 - m_{UV}^2))}$$

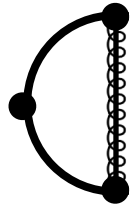
LOCAL RENORMALISATION OPERATOR $\mathbf{K}$

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$$\mathbf{T}(\gamma) = \mathbf{T}_{\text{dod}(\gamma)}(\gamma^\lambda) = \sum_{j=0}^{\text{dod}(\gamma)} \frac{1}{j!} \frac{d^j}{d\lambda^j} \gamma^\lambda \Big|_{\lambda=0}, \quad \mathbf{T}_0(\gamma_1) = \frac{\mathcal{N}(k, 0, 0, 0)}{(k^2 - m_{\text{UV}}^2)^3} \sim \text{Diagram}$$


LOCAL RENORMALISATION OPERATOR K

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$T(\gamma) :=$ **Local CT** : Taylor expansion around the “UV point” up to $\text{dod}(\gamma)$

$$\gamma_1 = \text{Diagram} = \frac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{((k - p_1)^2 - m^2)(k^2)((k + p_2)^2 - m^2)}$$

$$\gamma_1^\lambda := \frac{\mathcal{N}_{\gamma_1}(k, \lambda p_1, \lambda p_2, \lambda m)}{(k - \lambda p_1)^2 - m_{UV}^2 - \lambda^2(m^2 - m_{UV}^2))(k^2 - m_{UV}^2)((k + \lambda p_2)^2 - m_{UV}^2 - \lambda^2(m^2 - m_{UV}^2))}$$

$$T(\gamma) = T_{\text{dod}(\gamma)}(\gamma^\lambda) = \sum_{j=0}^{\text{dod}(\gamma)} \frac{1}{j!} \frac{d^j}{d\lambda^j} \gamma^\lambda \Big|_{\lambda=0}, \quad T_0(\gamma_1) = \frac{\mathcal{N}(k, 0, 0, 0)}{(k^2 - m_{UV}^2)^3} \sim \text{Diagram}$$

$$[T](\gamma) := \text{Integrated CT}, \quad [T](\gamma_1) = \left(\frac{\mu_r^2}{4\pi e^{-\gamma_E}} \right)^\epsilon \int d^{4-2\epsilon} k \text{Diagram} = \sum_{k=-\infty}^{+\infty} \alpha_k \epsilon^k$$

LOCAL RENORMALISATION OPERATOR $\mathbf{K}$

Our solution: $\mathbf{K}(\gamma) := \mathbf{T}(\gamma) - [\mathbf{T}](\gamma) + \delta^{\overline{\text{MS}}+\text{OS}}(\gamma)$

$\mathbf{T}(\gamma) := \text{Local CT}$: Taylor expansion around the “UV point” up to $\text{dod}(\gamma)$

$$\gamma_1 = \text{Diagram} = \frac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{((k - p_1)^2 - m^2)(k^2)((k + p_2)^2 - m^2)}$$

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$$\delta^X(\gamma) := \text{Renormalisation CT in scheme X}, \quad (-[\mathbf{T}] + \delta^{\overline{\text{MS}}}) := \bar{\mathbf{K}}, \quad \bar{\mathbf{K}}(\gamma_1) = \sum_{k=0}^{+\infty} \alpha_k \epsilon^k$$

R-OPERATOR UNFOLDING

$$\Gamma = \text{diagram} \quad \text{with UV subgraphs} \quad \gamma_1 = \text{diagram} \quad \gamma_2 = \text{diagram} \quad \gamma_3 = \text{diagram} \quad \gamma_4 = \text{diagram}$$

The diagrams represent Feynman graphs. Γ is a circle with a vertical wavy line and four external wavy lines. γ_1 is a semi-circle with a vertical wavy line and two external wavy lines. γ_2 is a semi-circle with a vertical wavy line and two external wavy lines, oriented differently. γ_3 is a circle with a vertical wavy line and four external wavy lines. γ_4 is a circle with a vertical wavy line and four external wavy lines, oriented differently.

$$R(\Gamma) = \Gamma - T_0(\gamma_1) * \Gamma \setminus \gamma_1 - T_0(\gamma_2) * \Gamma \setminus \gamma_2 - T_0(\gamma_3) * \Gamma \setminus \gamma_3 - T_2(\gamma_4) * \Gamma \setminus \gamma_4 \\ + T_2(T_0(\gamma_1) * \Gamma \setminus \gamma_1) + T_2(T_0(\gamma_2) * \Gamma \setminus \gamma_2) + T_2(T_0(\gamma_3) * \Gamma \setminus \gamma_3) + \bar{K} \text{ terms}$$

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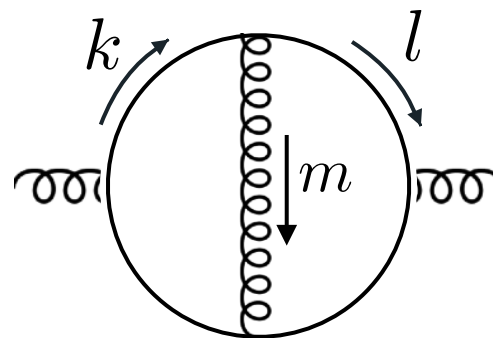
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The four different types of UV limits are now **finite** !



$$\begin{aligned} k, m &\rightarrow \infty, l \text{ finite} \\ l, m &\rightarrow \infty, k \text{ finite} \\ k, l &\rightarrow \infty, m \text{ finite} \\ k, l, m &\rightarrow \infty \end{aligned}$$

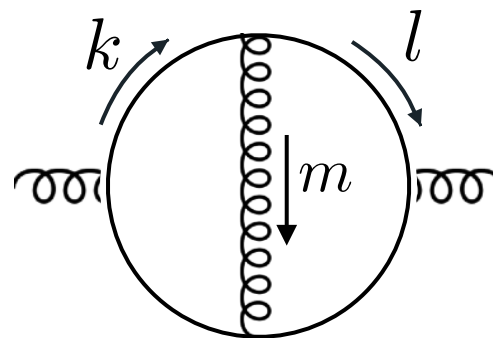
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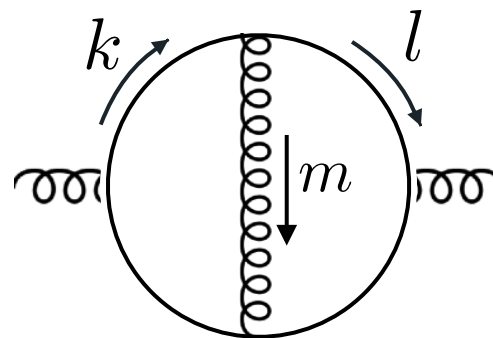
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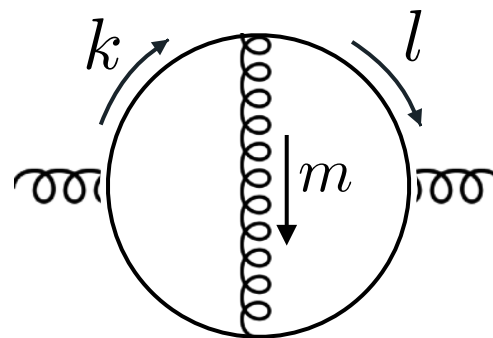
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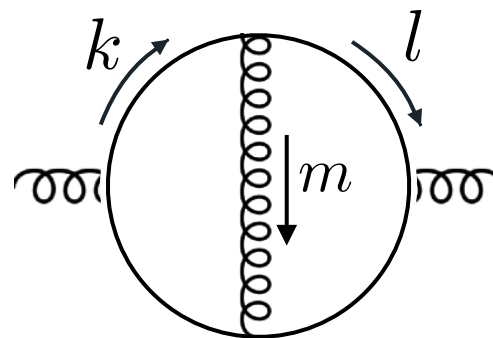
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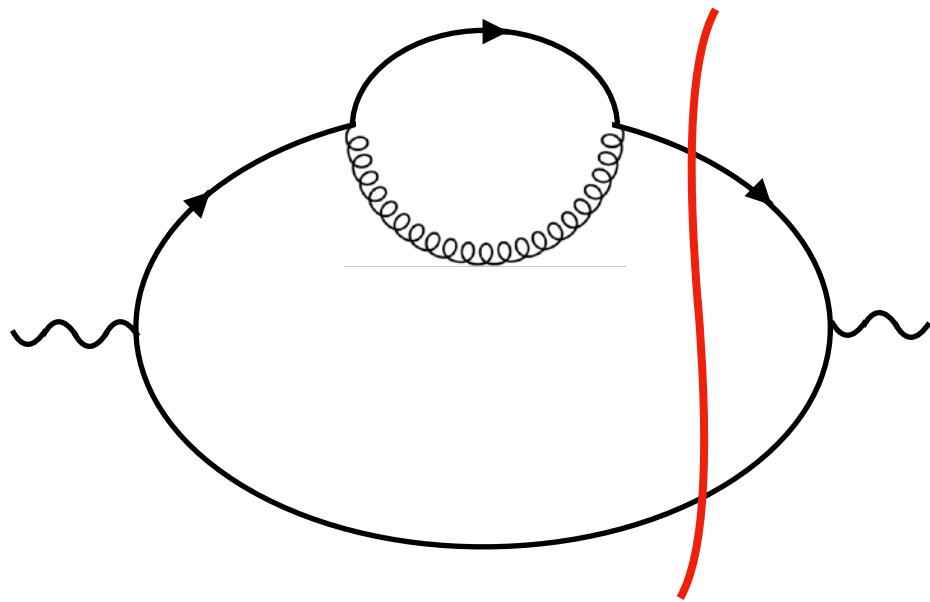
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LOCALITY UNITARITY: RAISED PROPAGATORS

[Capatti, VH, Ruijl, arxiv : 2203.11038]

In **LU**, we cannot consider *truncated* amplitudes only :



Traditional Cutkosky rule

$$\overrightarrow{p} \text{ --- } \text{red cut} = -2\pi i \frac{\delta(p^0 - E(\vec{p}))}{2E(\vec{p})}$$

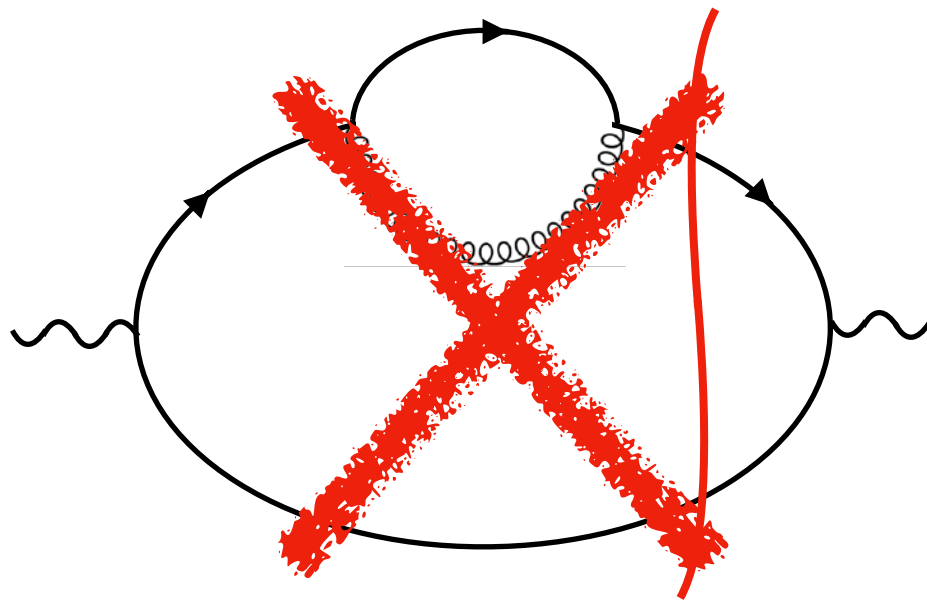
$$E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$$

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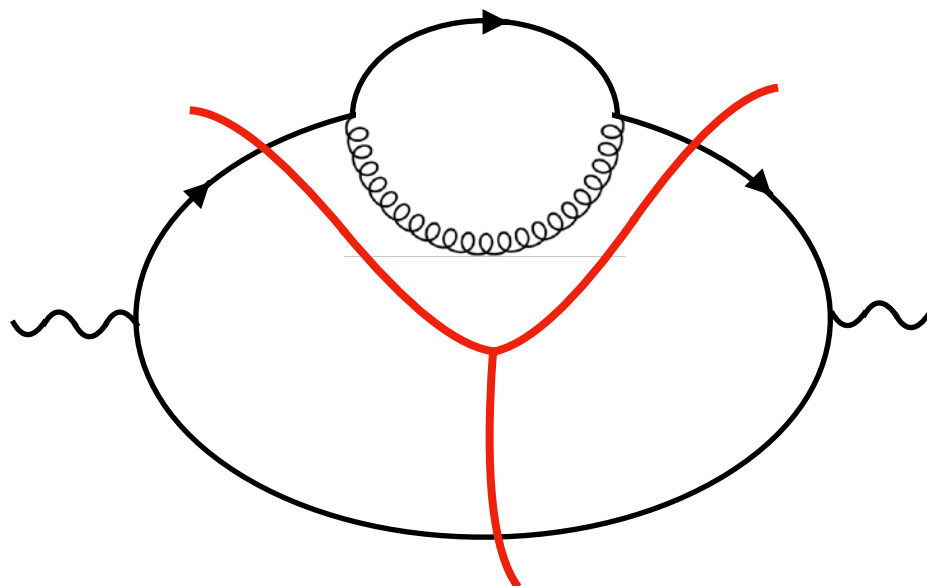
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would not apply here !

So consider this Cutkosky cut as a **higher-order residue** → **Generalised cutting rule**



$$\text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} = -2\pi i \frac{\delta^{(n)} [p^0 - E(\vec{p})]}{(p^0 + E(\vec{p}))^2}$$

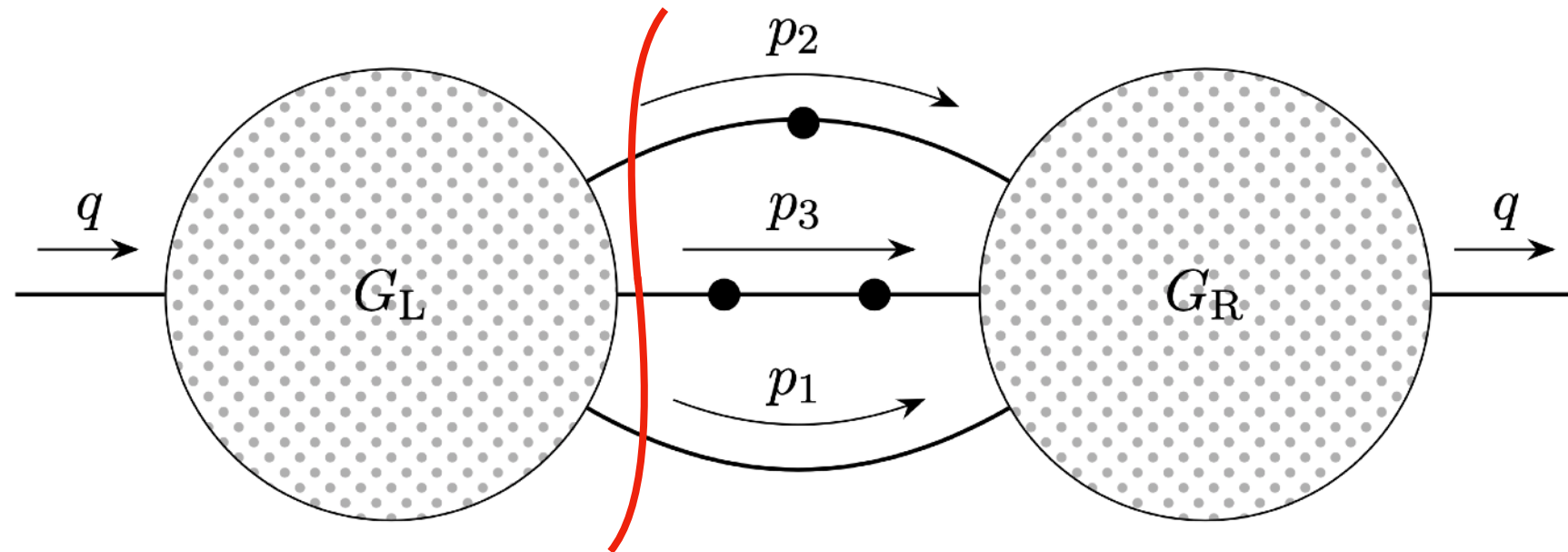
n – times

$$\int dx \delta^{(n+1)}[x] f(x) = \frac{1}{n!} \frac{d^n f}{dx^n} \Big|_{x=0}$$

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This is well understood for **raised loop propagators**,
but for **raised external propagators** of supergraphs, there are subtleties :

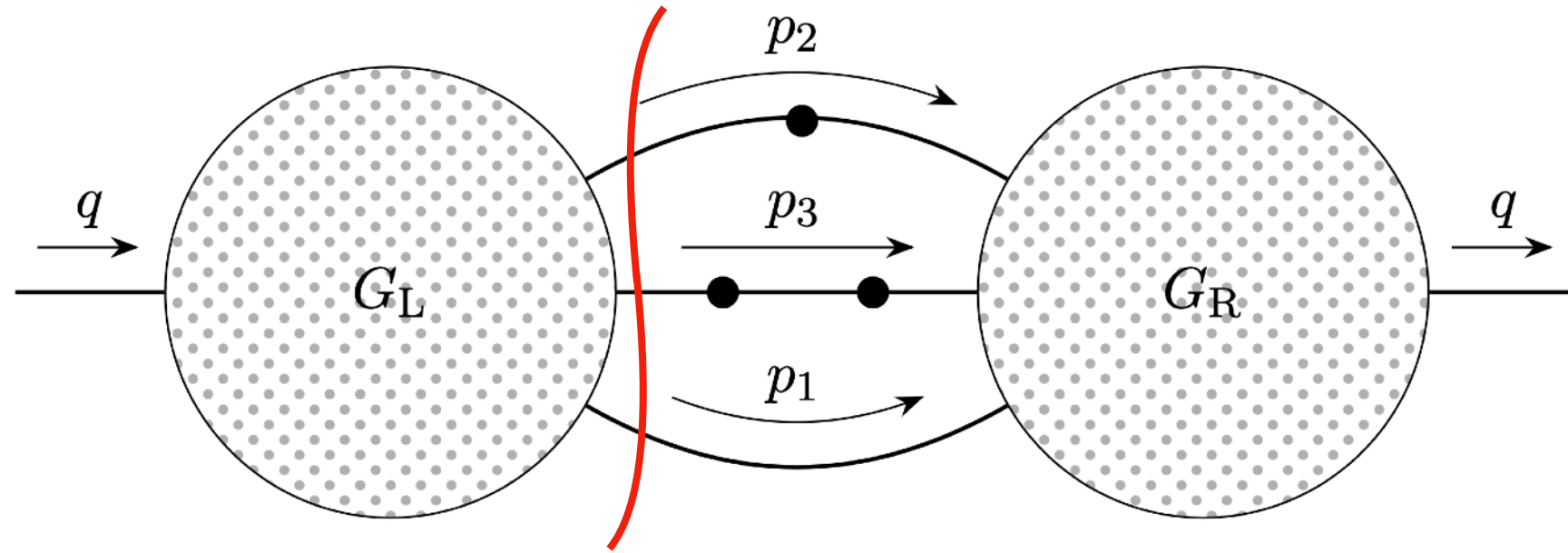


$$\propto \delta^{(1)}[p_1^0 - E(\vec{p}_1)] \delta^{(2)}[p_2^0 - E(\vec{p}_2)] \delta^{(3)}[p_3^0 - E(\vec{p}_3)]$$

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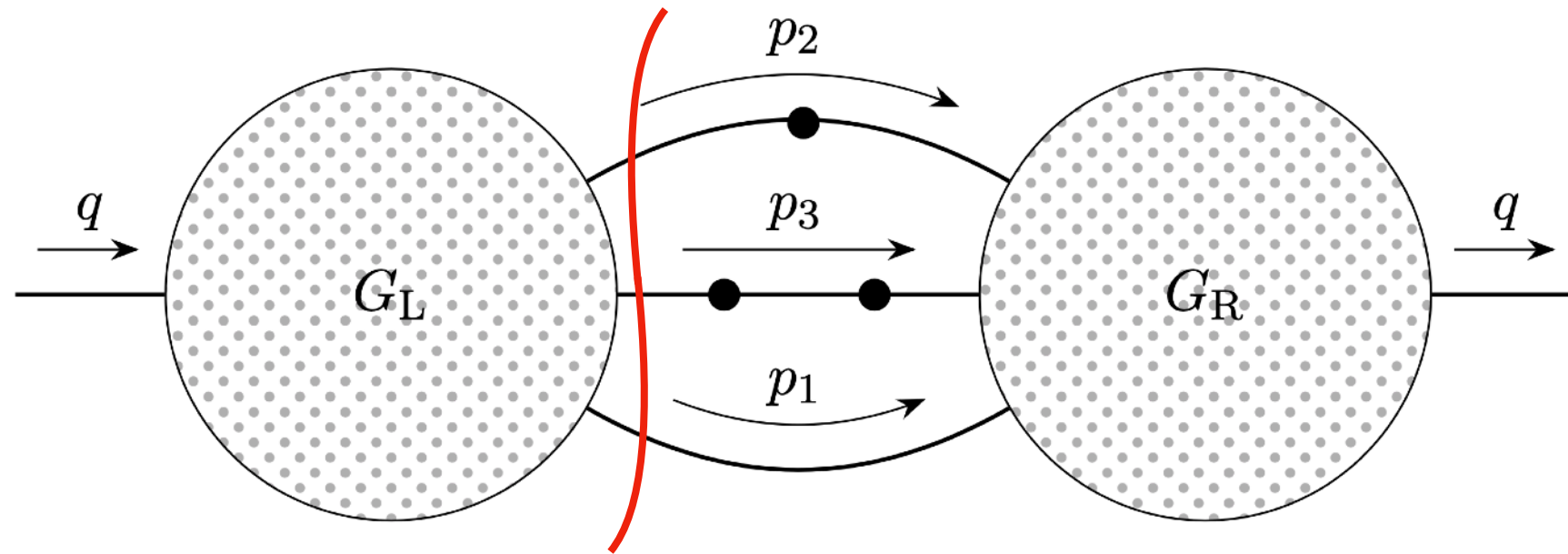
Derivatives can act on each other because: $p_3 = q - p_1 - p_2$

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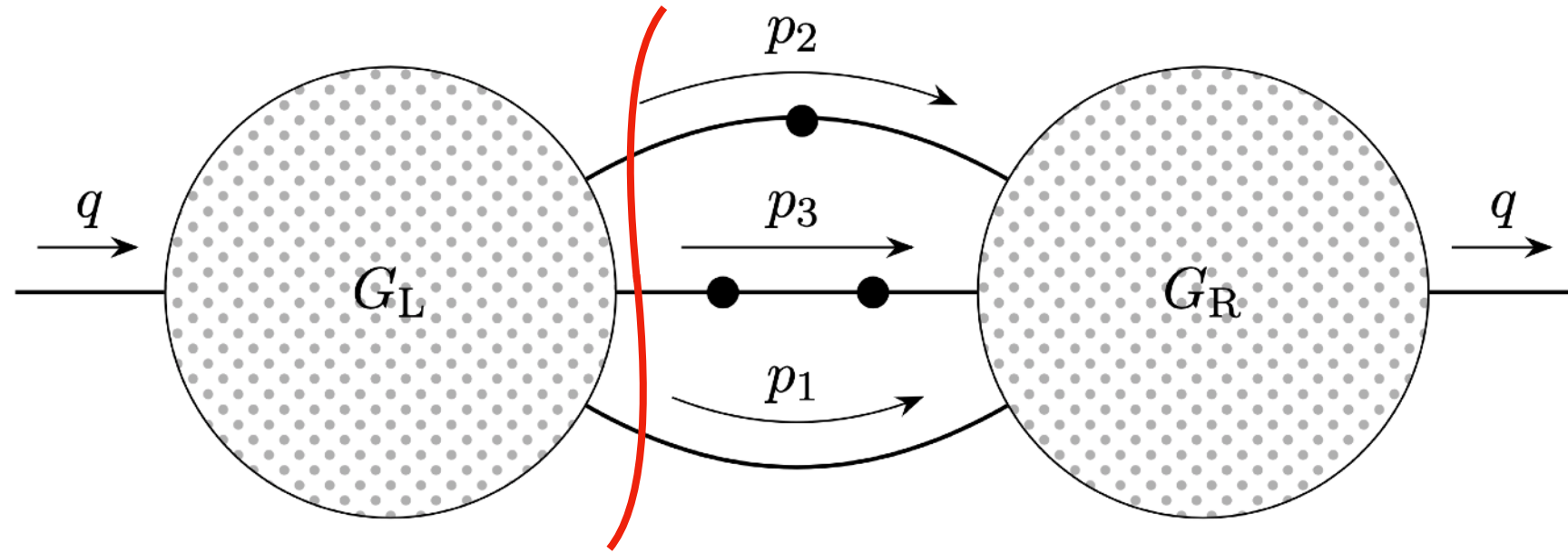
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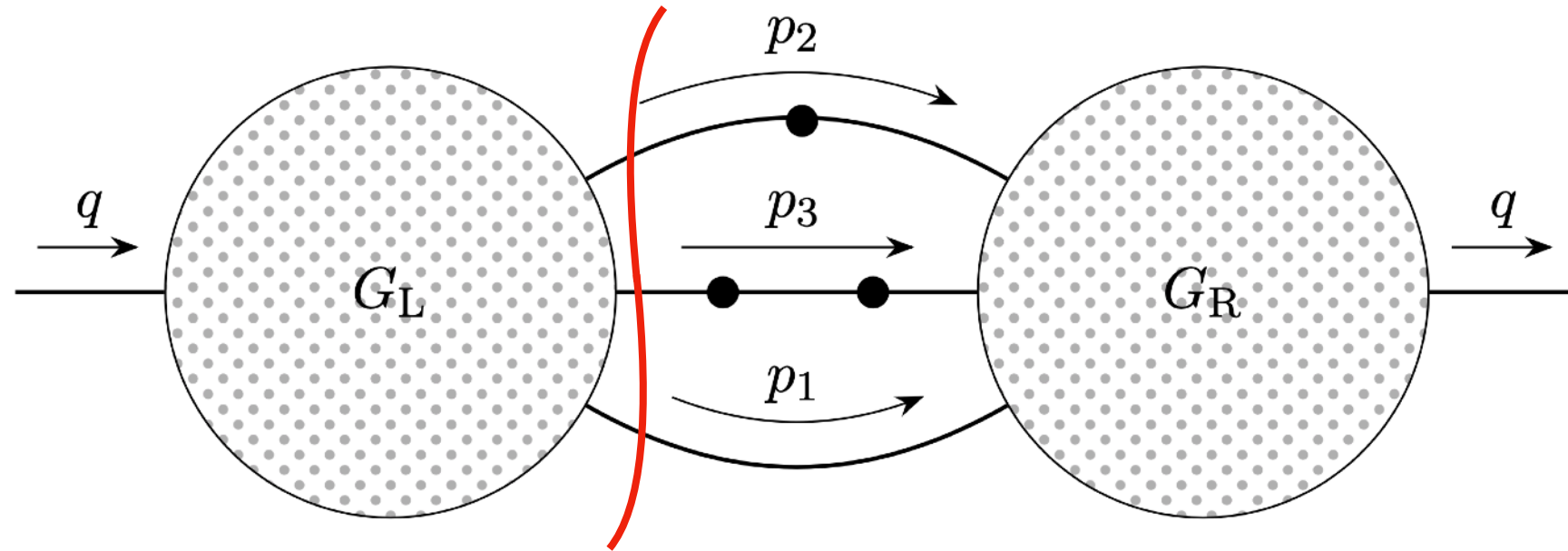
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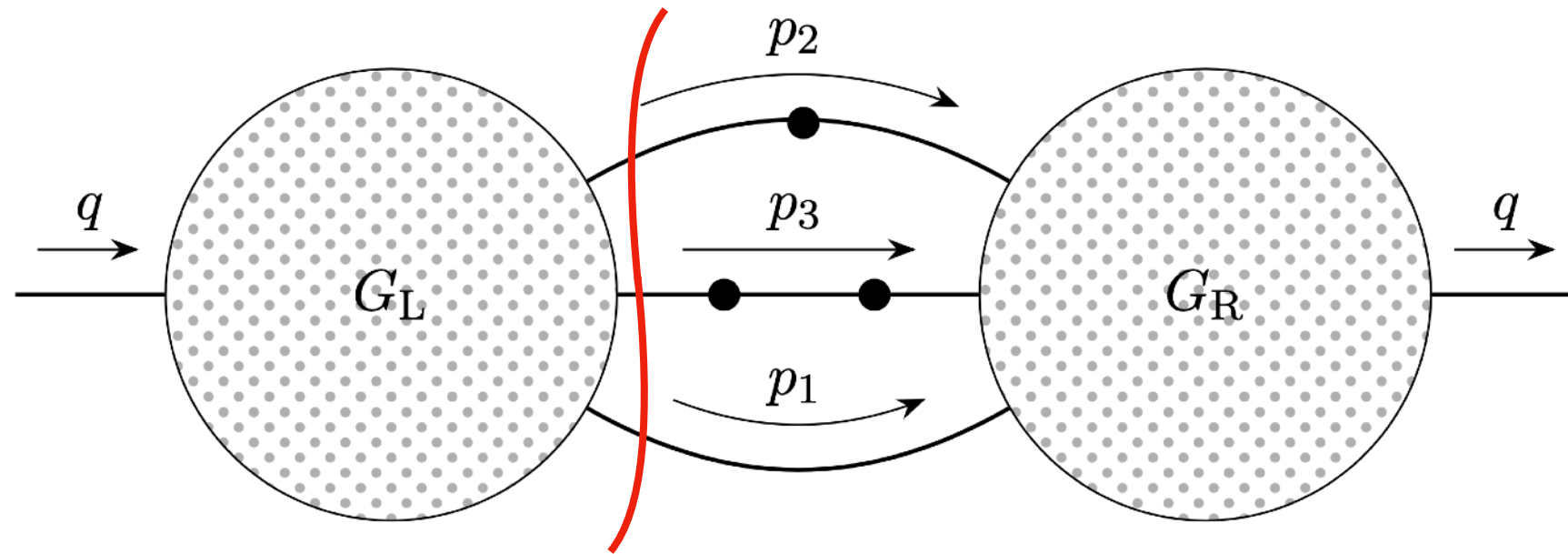
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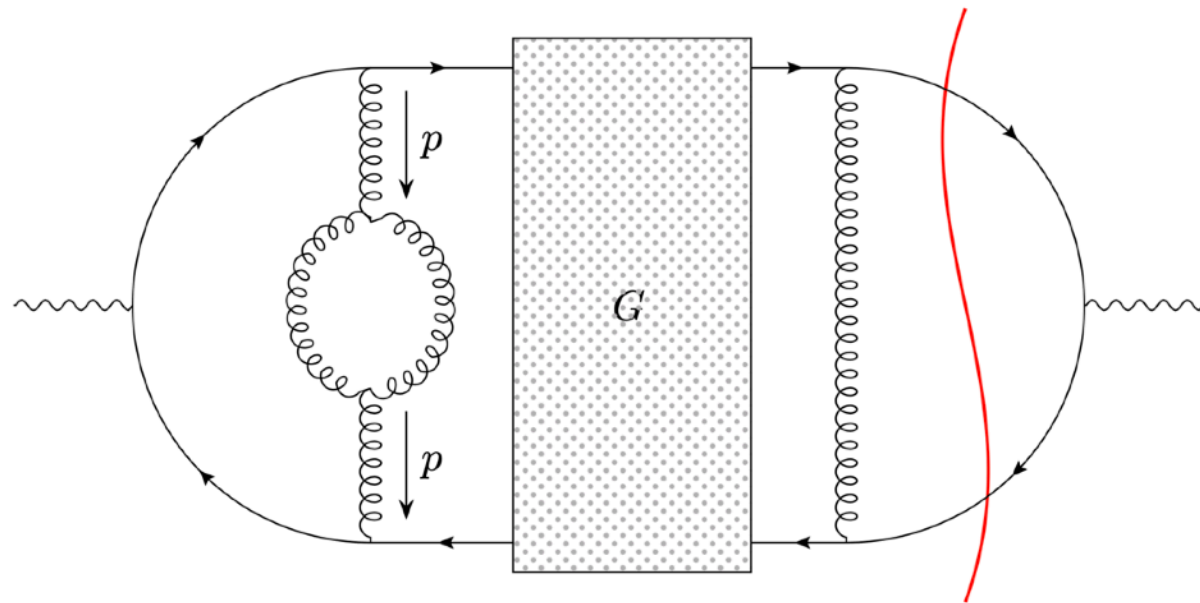
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Use **multivariate dual numbers** (auto-differentiation) in order to **efficiently compute amplitude derivatives** of G_L and G_R in p_2^0 and t (in this example)

SPURIOUS SOFT SINGULARITIES

[Capatti, VH, Ruijl, arxiv : 2203.11038]



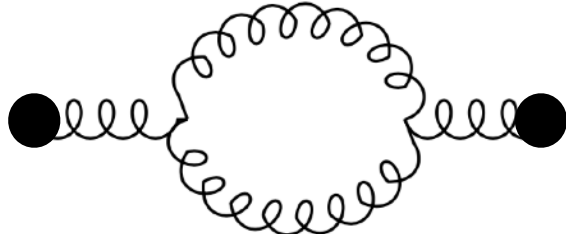
$$\propto \frac{1}{(p^2)^2}$$

For $p = 0$

this induces a spurious soft divergence whose cancellation has nothing to do with KLN!

