Progress in algorithms and numerical tools for QCD, Paris, 10 June 2026

Exploring Feynman Integrals in Parameter Space

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[2506.24073]

Gardi, Herzog, Ma
[2211.14845] [2407.13738] [26xx.xxxxx]

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THE ROYAL SOCIETY

Artwork: Milad Fakurian on Unsplash

Outline

Motivation

Calculation or expansion of Feynman Integrals

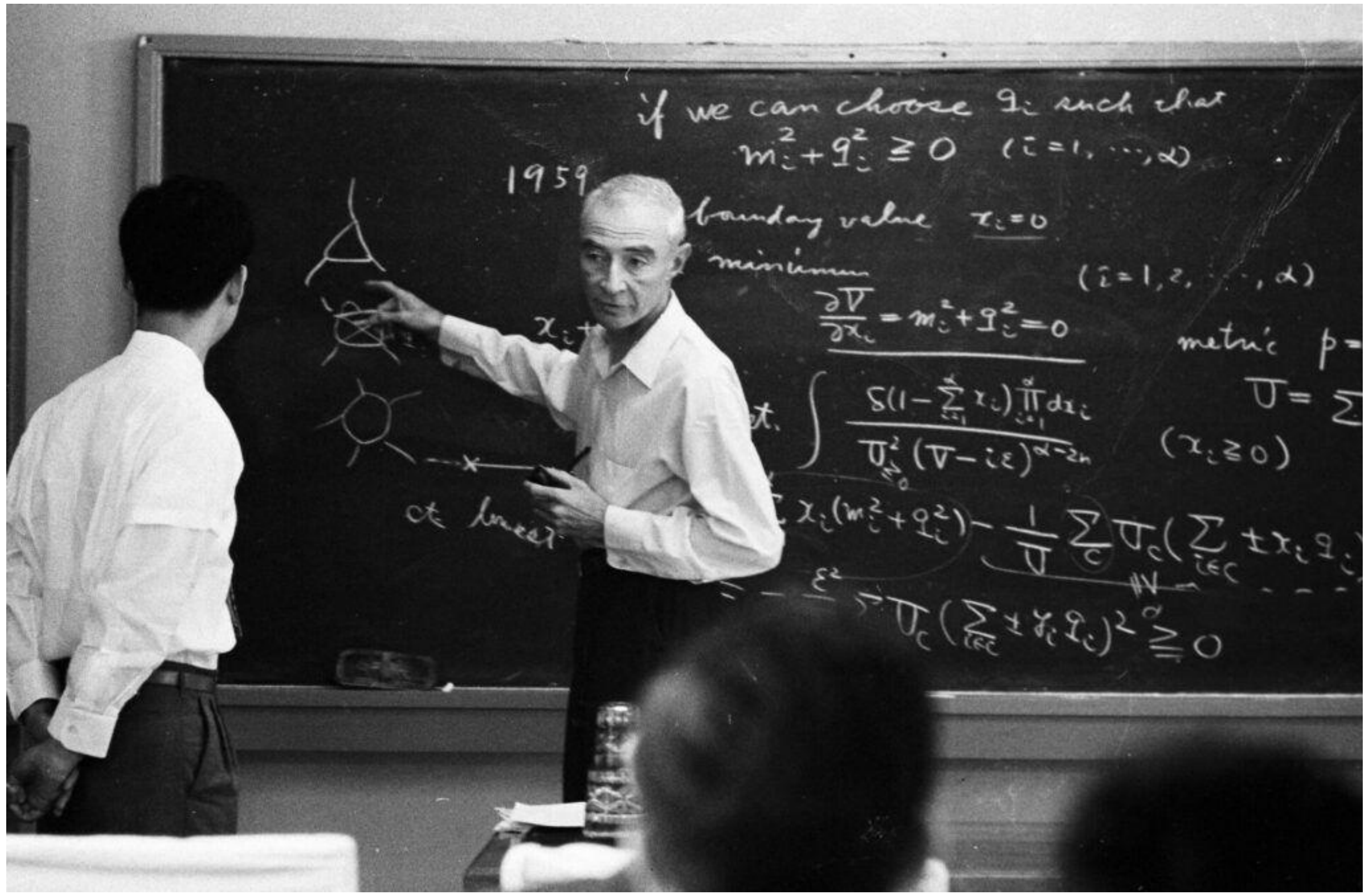
Question

Can we identify the threshold structure of Feynman integrals in parameter space?

Goals

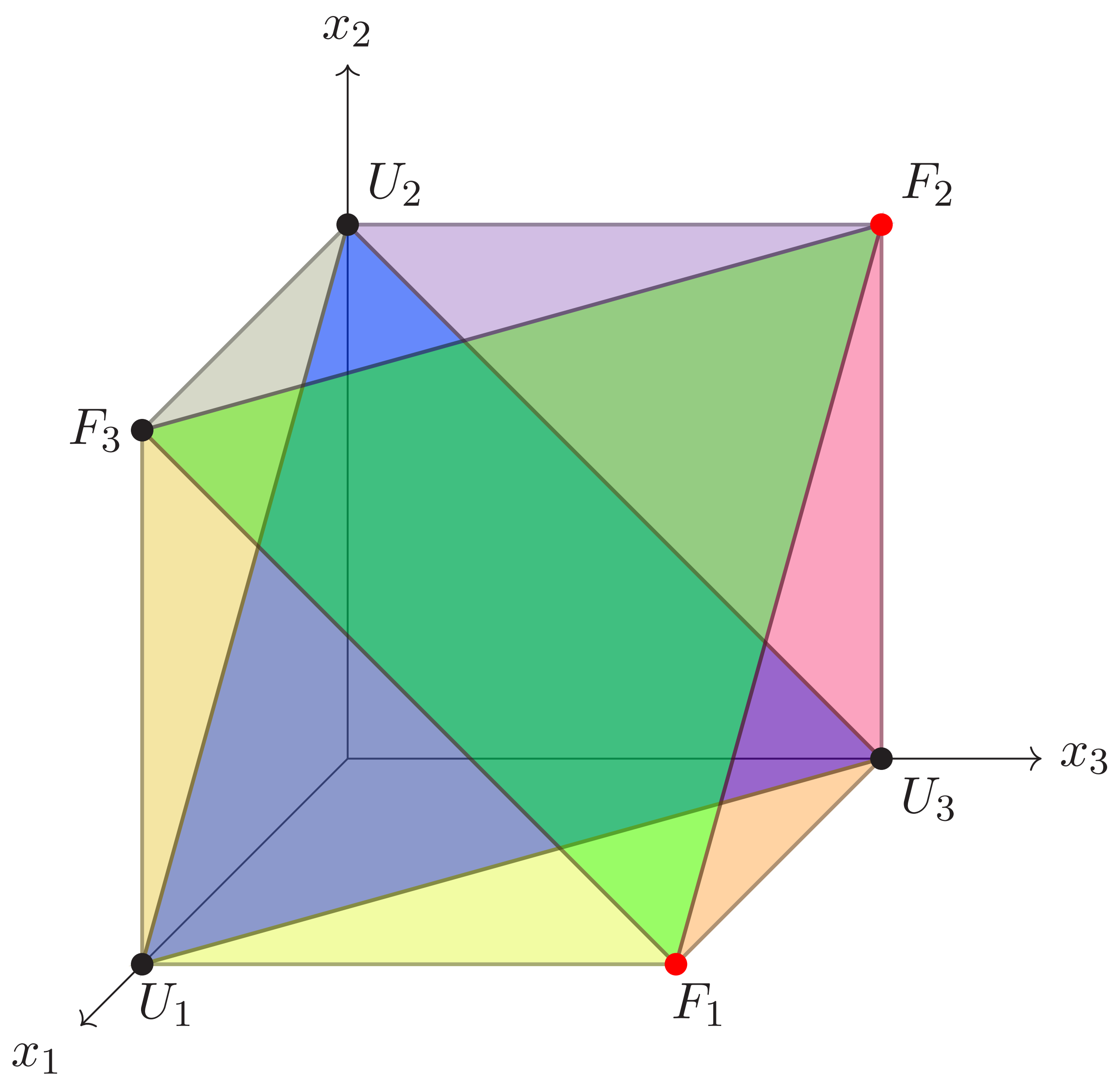
1. Recast physical Feynman integrals as non-negative ``Euclidean-like'' integrals

2. Find all singular surfaces, generalising the parametric approach to the Method of Regions



Oppenheimer (Kyoto University, Japan), 1960

Introduction



Feynman Integrals in Parameter Space

Loop momenta

$$[\mathrm{d}\mathbf{x}] = \prod_{e \in G} \frac{\mathrm{d}x_e}{x_e}$$

$$I(\mathbf{s}) \sim \int \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_l \frac{1}{\prod_{i=1}^{N} (q_i^2 - m_i^2 + i\delta)^{\nu_i}} \leftrightarrow \int_{\mathbb{R}^N_{>0}} [\mathrm{d}\mathbf{x}] \; \frac{[\mathcal{U}(\mathbf{x})]^{N-(L+1)D/2}}{[\mathcal{F}(\mathbf{x};\mathbf{s}) - i\delta]^{N-LD/2}} \delta(1 - \alpha(\mathbf{x}))$$

Propagator momentum (depend on $k_1, \ldots, k_l$)

$\mathcal{U}, \mathcal{F}$ *polynomials* in Feynman parameters

$$\mathbf{x} = (x_1, x_2, \ldots, x_N)$$

Linear in x_e

$$\mathcal{U}(\mathbf{x}) = \sum_{T^1} \prod_{e \notin T^1} x_e,$$

$$\mathcal{F}_0(\mathbf{x};\mathbf{s}) = \sum_{T^2} (-s_{T^2}) \prod_{e \notin T^2} x_e, \qquad \mathcal{F}(\mathbf{x};\mathbf{s}) = \mathcal{F}_0(\mathbf{x};\mathbf{s}) + \mathcal{U}(\mathbf{x}) \sum_e m_e^2 x_e,$$

Linear in x_e

Quadratic in x_e

The signs of the monomials of $\mathcal{F}$ depend on **kinematic invariants** and **masses**

Landau Equations

Third representation of Analytic S-Matrix

Eden, Landshoff, Olive, Polkinghorne 66

1 $$\mathscr{F}(\mathbf{x};\mathbf{s}) = 0$$

2 $$x_j \frac{\partial \mathscr{F}(\mathbf{x};\mathbf{s})}{\partial x_j} = 0 \quad \forall j$$