Only beyond NLO (needs a soft propagator dressed with a self-energy correction)

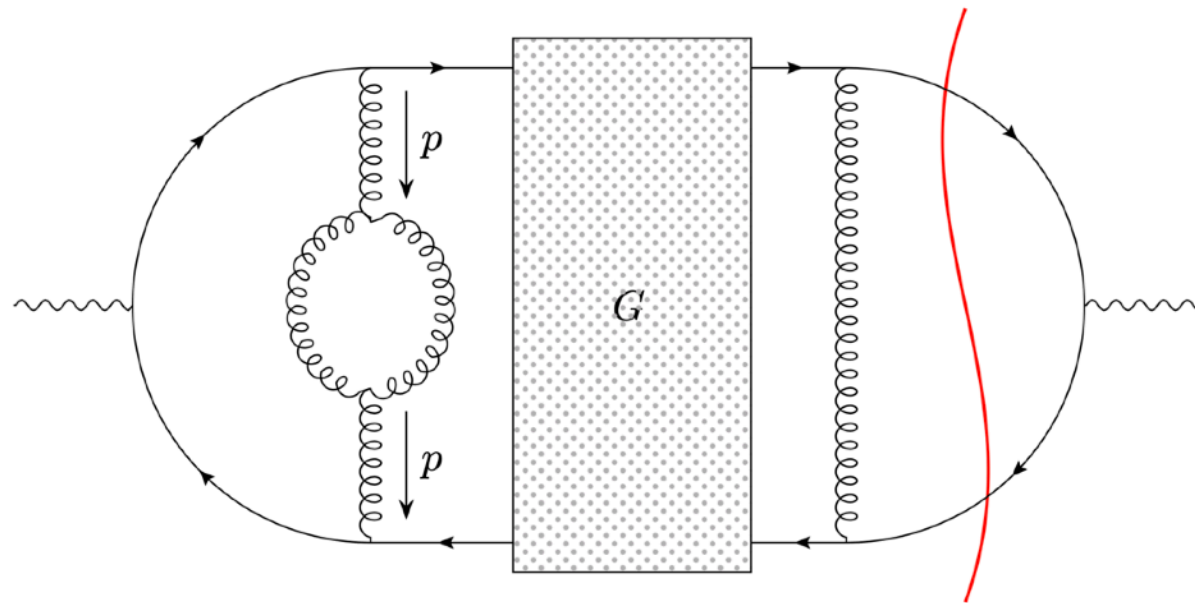
At the integrated level, we have



$$\propto \frac{1}{p^2} (p^2 g^{\mu\nu} - p^\mu p^\nu) \frac{1}{p^2}$$

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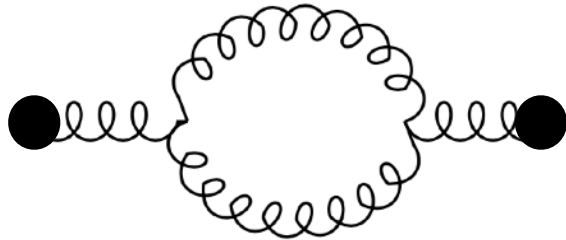
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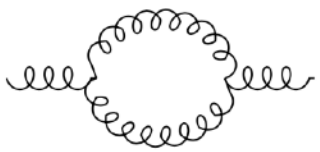
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At the integrated level, we have



$$\propto \frac{1}{p^2} (p^2 g^{\mu\nu} - p^\mu p^\nu) \frac{1}{p^2}$$

but not at the local level; we must introduce **spurious soft counterterms** :



$$-\tilde{T}_1 \left(\text{gluon loop diagram} \right), \quad \tilde{T}_{\text{soft_dod}}(\gamma) = \sum_{j=0}^{\text{soft_dod}(\gamma)} \frac{1}{j!} \frac{d^j}{d\lambda^j} \gamma(\lambda p) \Big|_{\lambda=0}, \quad [\tilde{T}] = 0$$

COMBINED UV AND SPURIOUS IR FOREST

Eureka moment:

- Remarkably, we always have : $\text{soft_dod} = \text{UV_dod} - 1$
- Spurious soft expansion also valid as UV counter term.
- Spurious soft IR forest similar to the one produced by the R-operation

so that we can combine the UV and spurious soft subtraction as one !

$$\hat{T}_{\text{dod}} = T_{\text{dod}} + \tilde{T}_{\text{dod}-1} - T_{\text{dod}}\tilde{T}_{\text{dod}-1}$$

until we realised that we had just re-invented the wheel: [J. H. Lowenstein, 1976]

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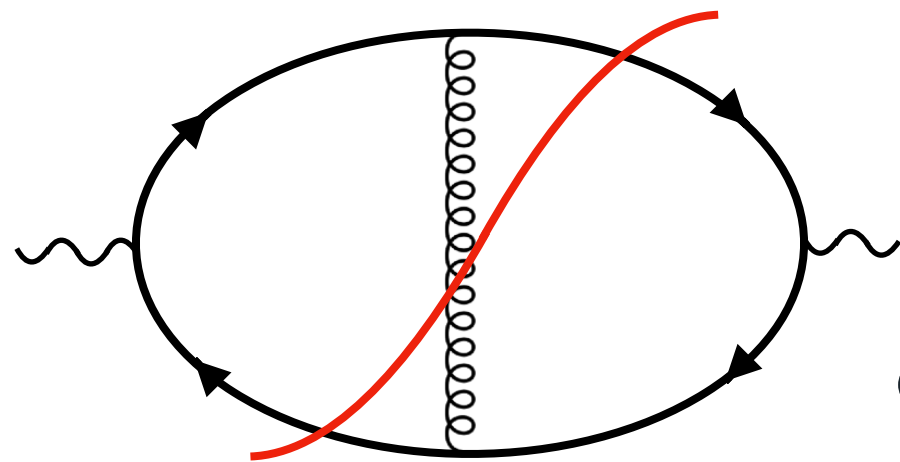
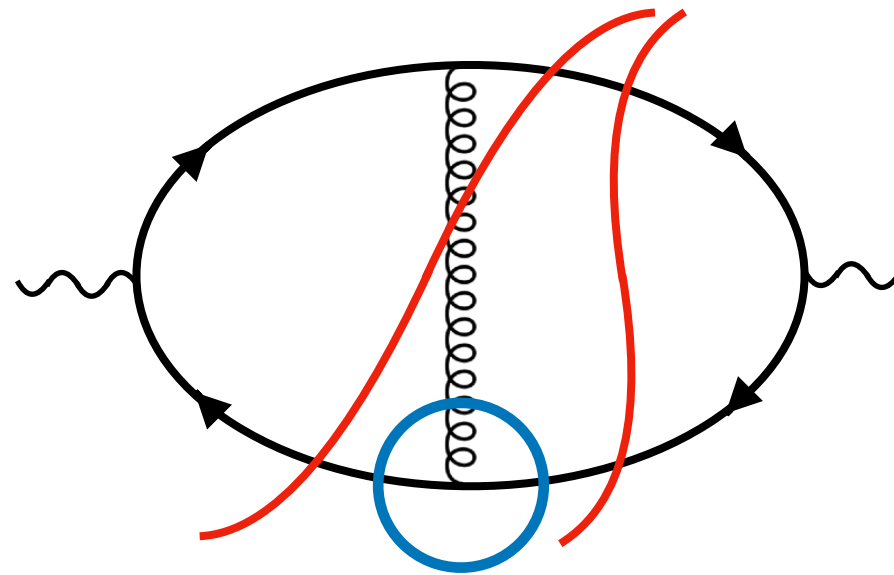
Novelty though: **automatic renormalisation** of fermion **masses** in the **OS** scheme:

$$T^{\text{os}\pm} \left(\Sigma = \xrightarrow{p} \text{---} \right) = (1 \pm \gamma^0) \Sigma(p = \pm p^{\text{os}}), \quad p^{\text{os}} = (m, 0, 0, 0)$$

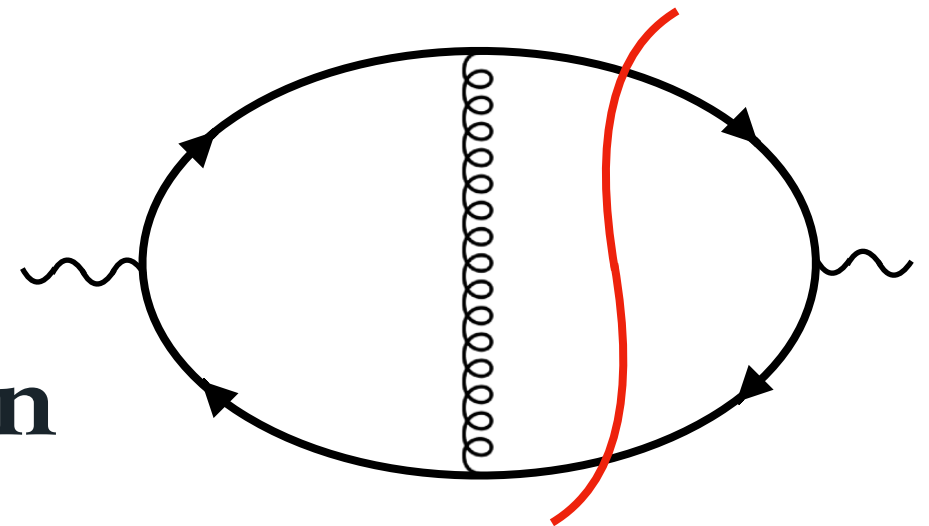
$$\frac{1}{2} \left([T^{\text{os}+}(\Sigma)] + [T^{\text{os}-}(\Sigma)] \right) = \delta m^{\text{os}}$$

Implying that our local UV counterterm T^{os} automatically generates the OS mass renormalisation counterterm !

LOCAL UNITARITY



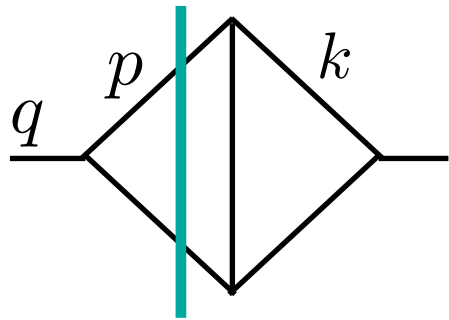
local
↔
cancellation



LOCALITY UNITARITY

We convert the four-dimensional Minkowski loop integration measure into a three-dimensional Euclidean phase-space measure:

$$\frac{d^3\vec{p}}{2|\vec{p}|} d^4k \delta(|\vec{p}| + |\vec{p} - \vec{q}| - Q_0)$$



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$$= \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3}$$

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But the measure is **not yet fully aligned**:

$$\left| \begin{array}{c} \text{Diagram with edges } E_1, E_2, E_3, E_4, E_5 \\ \hline \text{Diagram with teal vertical line} \\ \text{Diagram with teal diagonal line} \end{array} \right| = \int d^3\vec{k} d^3\vec{p} (\delta(E_1 + E_2 - Q_0) f_{\text{virt}} + \delta(E_1 + E_3 + E_5 - Q_0) f_{\text{real}})$$

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$\eta_v(\vec{k}, \vec{p})$

$\eta_r(\vec{k}, \vec{p})$

(on-shell energies: $E_i(\vec{k}_i) = \sqrt{\vec{k}_i^2 + m_i^2 - i\delta}$)

CAUSAL FLOW

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The measure now differs only in the **delta** enforcing on shell energy conservation

$$\text{Diagram 1} \sim \delta(E_1 + E_2 - Q_0)$$

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Solution: introduce an auxiliary variable in which to solve the delta

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Soper,
arXiv: [9804454](#) (1998)

Soper,
arXiv: [0102031](#) (2001 @ RADCOR)

ZC, Hirschi, Pelloni, Ruijl
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General FSR cancellations
For N to M N^kLO processes

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**General FSR cancellations
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A toy example:

$$\int d^3\vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k})$$

CAUSAL FLOW : TOY INTEGRAL

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$$= \int d^3 \vec{k} \int dt h(t) \delta(|\vec{k}| - Q_0) f(\vec{k}) \quad \text{using} \quad 1 = \int dt h(t)$$

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$$= \int d^3 \vec{k} \frac{Q_0^3}{|\vec{k}|^4} h(Q_0/|\vec{k}|) f(Q_0 \vec{k}/|\vec{k}|) \quad \text{with} \quad t^* = Q_0/|\vec{k}|$$

Solve all deltas in the common scaling variable. This completes the alignment of the measure!

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When applying this construction to LU we get:

$$\text{Diagram} = \int d^3 \vec{k} d^3 \vec{p} \delta(E_1 + E_2 - Q_0) f_{\text{virt}} = \int d^3 \vec{k} d^3 \vec{p} g_v(t_v^*)$$

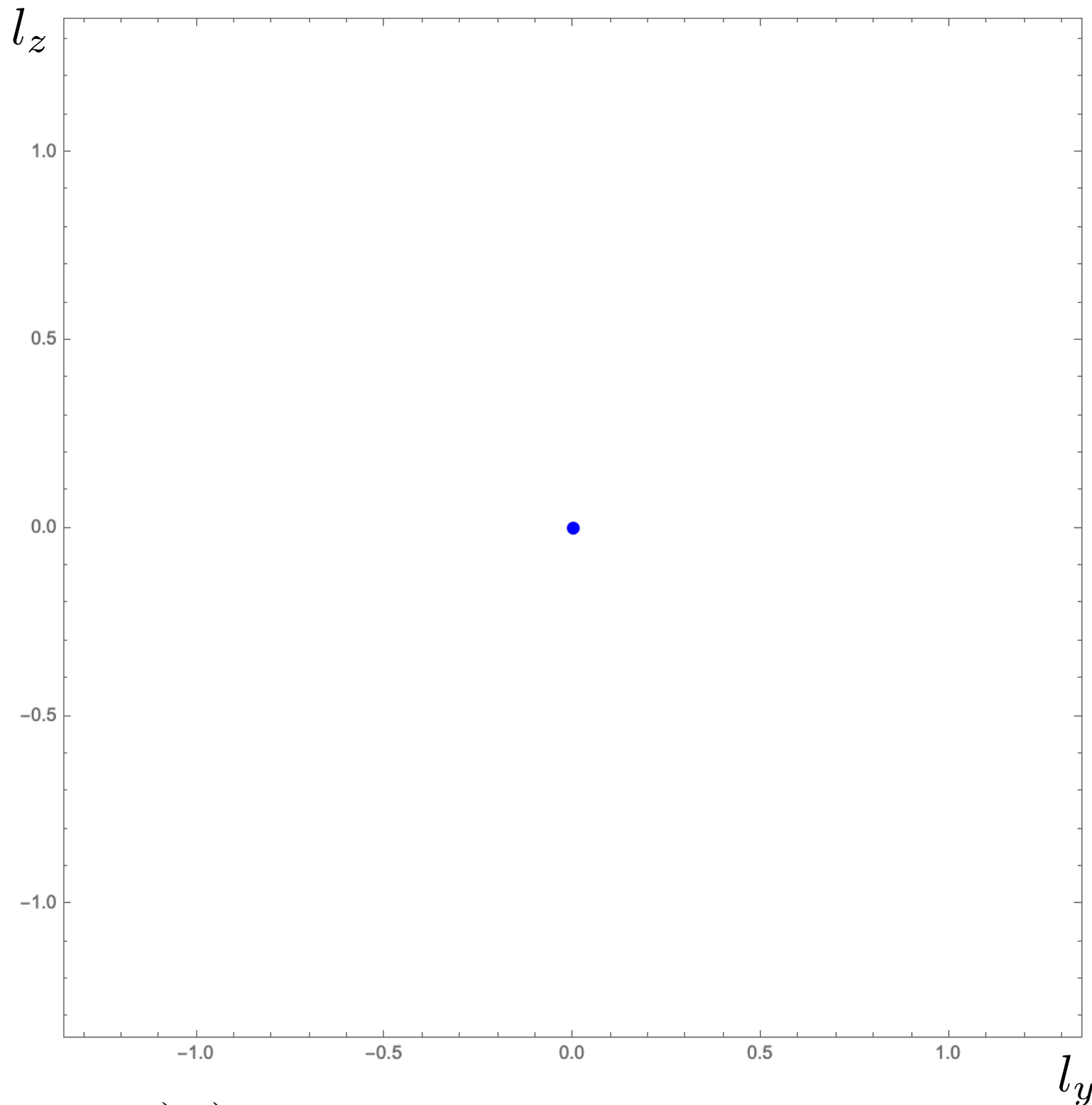
where $t_v^* = t_v^*(\vec{k}, \vec{p}) = \frac{Q_0}{E_1 + E_2}$

$(\vec{p}, \vec{k}) \rightarrow \vec{\phi}(t, (\vec{p}, \vec{k}))$

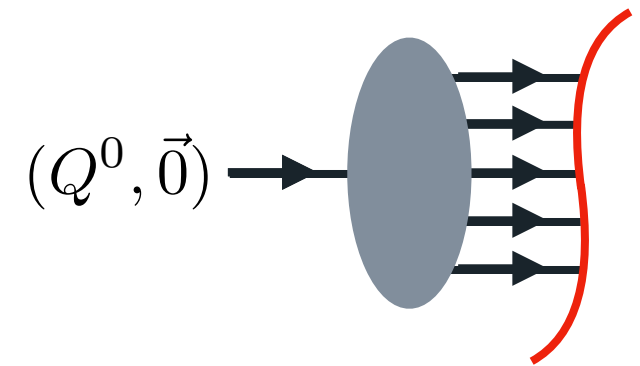
“Causal flow” is called like this because it is the generalisation of the Soper, derive from a contour deformation field satisfying the causal constraints.

$$\begin{cases} \partial_t \vec{\phi} = \vec{\kappa} \circ \vec{\phi} \\ \vec{\phi}(0, (\vec{k}, \vec{l})) = (\vec{k}, \vec{l}) \end{cases}$$

LOCALITY UNITARITY: VISUALISATION

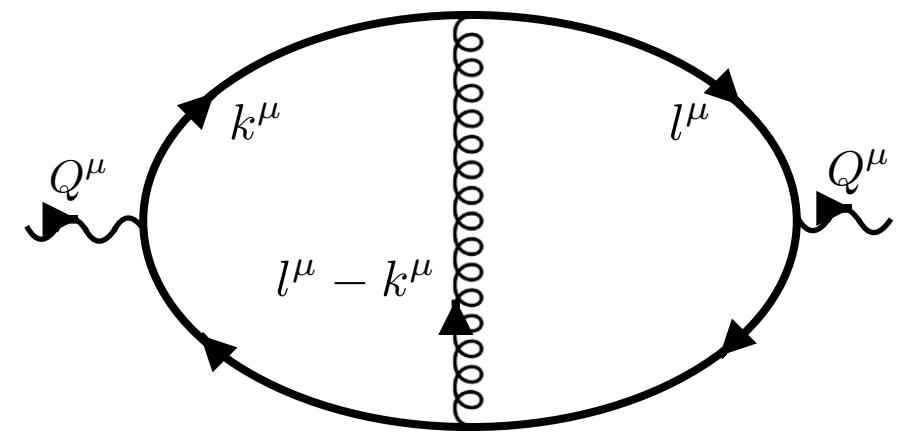


$(\vec{k}, \vec{l}) \in \mathbb{R}^3 \times \mathbb{R}^3$ projected to $(l_y, l_z) \in \mathbb{R}^2$



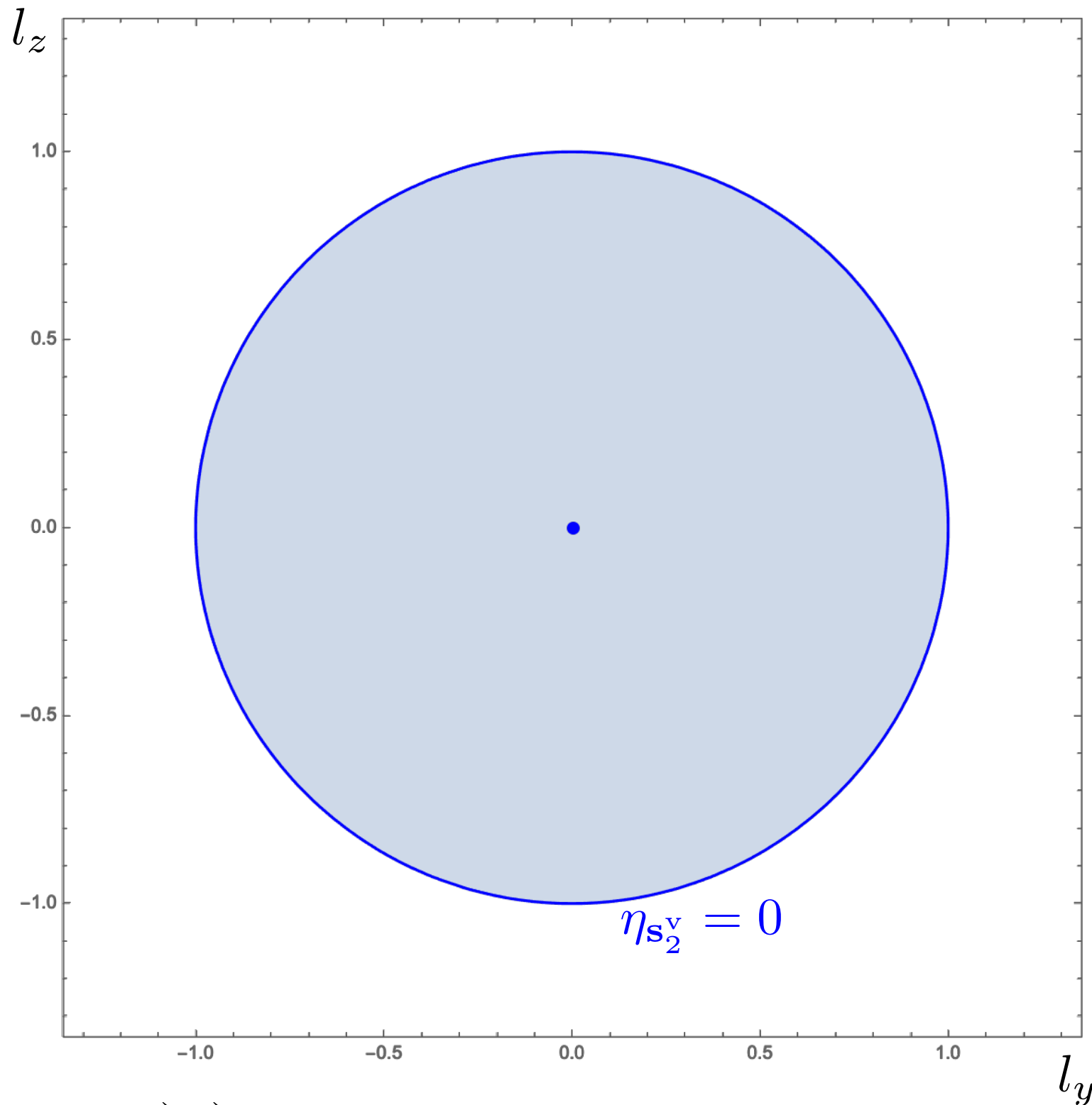
$$Q^\mu = (2, 0, 0, 0)$$

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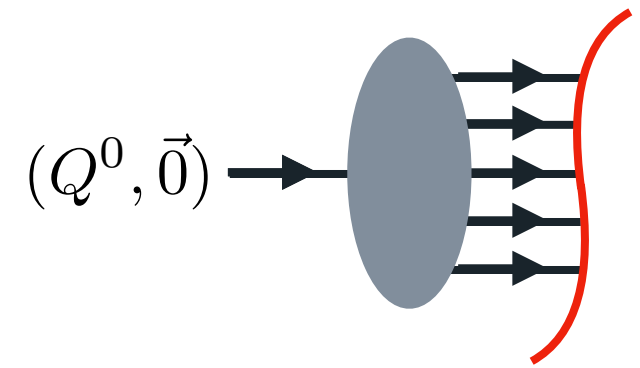


— = Cutkosky cut $\equiv$ threshold

LOCALITY UNITARITY: VISUALISATION



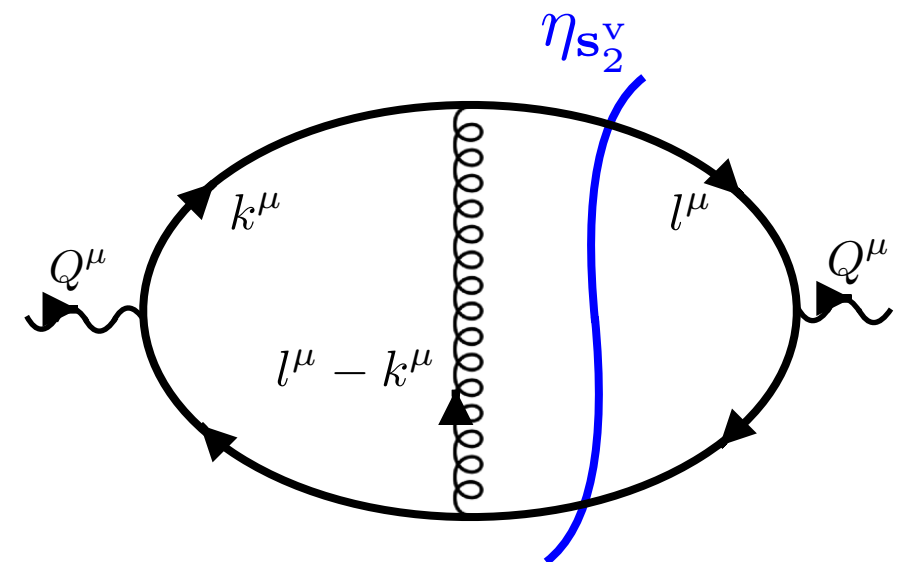
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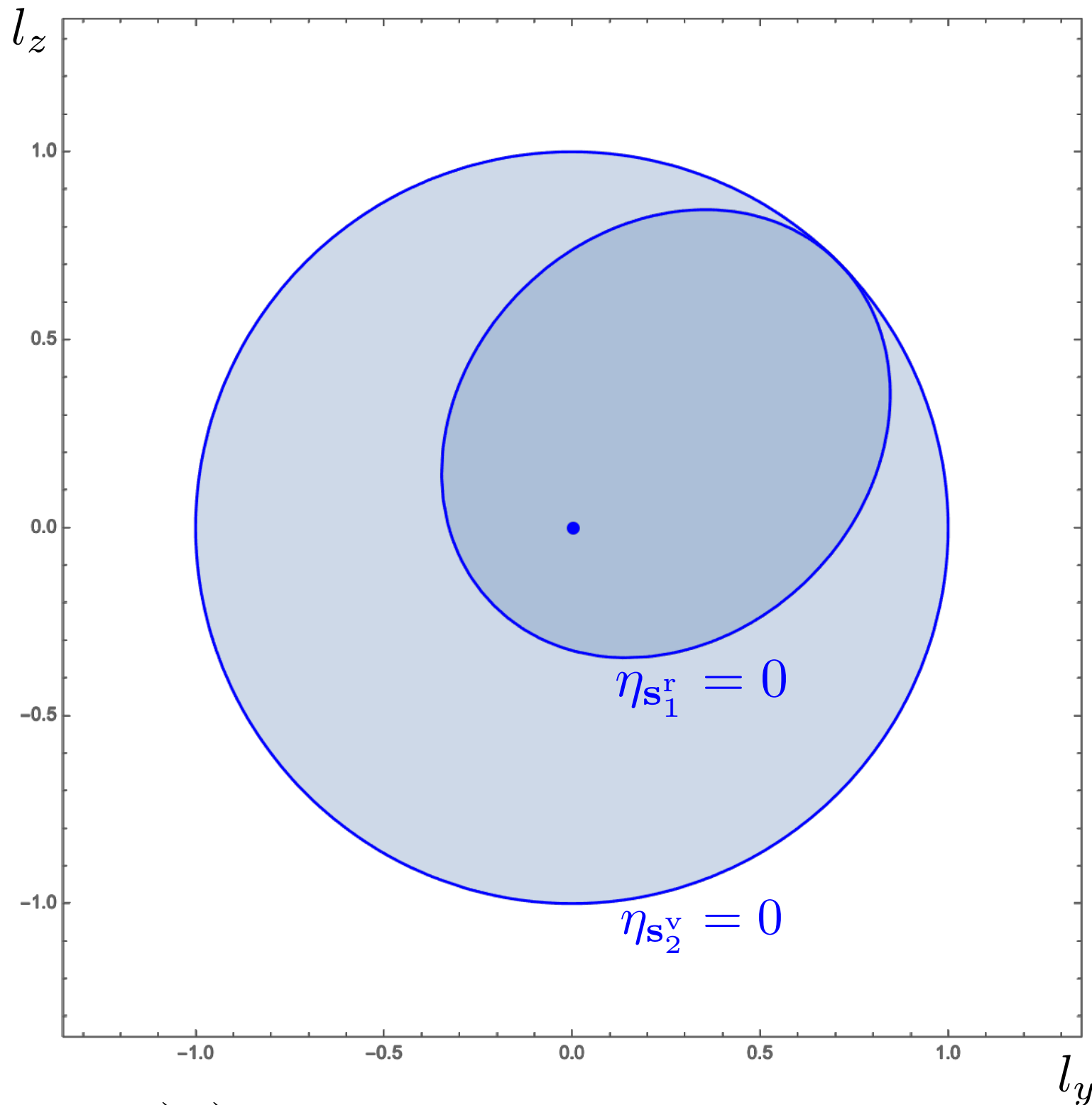
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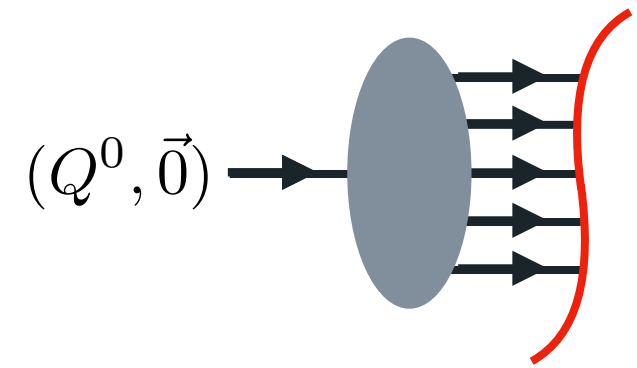


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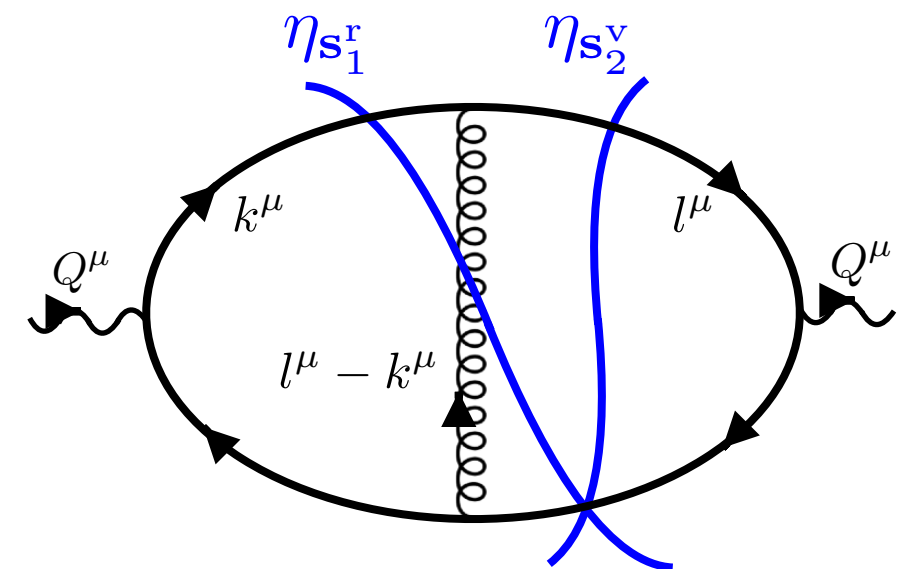


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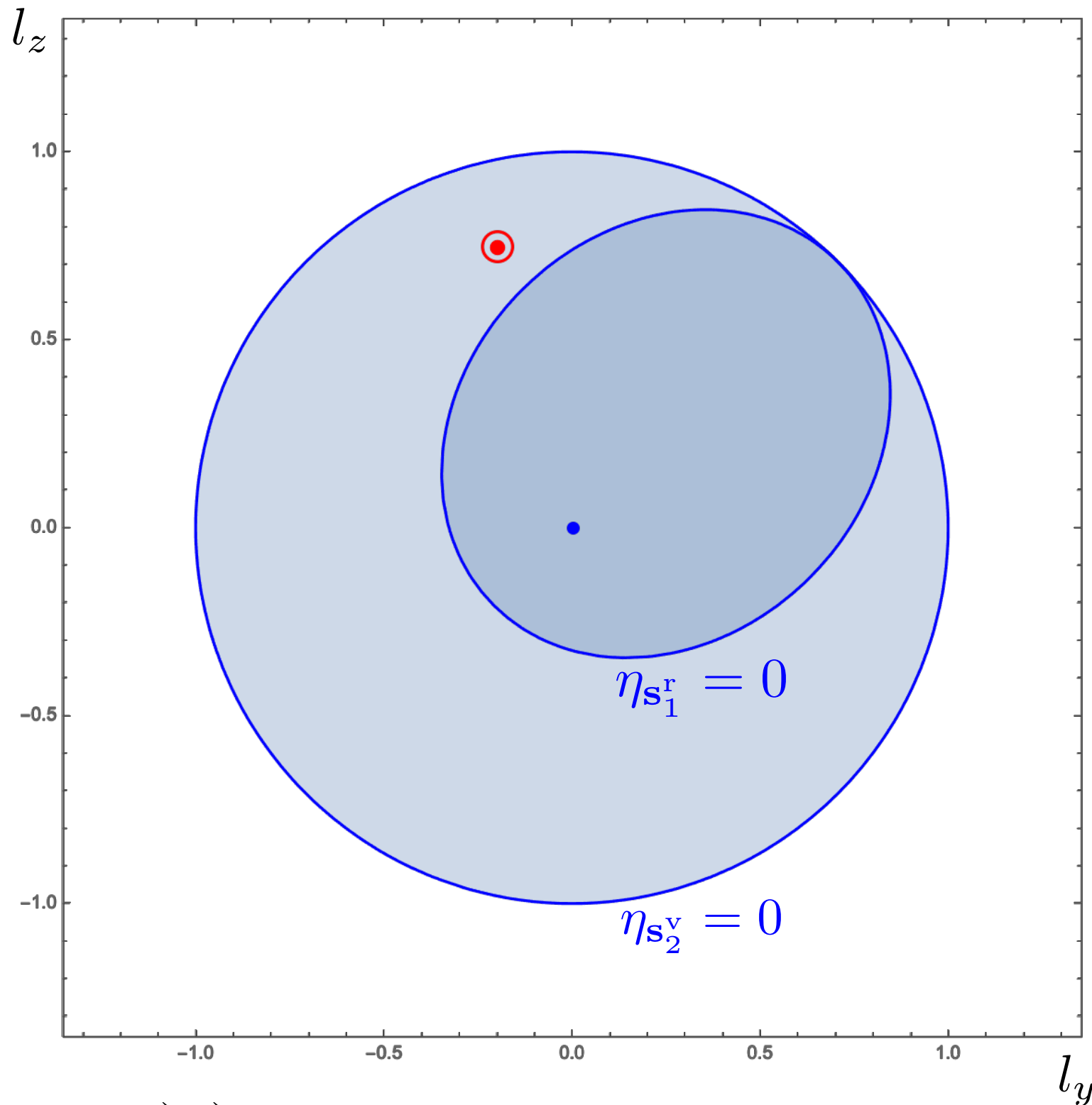
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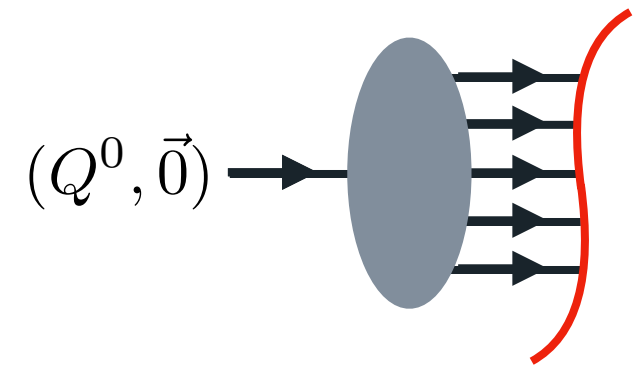


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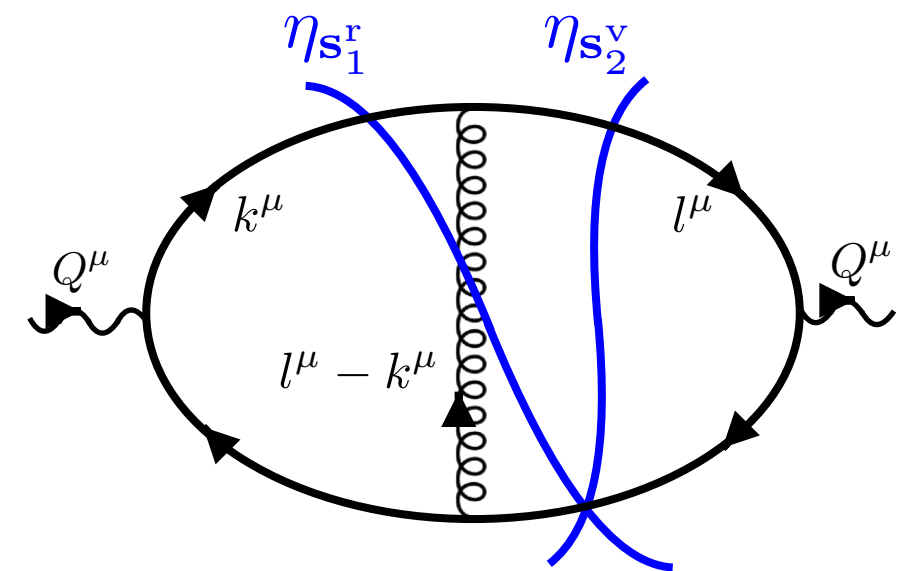


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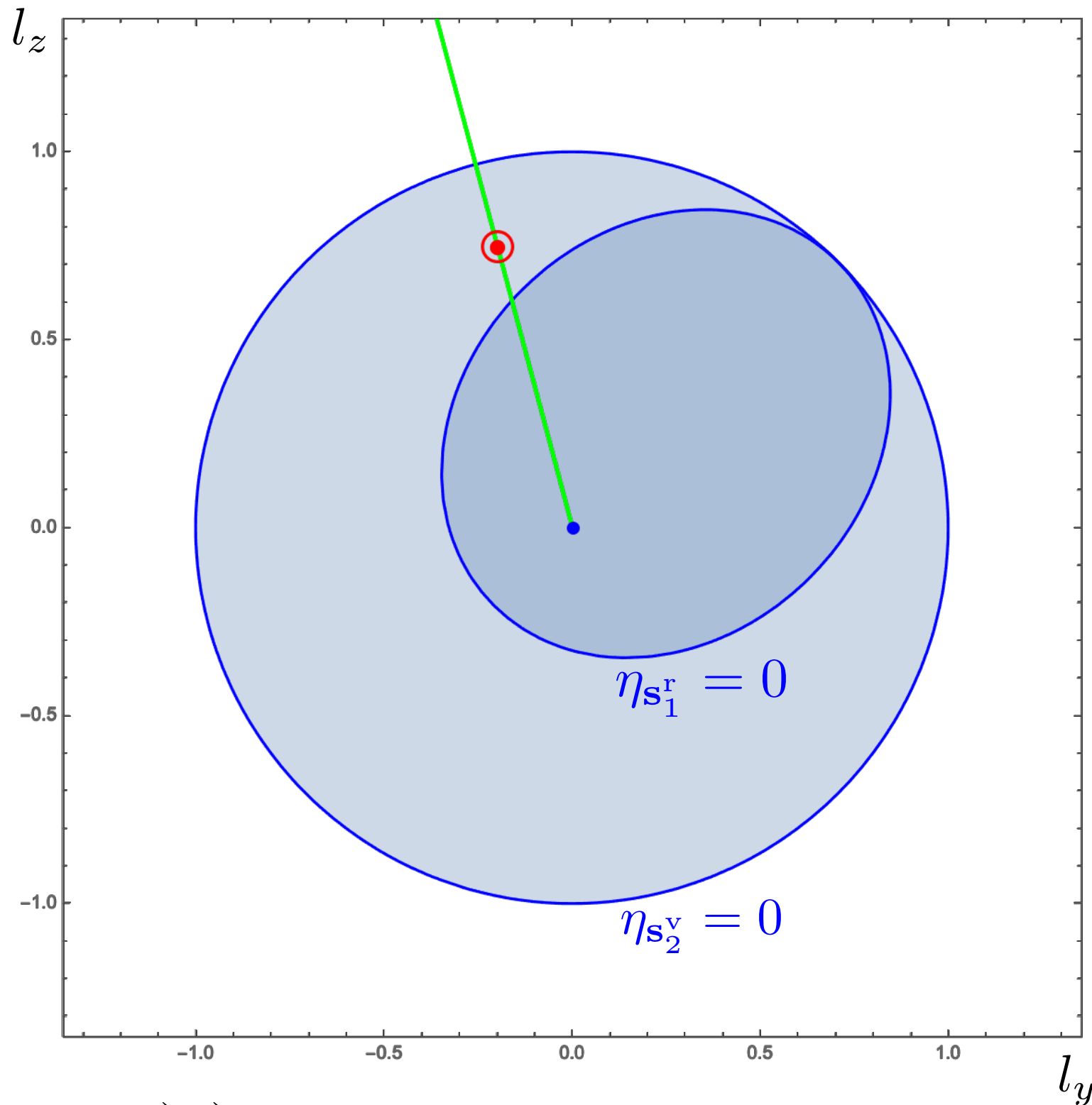
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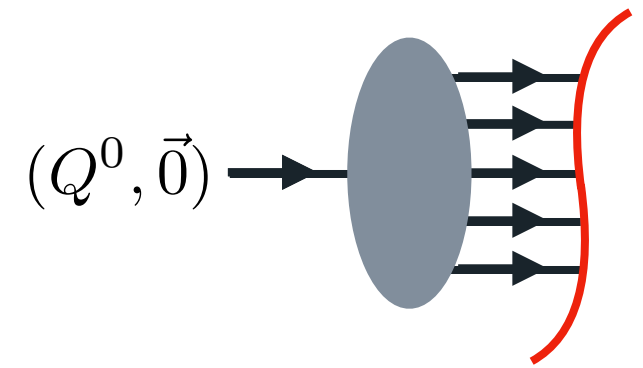


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LOCALITY UNITARITY: VISUALISATION



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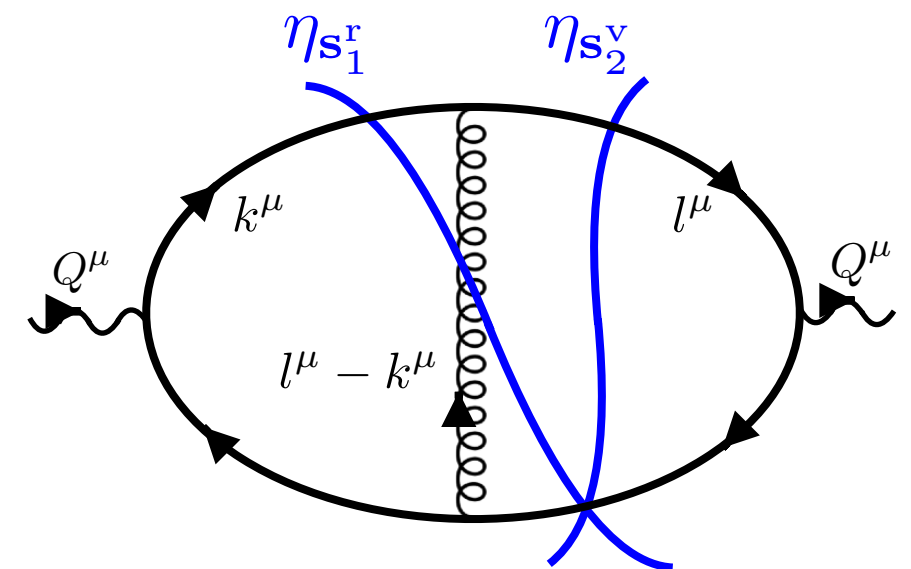


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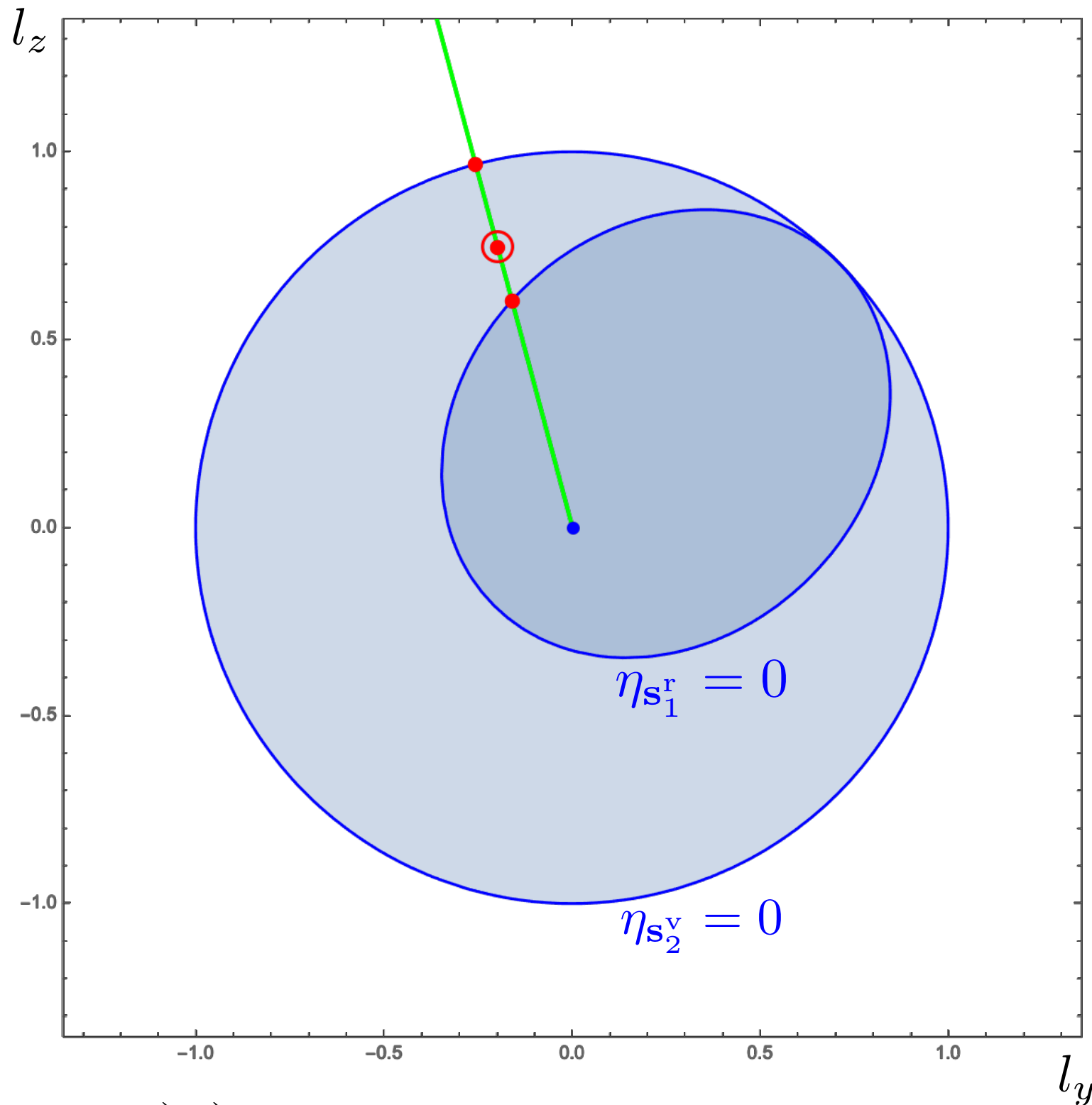
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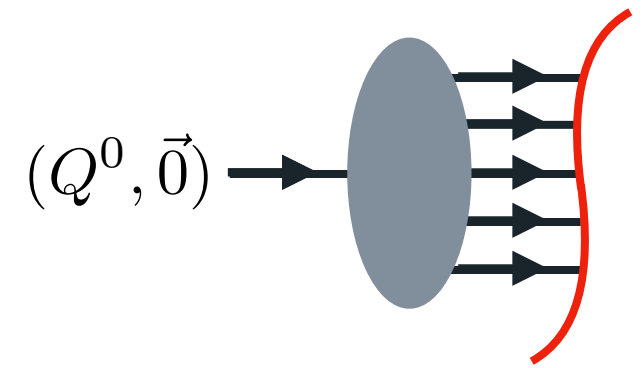


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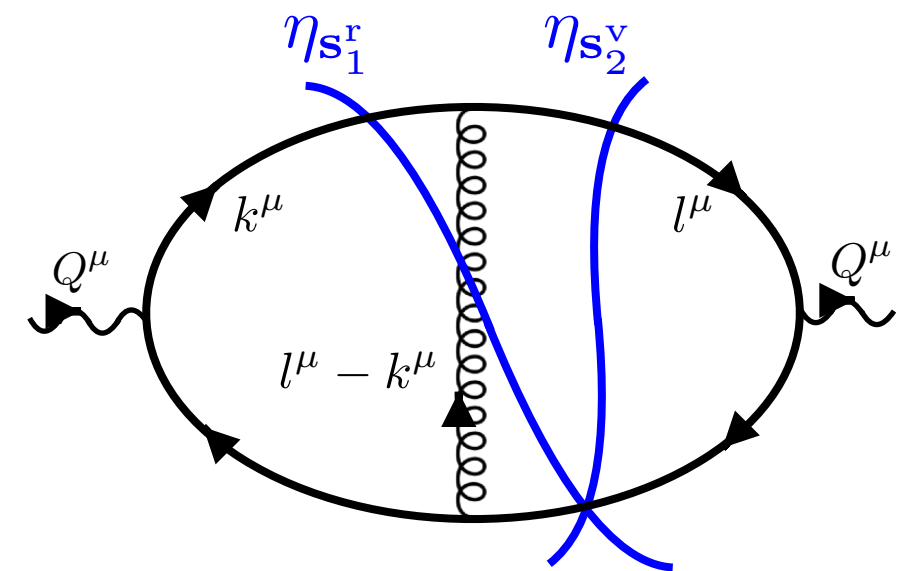


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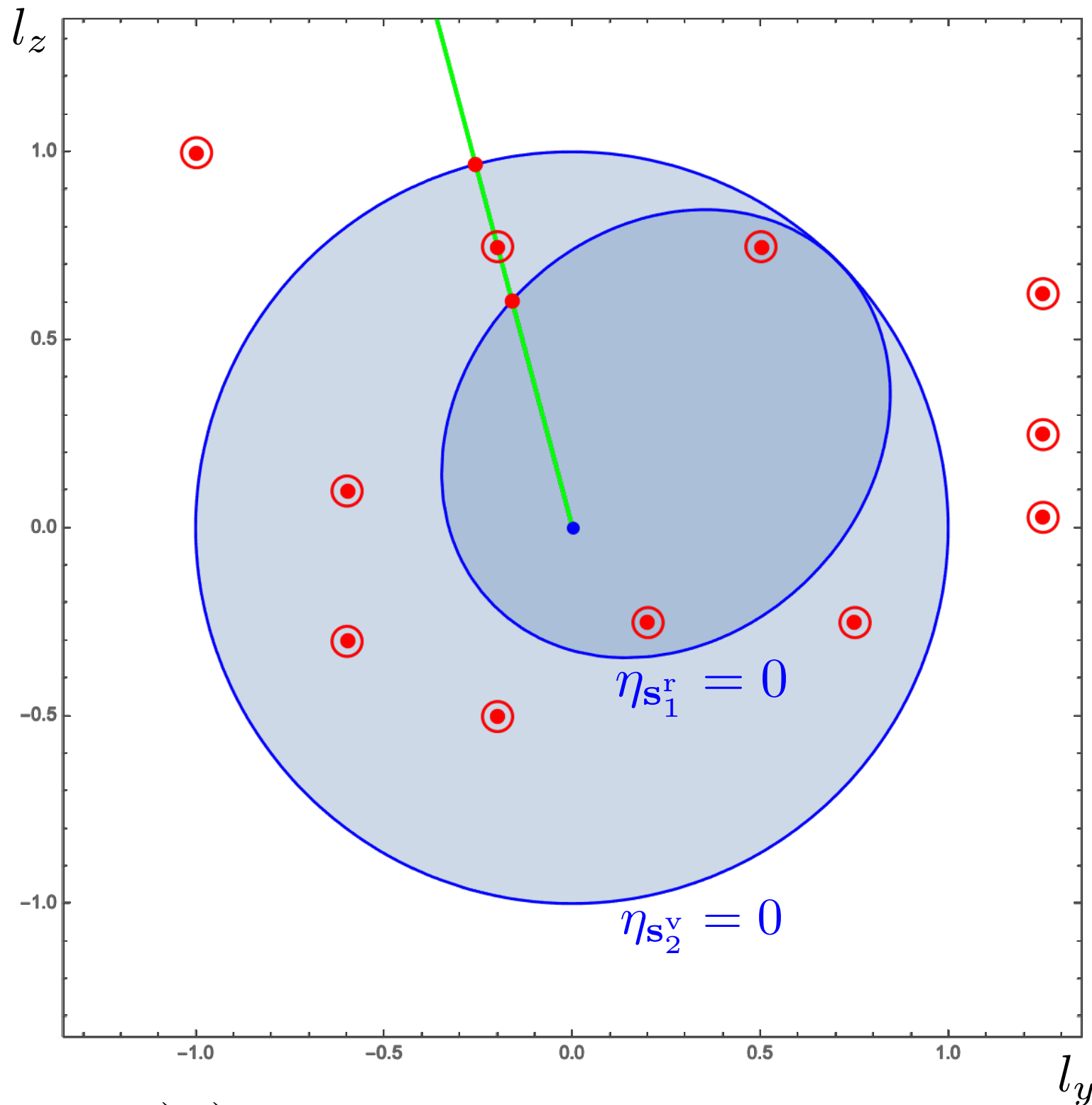
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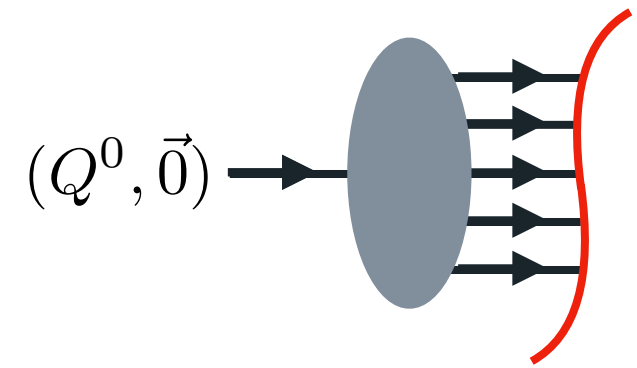


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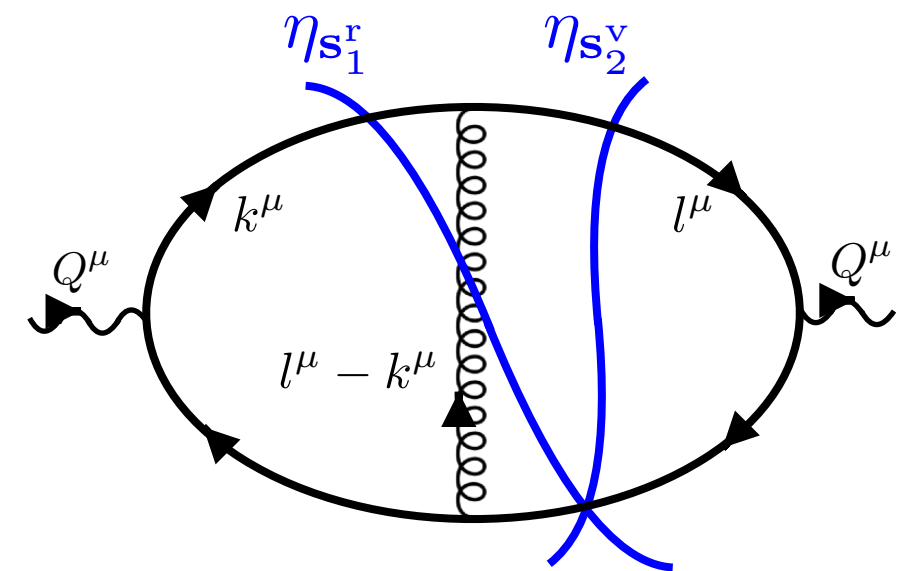


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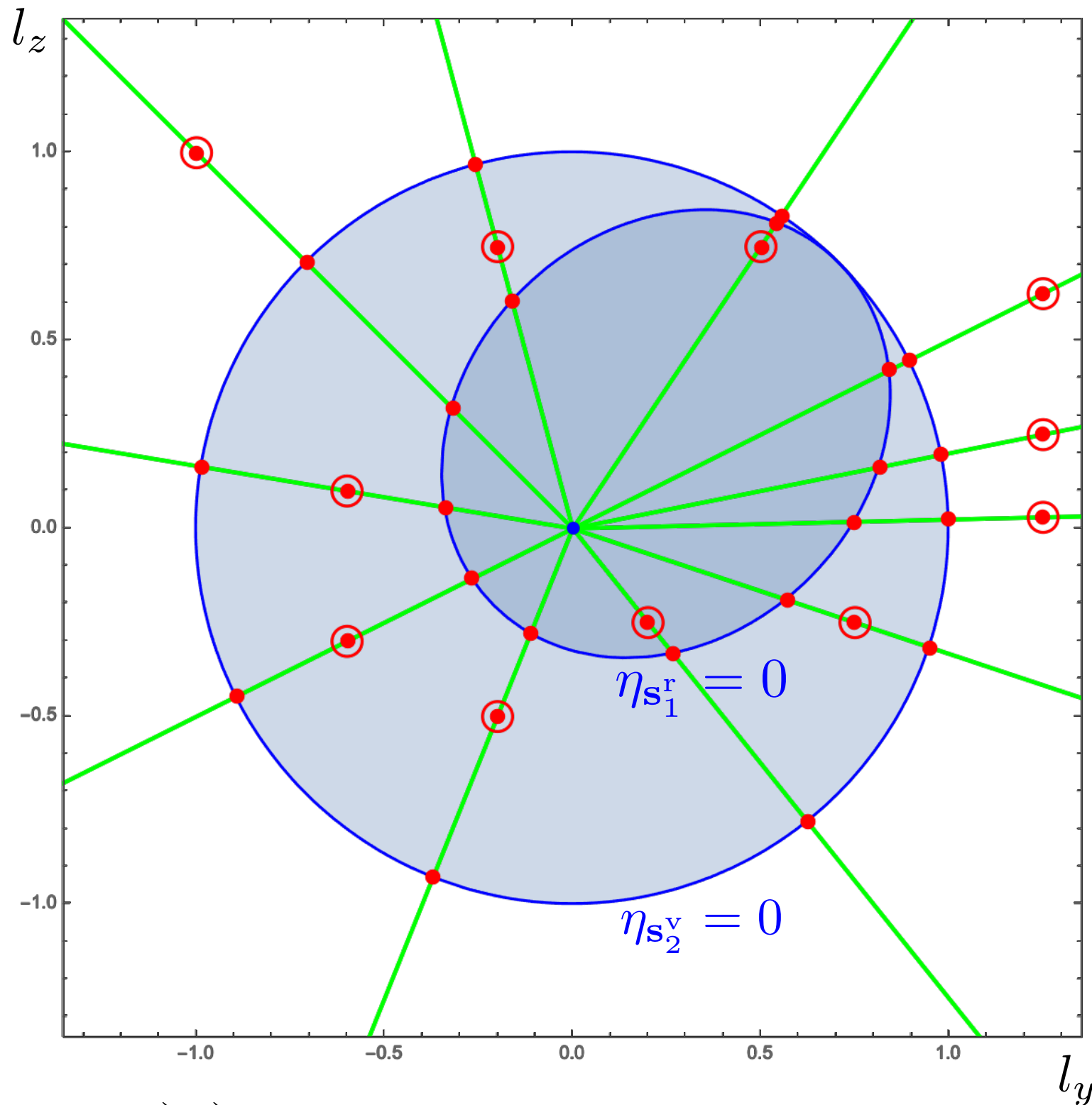
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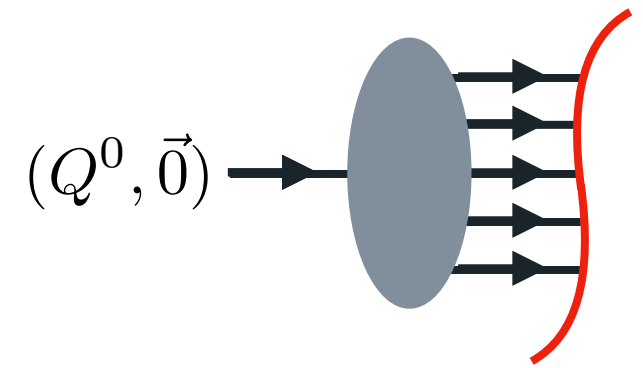


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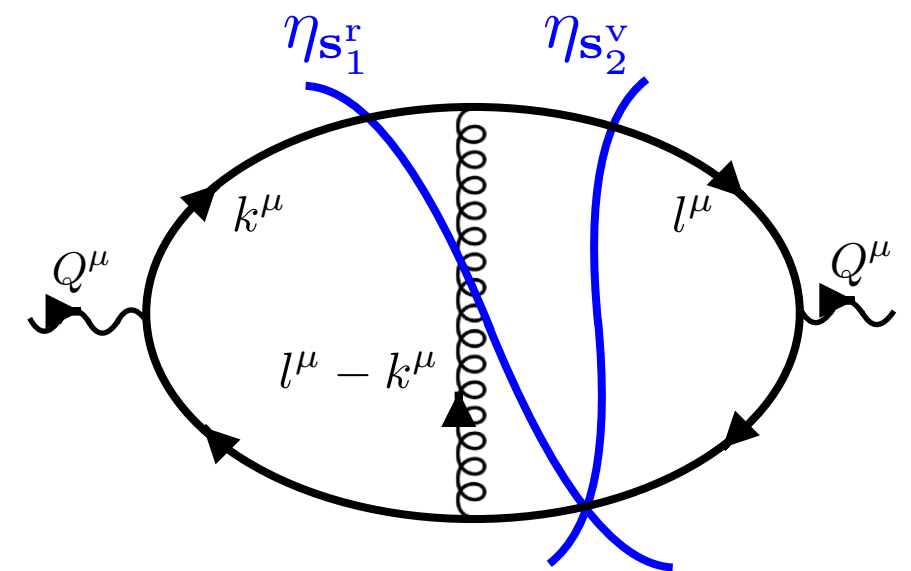


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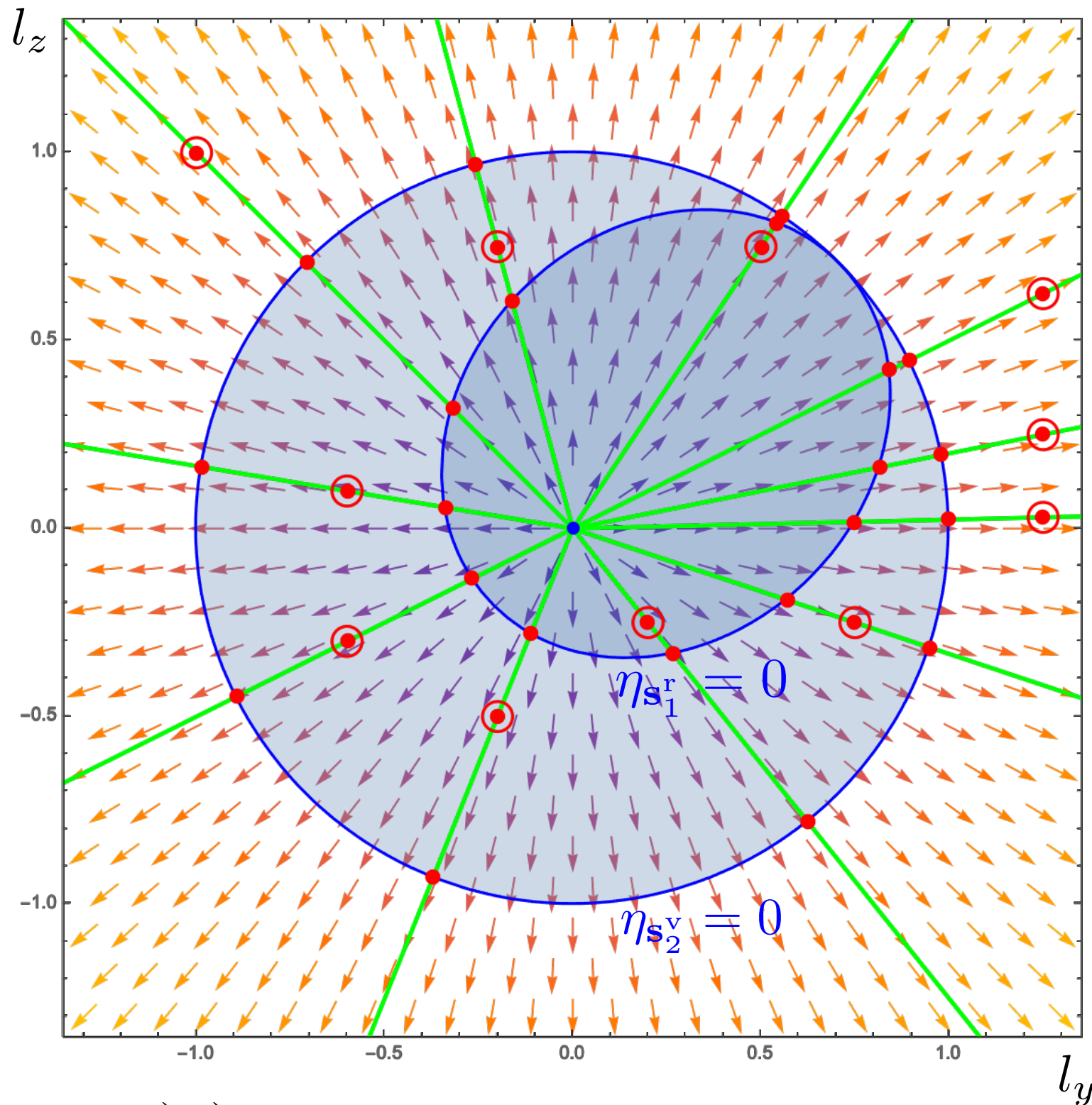
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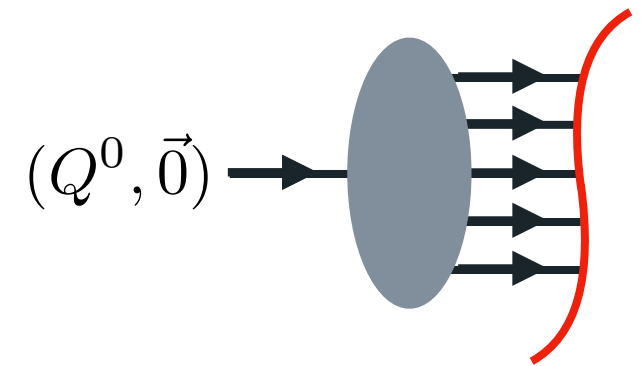


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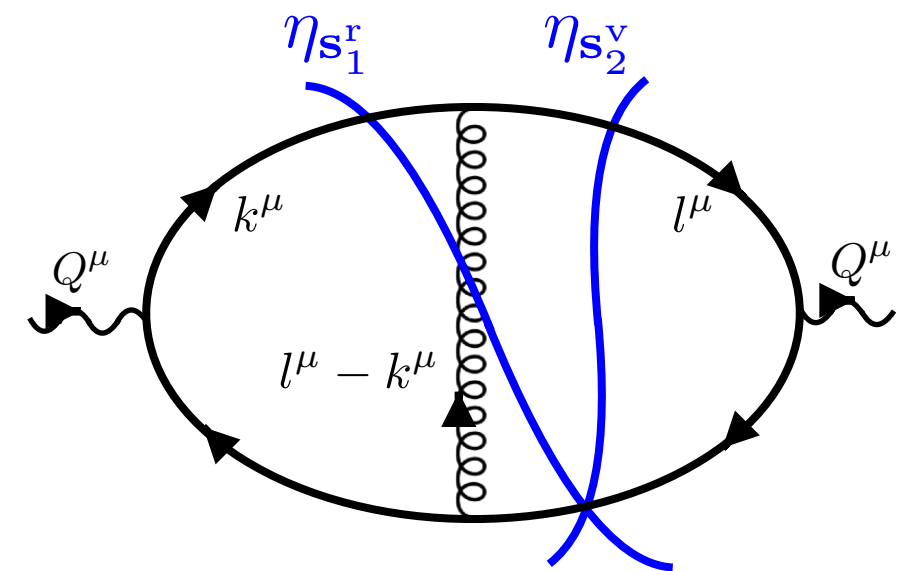


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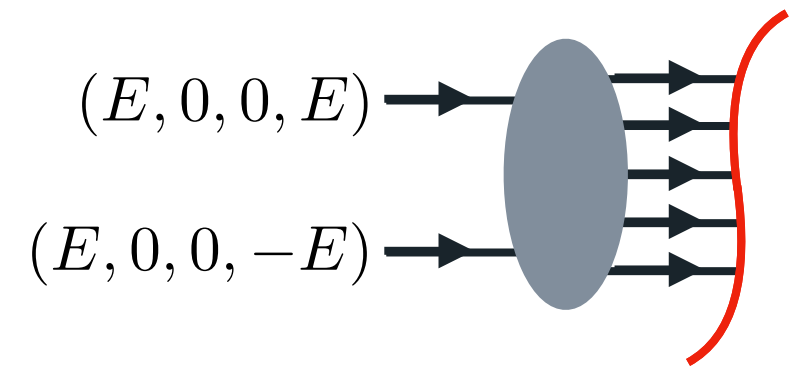
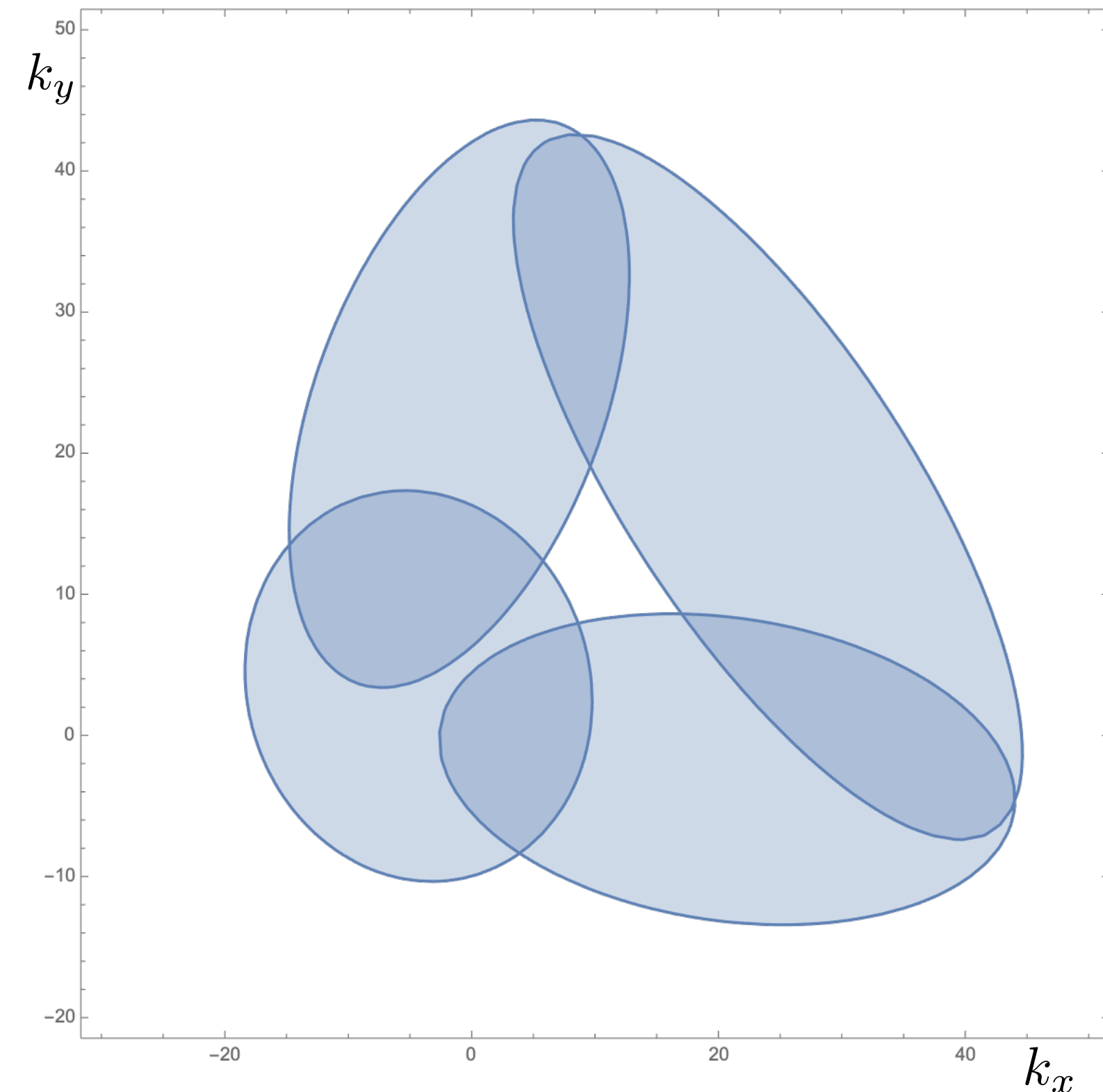


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LOCALITY UNITARITY

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

The **rescaling** change of variables is however **not general** :

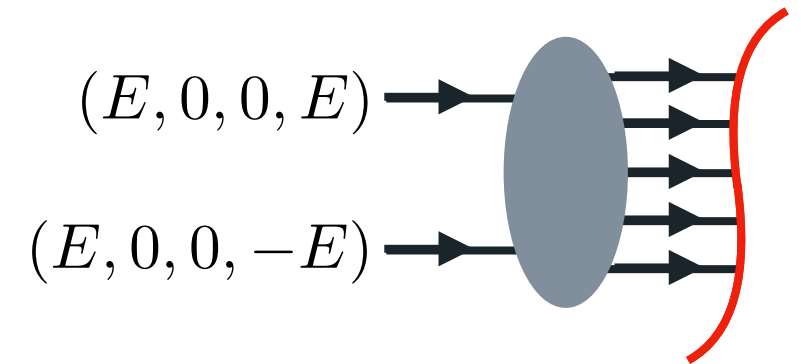
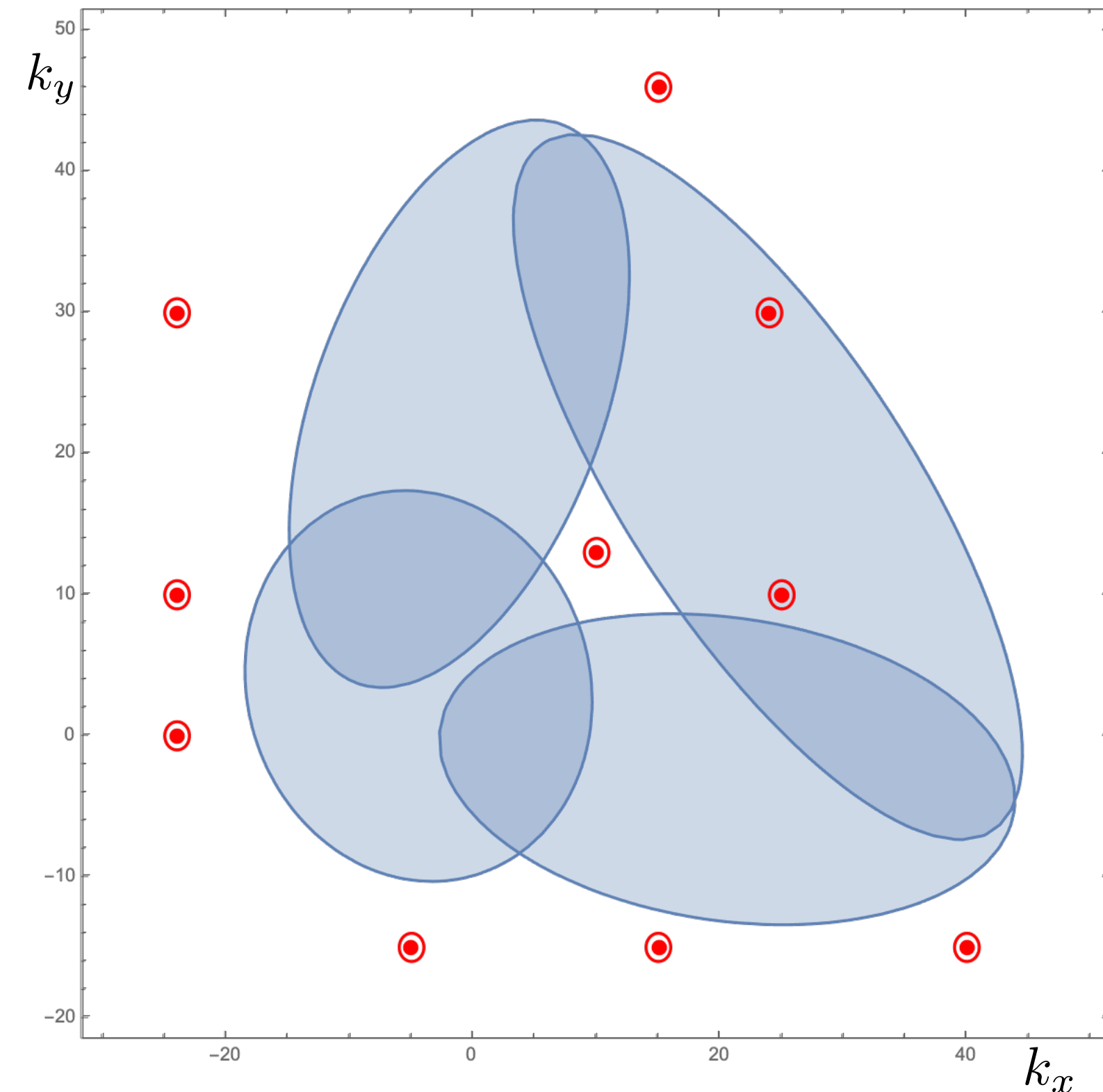


Ex: **Box_4E** from sect. 3.1 of
[Capatti, VH, Kermnashah, Pelloni, Ruijl, arxiv:1912.09291]