$x_j = 0$ or $\dfrac{\partial \mathscr{F}(\mathbf{x};\mathbf{s})}{\partial x_j}$

For fixed $\mathbf{s} = (s_1, \ldots, s_M, m_1^2, \ldots, m_N^2)$ each equation (variety of $\mathscr{F}$ or $\partial\mathscr{F}/\partial x_j$) defines a codim-1 hypersurface

Landau singularities dictate the singularity structure of the integral and properties of asymptotic expansions

Homogeneity / Euler's Theorem

$$\mathscr{F}(\mathbf{x};\mathbf{s}) \propto \sum_i x_i \frac{\partial \mathscr{F}(\mathbf{x};\mathbf{s})}{\partial x_j}$$

Singularities

Propagators

Loops

Hyperplane bounding integral for at least one $x_i \geq 0$

$$I(\mathbf{s}) \sim \int_{\mathbb{R}^N_{\geq 0}} [\mathrm{d}\mathbf{x}] \; \frac{[\mathscr{U}(\mathbf{x})]^{N-(L+1)D/2}}{[\mathscr{F}(\mathbf{x};\mathbf{s}) - i\delta]^{N-LD/2}} \, \delta(1-\alpha(\mathbf{x}))$$

Dimensional regulator $D = 4 - 2\epsilon$

Singularities

1. UV/IR singularities when some $x \to 0$ (or $x \to \infty$) simultaneously
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Solve Landau Equations on the Boundary

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Solve Landau Equations on the Boundary

Dealing with Singularities

Let us first consider singularities on the boundary (i.e. for $x_j \to 0$ or $x_j \to \infty$)

$t_j \in \mathbb{C}$

$$I(\mathbf{s}) = \int_{\mathbb{R}^N_{\geq 0}} [\mathrm{d}\boldsymbol{x}] \left(\prod_i^N x_i^{p_i} \right) \left[\prod_j^M f_j(\mathbf{x};\mathbf{s})^{t_j} \right]$$

$\mathbf{v}_{ji} \in \mathbb{Z}^N$

$$f_j(\mathbf{x}) = \sum_i^m c_{ji}(\mathbf{s})\, \mathbf{x}^{\mathbf{v}_{ji}}$$

Assume: $c_{ji}(\mathbf{s}) \in \mathbb{R}_{\geq 0}$ **``Euclidean region"**

~ tropicalisation

Change variables to $z_i = \log(x_i)$

Singularities encoded by Newton Polytope

$\mathcal{N}(f) = \mathrm{conv}(\mathbf{v}_{j1}, \ldots, \mathbf{v}_{jm})$

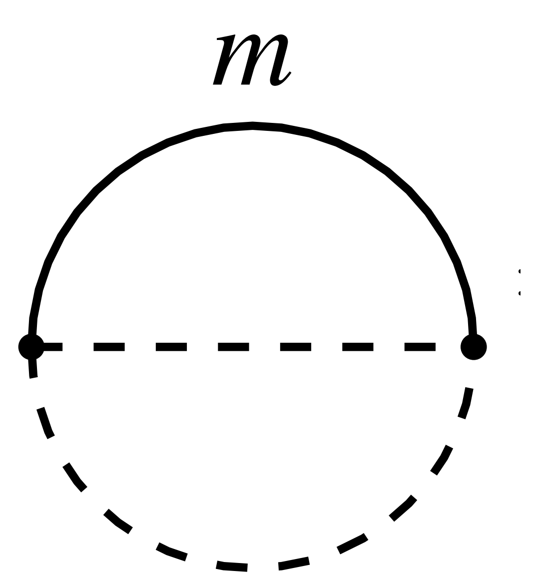
$$I(\mathbf{s}) = \int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{z}\; e^{\langle \mathbf{p},\mathbf{z}\rangle} \left[\prod_j^M f_j(\mathbf{z};\mathbf{s})^{t_j} \right]$$

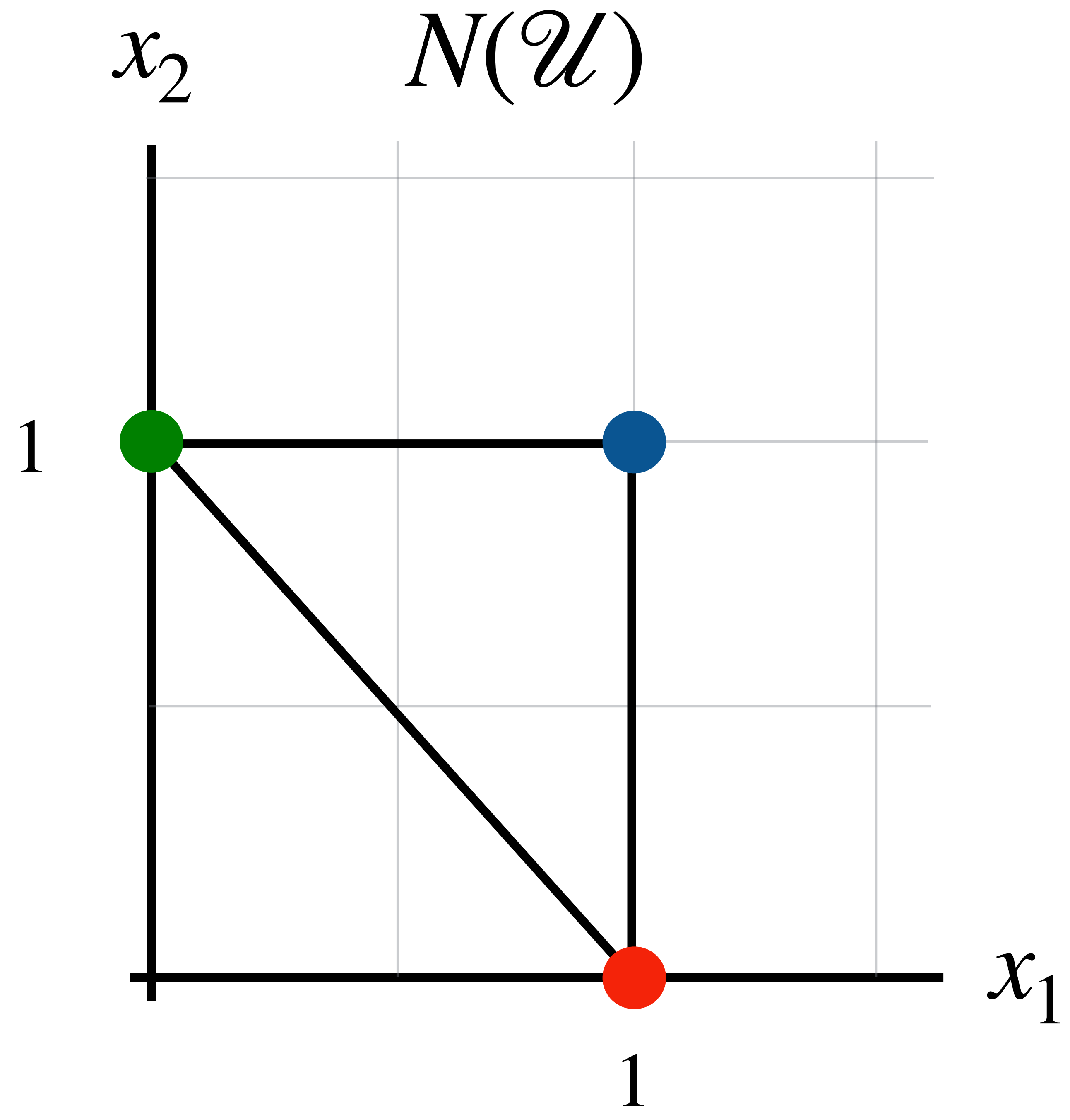
$$f_j(\mathbf{z}) = \sum_i^m c_{ji}(\mathbf{s})\, e^{\langle \mathbf{v}_{ji},\mathbf{z}\rangle}$$

Dealing with Singularities

$$\mathcal{U}(\mathbf{x}) = x_1^1 x_2^0 \quad + x_1^1 x_2^1 \quad + x_1^0 x_2^1$$

Used $\delta(1 - x_3)$ to integrate out x_3





Dealing with Singularities

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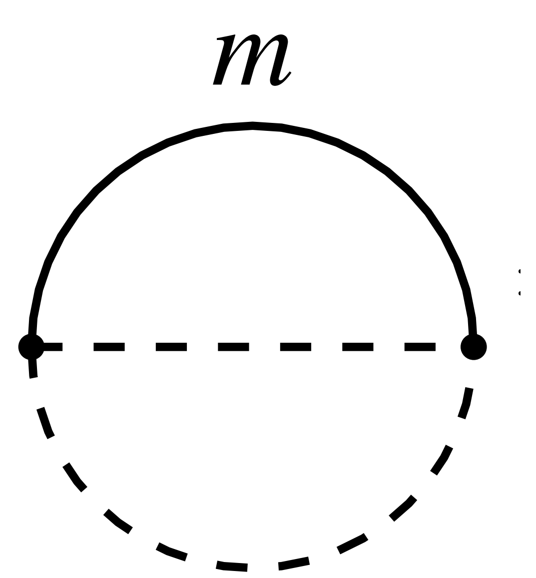
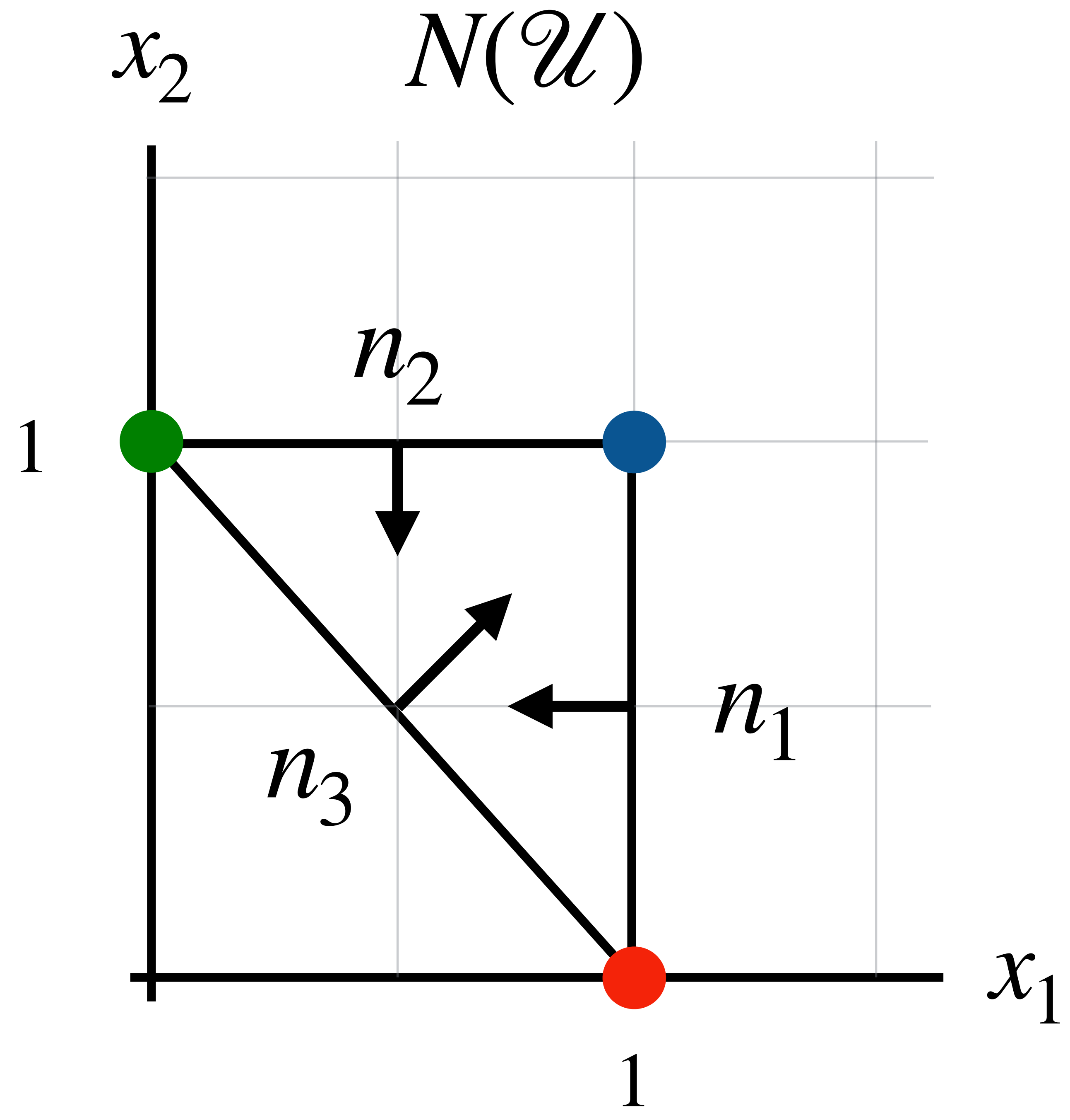