LOCALITY UNITARITY

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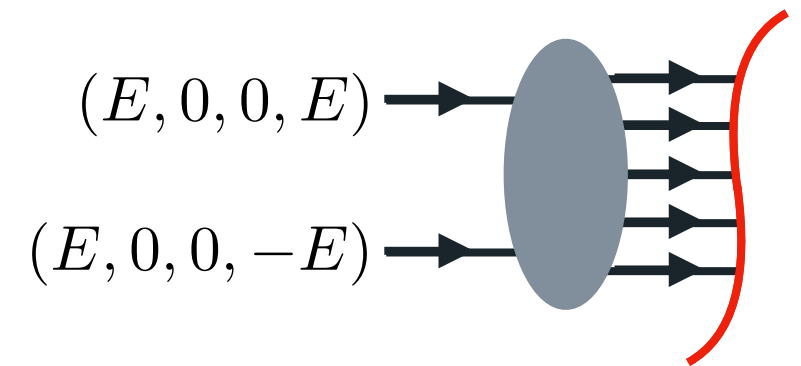
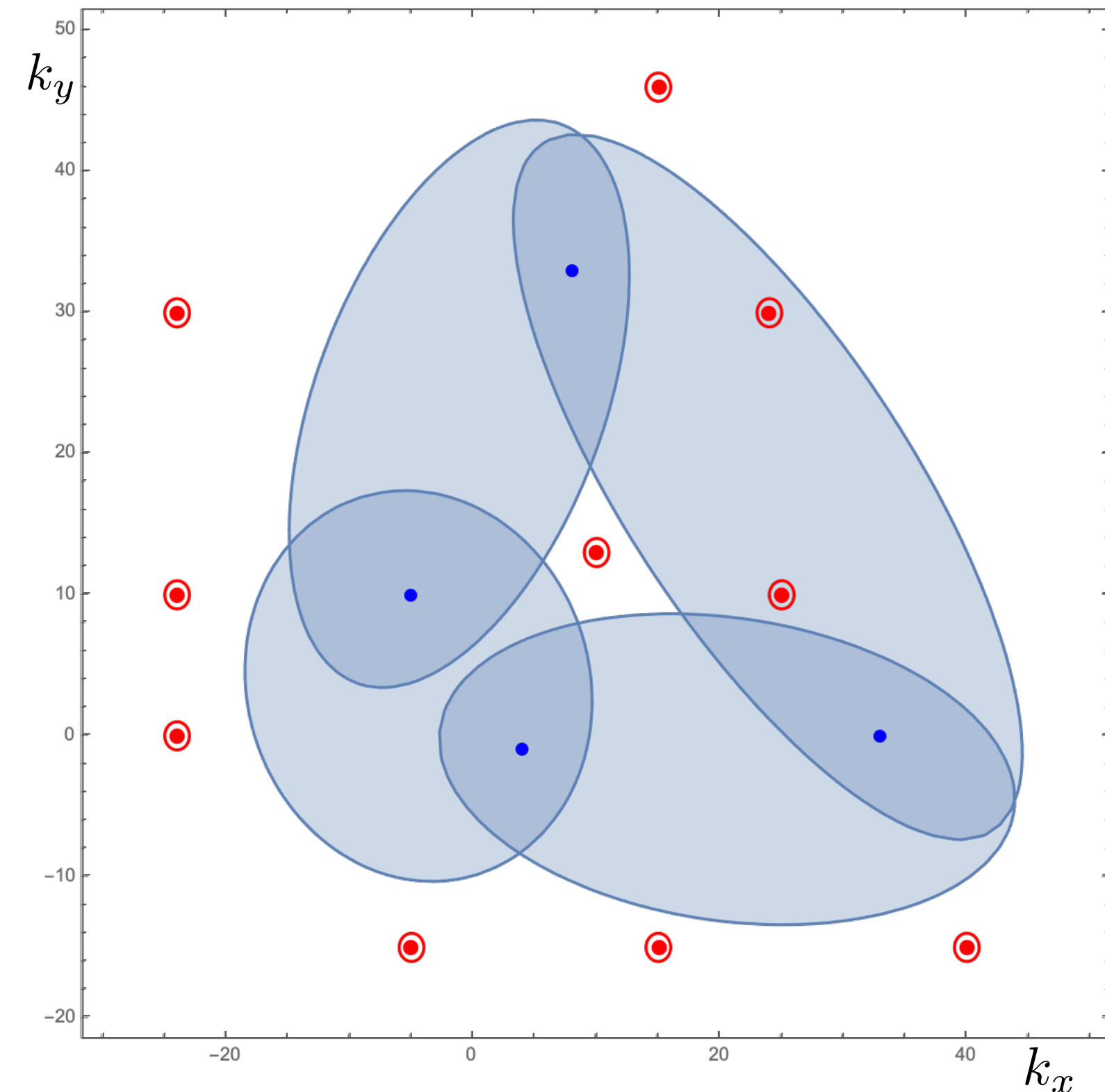
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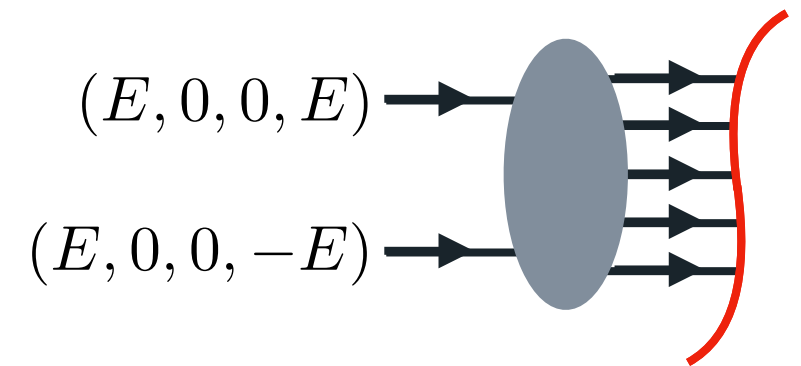
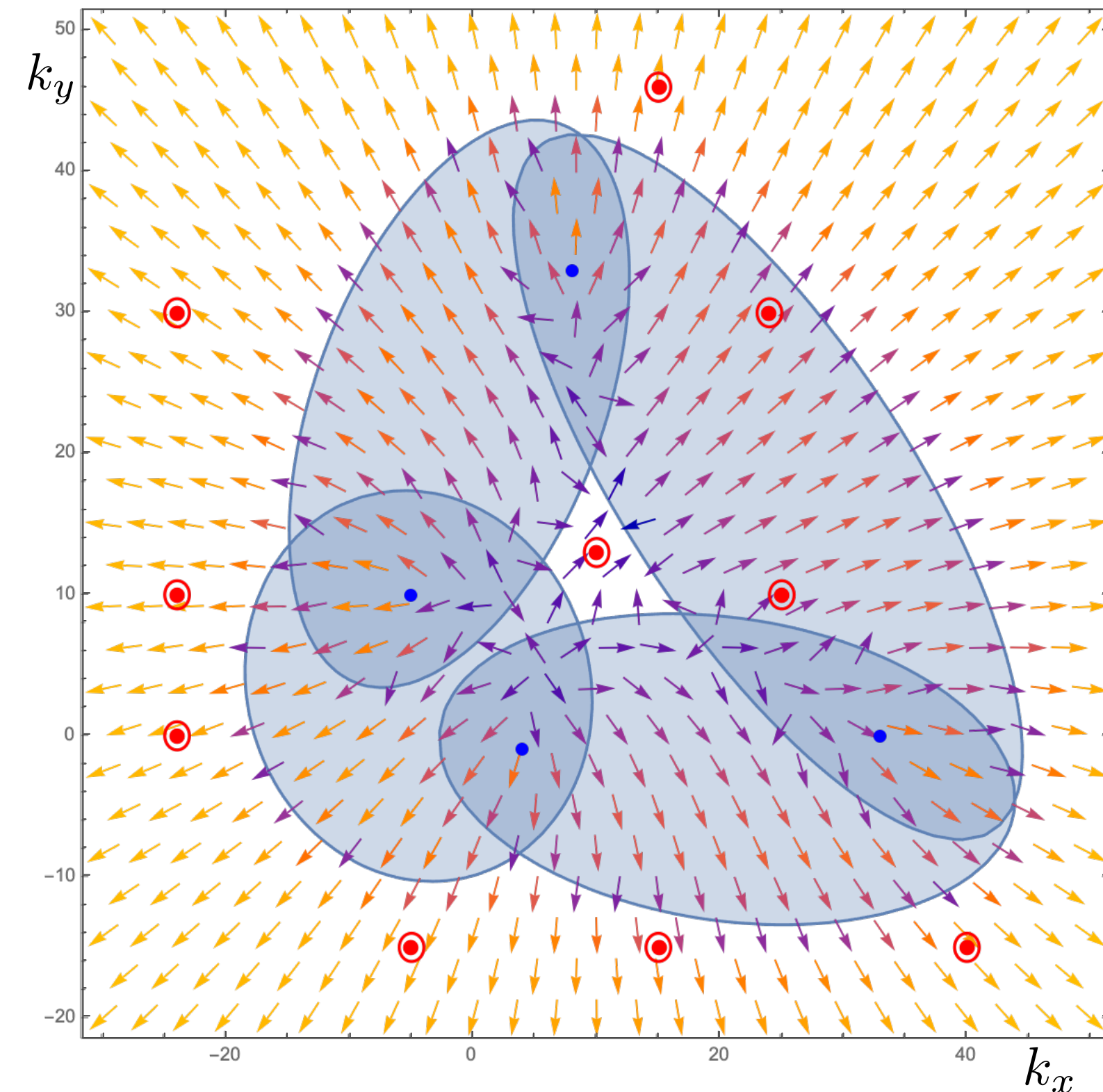


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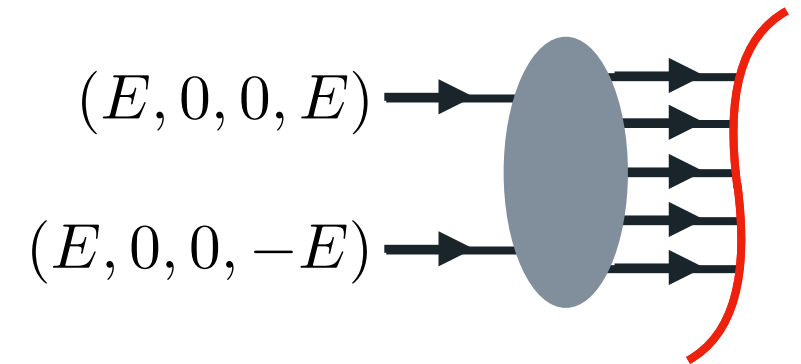
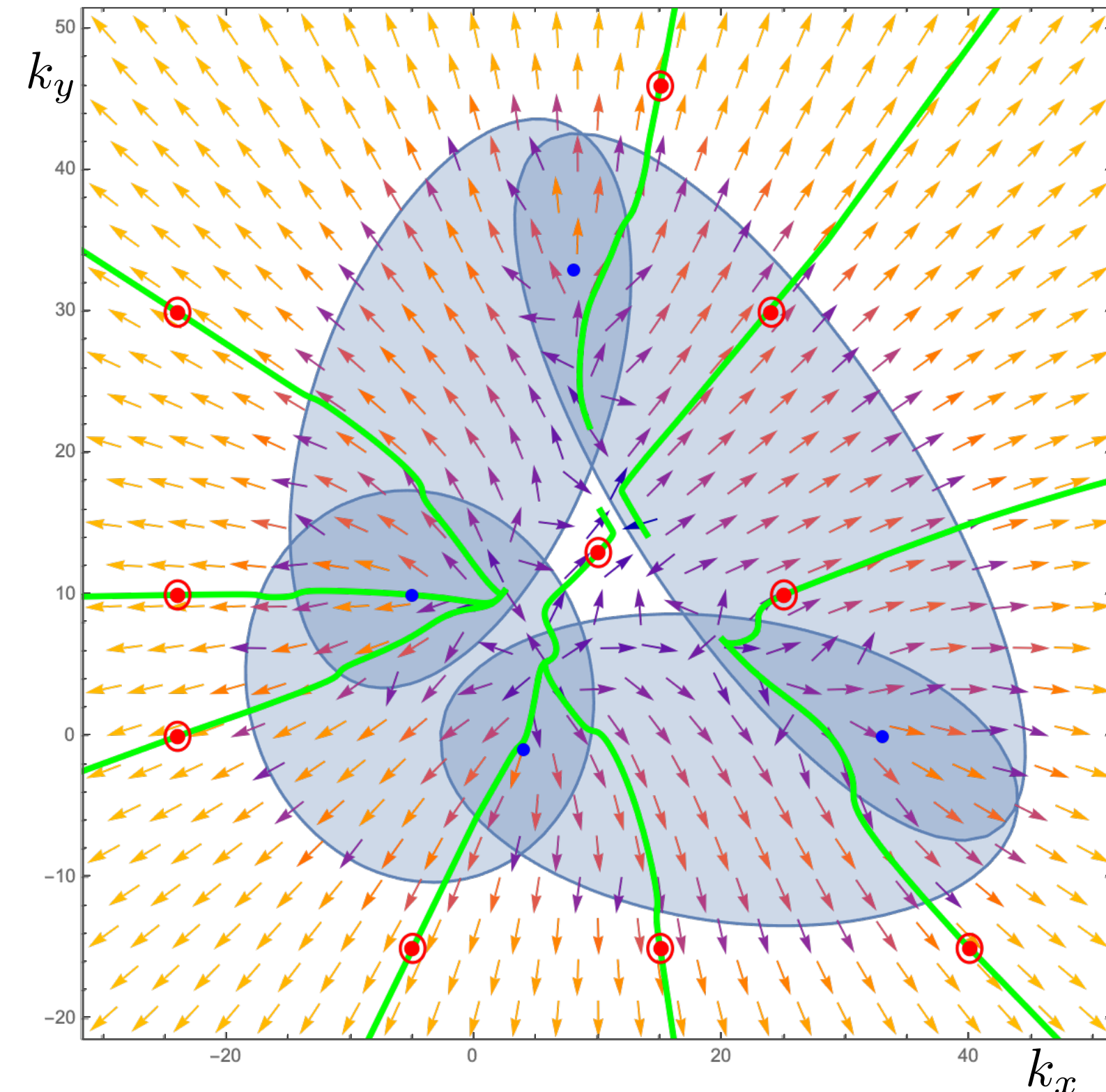
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[Capatti, VH, Kermnashah, Pelonni, Ruijl, arxiv:1912.09291]

LOCALITY UNITARITY

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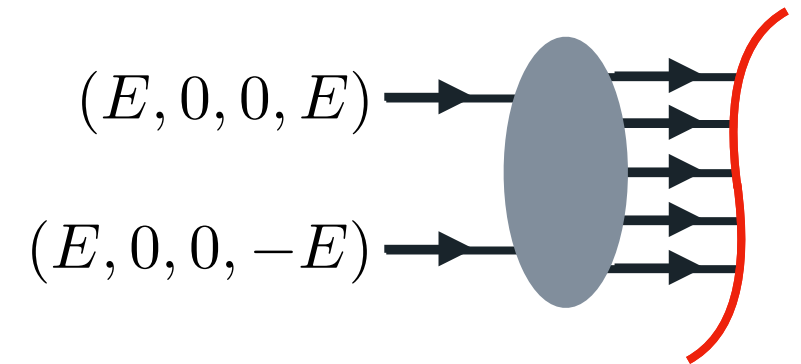
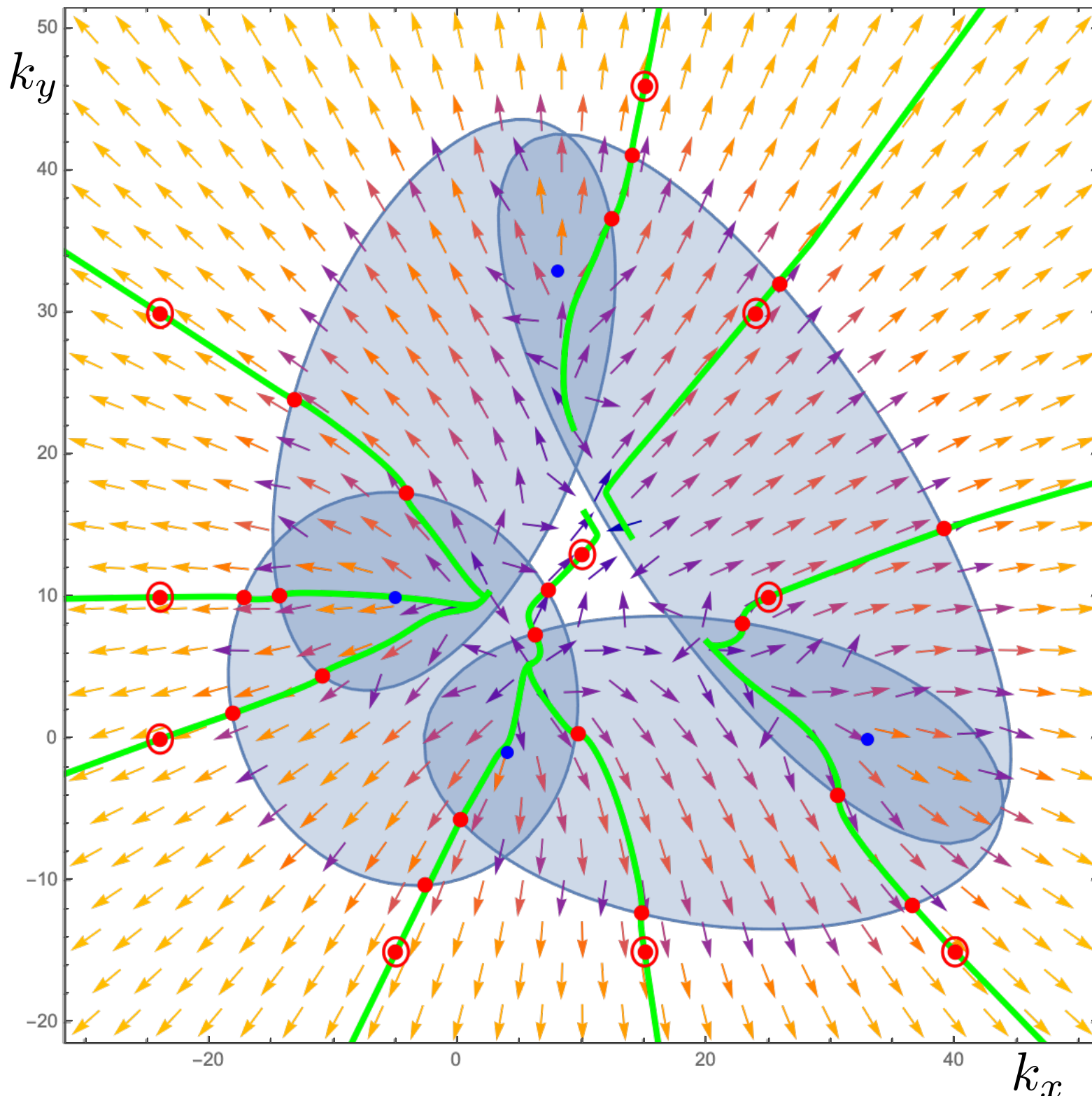
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LOCALITY UNITARITY: ALL-ORDERS PROOF

[Z. Capatti, VH, A. Pelloni, B. Ruijl, arXiv : [2010.01068](https://arxiv.org/abs/2010.01068)] [Summary in proceedings, arXiv : [2110.15662](https://arxiv.org/abs/2110.15662)]

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The **LTD representation** of the double triangle with rescaled momenta is

$$\begin{array}{c} tq_1 \\ \diagup \\ Q_0 - \text{---} \triangleleft \quad \triangleright \text{---} \\ \diagdown \\ tq_4 \end{array} \quad \begin{array}{c} tq_2 \\ \diagdown \\ tq_3 \\ \diagup \\ tq_5 \end{array} \quad f_{\text{ltd}} \left(\text{---} \triangleleft \triangleright \text{---} \right) \Big|_{tq_i} = \left[\begin{array}{cccc} \text{---} \triangleleft \triangleright \text{---} & + & \text{---} \triangleleft \triangleright \text{---} & + & \text{---} \triangleleft \triangleright \text{---} & + & \text{---} \triangleleft \triangleright \text{---} \\ + & \text{---} \triangleleft \triangleright \text{---} & + & \text{---} \triangleleft \triangleright \text{---} & + & \text{---} \triangleleft \triangleright \text{---} & + & \text{---} \triangleleft \triangleright \text{---} \end{array} \right] q_i \rightarrow tq_i$$

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Then one can capture the **thresholds** of this forward-scattering graphs with

$$\begin{array}{c} \text{diagram with blue line} \\ \text{diagram with red line} \end{array} = \int d^3\vec{p} d^3\vec{k} \left[\lim_{t \rightarrow t_v^*} (t - t_v^*) f_{\text{ltd}} \left(\triangleleft \right) \Big|_{tq_i} + \lim_{t \rightarrow t_r^*} (t - t_r^*) f_{\text{ltd}} \left(\triangleleft \right) \Big|_{tq_i} \right]$$

g_v , g_r can be written as different limits of the same function!

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Solving delta in the scaling variable $\Rightarrow$ **1d residue theorem along the line** $\gamma(t) = (t\vec{k}, t\vec{p})$

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LU representation

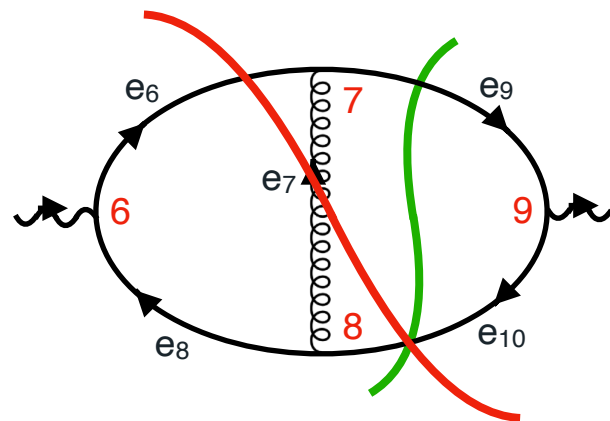
Cutkosky, but at the local level! We prove cancellations by studying the limit $t_r^* \rightarrow t_v^*$

LOCALITY UNITARITY

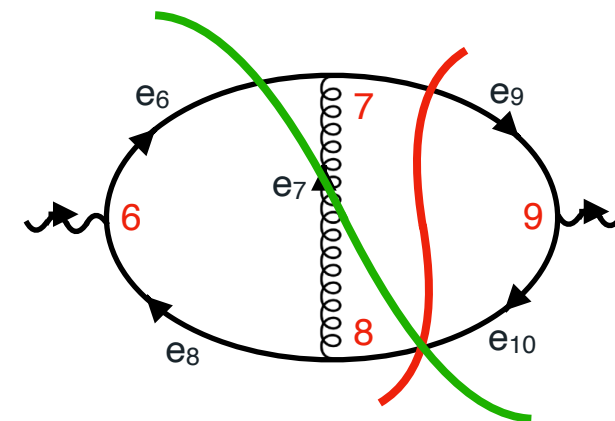
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This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :

— = Cutkosky cut — = threshold singularity



cancel

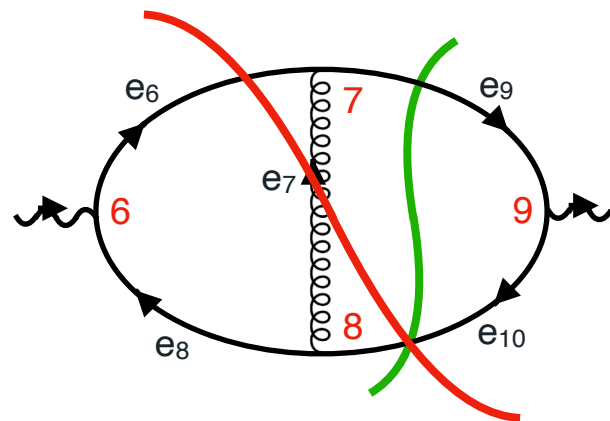


LOCALITY UNITARITY

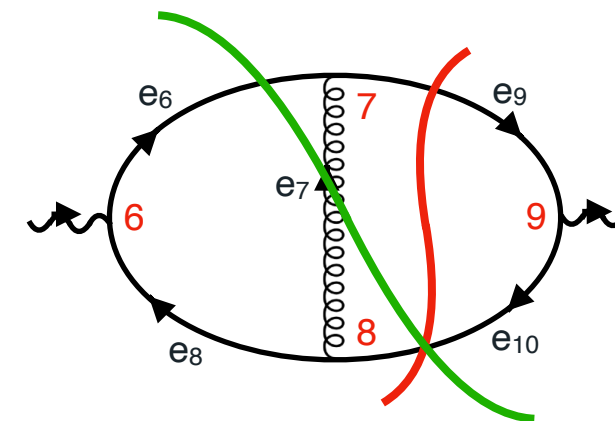
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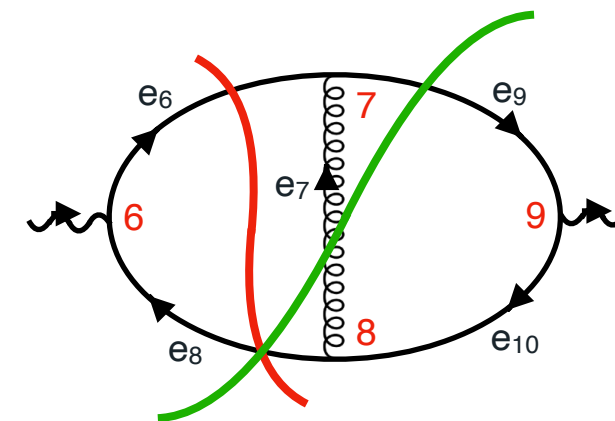
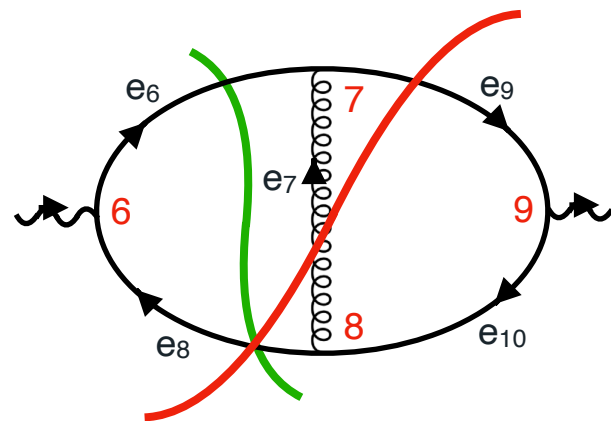
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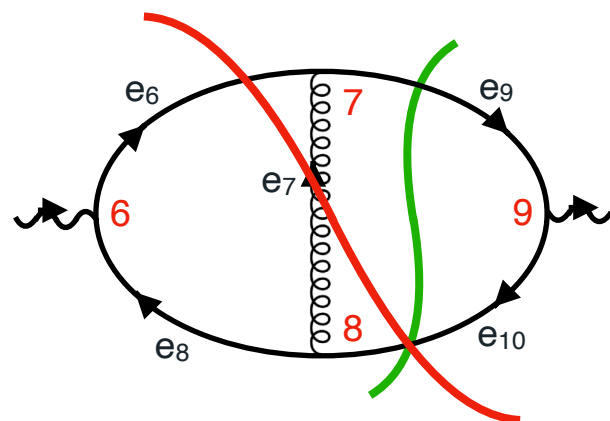


LOCALITY UNITARITY

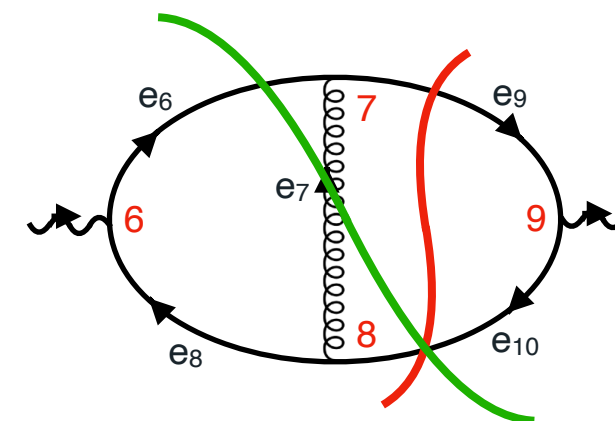
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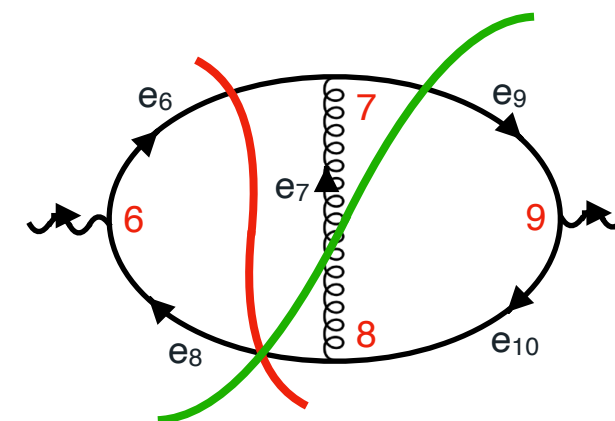
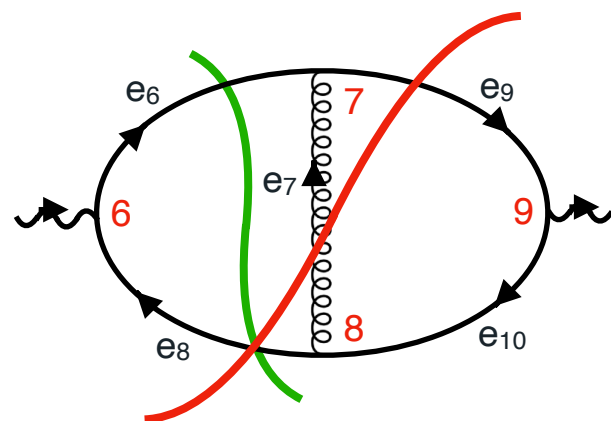
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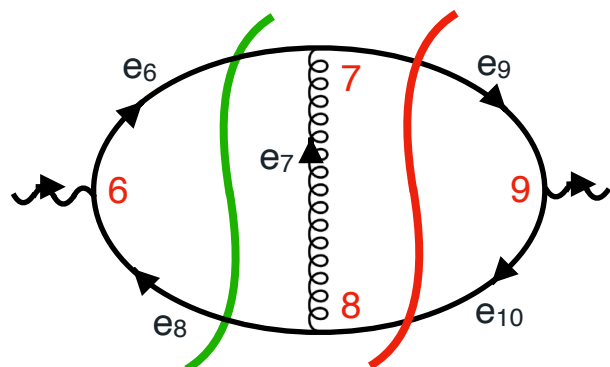
cancels



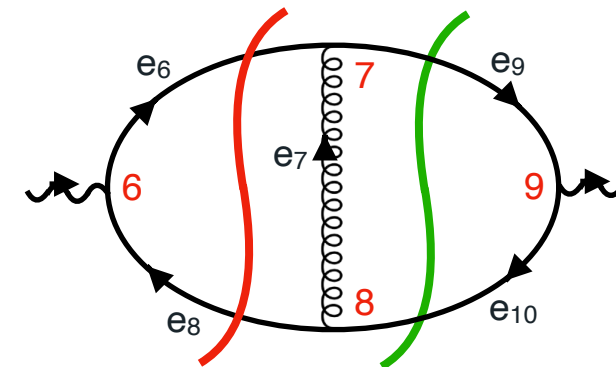
cancels



Even for **non-pinched singular threshold** ! (when $\mathcal{O}_s \equiv 1$) :



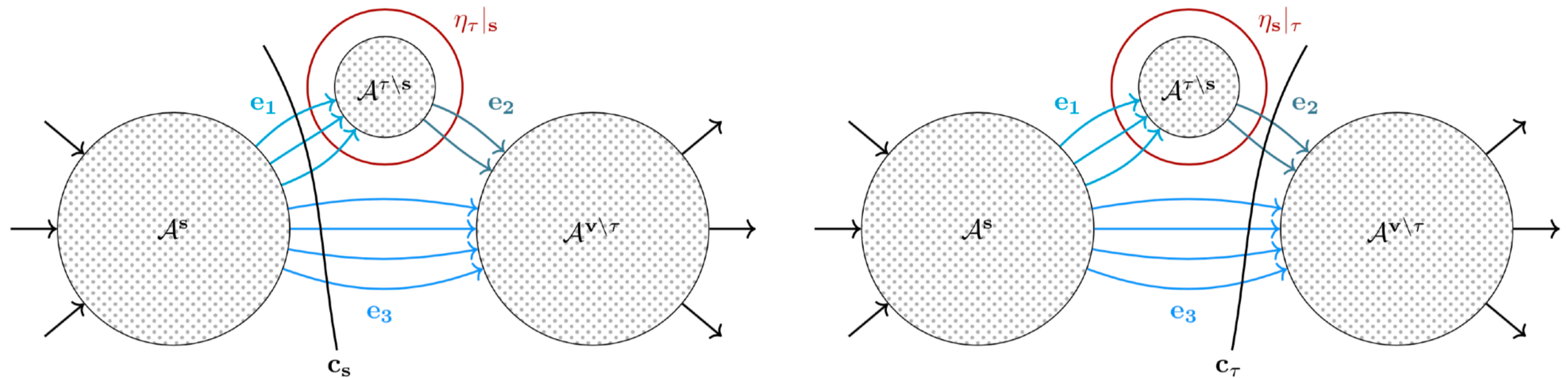
cancels



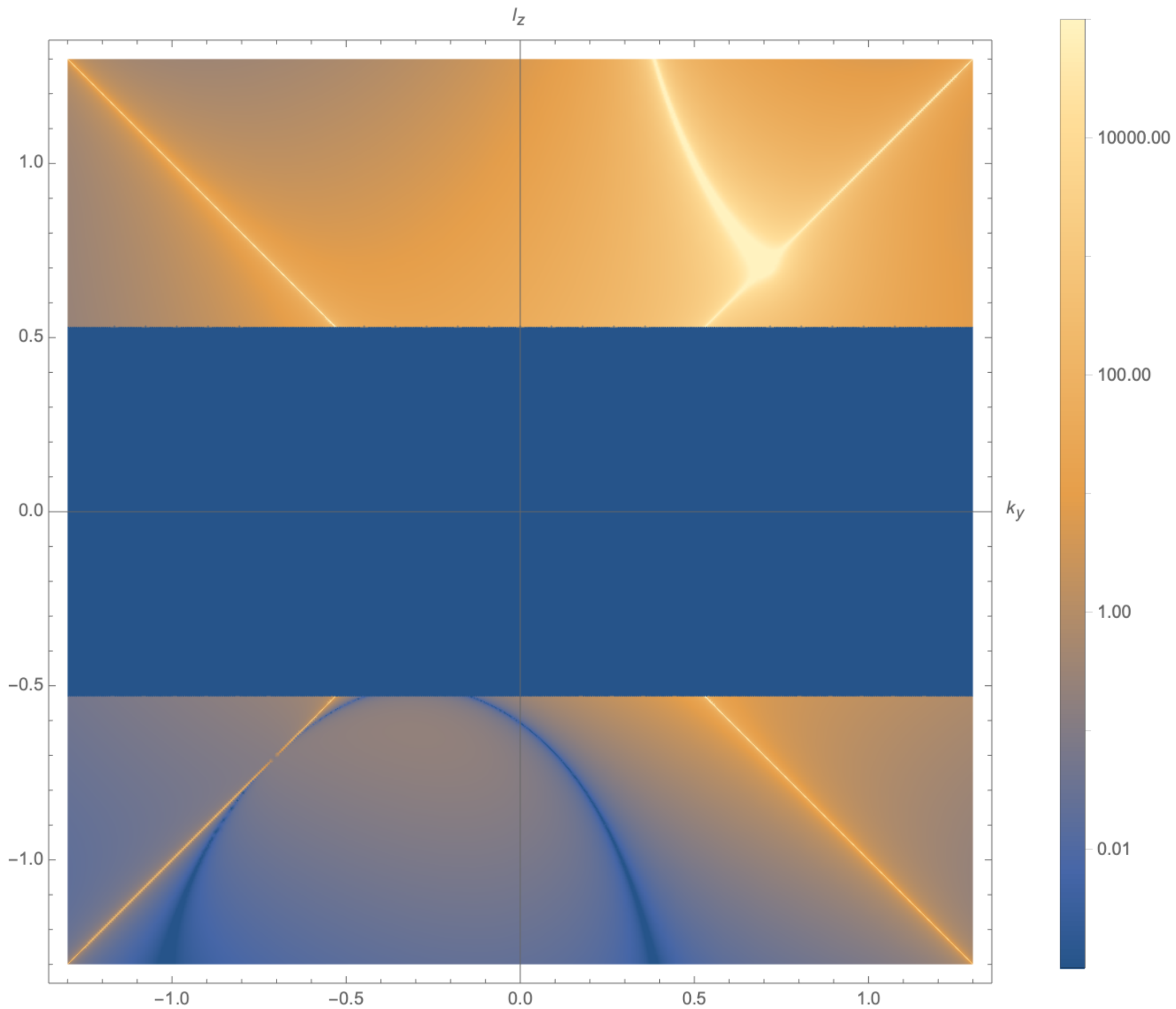
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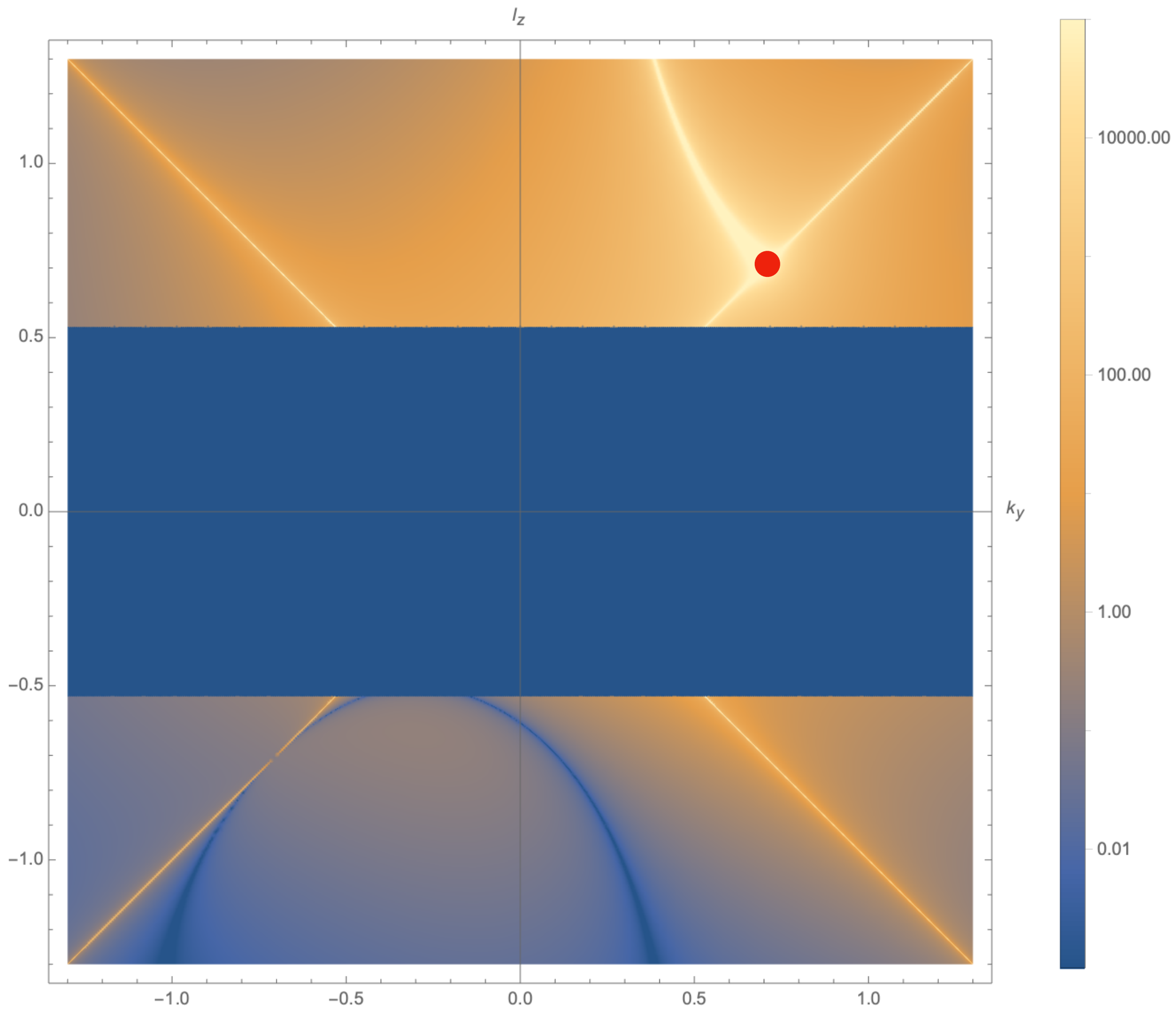


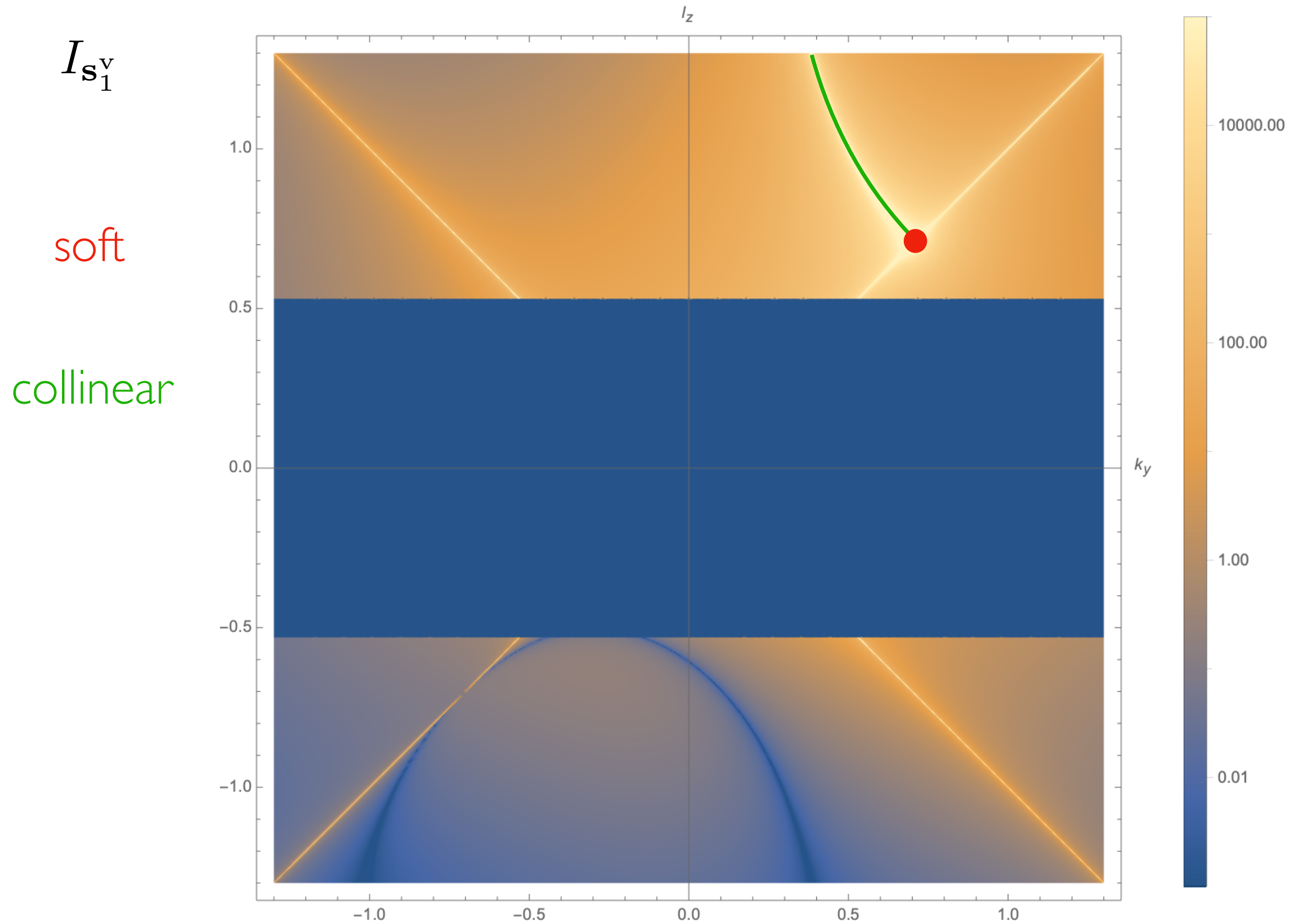
$I_{\mathbf{s}_1^v}$

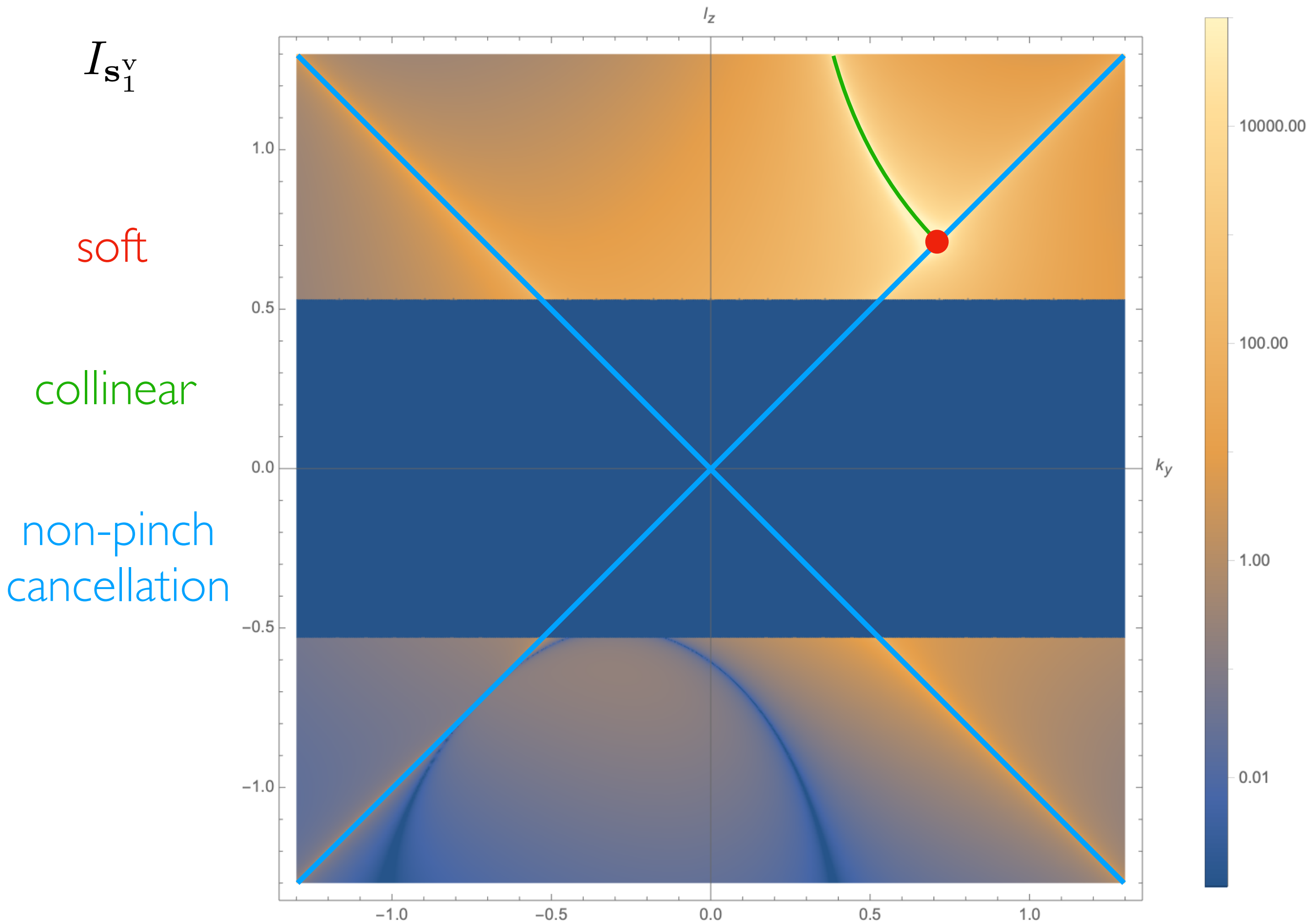


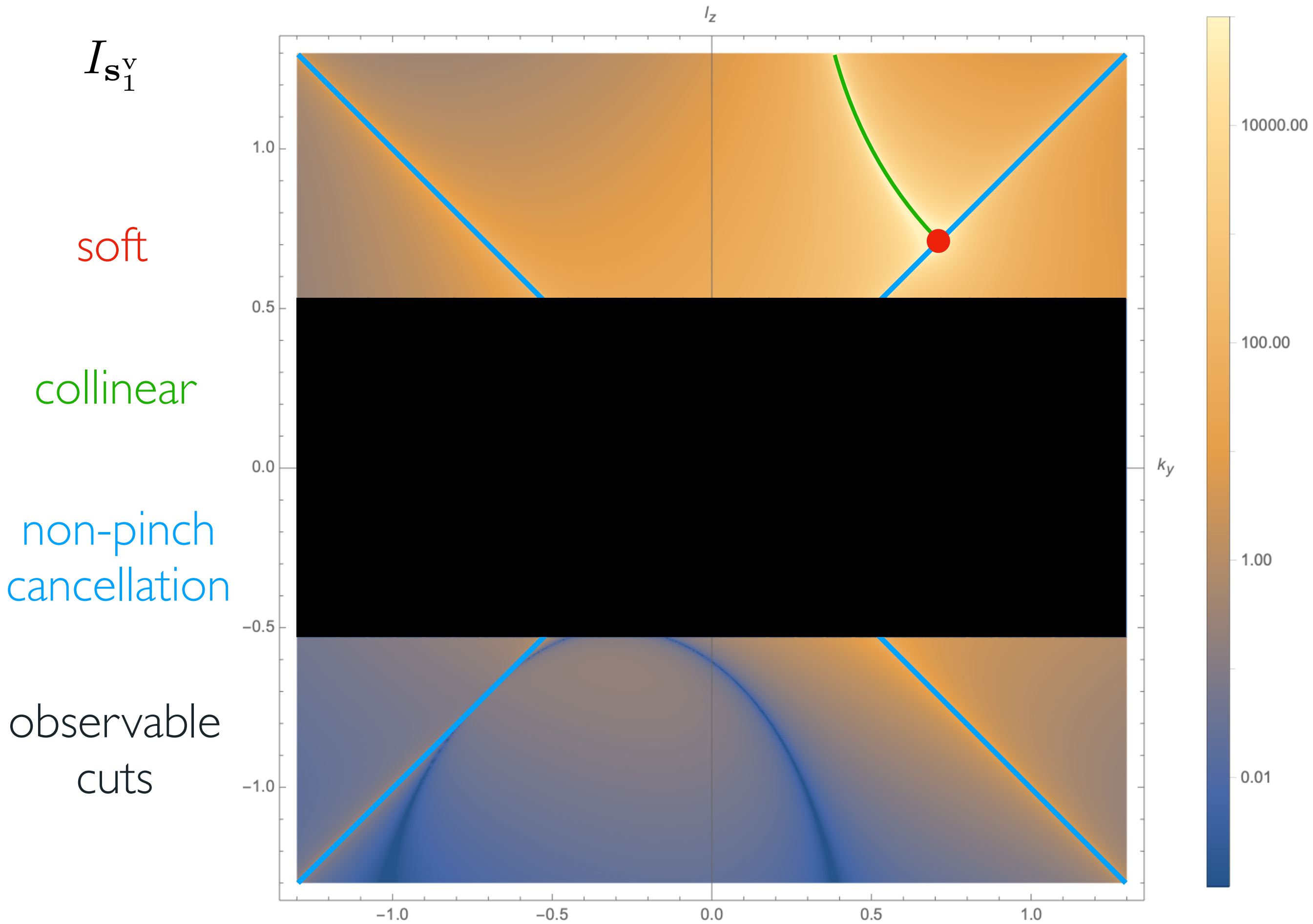
$I_{\mathbf{s}_1^v}$

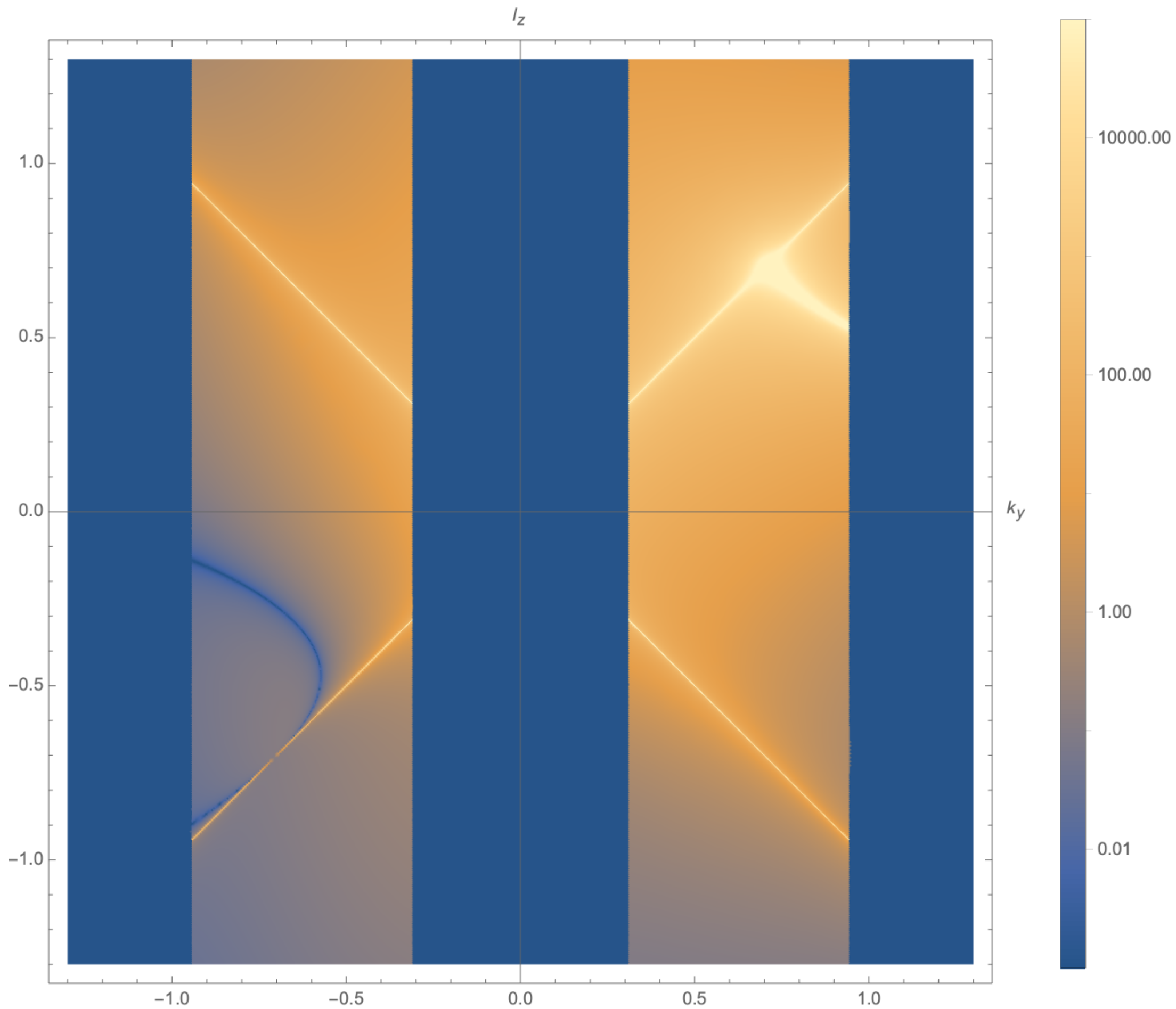
soft



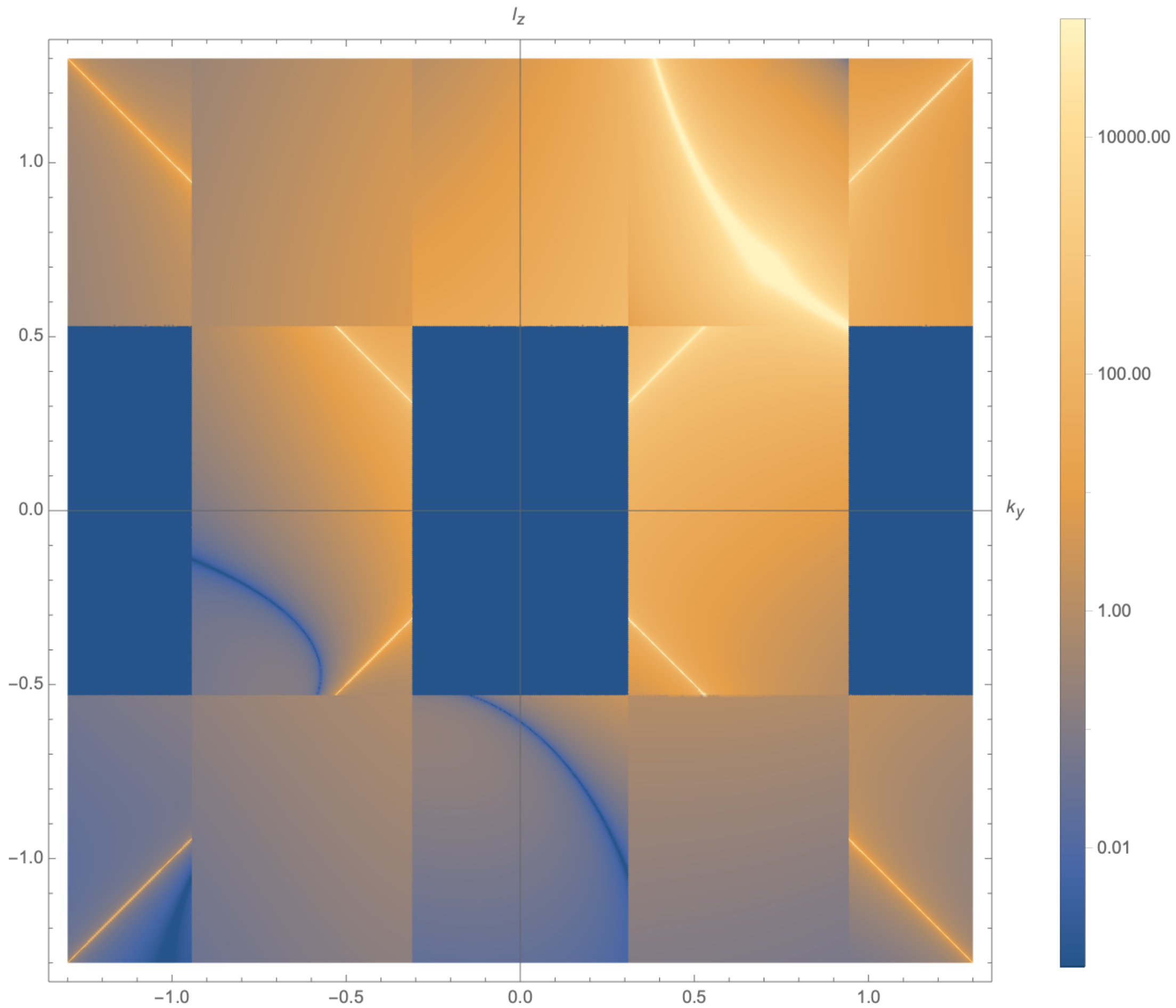




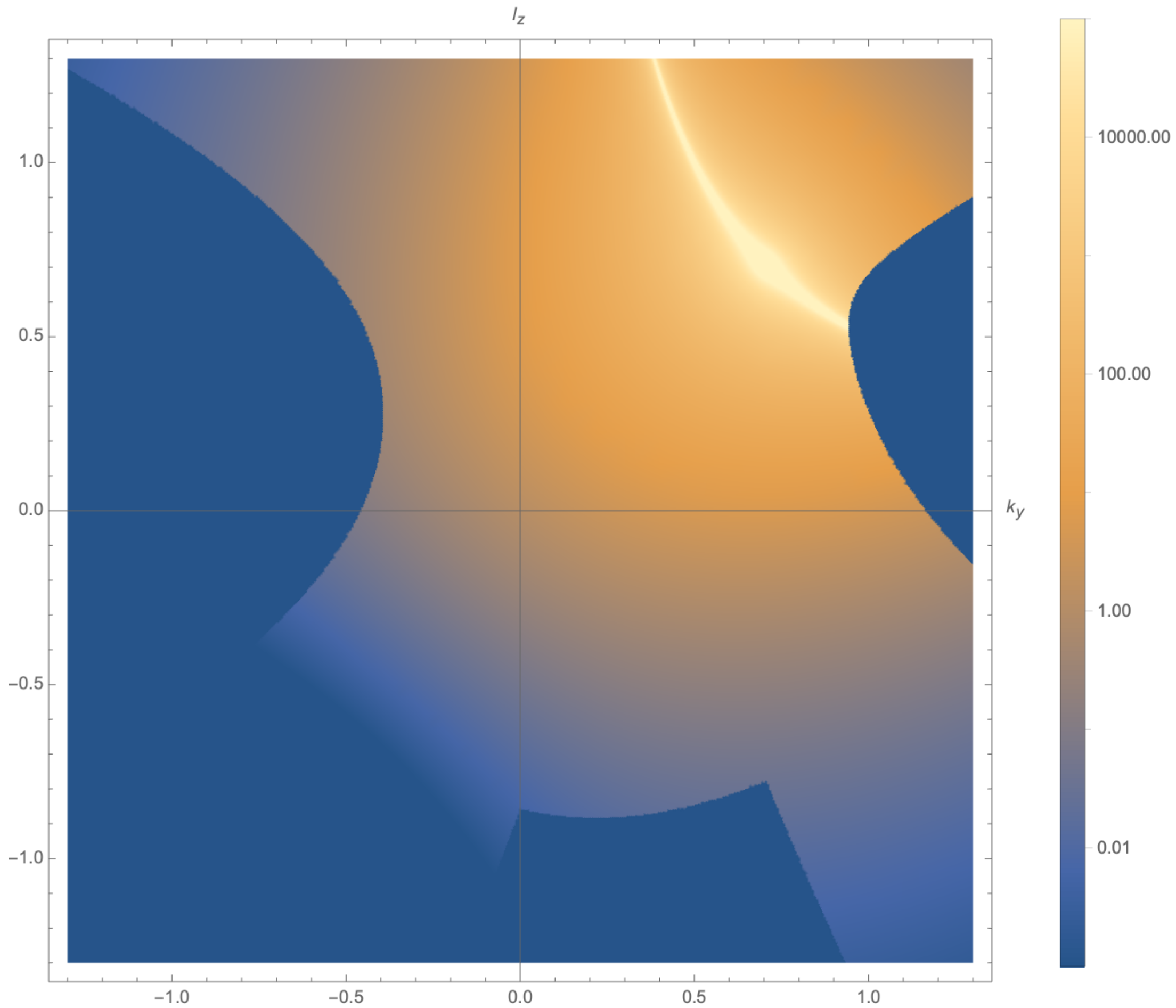


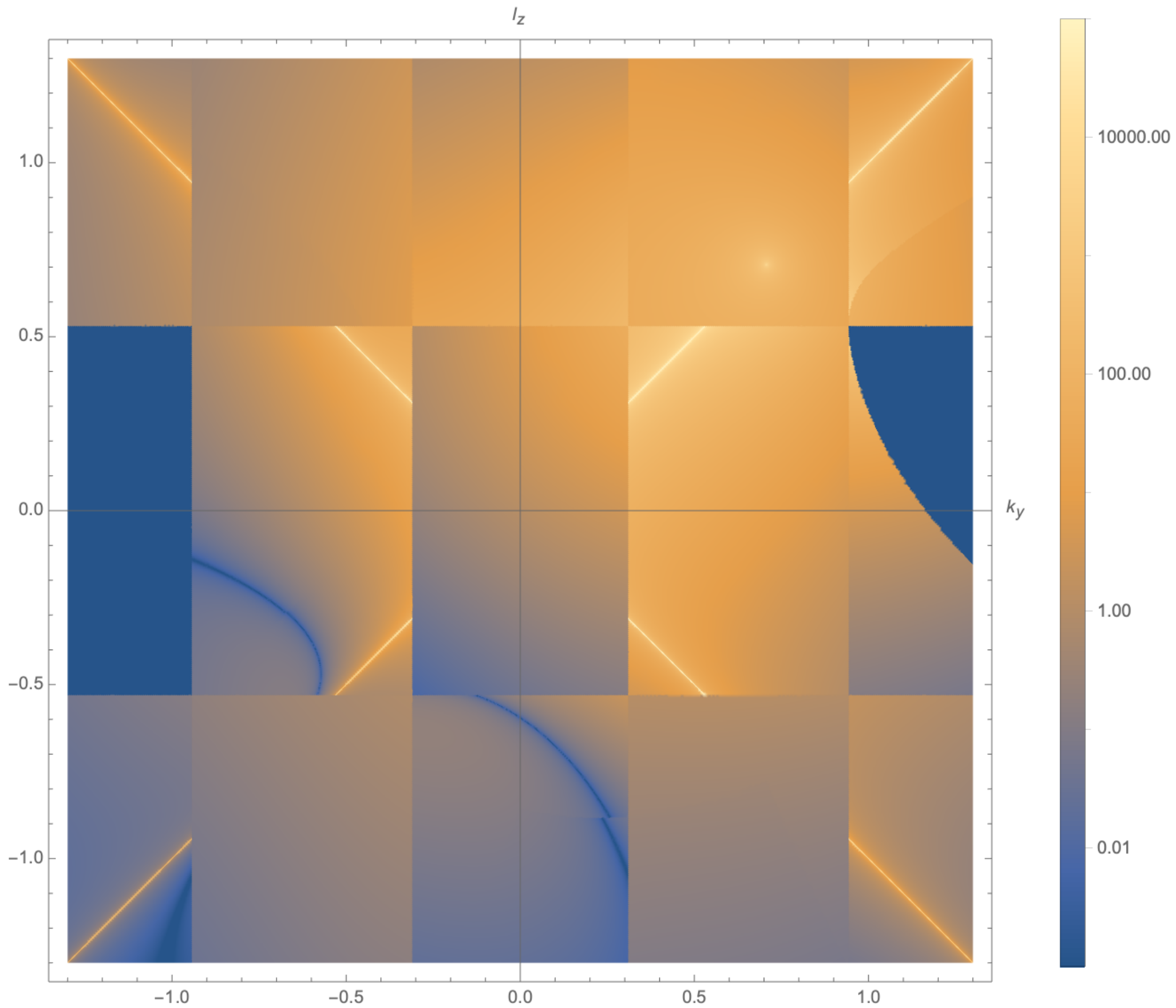
$I_{\mathbf{s}_2^v}$ 

$$I_{\mathbf{s}_1^v} + I_{\mathbf{s}_2^v}$$

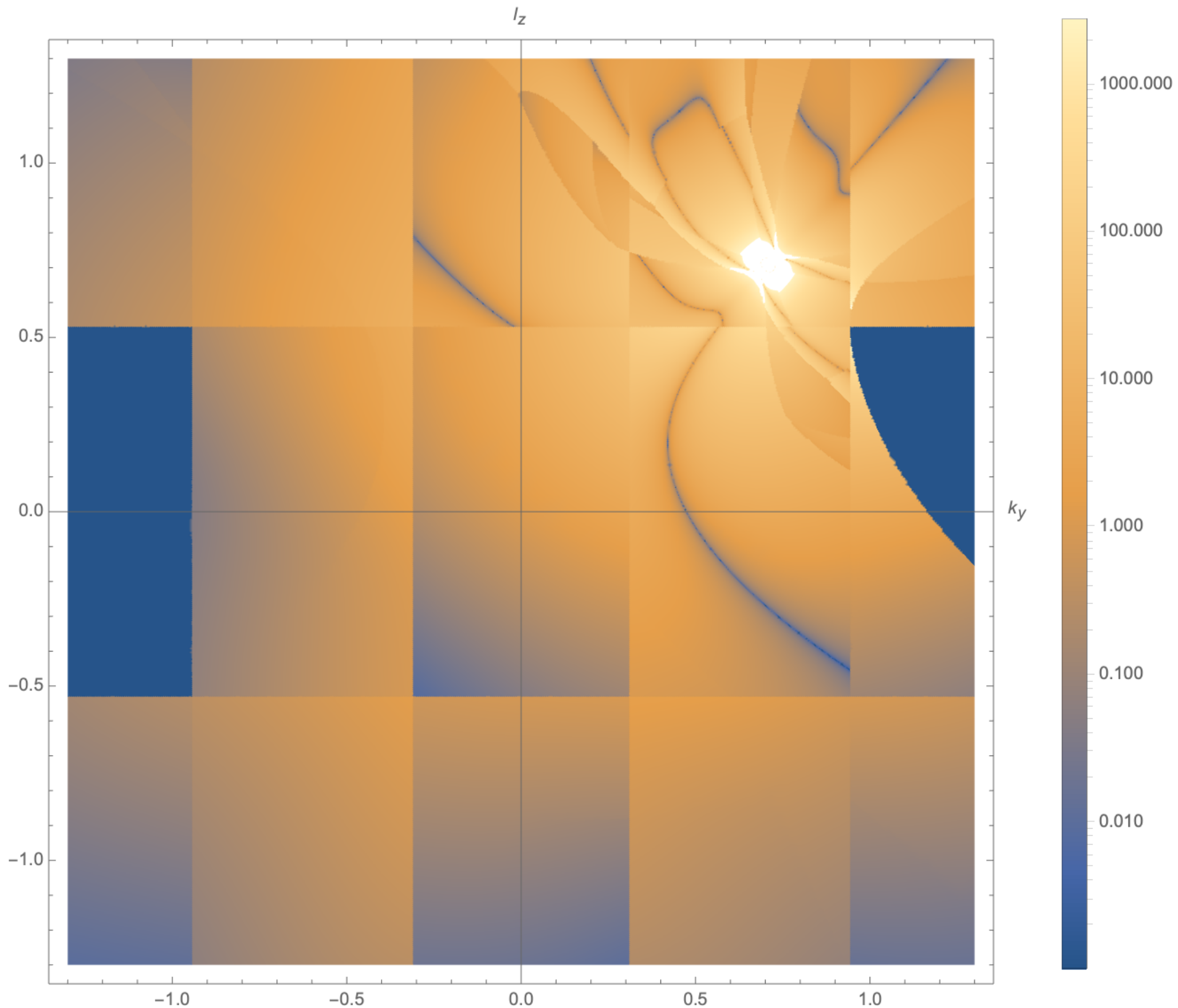


$$I_{\mathbf{s}_1^r} + I_{\mathbf{s}_2^r}$$

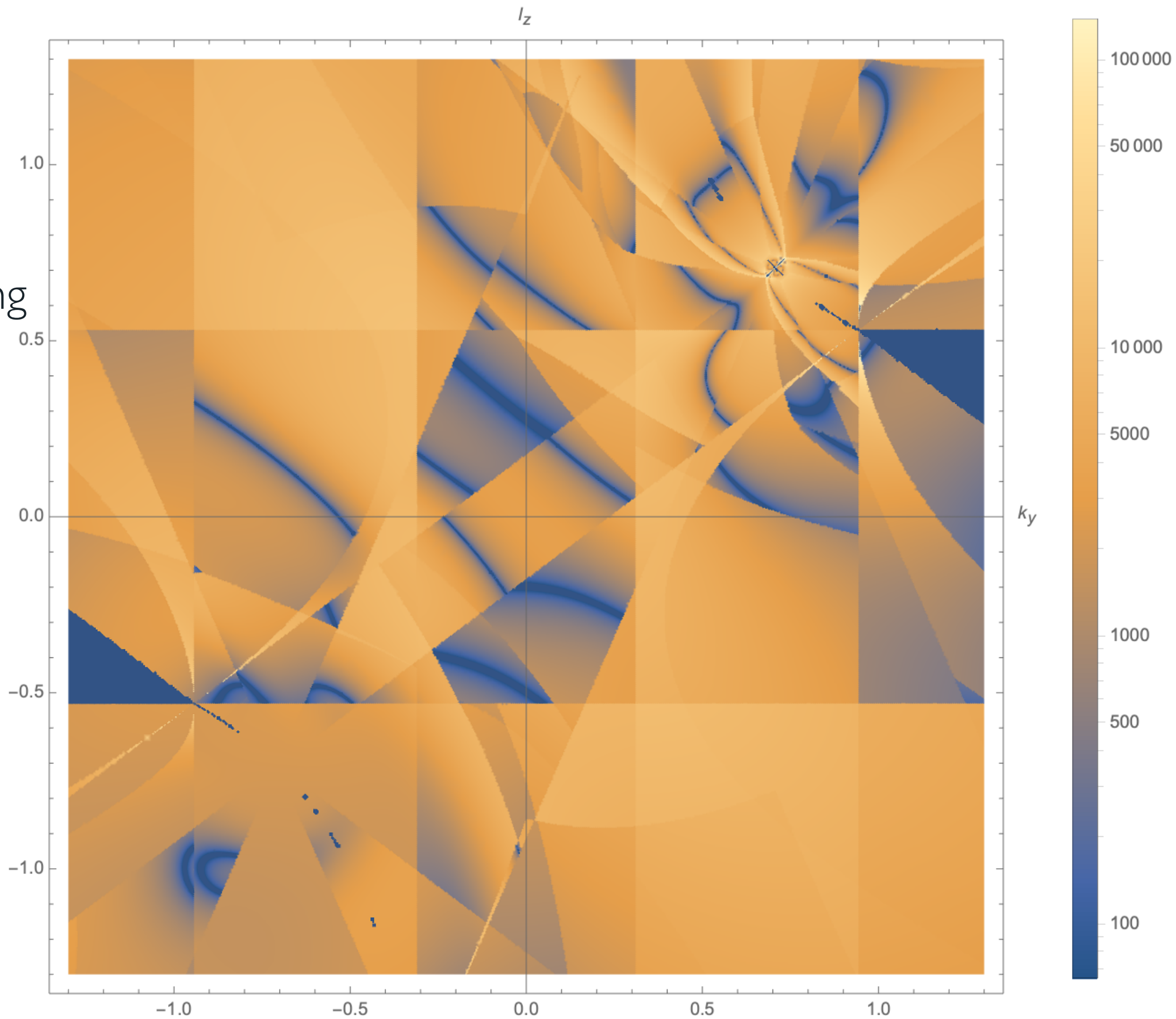


I_Σ 

$\text{Re} [I_\Sigma]$
with
deformation



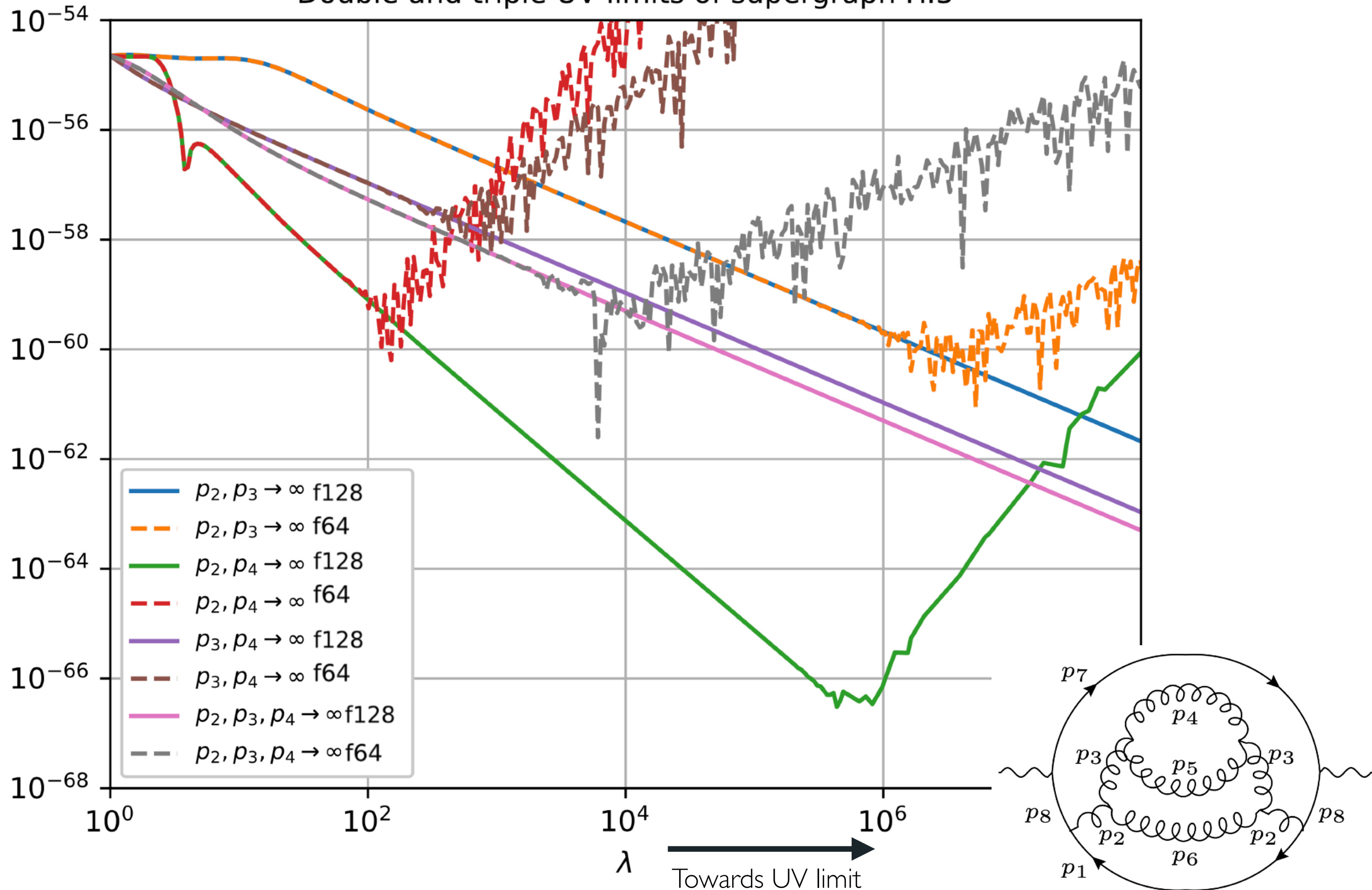
$\text{Re} [I_\Sigma]$
with
deformation
and
multichanneling