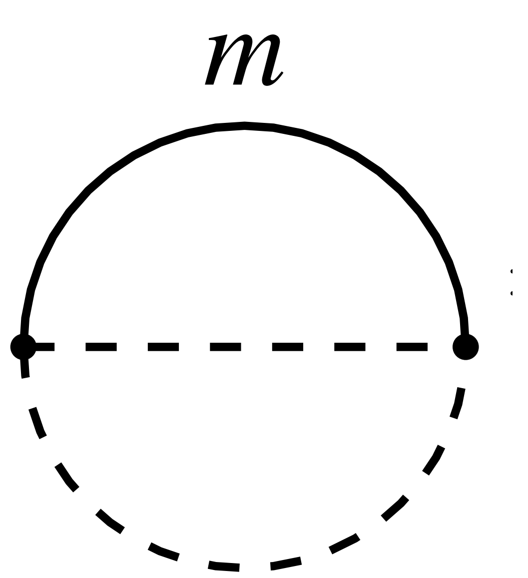


Sector decomposition

Facets define a local change of coordinates for each singularity that factorises it (blow-up)

Hepp 66; Roth, Denner 96; Binoth, Heinrich 00; Heinrich 08; Kaneko, Ueda 10; Schlenk 16

Dealing with Singularities



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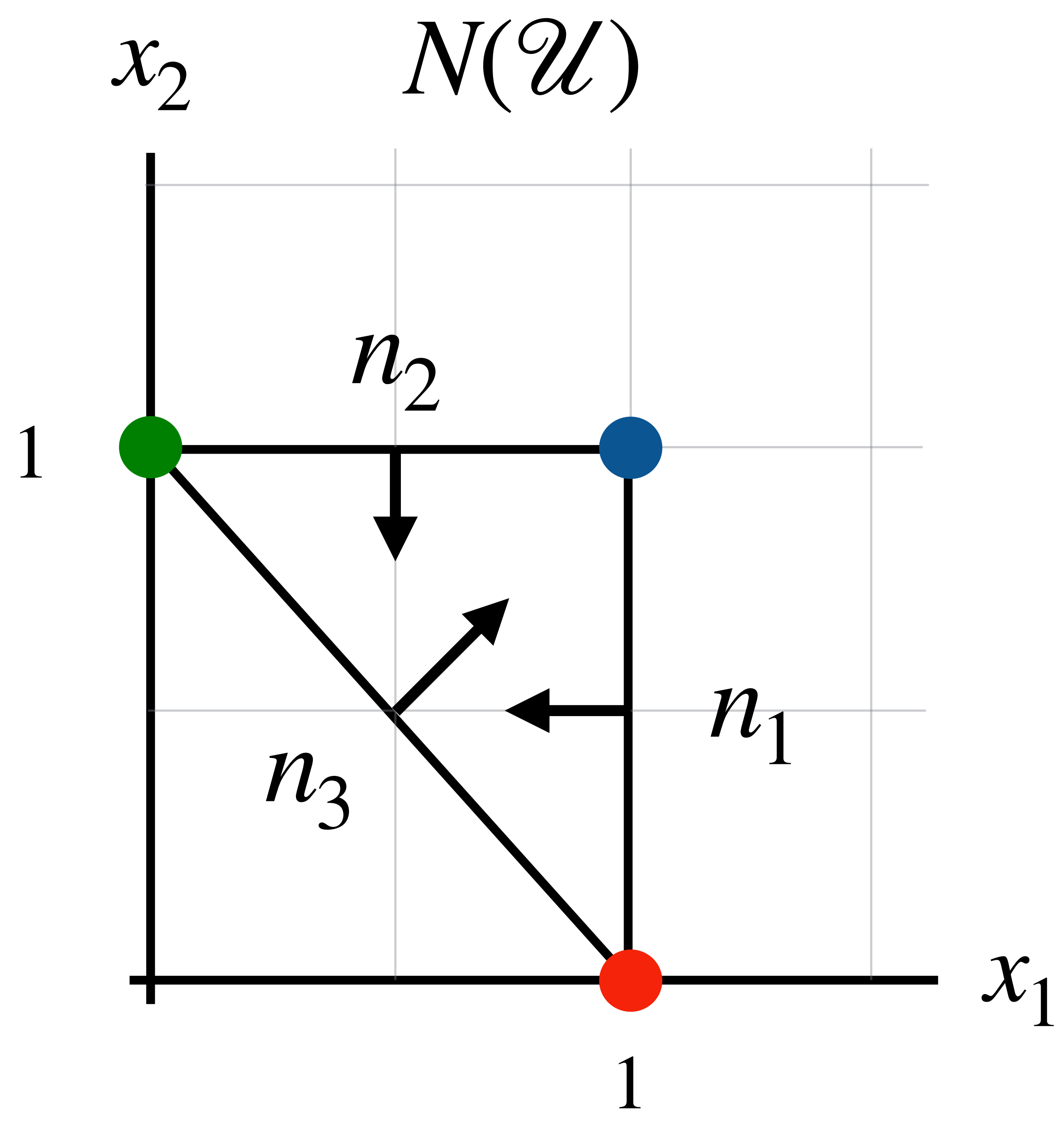
$$\mathcal{U}(\mathbf{x}) = x_1^1 x_2^0 \; + x_1^1 x_2^1 \; + x_1^0 x_2^1$$



$$x_1 = y_1^{-1} y_3 \quad x_2 = y_3 \qquad \mathcal{U}_1(\mathbf{x}) = y_3 y_1^{-1}(1 + y_1 + y_3)$$

$$x_1 = y_1^{-1} \quad x_2 = y_2^{-1} \qquad \mathcal{U}_2(\mathbf{x}) = y_1^{-1} y_2^{-1}(1 + y_1 + y_2)$$

$$x_1 = y_3 \quad x_2 = y_2^{-1} y_3 \qquad \mathcal{U}_3(\mathbf{x}) = y_3 y_2^{-1}(1 + y_2 + y_3)$$

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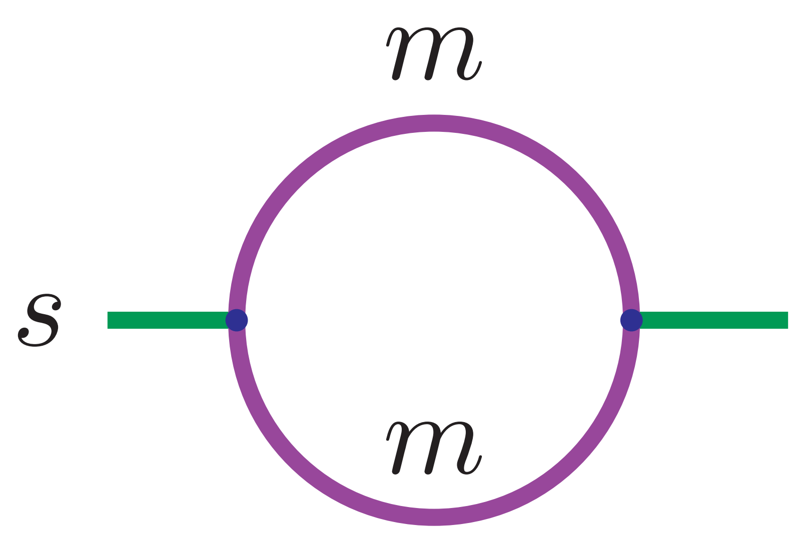
Thresholds



Singularities II

$$I(\mathbf{s}) \sim \int_{\mathbb{R}^N_{\geq 0}} [\mathrm{d}\mathbf{x}] \, \frac{[\mathscr{U}(\mathbf{x})]^{N-(L+1)D/2}}{[\mathscr{F}(\mathbf{x};\mathbf{s}) - i\delta]^{N-LD/2}} \delta(1 - \alpha(\mathbf{x}))$$

$$\mathscr{F}(\boldsymbol{x};\mathbf{s}) = 0$$

$$x_j \frac{\partial \mathscr{F}(\boldsymbol{x};\mathbf{s})}{\partial x_j} = 0 \quad \forall j$$

Solve Landau Equations on the Boundary

Singularities

1. UV/IR singularities when some $x \to 0$ (or $x \to \infty$) simultaneously

2. Thresholds when $\mathscr{F}$ vanishes inside integration region, taking $\lim_{\delta \to 0^+}$ gives causal Feynman prescription

$$\mathscr{F}(\boldsymbol{x};\mathbf{s}) = 0$$

Beyond Euclidean Region

$$f_j(\mathbf{x}) = \sum_i^m c_{ji}(\mathbf{s})\, \mathbf{x}^{\mathbf{v}_{ji}}$$

Allows ``Minkowski region”

What happens to the polytope picture if we relax the assumption $c_{ij}(\mathbf{s}) \in \mathbb{R}^N_{>0}$ to $c_{ij}(\mathbf{s}) \in \mathbb{R}^N$?