TESTING N3LO UV LIMITS

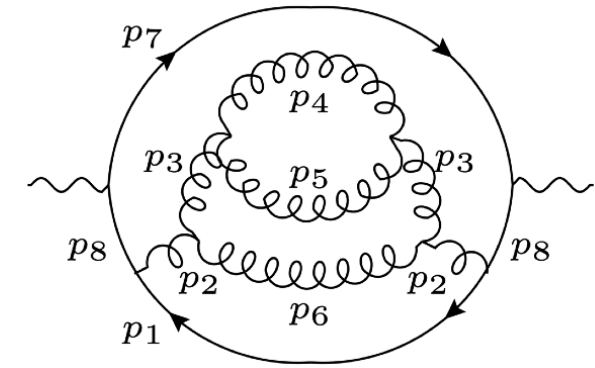
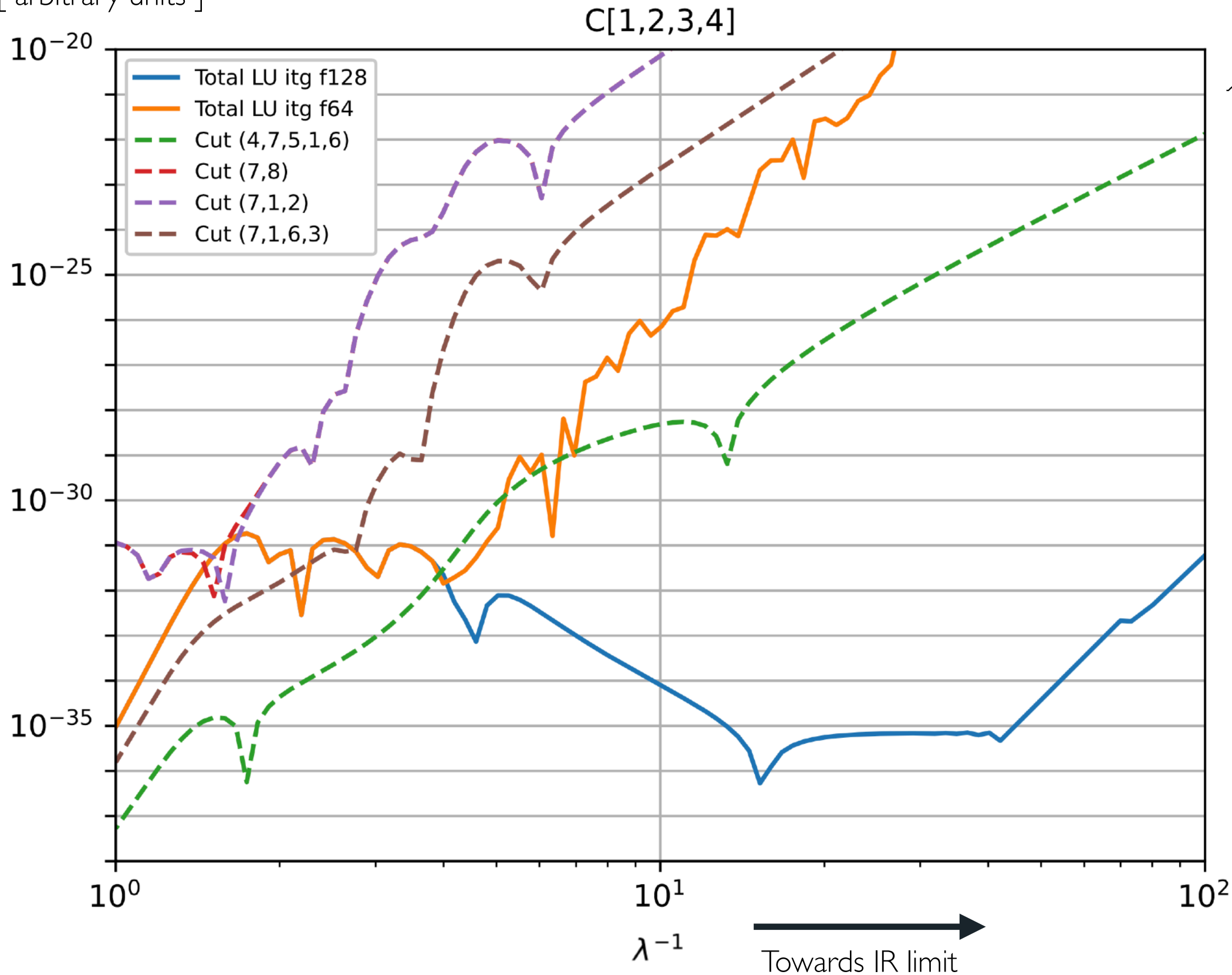
[arbitrary units]

Double and triple UV limits of supergraph H.3



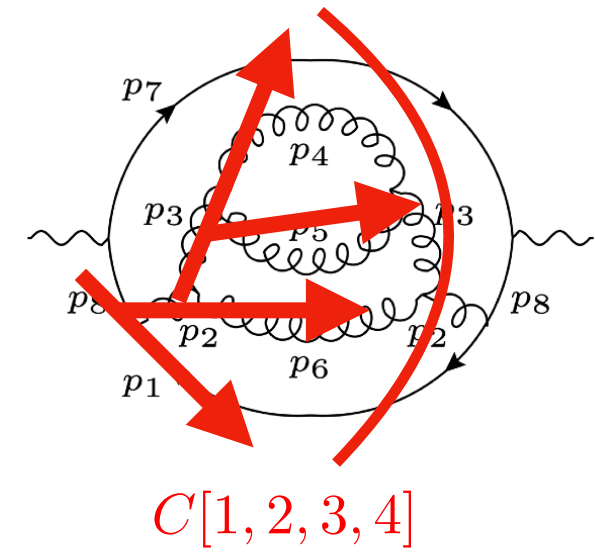
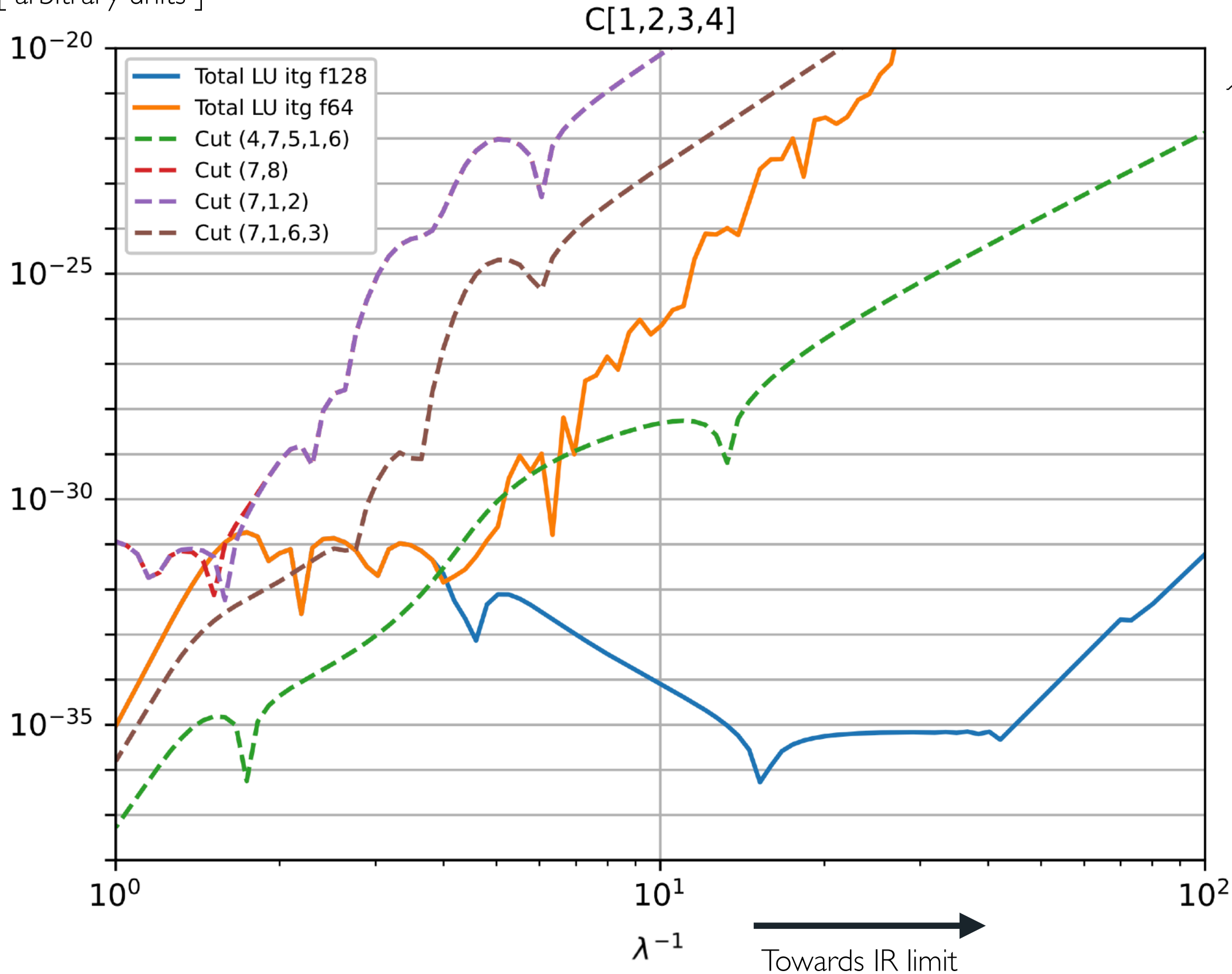
TESTING IR QUADRUPLE COLLINEAR LIMITS

[arbitrary units]



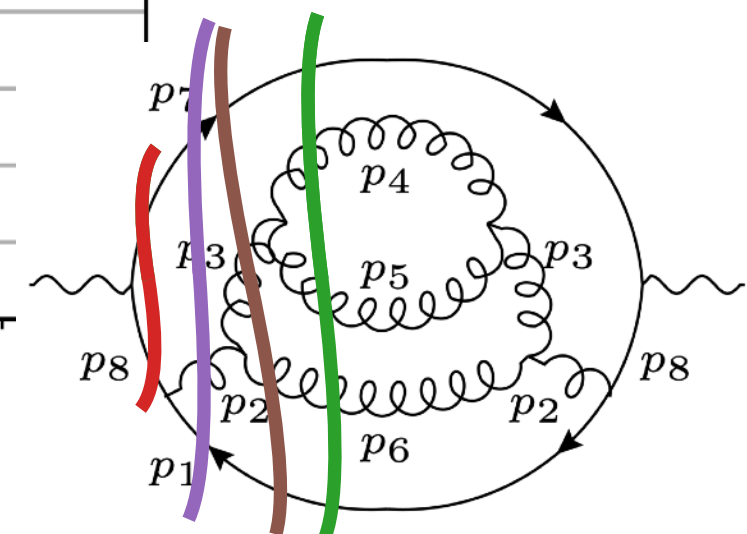
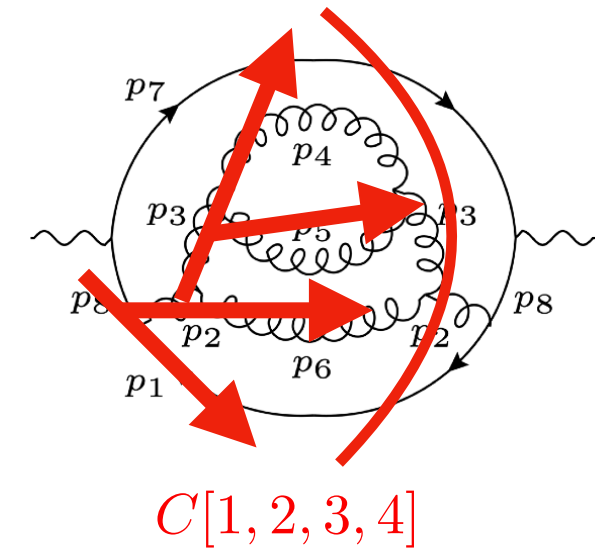
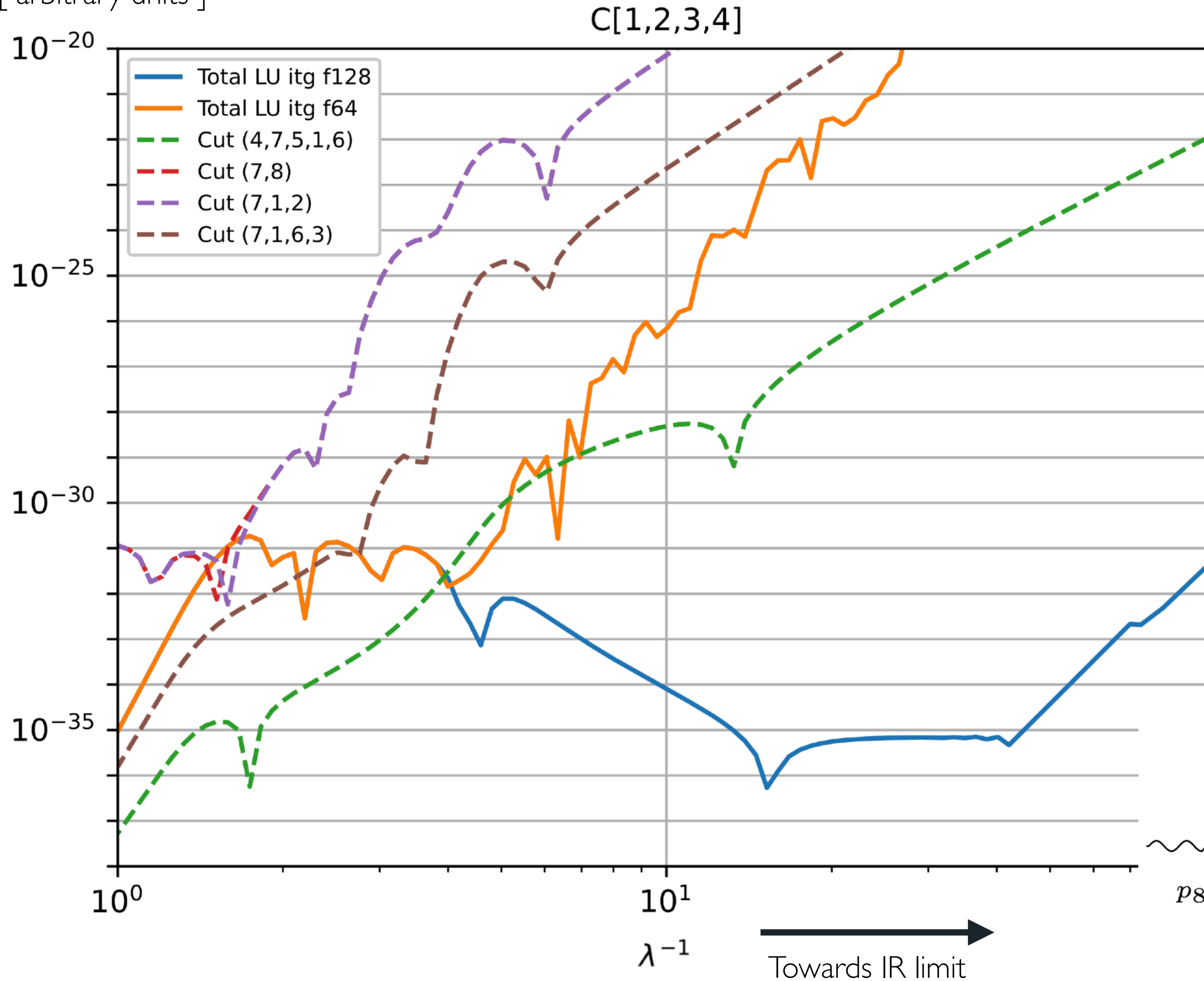
TESTING IR QUADRUPLE COLLINEAR LIMITS

[arbitrary units]



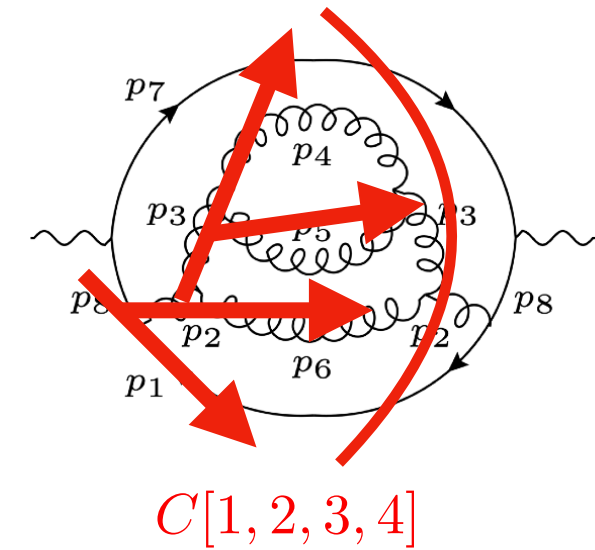
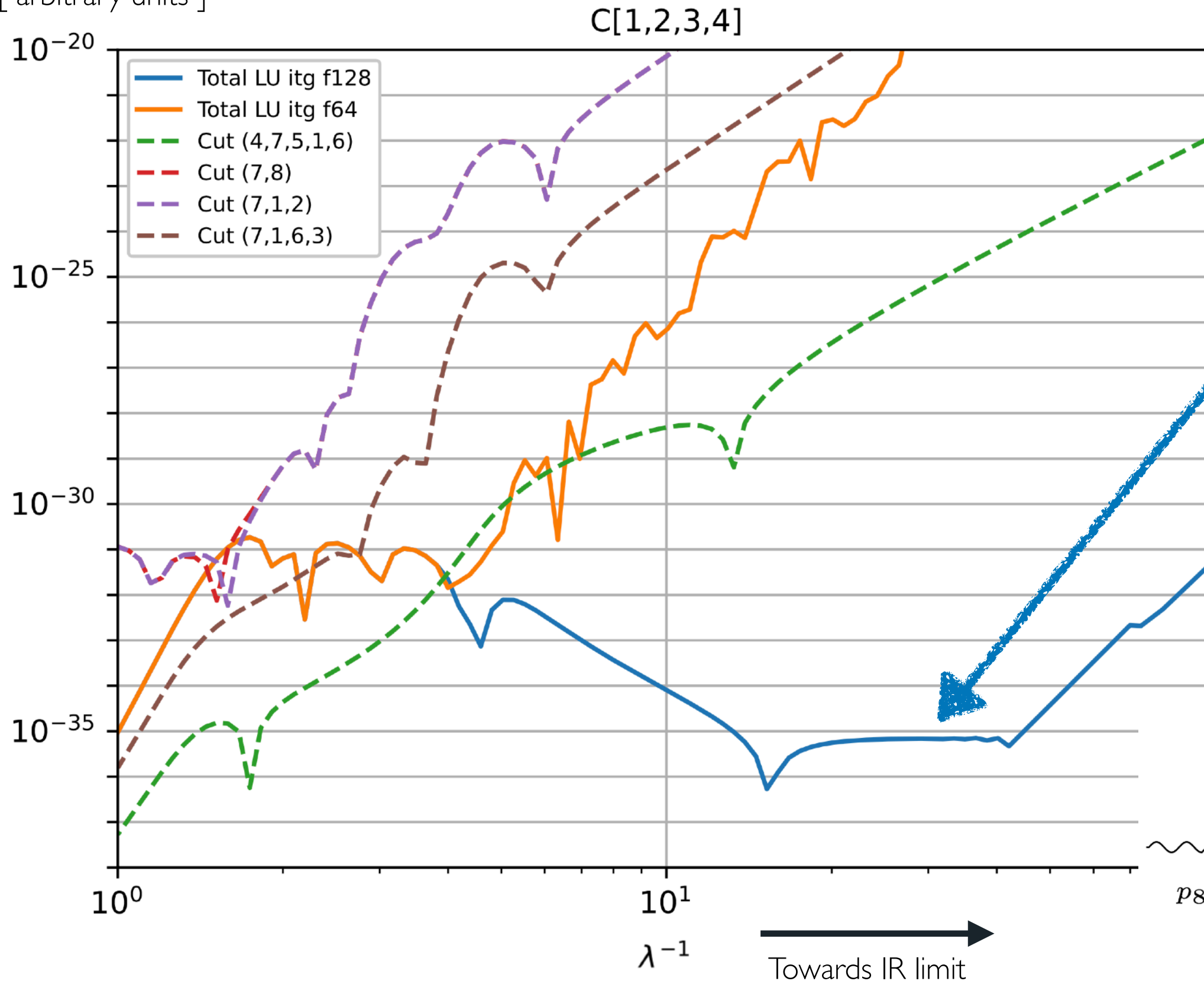
TESTING IR QUADRUPLE COLLINEAR LIMITS

[arbitrary units]

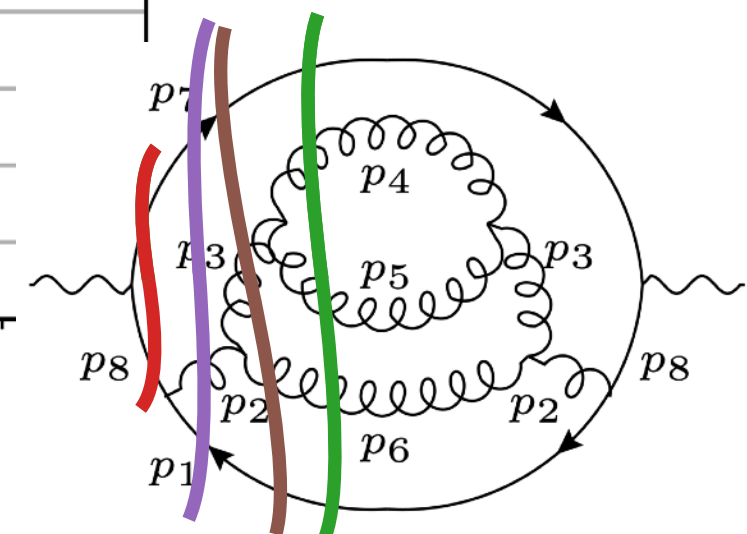


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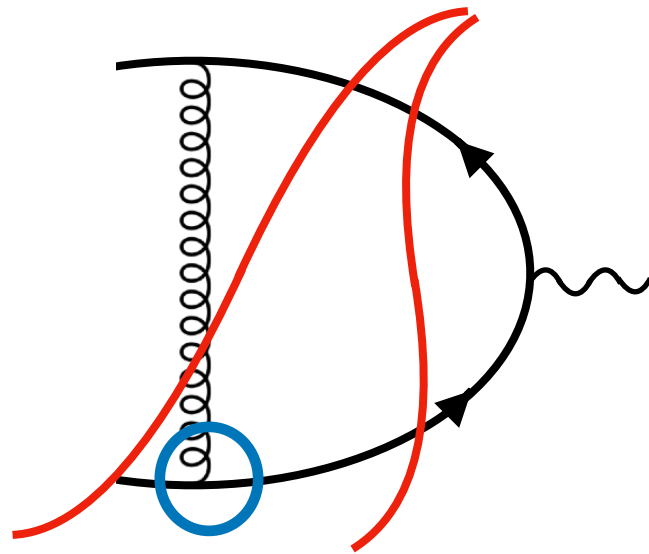
[arbitrary units]



LU integrand always goes to a constant on **collinear limits** without incl. *any* scaling of the measure !
No residual integrable singularity.



INITIAL-STATE SINGULARITIES

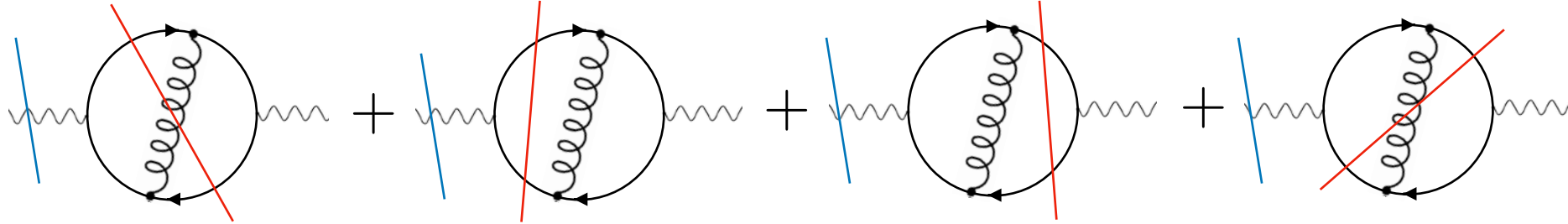


KLN CAN WORK FOR INITIAL-STATE !

INITIAL-STATE SINGULARITIES: IDEA

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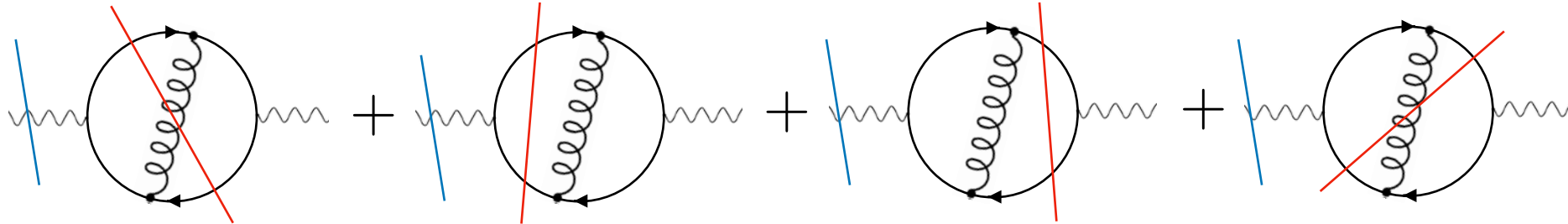
For $e^+e^- \rightarrow 2j$ @ NLO we used



(Including all degenerate configurations, higher final-state multiplicities)

INITIAL-STATE SINGULARITIES: IDEA

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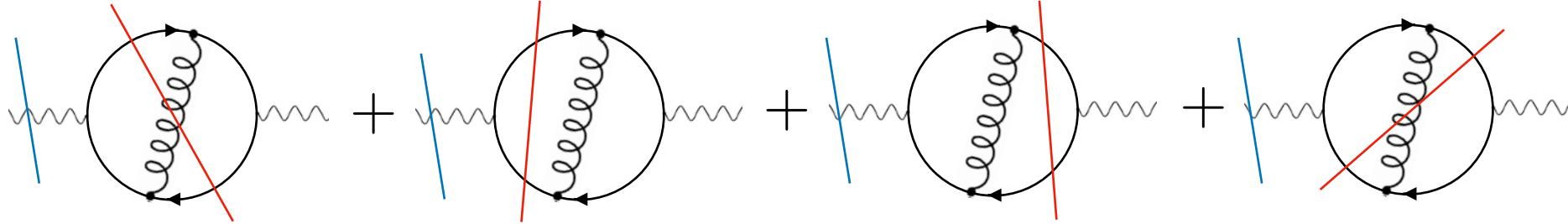


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Flip it, and obtain the answer for Drell-Yan, $2j \rightarrow e^+e^-$

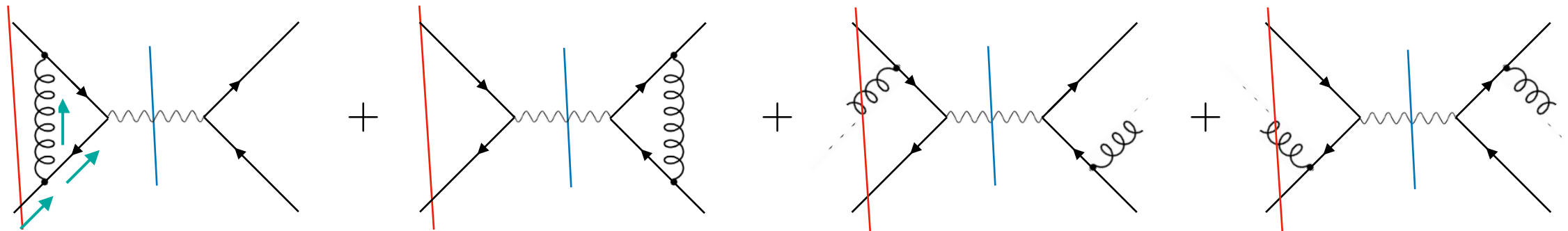
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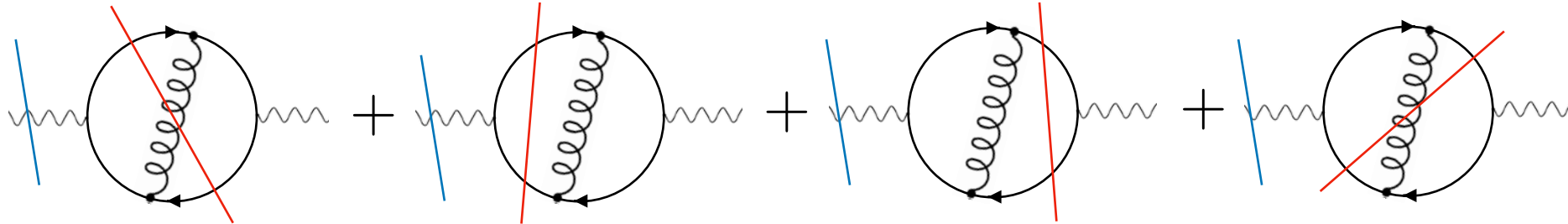
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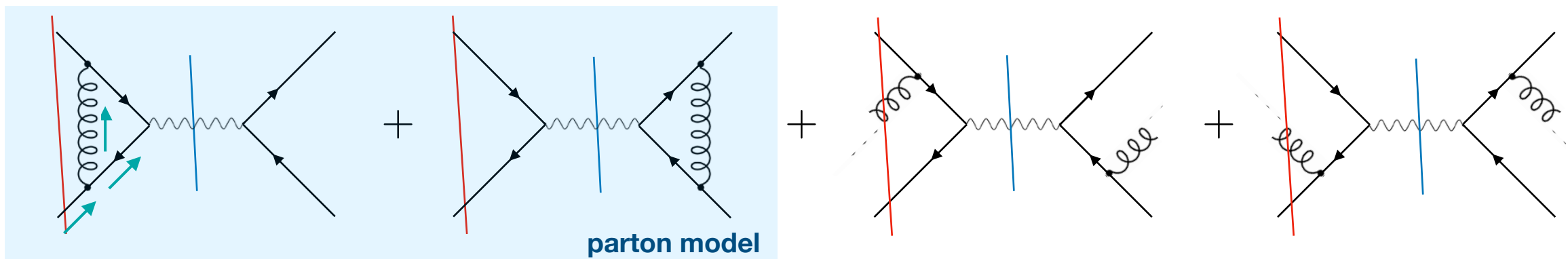
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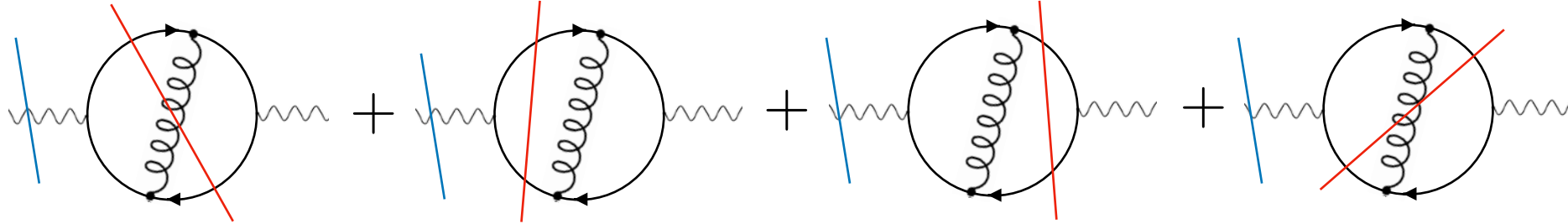
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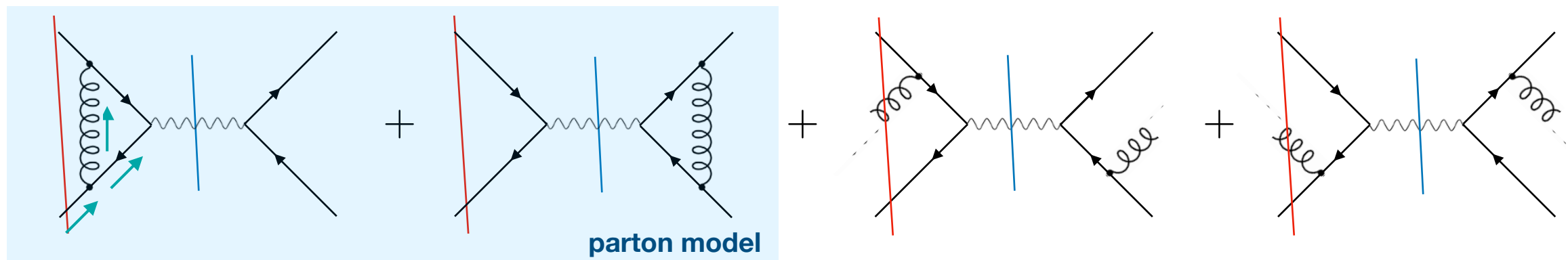
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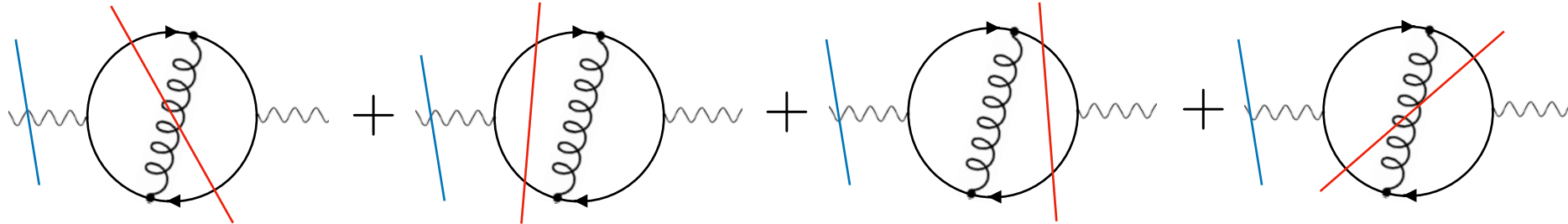
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The initial state singularity is now absent!