Solutions of $\mathcal{F}(\mathbf{x};\mathbf{s}) = 0$ or $x_j \dfrac{\partial \mathcal{F}(\mathbf{x};\mathbf{s})}{\partial x_j} = 0$ not all fully characterised by some $x_j \to 0$

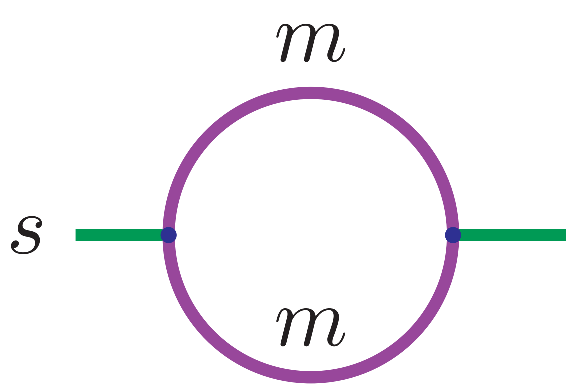
Example:

$x_1 - x_2$ vanishes for $x_1 \to 0, x_2 \to 0$, **but also** $x_1 = x_2$

Start by considering solutions of just the 1st Landau Equation

1 $$\mathcal{F}(\mathbf{x};\mathbf{s}) = 0$$

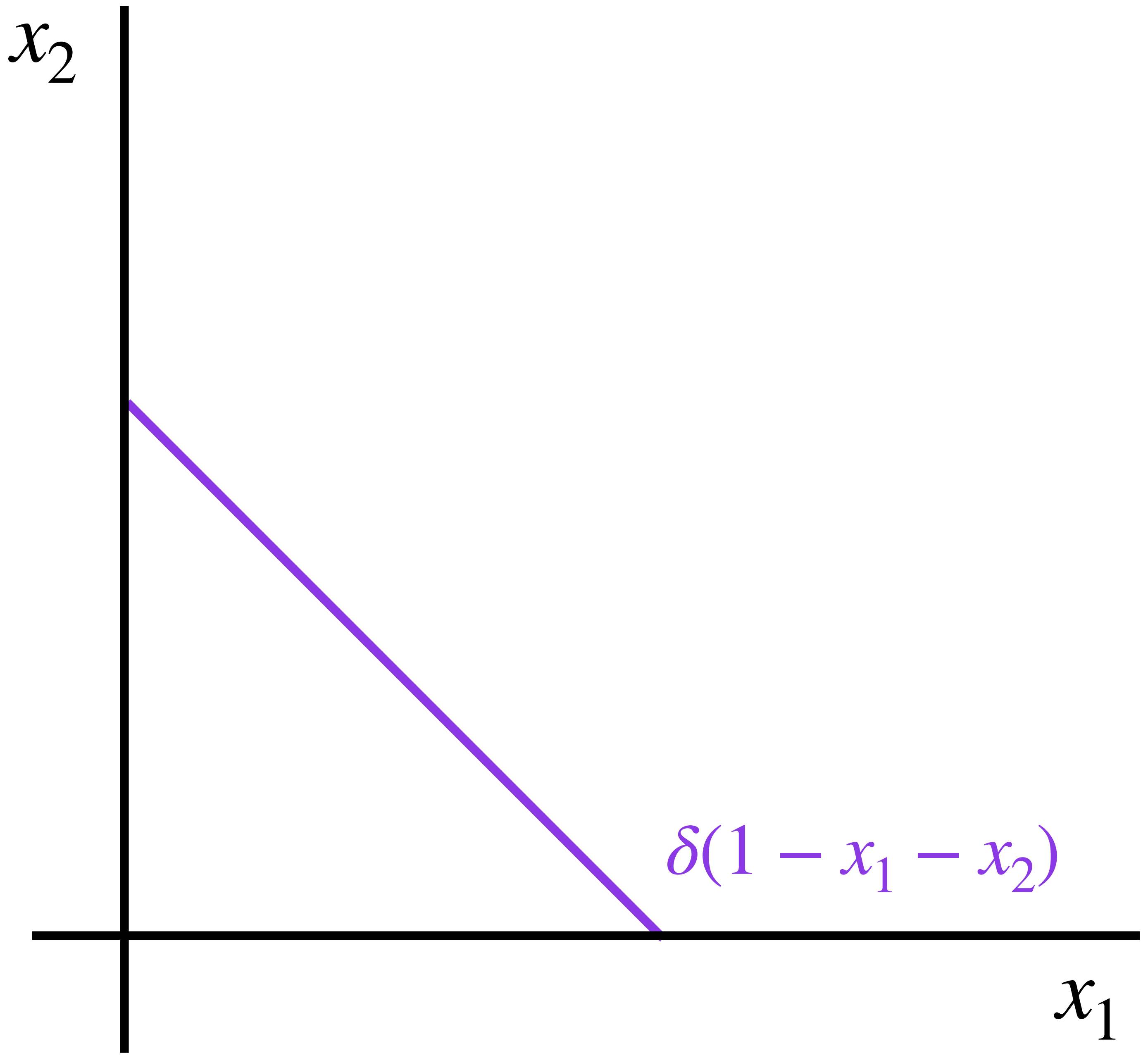
Thresholds

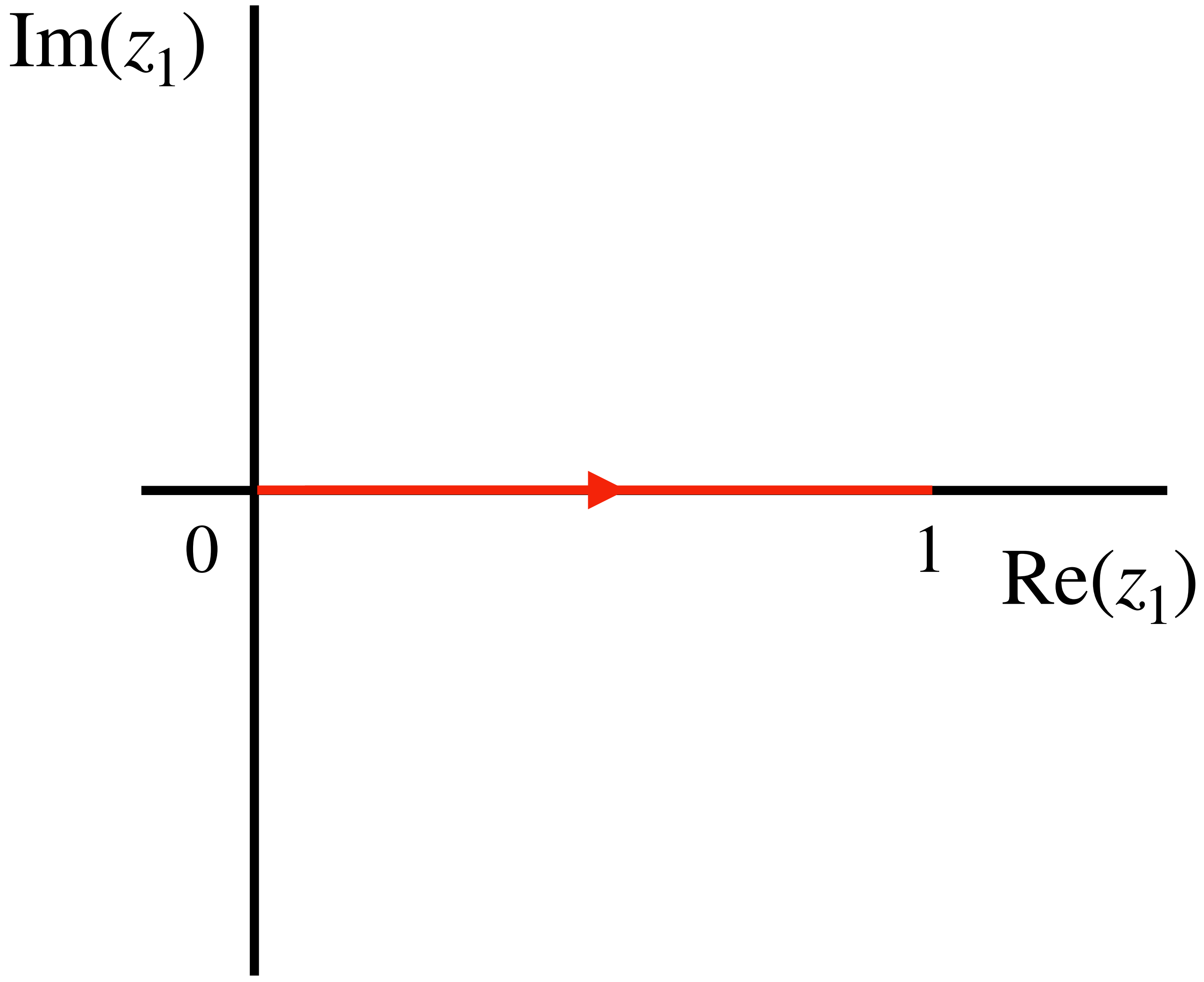


$$= \int_0^\infty \mathrm{d}x_1\,\mathrm{d}x_2\,\frac{\mathscr{U}(\mathbf{x})^{-2+2\epsilon}}{(\mathscr{F}(\mathbf{x};\mathbf{s})-i\delta)^\epsilon}\delta(1-x_1-x_2) \to \int \mathrm{d}z_1 \frac{\mathscr{U}(z_1)^{-2+2\epsilon}}{\mathscr{F}(z_1;s,m)^\epsilon} = \int_0^1 \mathrm{d}x\,|J_z|\,\frac{\mathscr{U}(z_1(x))^{-2+2\epsilon}}{\mathscr{F}(z_1(x);s,m)^\epsilon}$$

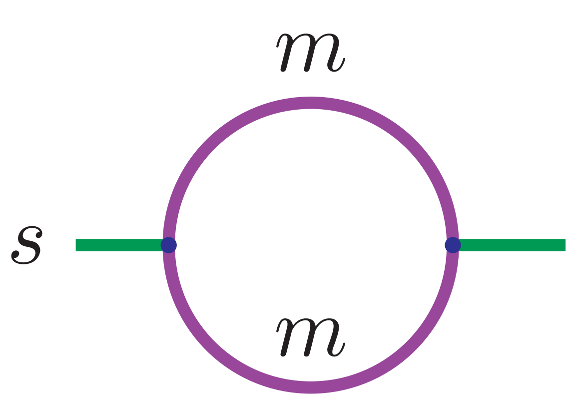
$$\mathscr{F}(\mathbf{x};\mathbf{s}) = -s x_1 x_2 + (m^2 x_1 + m^2 x_2)\,(x_1 + x_2)$$

Fix $m^2 = 1$ and $s = 0$





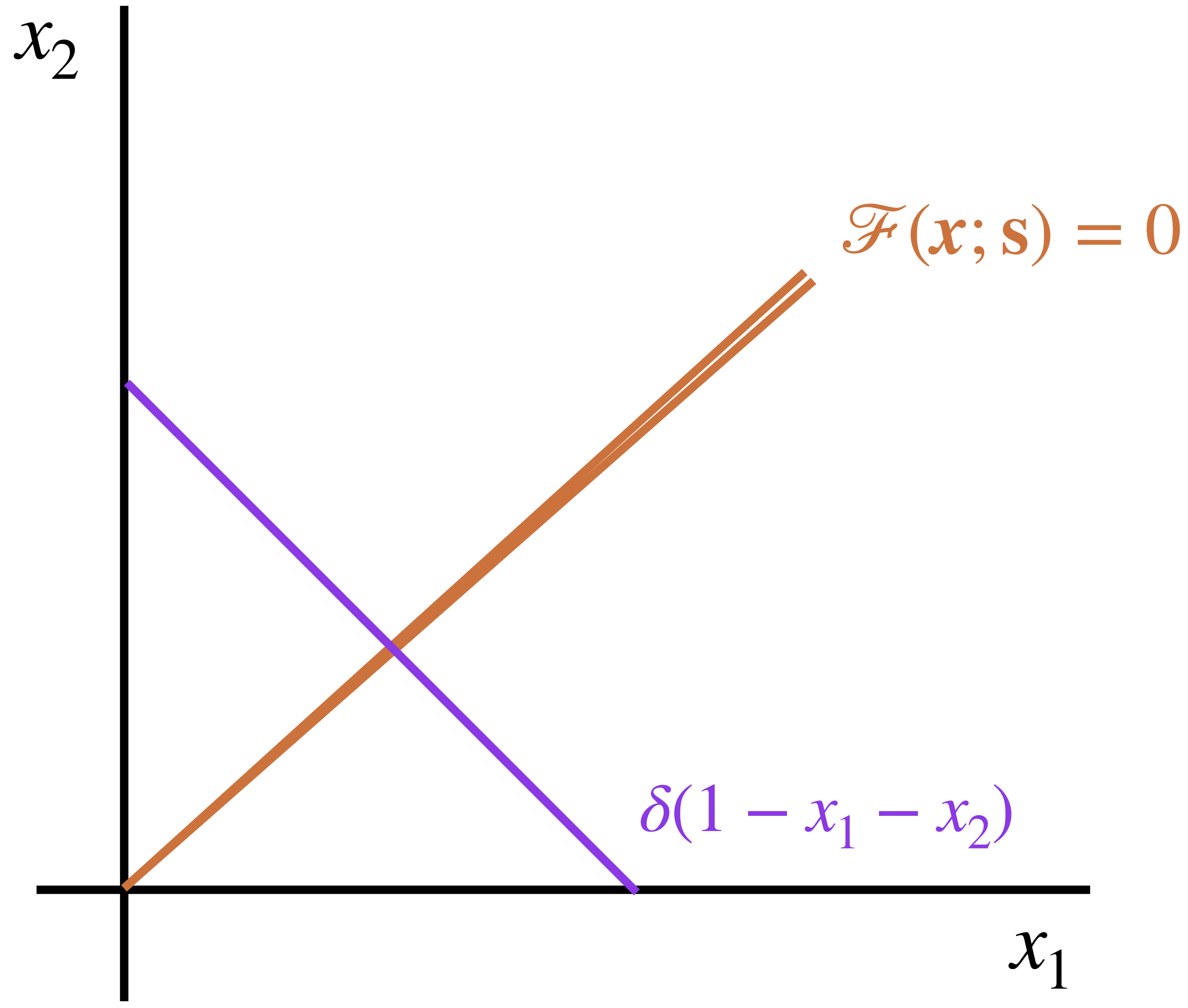
Thresholds

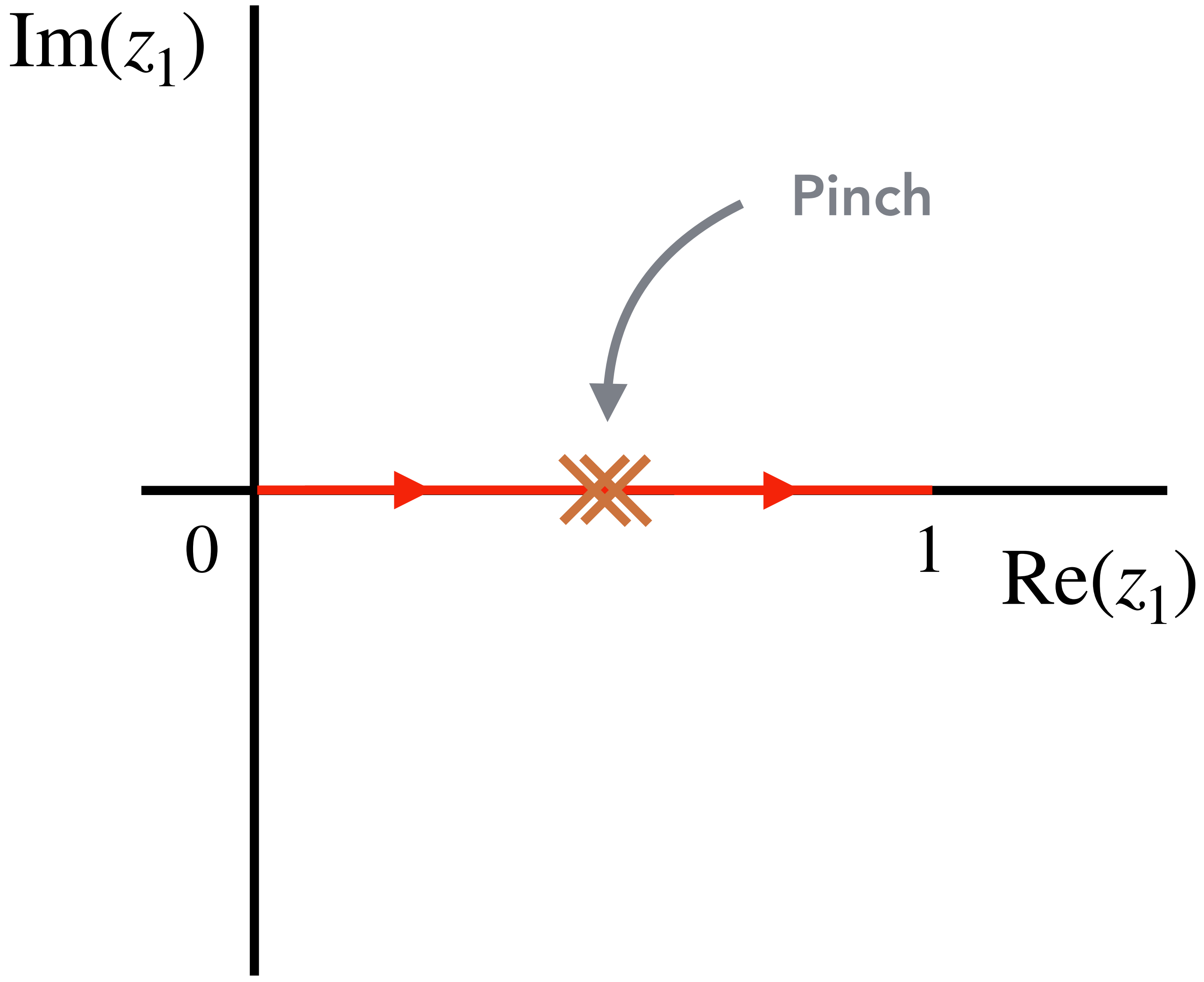


$$= \int_0^\infty \mathrm{d}x_1\,\mathrm{d}x_2 \frac{\mathcal{U}(\mathbf{x})^{-2+2\epsilon}}{(\mathcal{F}(\mathbf{x};\mathbf{s}) - i\delta)^\epsilon}\delta(1 - x_1 - x_2) \to \int \mathrm{d}z_1 \frac{\mathcal{U}(z_1)^{-2+2\epsilon}}{\mathcal{F}(z_1; s, m)^\epsilon} = \int_0^1 \mathrm{d}x\,|J_z|\frac{\mathcal{U}(z_1(x))^{-2+2\epsilon}}{\mathcal{F}(z_1(x); s, m)^\epsilon}$$

$$\mathcal{F}(\mathbf{x};\mathbf{s}) = -s x_1 x_2 + (m^2 x_1 + m^2 x_2)(x_1 + x_2)$$