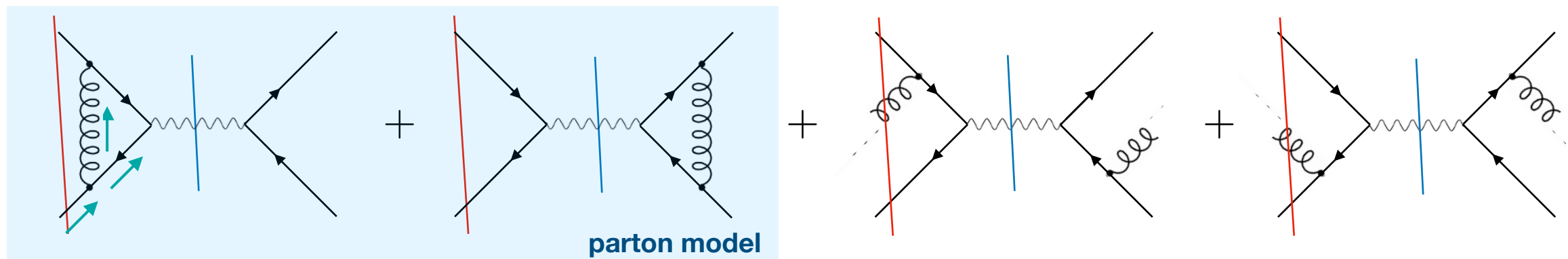
INITIAL-STATE SINGULARITIES: IDEA

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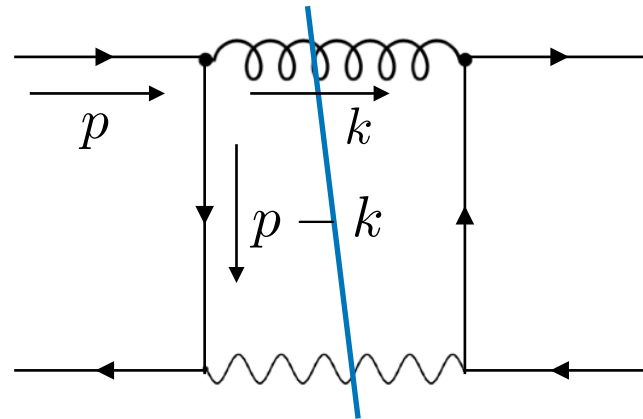
The initial state singularity is now absent!

Include degenerate initial states $\rightarrow$ Higher multiplicity initial states

INITIAL-STATE SINGULARITIES: IDEA

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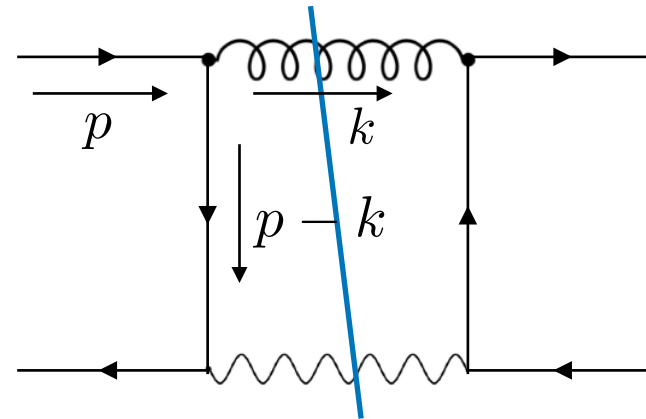
What about
this diagram?



Also has collinear
singularity at $k = xp$

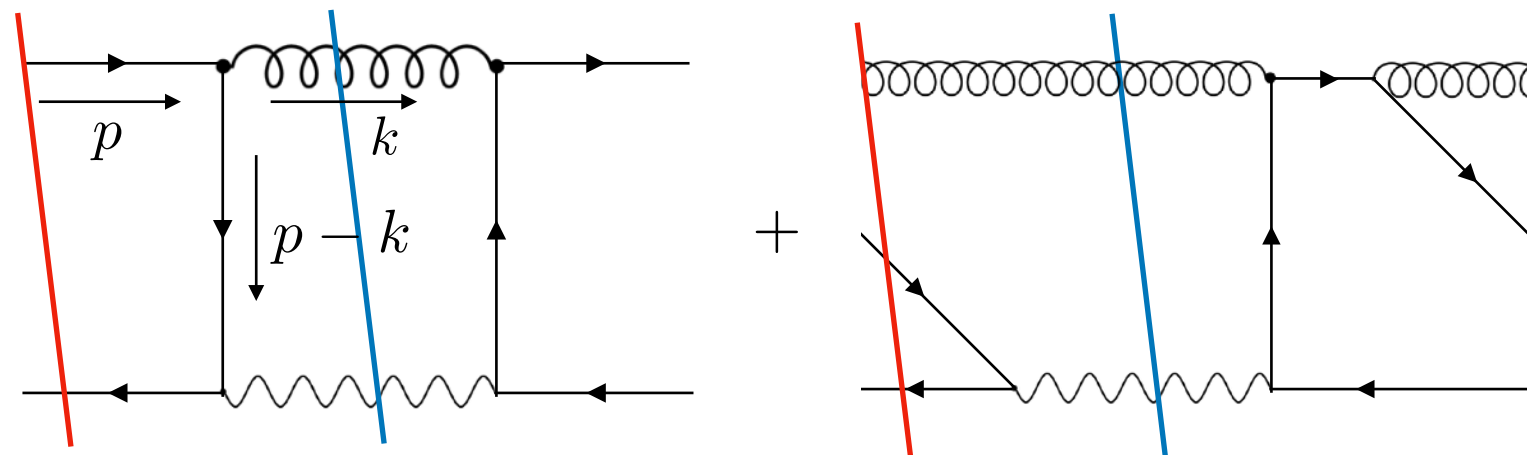
INITIAL-STATE SINGULARITIES: IDEA

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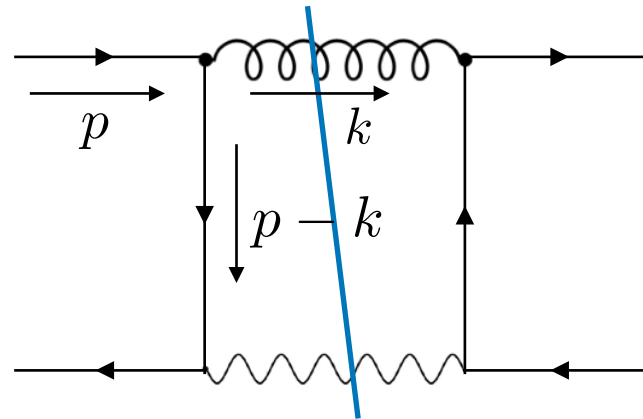
In this case, the cancelling partner is



Higher multiplicity initial states, but also **disconnected!** **Free travelling gluon!**

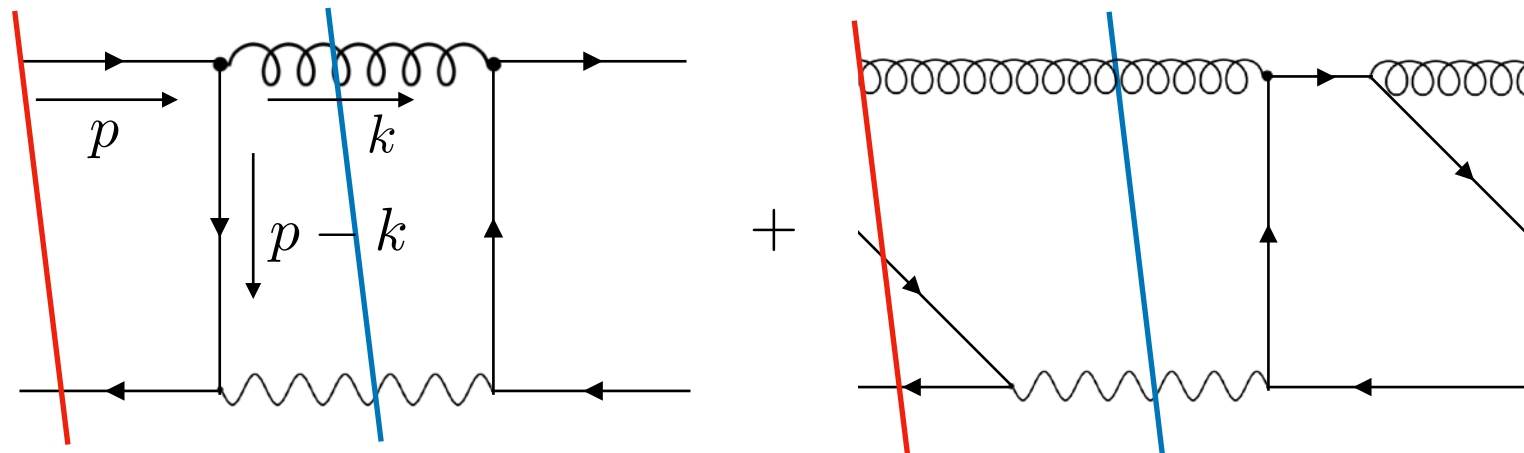
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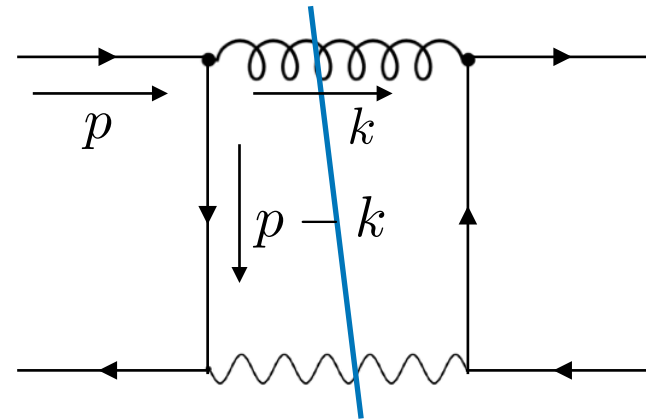


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The sum of these two diagrams is finite everywhere in phase space

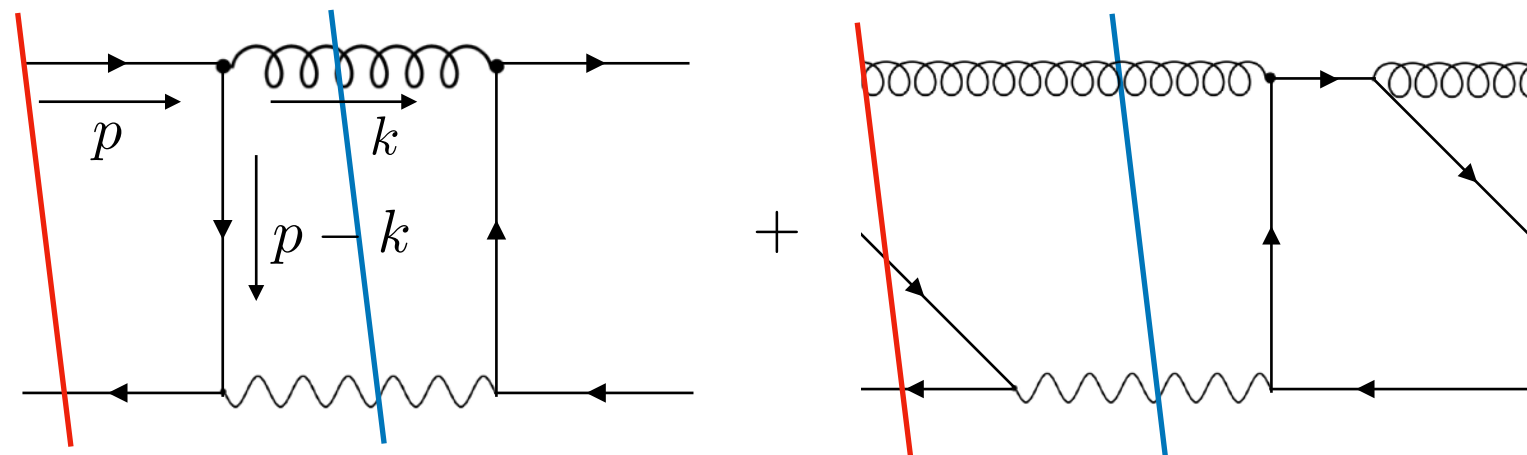
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Looks radical, but actually not a new idea, e.g. :

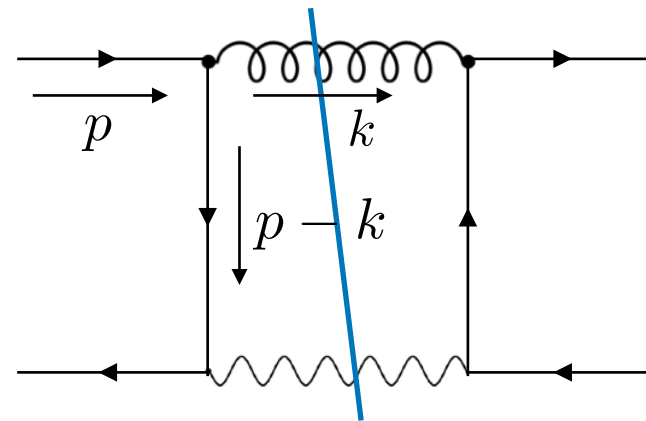
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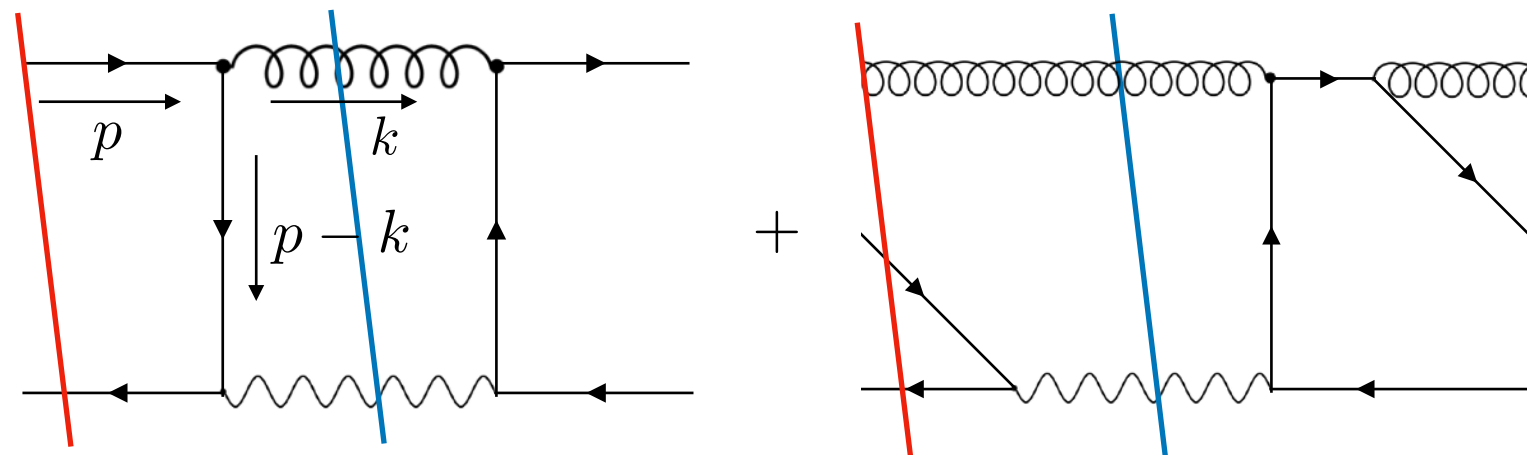
INITIAL-STATE SINGULARITIES: IDEA

What about this diagram?



Also has collinear singularity at $k = xp$

In this case, the cancelling partner is



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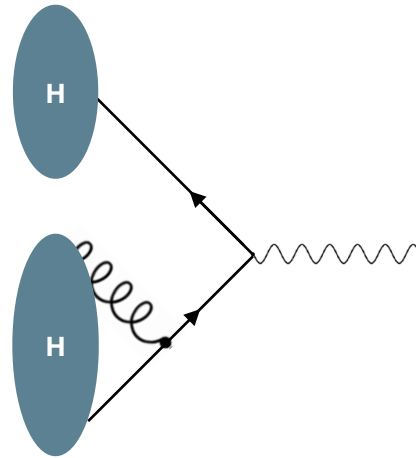
But also more recently, they were studied in:

Frye, Hannesdottir, Paul, Schwartz, Yan
arXiv:1810.10022 (2019)

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

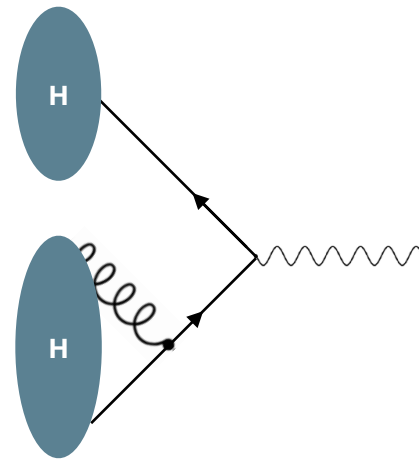
INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

This argument suggests that, in order to maintain IR-finiteness, one requires more than two initial state partons

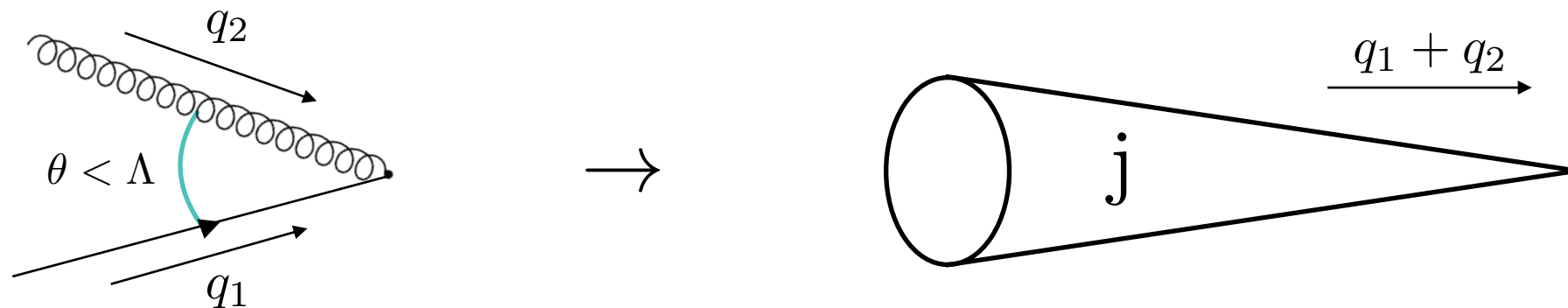


INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

This argument suggests that, in order to maintain IR-finiteness, one requires more than two initial state partons



and that the multiple partons should be clustered into **two jet-like objects** that resemble boosted hadrons



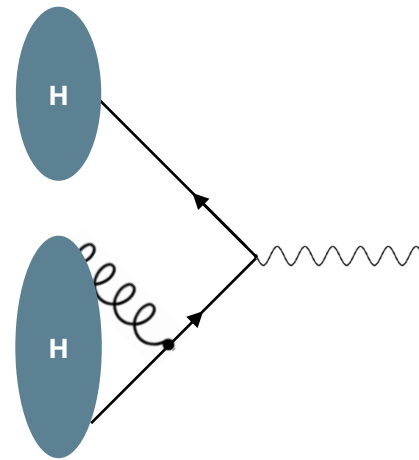
After clustering, we get two jets with momenta

$$P_1^j$$

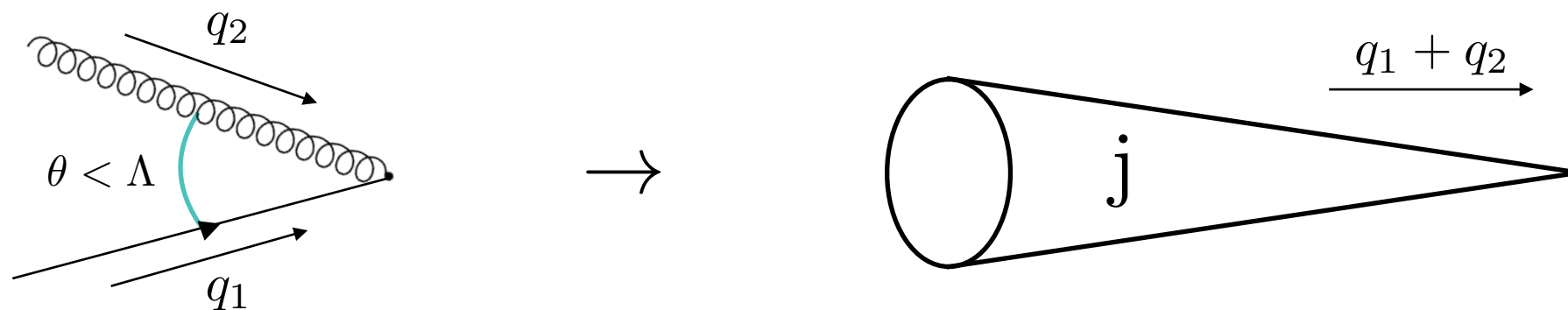
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$$P_1^j$$

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Cluster initial states analogously to final states: symmetry initial-final state

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

There are two relevant scales for the two initial state jets reconstructed:

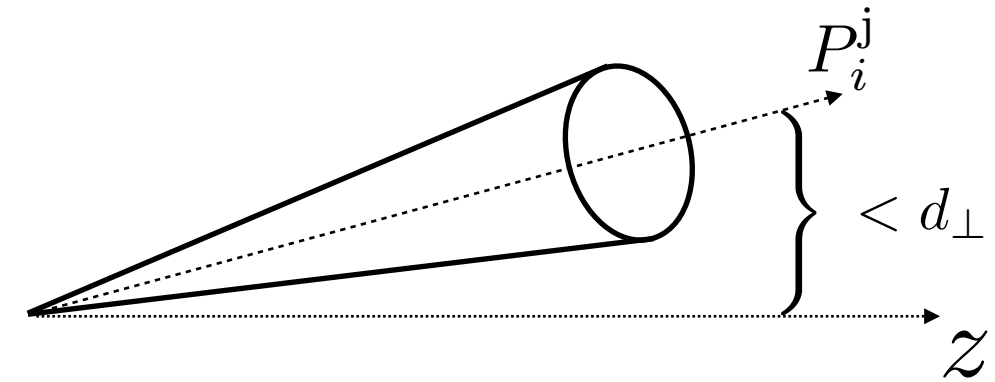
INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

There are two relevant scales for the two initial state jets reconstructed:

- One measuring the allowed phase space for the **total momentum** of the jet

$$(P_i^j)_\perp < d_\perp$$

If the scale is zero, the jet lies **exactly** on the z axis



If $d_\perp = 0$ the two jets are exactly back-to-back. This is equivalent to the parton's model

$$p_1 = (x_1\sqrt{s}, 0, 0, x_1\sqrt{s}), \quad p_2 = (x_2\sqrt{s}, 0, 0, -x_2\sqrt{s})$$

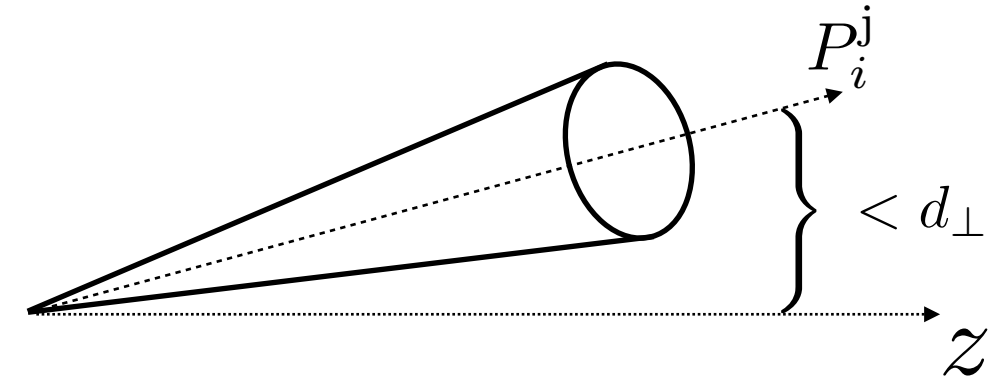
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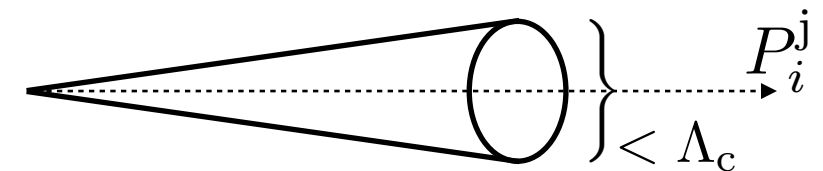


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The smaller this scale, the more **collinear** the partons are

The more collinear the partons, the more divergent the observable

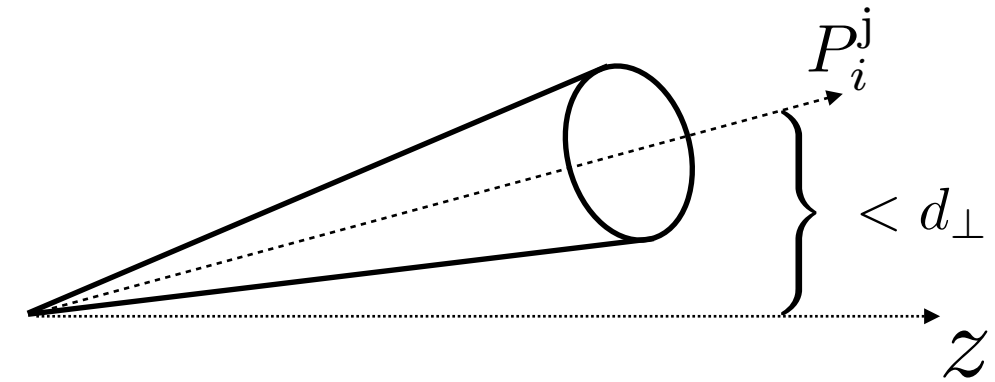
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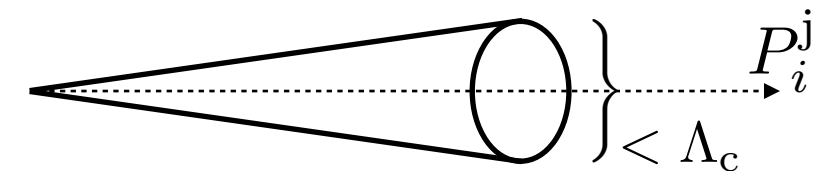


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Λ_c is the equivalent of the factorisation scale! $\approx \log(\Lambda_c)$

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

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- Take the limit $d_{\perp} \rightarrow 0$ **analytically** and obtain **exact back to back jets**

$$P_i^j = ((P_i^j)^0, 0, 0, (P_i^j)^3)$$

This allows us to define **Bjorken variables**

$$x_1 = \frac{(P_1^j)^0 + (P_1^j)^3}{2} \qquad x_2 = \frac{(P_2^j)^0 - (P_2^j)^3}{2}$$

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- Bin the distribution in the Bjorken variables $\rightarrow$ **Fit PDFs!**
(not in $\overline{\text{MS}}$ bar)
- Vary the factorisation scale Λ_c and interpolate the dependence on the factorisation scale **Numerical resummation?** [Banfi, Salam, Zanderighi, arXiv:0407286 \(2004\)](#)

INITIAL-STATE SINGULARITIES: “PDFs”

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We started with a very generic formalism for scattering

$$\sigma(HH \rightarrow X + nj) = \sum_m \int \left[\prod_{i=1}^m d^3 \vec{p}_i \right] f(p_1, \dots, p_m) \frac{d^m \sigma}{dp_1 \dots dp_m}(p_1, \dots, p_m \rightarrow X + nj)$$

Diagram illustrating the components of the scattering cross-section formula:

- Sum over number of initial state partons** (points to $\sum_m$)
- Integration over initial state partons momenta** (points to $\int \left[\prod_{i=1}^m d^3 \vec{p}_i \right]$)
- Weight** (points to $f(p_1, \dots, p_m)$)
- Cross-sections for m initial state partons** (points to $\frac{d^m \sigma}{dp_1 \dots dp_m}(p_1, \dots, p_m \rightarrow X + nj)$)

INITIAL-STATE SINGULARITIES: “PDFs”

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Diagram illustrating the components of the generic scattering formalism:

- Sum over number of initial state partons**: Points to the summation index m .
- Integration over initial state partons momenta**: Points to the integration measure $\left[\prod_{i=1}^m d^3 \vec{p}_i \right]$.
- Weight**: Points to the function $f(p_1, \dots, p_m)$.
- Cross-sections for m initial state partons**: Points to the differential cross-section $\frac{d^m \sigma}{dp_1 \dots dp_m}(p_1, \dots, p_m \rightarrow X + n j)$.

And we “forced” the initial-state observable to reproduce the usual factorised structure:

$$\sigma(HH \rightarrow X + n j) = \int dx_1 dx_2 f(x_1, \Lambda_c) f(x_2, \Lambda_c) \frac{d^2 \sigma_p}{dx_1 dx_2}(2 j \rightarrow X + n j, \Lambda_c)$$

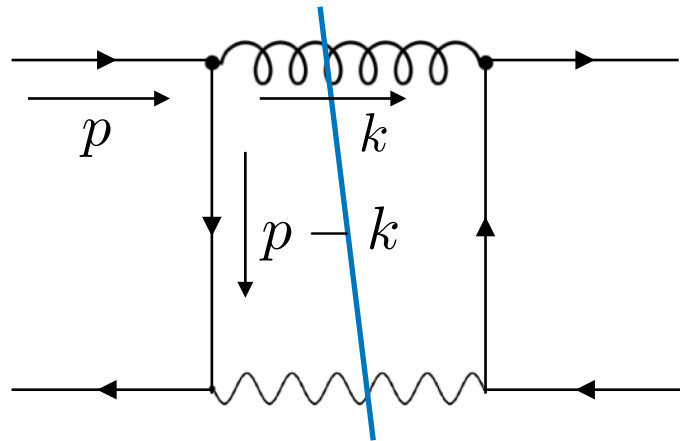
But we did not need to start from this factorised ansatz!

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

Numerical example result for this finite sum of two interference diagrams:

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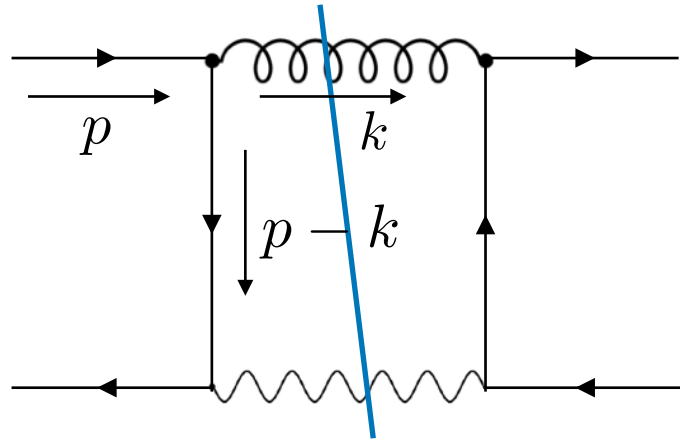


Always included!

This is the usual contribution.

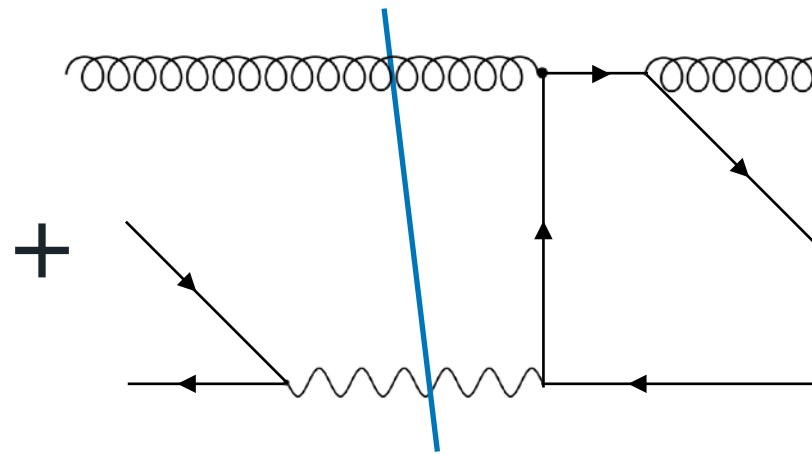
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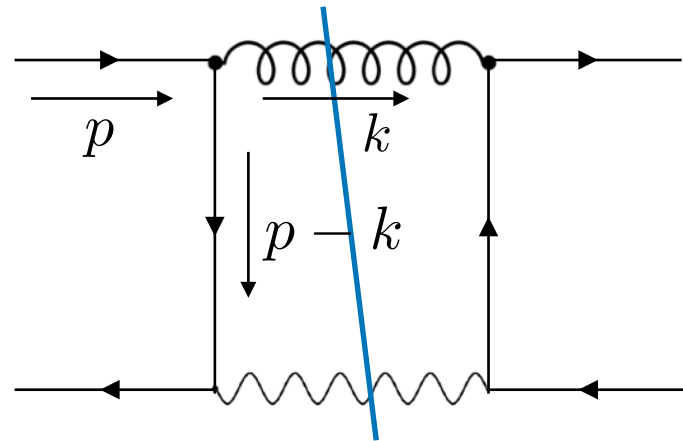
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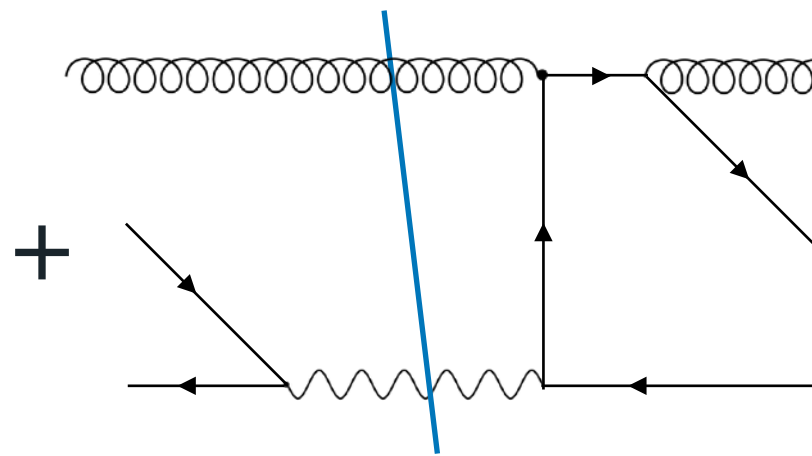
Included **only when** two initial-state quarks with collinearity $< \Lambda_c$

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

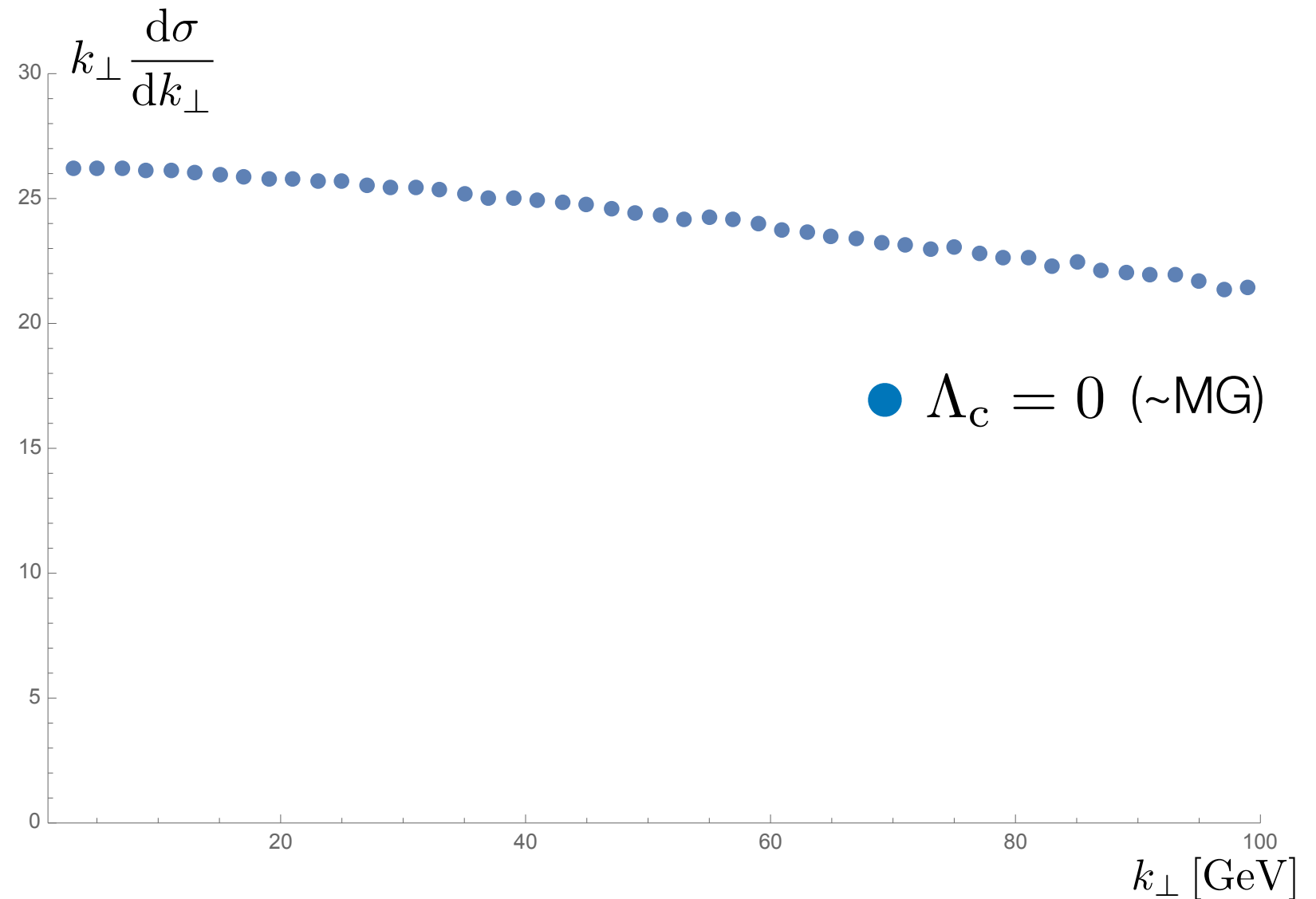
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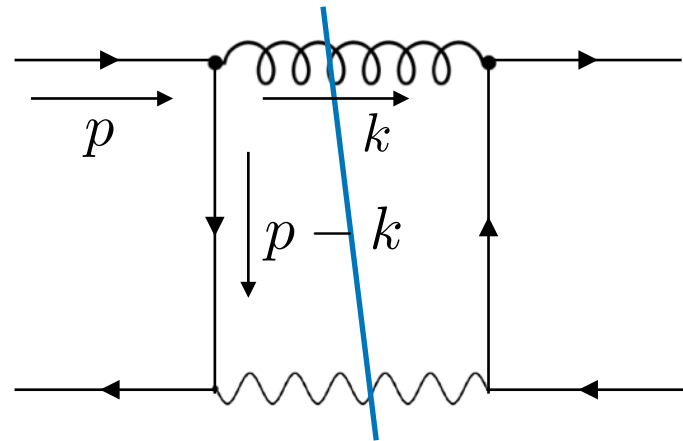