Fix $m^2 = 1$ and $s \gtrsim 4$ ← Threshold





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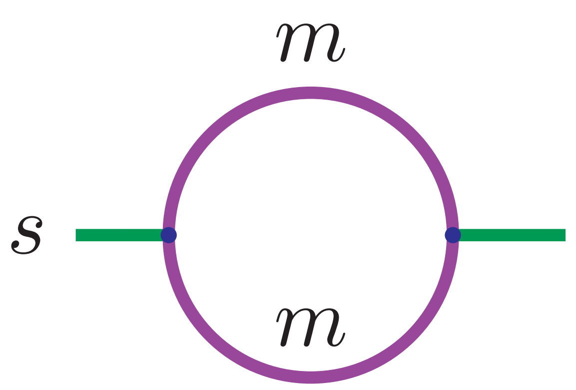
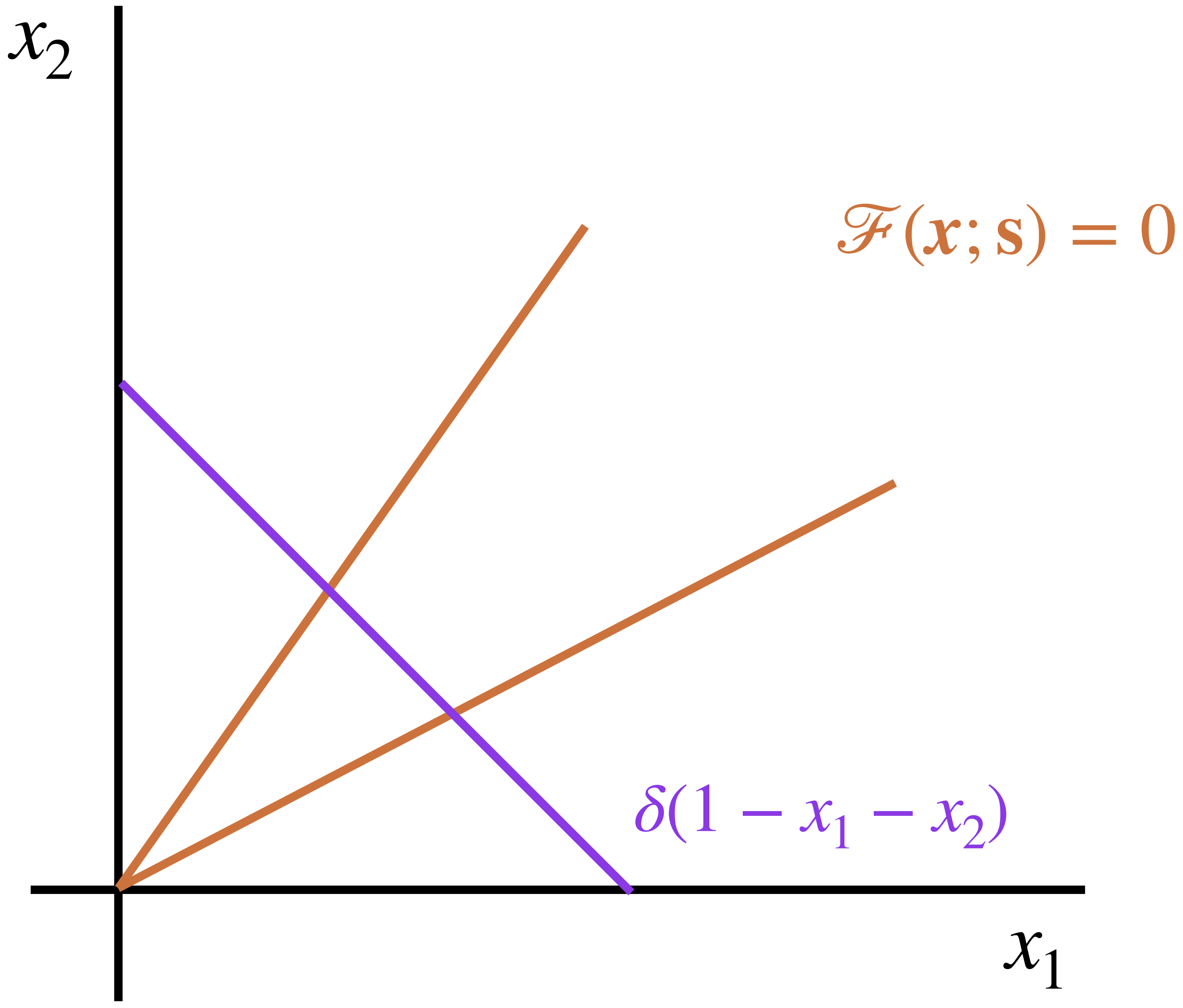
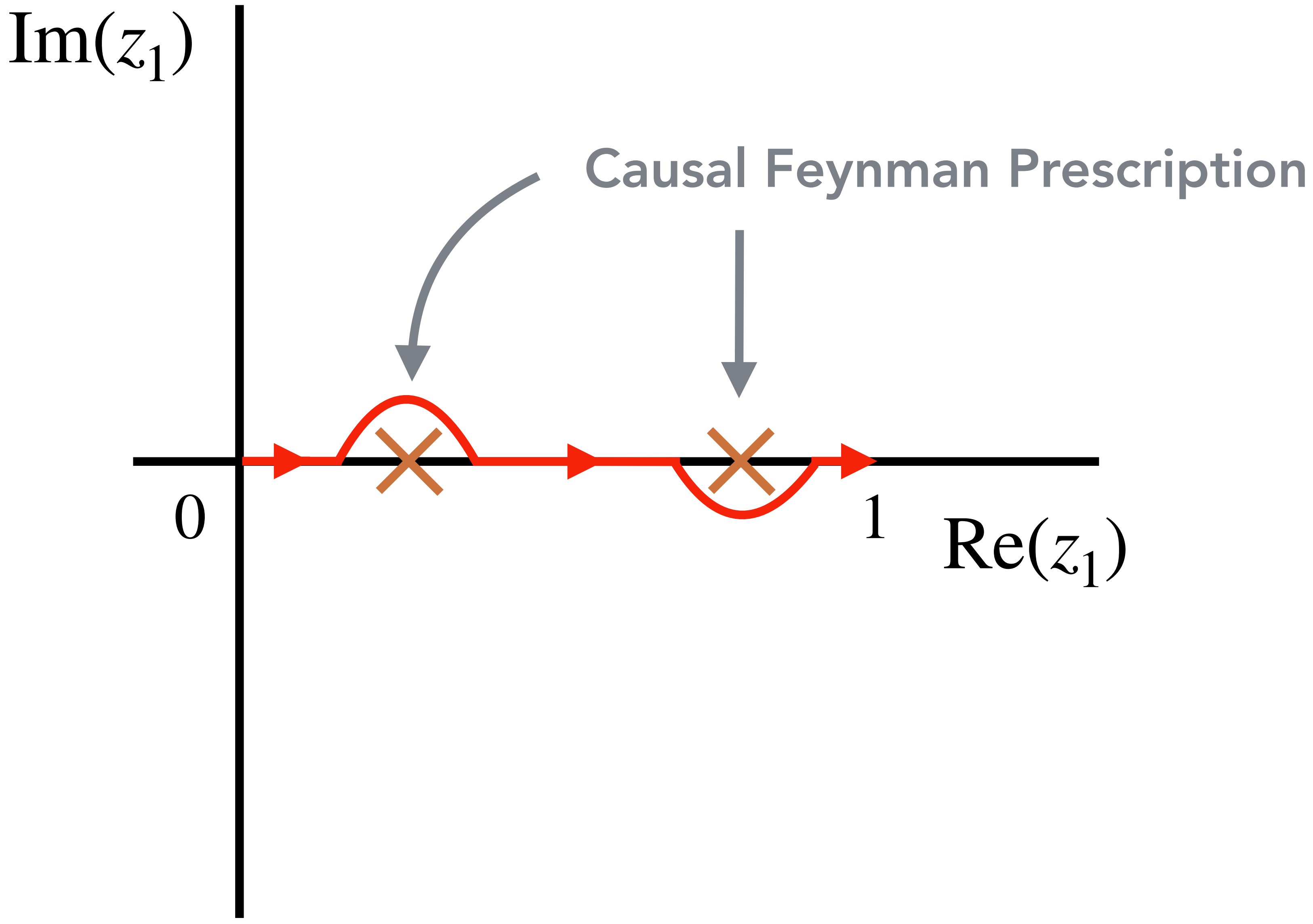


$$= \int_0^\infty \mathrm{d}x_1\, \mathrm{d}x_2 \frac{\mathscr{U}(\mathbf{x})^{-2+2\epsilon}}{(\mathscr{F}(\mathbf{x};\mathbf{s}) - i\delta)^\epsilon} \delta(1 - x_1 - x_2) \to \int \mathrm{d}z_1 \frac{\mathscr{U}(z_1)^{-2+2\epsilon}}{\mathscr{F}(z_1; s, m)^\epsilon} = \int_0^1 \mathrm{d}x\, |J_z| \frac{\mathscr{U}(z_1(x))^{-2+2\epsilon}}{\mathscr{F}(z_1(x); s, m)^\epsilon}$$

$$\mathscr{F}(\mathbf{x};\mathbf{s}) = -s x_1 x_2 + (m^2 x_1 + m^2 x_2)\,(x_1 + x_2)$$

Fix $m^2 = 1$ and $s \gg 4$ ← Above threshold





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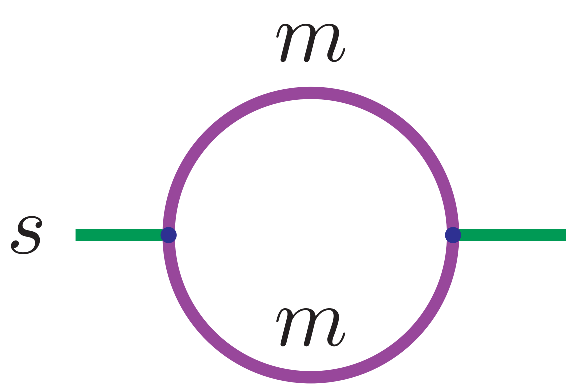

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Jacobian determinant

$$\mathcal{F}(\mathbf{x};\mathbf{s}) = -s x_1 x_2 + (m^2 x_1 + m^2 x_2)\,(x_1 + x_2)$$

Realise this integration contour via change of variables

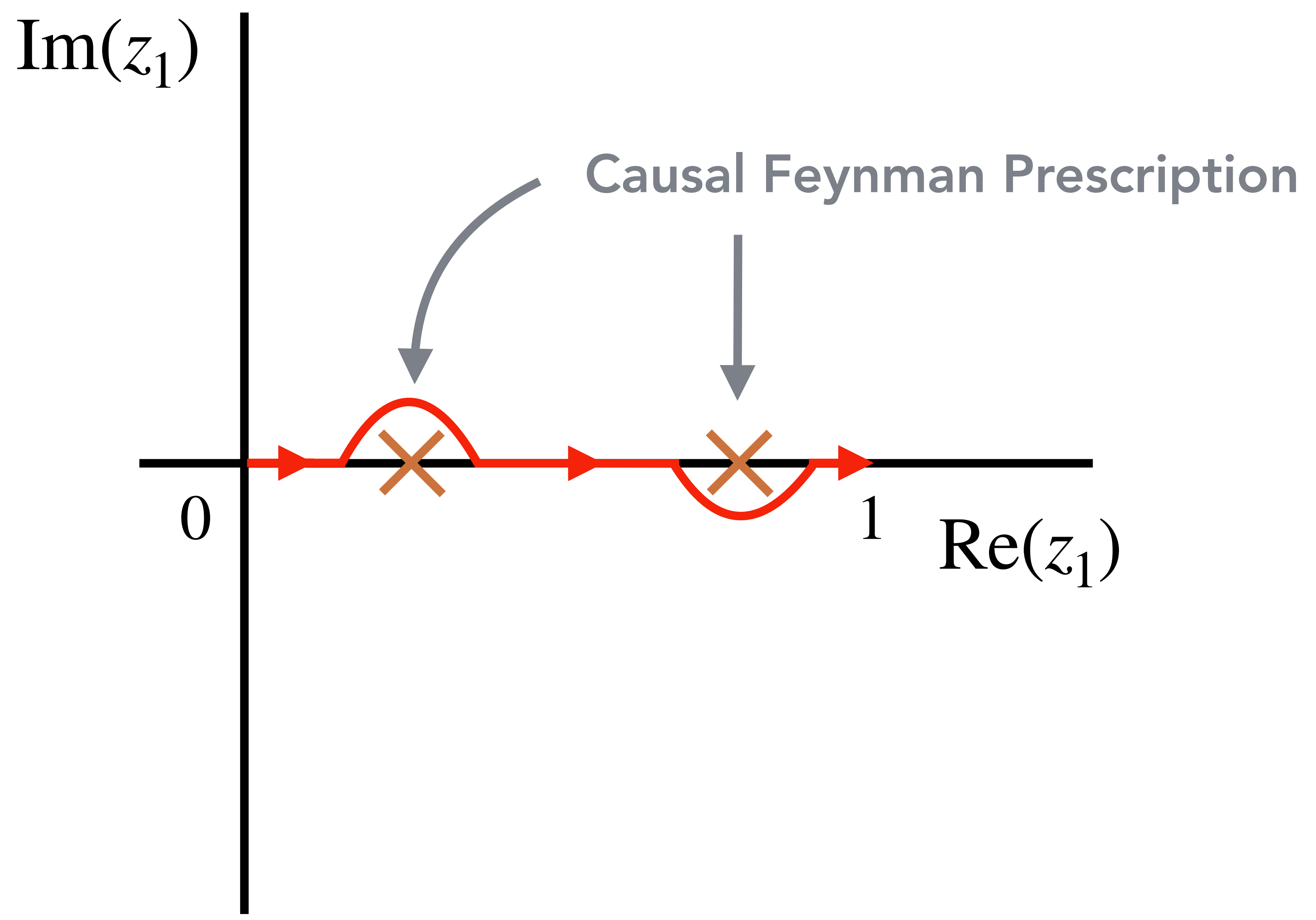
$$\mathbf{z} = \mathbf{x} - i\boldsymbol{\tau}: \quad \mathcal{F}(\mathbf{z};\mathbf{s}) = \mathcal{F}(\mathbf{x};\mathbf{s}) - i \sum_j \tau_j \frac{\partial \mathcal{F}(\mathbf{x};\mathbf{s})}{\partial x_j} + \mathcal{O}(\tau^2)$$

$$\tau_j = \lambda_j\, x_j (1 - x_j) \frac{\partial \mathcal{F}(\mathbf{x};\mathbf{s})}{\partial x_j} \text{ with small } \lambda_j > 0$$

$$-i \sum_j \left(\frac{\partial \mathcal{F}}{\partial x_j} \right)^2$$

correct deformation sign

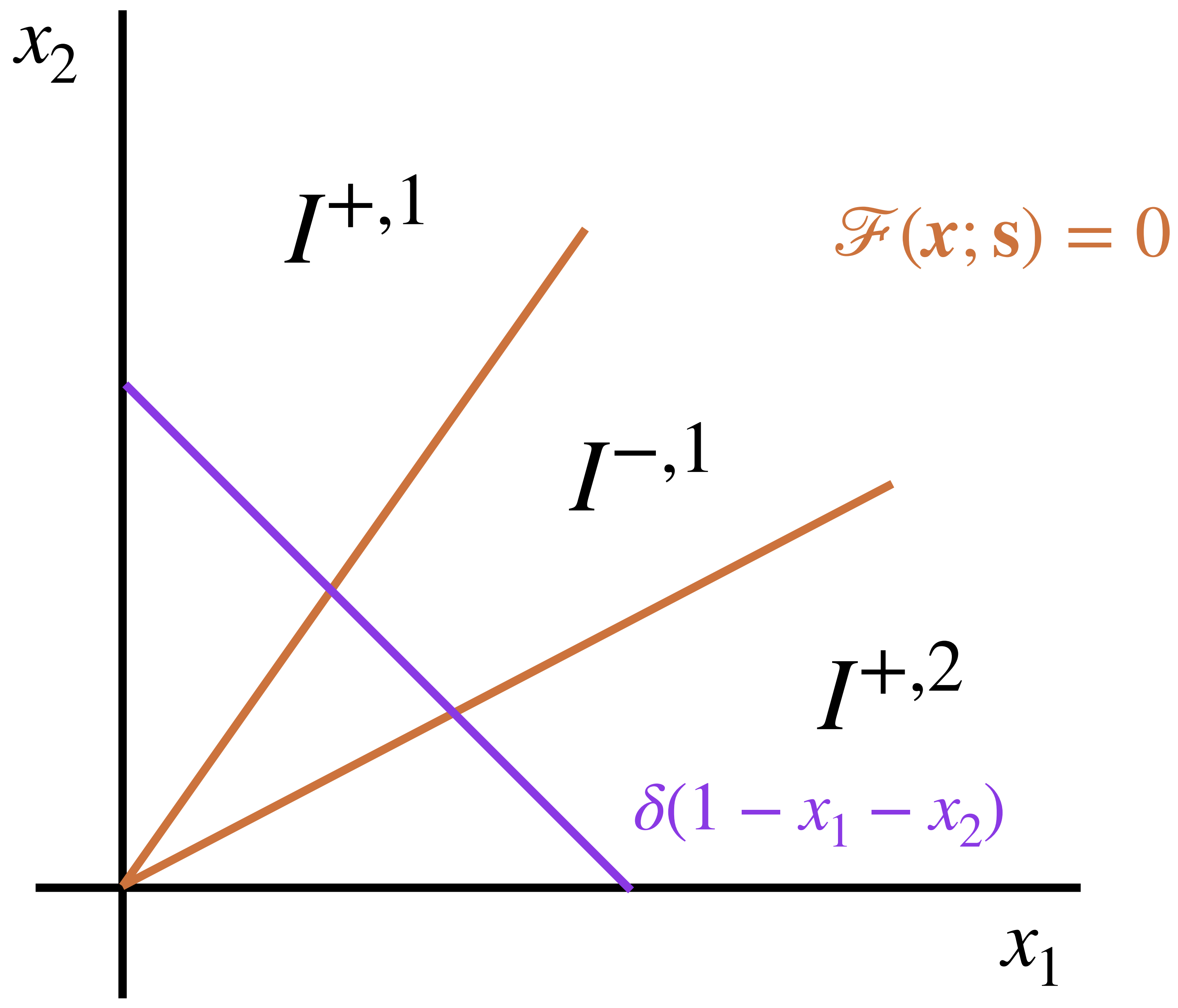
Soper 99; Binoth, Guillet, Heinrich, Pilon, Schubert 05; Nagy, Soper 06; Anastasiou, Beerli, Daleo 07, 08; Beerli 08; Borowka, Carter, Heinrich 12; Borowka 14; Borinsky, Munch, Tellander 23; …



Thresholds and Discontinuities

SPJ, Olsson, Stone 25

Instead, why not remap the $\mathscr{F}(\mathbf{x};\mathbf{s}) = 0$ hypersurface to a boundary of integration?



Manifestly non-negative integrands

$$I(\mathbf{s}) = \sum_{n_+=1}^{N_+} I^{+,n_+}(\mathbf{s}) + \lim_{\delta\to 0^+} (-1 - i\delta)^{-(N-LD/2)} \sum_{n_-=1}^{N_-} I^{-,n_-}(\mathbf{s})$$

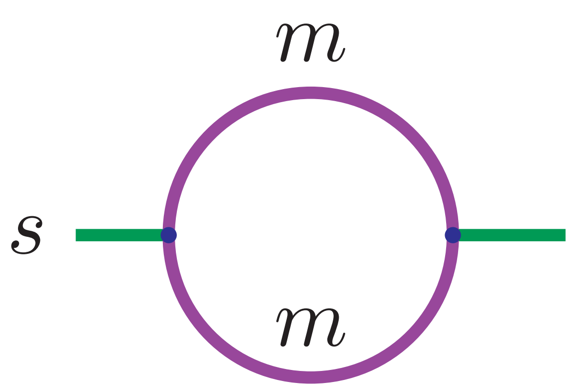
Generates imaginary part of integral

Corresponds to considering discontinuities of the integral, allows to define ``Euclidean'' like integrals even above physical thresholds