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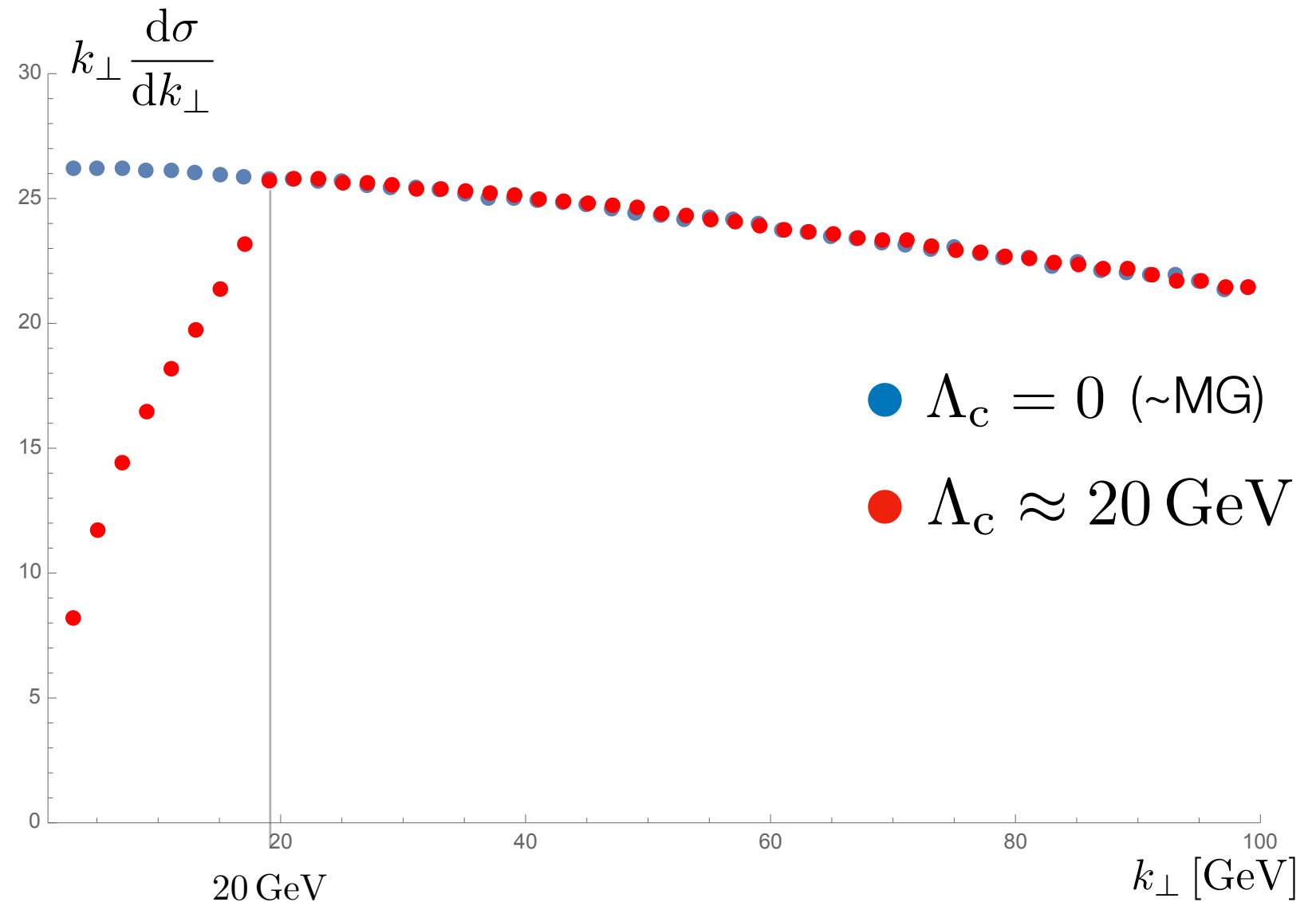
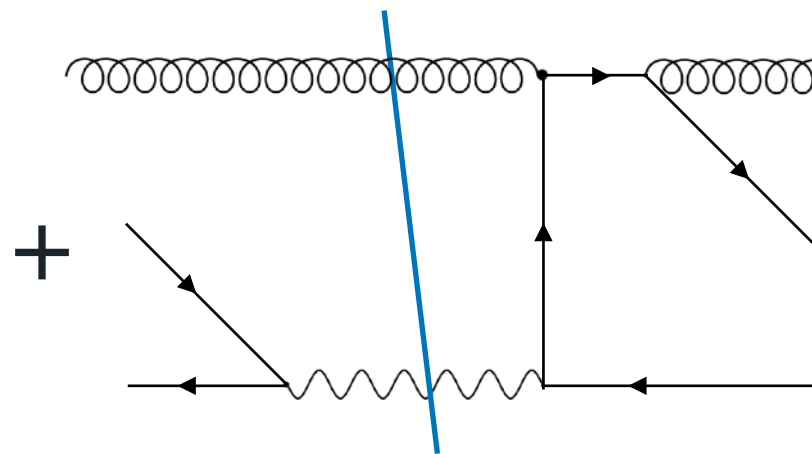


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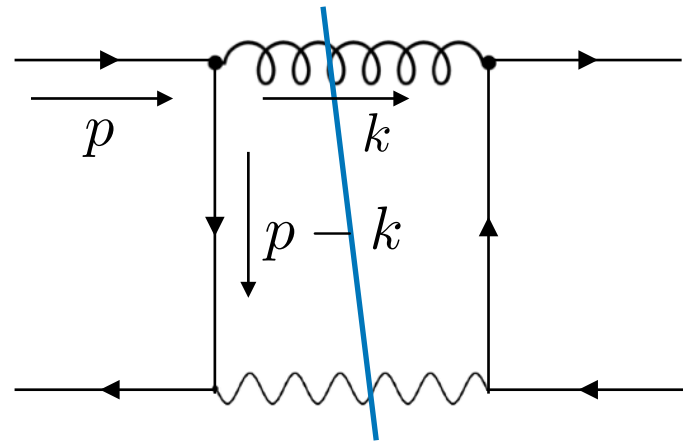
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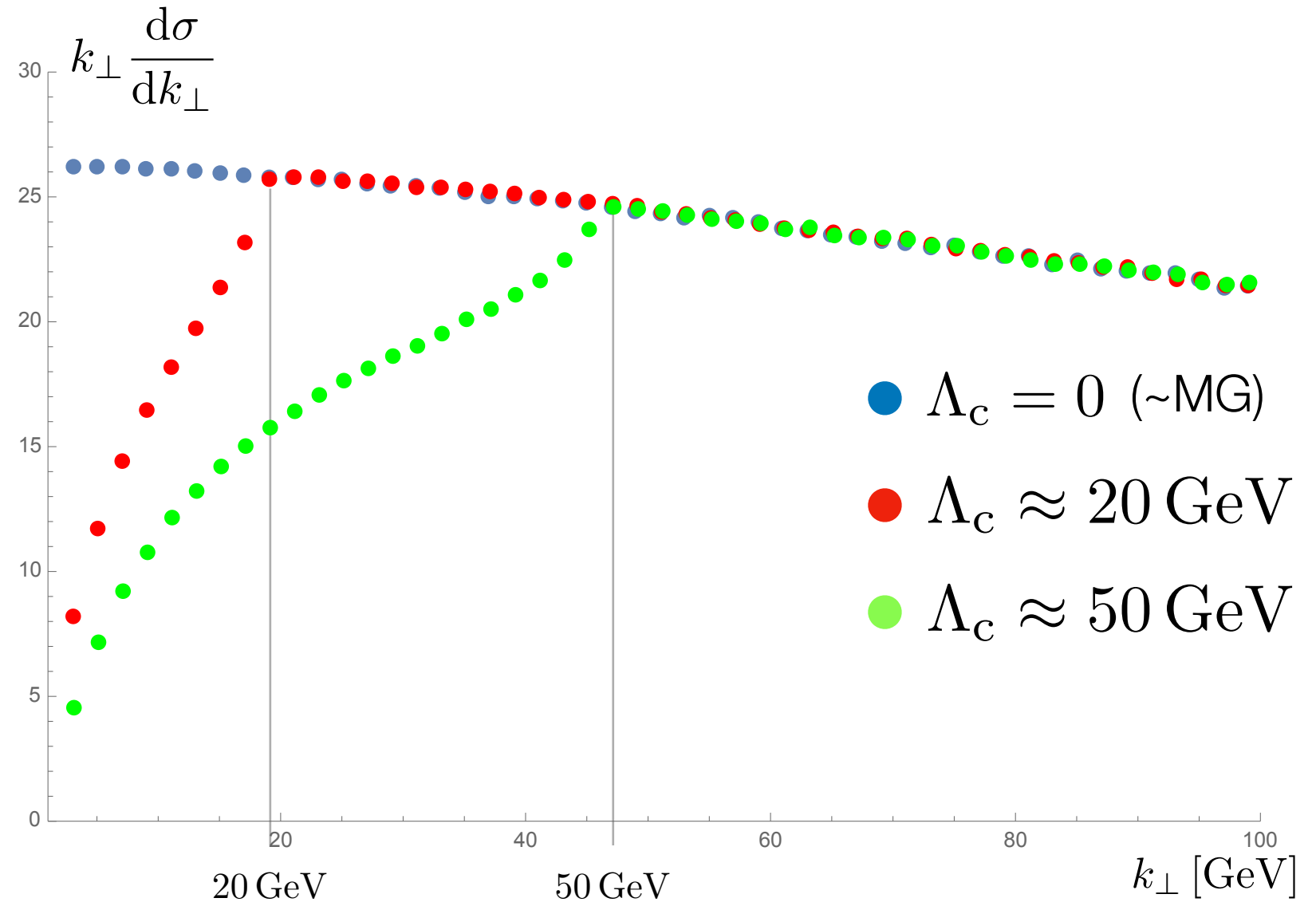
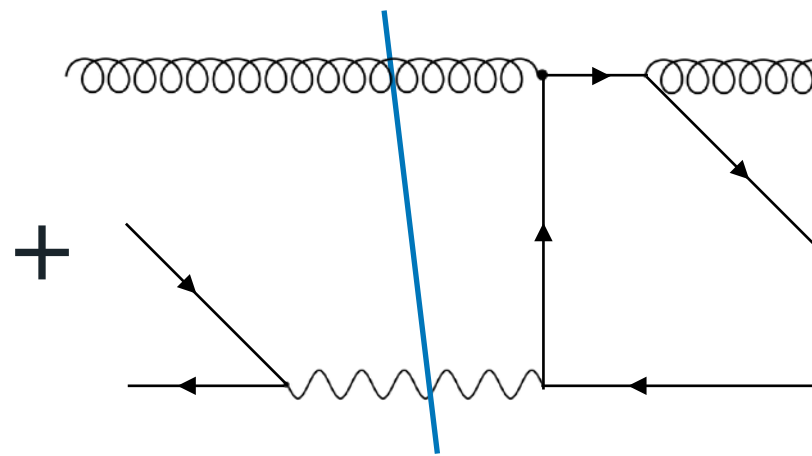
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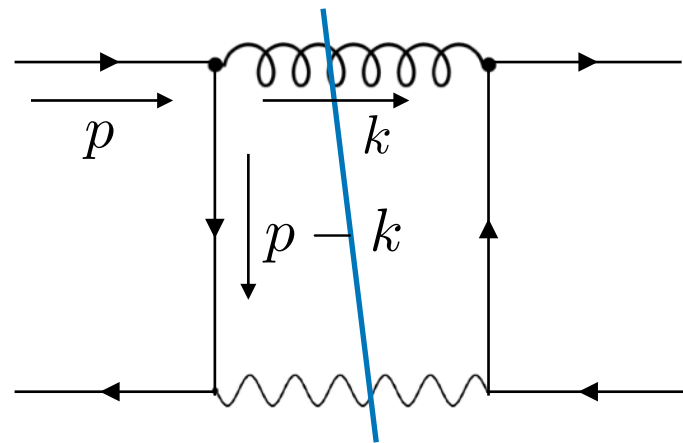
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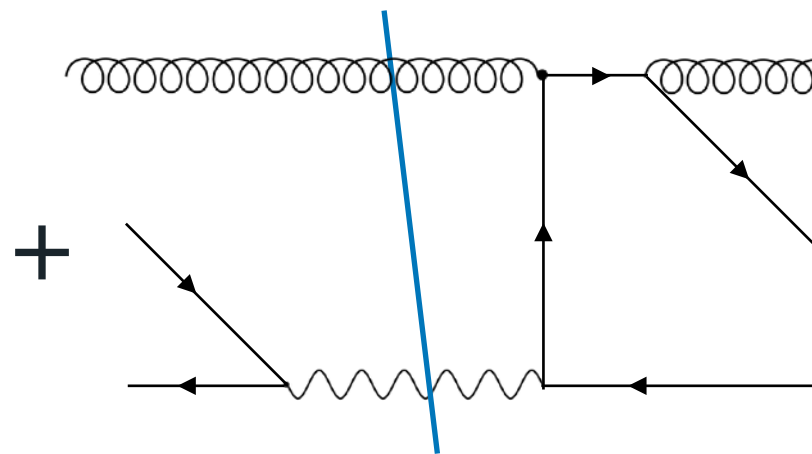
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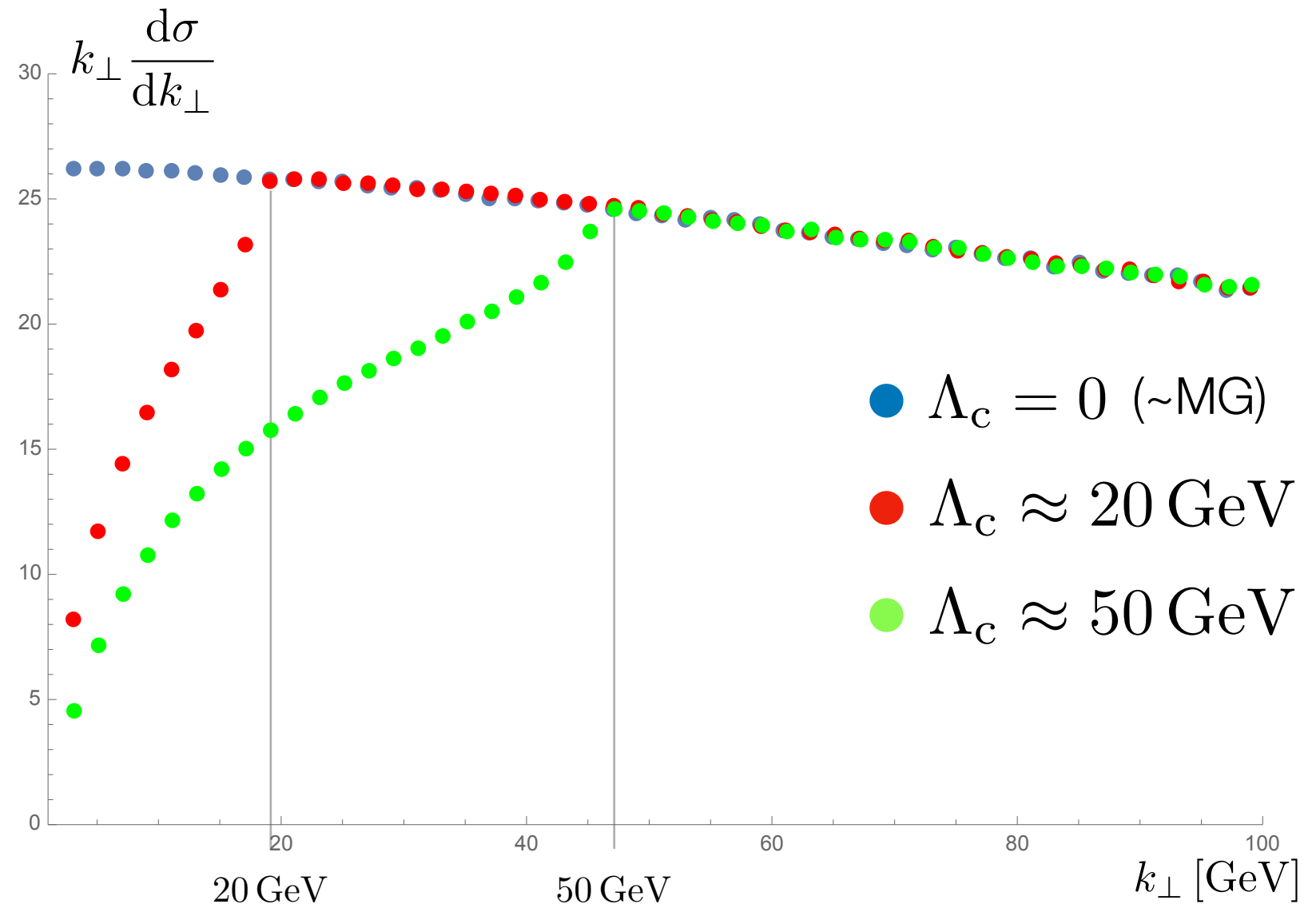
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Note that : for $\Lambda_c > 50 \text{ GeV}$
the distribution does not change anymore
because highest separation of two partons
in a jet is of **order of Z mass**

INITIAL-STATE SINGULARITIES: SCALE DEPENDENCE

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Consider now the inclusive integral

$$I(\Lambda_c) = \int_0^{100} dk_{\perp} \frac{d\sigma}{dk_{\perp}}(\Lambda_c)$$

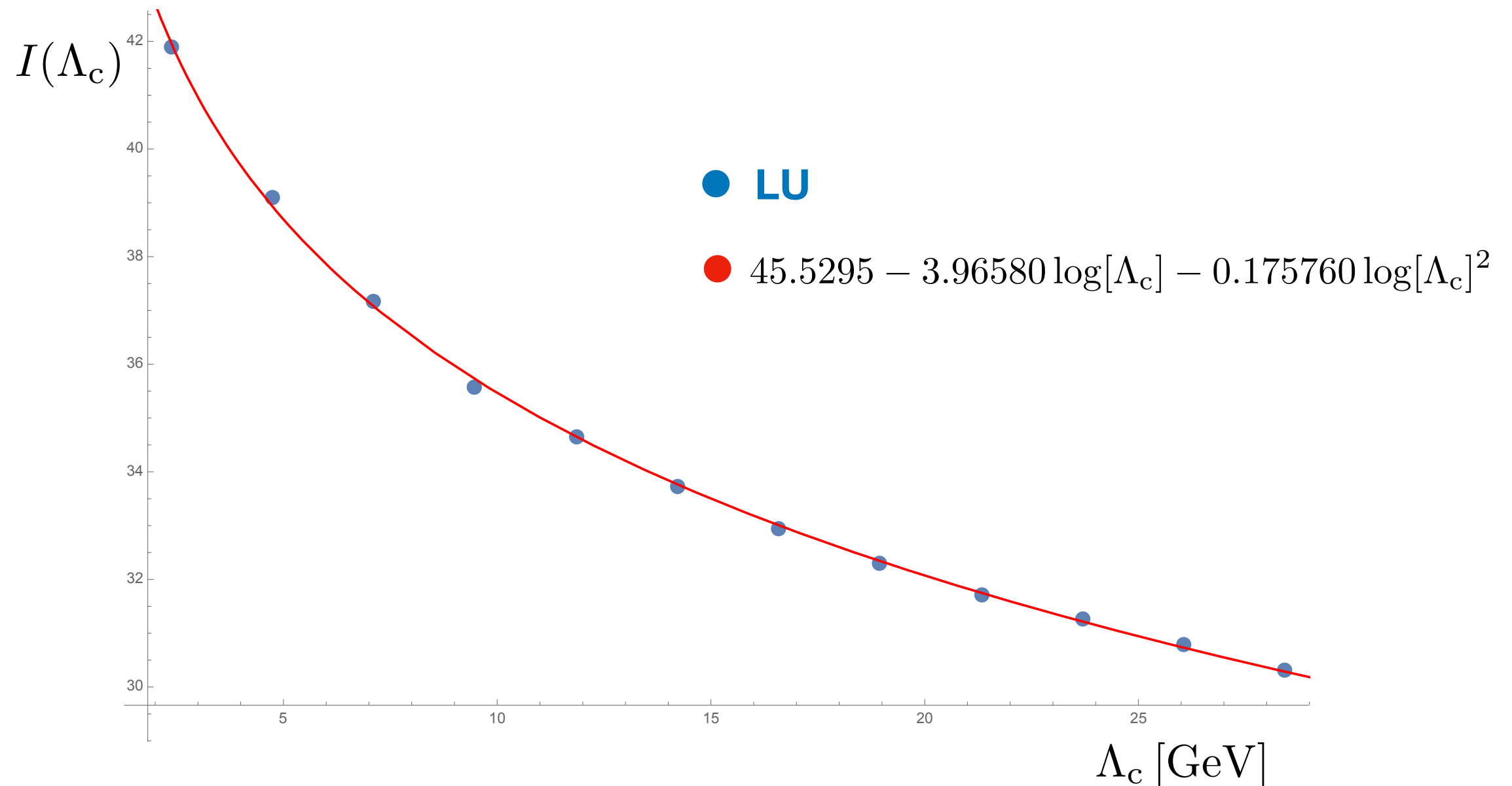
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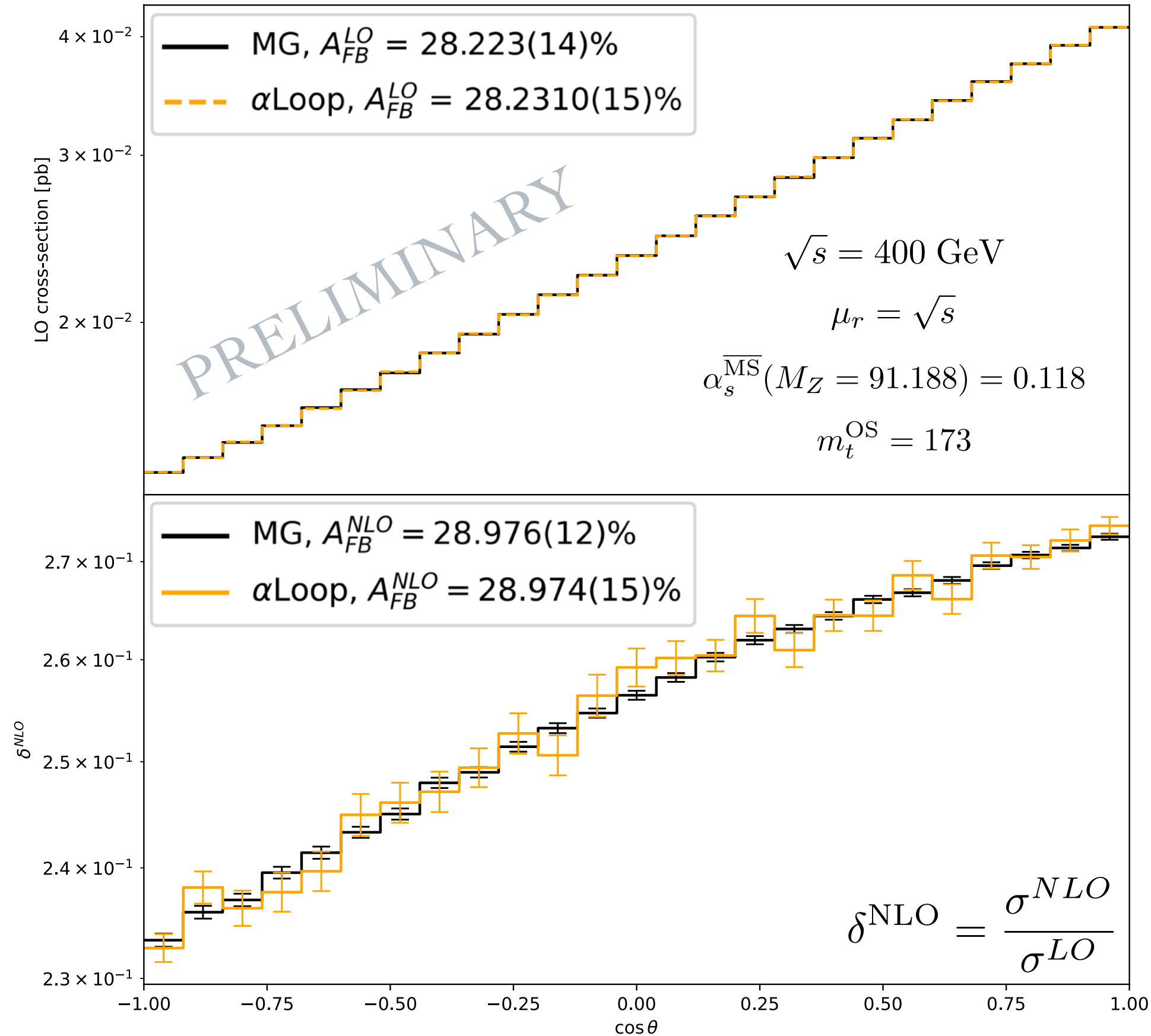
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APPLICATIONS

EXAMPLE II : NLO AFB FOR $e^+e^- \rightarrow \gamma^*/Z \rightarrow t\bar{t}$



First result in LU with γ^5 and EW-boson

Contour deformation well-behaved in this case

Credits to ETHZ student

Max Hofer

NUMERICAL RESULTS

SG id	Ξ	$\sigma_{\gamma^* \rightarrow jj}^{(\overline{\text{MS}})} [\text{GeV}^{-2}]$ $p_{\gamma^*}^2 = \mu_r^2 = (400 \text{ GeV})^2$	$\Delta [\%]$	$\sigma_{\gamma^* \rightarrow t\bar{t}}^{[\alpha_s^{(\overline{\text{MS}})}, m_t^{(\text{OS})}]} [\text{GeV}^{-2}]$ $\mu_r^2 = m_t^2, p_{\gamma^*}^2 = (400 \text{ GeV})^2$	$\Delta [\%]$
LO		$\mathcal{O}(\alpha_s^0)$			
A.1	1	$5.031049 \cdot 10^{-01}$	0.0018	$1.387586 \cdot 10^{+00}$	0.0011
Total		$5.031049 \cdot 10^{-01}$	0.0018	$1.387586 \cdot 10^{+00}$	0.0011
NLO		$\mathcal{O}(\alpha_s^1)$			
B.1	1	$5.03926 \cdot 10^{-02}$	0.0075	$2.52705 \cdot 10^{-01}$	0.034
B.2	2	$-3.14956 \cdot 10^{-02}$	0.018	$1.80050 \cdot 10^{-01}$	0.049
Total		$1.88970 \cdot 10^{-02}$	0.036	$4.3276 \cdot 10^{-01}$	0.028
Benchmark		$1.889690 \cdot 10^{-02}$	0.00053	$4.32831 \cdot 10^{-01}$	-0.018
NNLO		$\mathcal{O}(\alpha_s^2 n_f)$			
C.1	1	$-4.66342 \cdot 10^{-04}$	0.019	$-1.0022 \cdot 10^{-03}$	0.17
C.2	2	$3.8448 \cdot 10^{-04}$	0.036	$-4.6982 \cdot 10^{-03}$	0.081
Total		$-8.186 \cdot 10^{-05}$	0.20	$-5.7004 \cdot 10^{-03}$	0.073
Benchmark		$-8.1834 \cdot 10^{-05}$	0.036	$-5.6982 \cdot 10^{-03}$	0.038
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D.1	2	$-2.30886 \cdot 10^{-03}$	0.017	$3.8886 \cdot 10^{-02}$	0.031
D.2	2	$6.42018 \cdot 10^{-03}$	0.0055	$5.6351 \cdot 10^{-03}$	0.14
D.3	2	$-6.91254 \cdot 10^{-03}$	0.0046	$1.76075 \cdot 10^{-02}$	0.055
D.4	1	$3.20278 \cdot 10^{-03}$	0.0084	$8.8163 \cdot 10^{-03}$	0.078
D.5	1	$1.68148 \cdot 10^{-03}$	0.013	$9.200 \cdot 10^{-04}$	0.79
D.6	2	$6.6698 \cdot 10^{-04}$	0.027	$5.1058 \cdot 10^{-03}$	0.15
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D.11	2	$2.34300 \cdot 10^{-04}$	0.031	$2.0845 \cdot 10^{-03}$	0.083
D.12	1	$4.11063 \cdot 10^{-04}$	0.017	$3.5114 \cdot 10^{-03}$	0.12
D.13	1	$2.41514 \cdot 10^{-04}$	0.026	$8.222 \cdot 10^{-04}$	0.19
D.14	2	$5.8386 \cdot 10^{-05}$	0.088	$1.76075 \cdot 10^{-02}$	0.055
D.15	1	$-1.75957 \cdot 10^{-04}$	0.022	$-7.242 \cdot 10^{-04}$	0.14
Total		$1.40910 \cdot 10^{-03}$	0.056	$1.04214 \cdot 10^{-01}$	0.024
Benchmark		$1.40941 \cdot 10^{-03}$	-0.022	$1.0386 \cdot 10^{-01}$	0.34

Analytic benchmarks :

$$\sigma_{\gamma^* \rightarrow X}^{\overline{\text{MS}}}(p^2, \mu_r^2 = p^2) \supset K_X n_f \left(\frac{\alpha_s}{4\pi} \right)^2 \sigma_{\gamma^* \rightarrow X}^{(\text{LO})}(p^2)$$

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D.13	1	$2.41514 \cdot 10^{-04}$	0.026	$8.222 \cdot 10^{-04}$	0.19
D.14	2	$5.8386 \cdot 10^{-05}$	0.088	$1.76075 \cdot 10^{-02}$	0.055
D.15	1	$-1.75957 \cdot 10^{-04}$	0.022	$-7.242 \cdot 10^{-04}$	0.14
Total		$1.40910 \cdot 10^{-03}$	0.056	$1.04214 \cdot 10^{-01}$	0.024
Benchmark		$1.40941 \cdot 10^{-03}$	-0.022	$1.0386 \cdot 10^{-01}$	0.34

Analytic benchmarks :

$$\sigma_{\gamma^* \rightarrow X}^{\overline{\text{MS}}}(p^2, \mu_r^2 = p^2) \supset K_X n_f \left(\frac{\alpha_s}{4\pi} \right)^2 \sigma_{\gamma^* \rightarrow X}^{(\text{LO})}(p^2)$$

$$\gamma^* \rightarrow jj$$

$$K_{jj} = -C_F [11 - 8\zeta_3]$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

NUMERICAL RESULTS

SG id	Ξ	$\sigma_{\gamma^* \rightarrow jj}^{(\overline{\text{MS}})} [\text{GeV}^{-2}]$ $p_{\gamma^*}^2 = \mu_r^2 = (400 \text{ GeV})^2$	$\Delta [\%]$	$\sigma_{\gamma^* \rightarrow t\bar{t}}^{[\alpha_s^{(\overline{\text{MS}})}, m_t^{(\text{OS})}]} [\text{GeV}^{-2}]$ $\mu_r^2 = m_t^2, p_{\gamma^*}^2 = (400 \text{ GeV})^2$	$\Delta [\%]$
LO		$\mathcal{O}(\alpha_s^0)$			
A.1	1	$5.031049 \cdot 10^{-01}$	0.0018	$1.387586 \cdot 10^{+00}$	0.0011
Total		$5.031049 \cdot 10^{-01}$	0.0018	$1.387586 \cdot 10^{+00}$	0.0011
NLO		$\mathcal{O}(\alpha_s^1)$			
B.1	1	$5.03926 \cdot 10^{-02}$	0.0075	$2.52705 \cdot 10^{-01}$	0.034
B.2	2	$-3.14956 \cdot 10^{-02}$	0.018	$1.80050 \cdot 10^{-01}$	0.049
Total		$1.88970 \cdot 10^{-02}$	0.036	$4.3276 \cdot 10^{-01}$	0.028
Benchmark		$1.889690 \cdot 10^{-02}$	0.00053	$4.32831 \cdot 10^{-01}$	-0.018
NNLO		$\mathcal{O}(\alpha_s^2 n_f)$			
C.1	1	$-4.66342 \cdot 10^{-04}$	0.019	$-1.0022 \cdot 10^{-03}$	0.17
C.2	2	$3.8448 \cdot 10^{-04}$	0.036	$-4.6982 \cdot 10^{-03}$	0.081
Total		$-8.186 \cdot 10^{-05}$	0.20	$-5.7004 \cdot 10^{-03}$	0.073
Benchmark		$-8.1834 \cdot 10^{-05}$	0.036	$-5.6982 \cdot 10^{-03}$	0.038
NNLO		$\mathcal{O}(\alpha_s^2)$			
D.1	2	$-2.30886 \cdot 10^{-03}$	0.017	$3.8886 \cdot 10^{-02}$	0.031
D.2	2	$6.42018 \cdot 10^{-03}$	0.0055	$5.6351 \cdot 10^{-03}$	0.14
D.3	2	$-6.91254 \cdot 10^{-03}$	0.0046	$1.76075 \cdot 10^{-02}$	0.055
D.4	1	$3.20278 \cdot 10^{-03}$	0.0084	$8.8163 \cdot 10^{-03}$	0.078
D.5	1	$1.68148 \cdot 10^{-03}$	0.013	$9.200 \cdot 10^{-04}$	0.79
D.6	2	$6.6698 \cdot 10^{-04}$	0.027	$5.1058 \cdot 10^{-03}$	0.15
D.7	2	$-1.30381 \cdot 10^{-03}$	0.013	$6.7284 \cdot 10^{-03}$	0.10
$\overline{\text{D.7}}$	2	$-1.30395 \cdot 10^{-03}$	0.013	$6.7300 \cdot 10^{-03}$	0.10
D.9	2	$-1.6661 \cdot 10^{-04}$	0.064	$2.3361 \cdot 10^{-03}$	0.12
D.10	2	$6.64155 \cdot 10^{-04}$	0.012	$3.7418 \cdot 10^{-03}$	0.14
D.11	2	$2.34300 \cdot 10^{-04}$	0.031	$2.0845 \cdot 10^{-03}$	0.083
D.12	1	$4.11063 \cdot 10^{-04}$	0.017	$3.5114 \cdot 10^{-03}$	0.12
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$$\gamma^* \rightarrow t\bar{t}$$

$$K_{t\bar{t}} = \delta^{(2)} = -\frac{(3-v^2)(1+v^2)}{6} \times \left\{ \text{Li}_3(p) - 2\text{Li}_3(1-p) - 3\text{Li}_3(p^2) - 4\text{Li}_3\left(\frac{p}{1+p}\right) - 5\text{Li}_3(1-p^2) + \frac{11}{2}\zeta(3) \right. \\ + \text{Li}_2(p) \ln\left(\frac{4(1-v^2)}{v^4}\right) + 2\text{Li}_2(p^2) \ln\left(\frac{1-v^2}{2v^2}\right) + 2\zeta(2) \left[\ln p - \ln\left(\frac{1-v^2}{4v}\right) \right] \\ - \frac{1}{6} \ln\left(\frac{1+v}{2}\right) \left[36 \ln 2 \ln p - 44 \ln^2 p + 49 \ln p \ln\left(\frac{1-v^2}{4}\right) + \ln^2\left(\frac{1-v^2}{4}\right) \right] \\ \left. - \frac{1}{2} \ln p \ln v \left[36 \ln 2 + 21 \ln p + 16 \ln v - 22 \ln(1-v^2) \right] \right\} \\ + \frac{1}{24} \left\{ (15 - 6v^2 - v^4) (\text{Li}_2(p) + \text{Li}_2(p^2)) + 3(7 - 22v^2 + 7v^4) \text{Li}_2(p) \right. \\ - (1-v)(51 - 45v - 27v^2 + 5v^3) \zeta(2) \\ + \frac{(1+v)(-9 + 33v - 9v^2 - 15v^3 + 4v^4)}{v} \ln^2 p \\ + \left[(33 + 22v^2 - 7v^4) \ln 2 - 10(3-v^2)(1+v^2) \ln v \right. \\ \left. - (15 - 22v^2 + 3v^4) \ln\left(\frac{1-v^2}{4v^2}\right) \right] \ln p \\ + 2v(3-v^2) \ln\left(\frac{4(1-v^2)}{v^4}\right) \left[\ln v - 3 \ln\left(\frac{1-v^2}{4v}\right) \right] \\ + \frac{237 - 96v + 62v^2 + 32v^3 - 59v^4}{4} \ln p - 16v(3-v^2) \ln\left(\frac{1+v}{4}\right) \\ \left. - 2v(39 - 17v^2) \ln\left(\frac{1-v^2}{2v^2}\right) - \frac{v(75 - 29v^2)}{2} \right\} \dots \quad (\text{B.3})$$

[Chetykrin, Kuehn, Steinhauser, arxiv : 9606230]