Britto 23; Britto, Hannesdottir 25

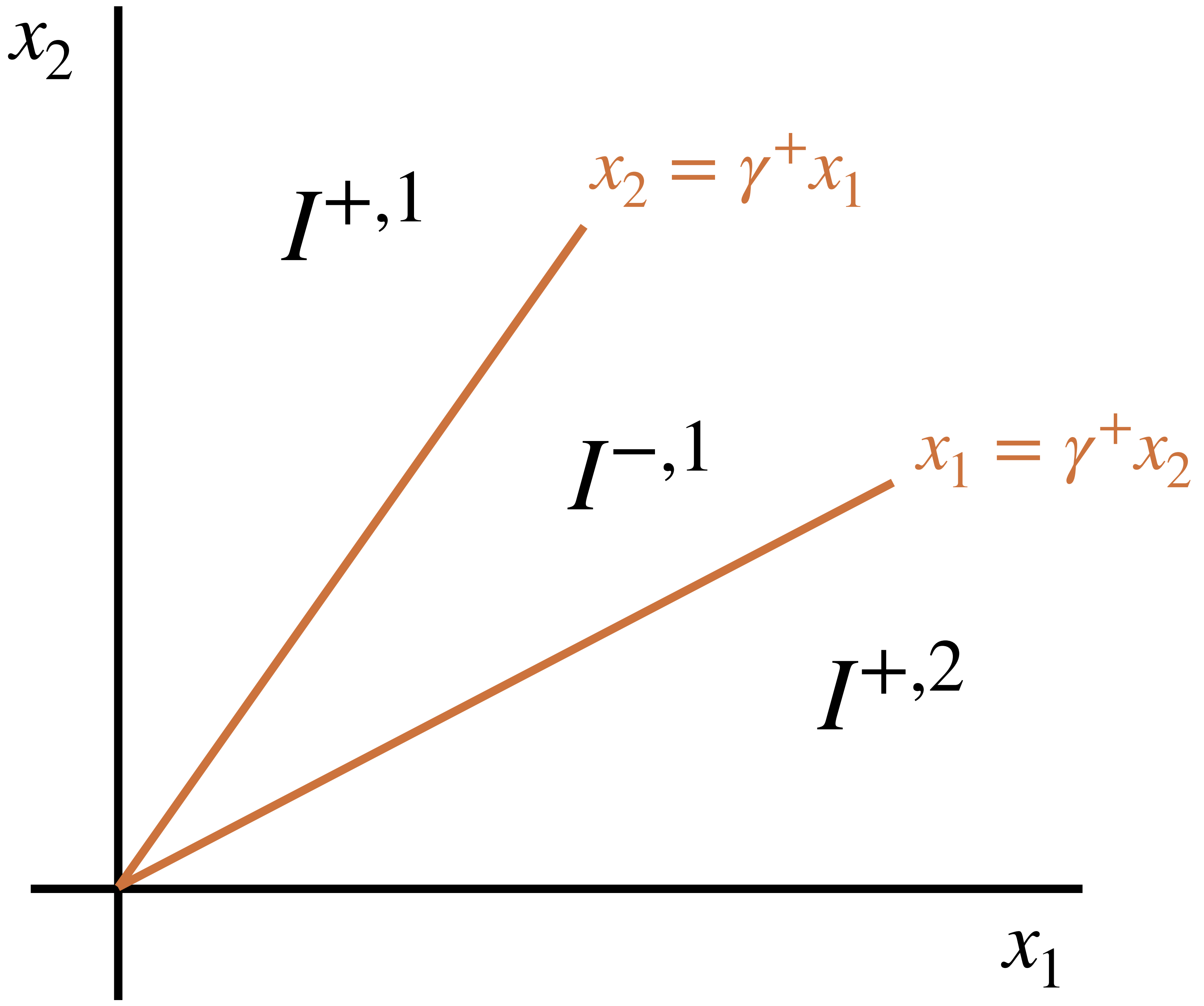
Bubble Example

SPJ, Olsson, Stone 25



$$\widetilde{\mathcal{F}}(x_1, x_2; \beta) = x_1^2 + x_2^2 - 2\frac{1+\beta^2}{1-\beta^2}x_1 x_2$$

$$\beta^2 = \frac{x-(m_1+m_2)^2}{x-(m_1-m_2)^2} \in (0,1), \quad \gamma^{\pm} = \frac{1}{\gamma^{\mp}} = \frac{1\pm\beta}{1\mp\beta}$$



$$\text{I}: \quad x_2 \to y_2 = x_2 + \gamma^+ x_1, \qquad \widetilde{\mathcal{F}}^{+,1} = x_2\left(x_2 + \frac{4\beta}{1-\beta^2}x_1\right),$$

$$\text{II}: \quad x_1 \to y_1 = x_1 + \gamma^+ x_2, \qquad \widetilde{\mathcal{F}}^{+,2} = x_1\left(x_1 + \frac{4\beta}{1-\beta^2}x_2\right),$$

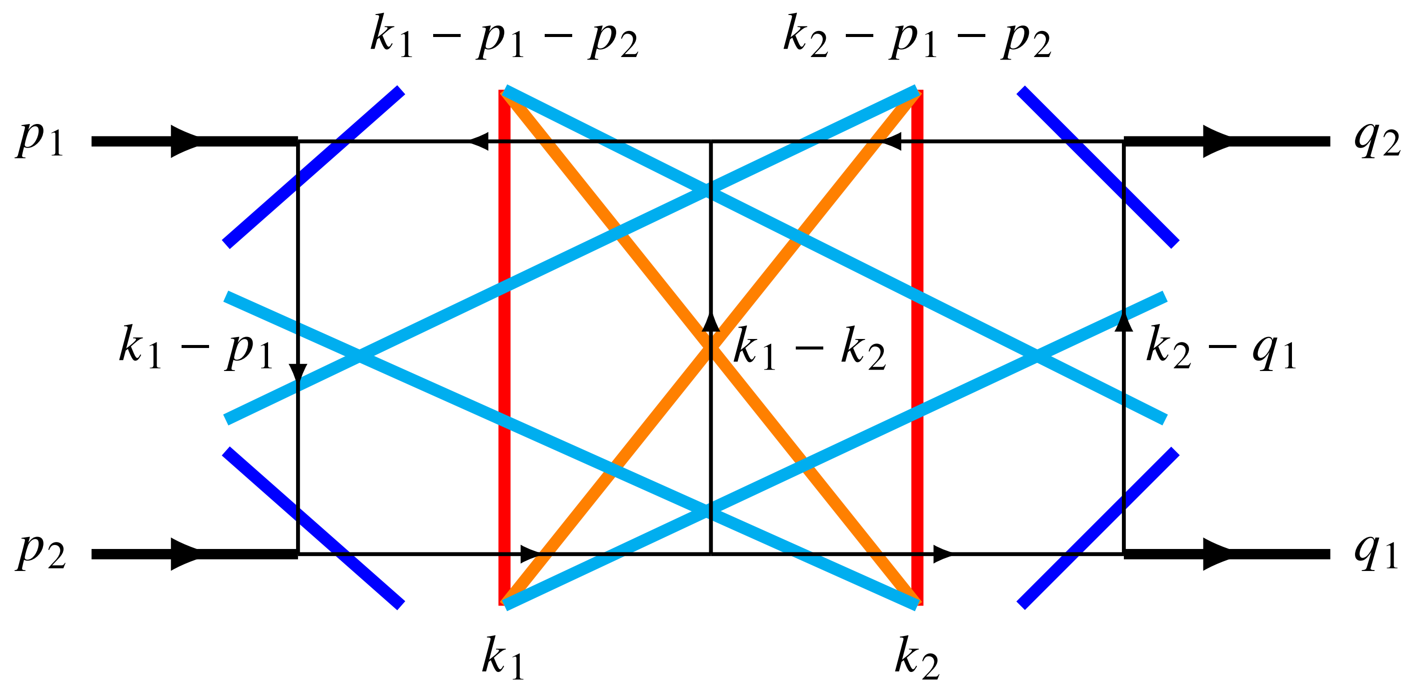
$$\text{III}: \quad \begin{array}{l} x_2 \to y_2 = x_2 + \gamma^- x_1, \\ x_1 \to y_1 = x_1 + \gamma^- x_2, \end{array} \qquad \widetilde{\mathcal{F}}^{-,1} = -\frac{16\beta^2}{(1-\beta)(1+\beta)^3}x_1 x_2,$$

$$I(\beta) = I^{+,1}(\beta) + I^{+,2}(\beta) + (-1 - i\delta)^{-\epsilon}\, I^{-,1}(\beta)$$

Avoiding* Contour Deformation

There have been various other efforts to avoid/mitigate contour deformation in numerical contexts

Threshold subtraction (used in context of Loop-Tree Duality)



Locate thresholds of integral/amplitude

Subtract using local counterterms

Dispersive/absorptive parts can be computed separately

Kermanschah 21; Kermanschah, Vicini 24, 25, 26, 26;
Locally finite amplitudes: Anastasiou, Haindl, Sterman, Yang, Zeng 20; Anastasiou, Sterman 22; Anastasiou, Karlen, Sterman, Venkata 24;

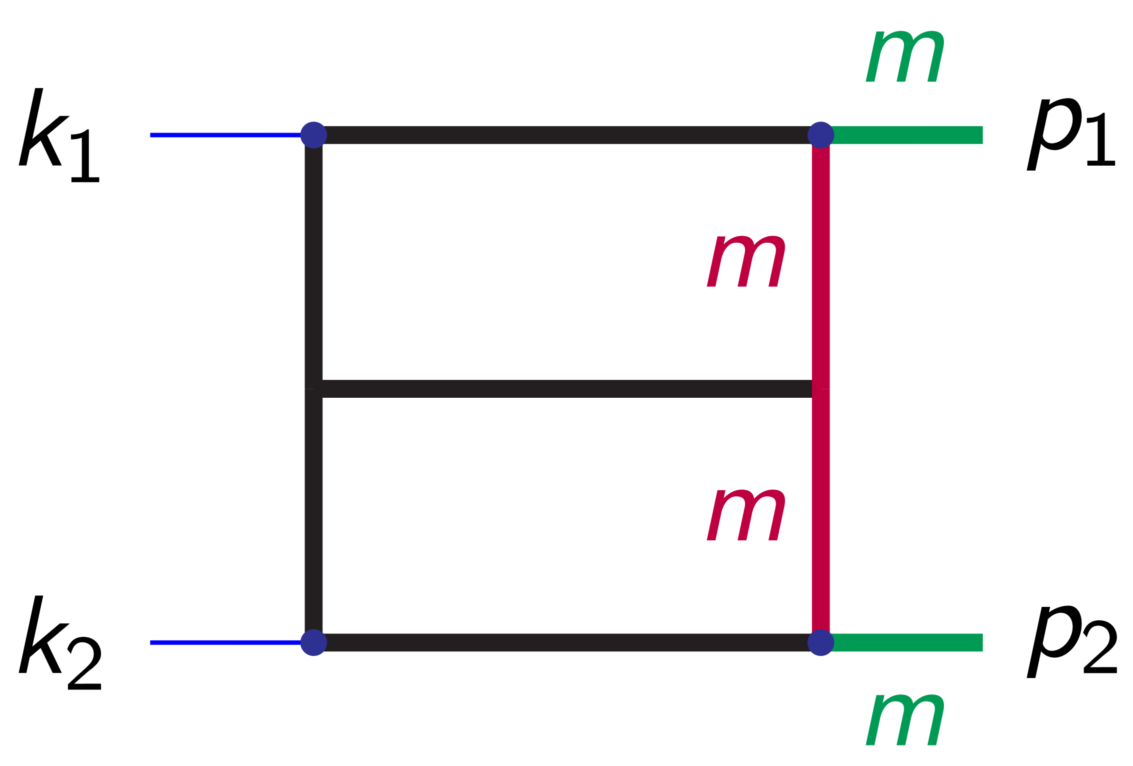
GLoop*

Finite $\delta \neq 0$ but flatten behaviour of integral near $\delta \to 0$ with variable changes

$$J = \lim_{\delta \to 0} \int_{-1}^{1} \mathrm{d}x \, \frac{F(x)}{x + i\delta} \to \frac{-i\pi}{g_\delta} \int_{\delta/\pi}^{1-\delta/\pi} \mathrm{d}\alpha \left(1 + i \frac{x(\alpha;\delta)}{\delta} \right) F(x(\alpha;\delta)), \qquad x = \frac{\delta}{\tan\left[\pi(1-\alpha)\right]}$$

GLoop: Pittau, Webber 22; Pittau 24, 26;

Finding the Hypersurfaces

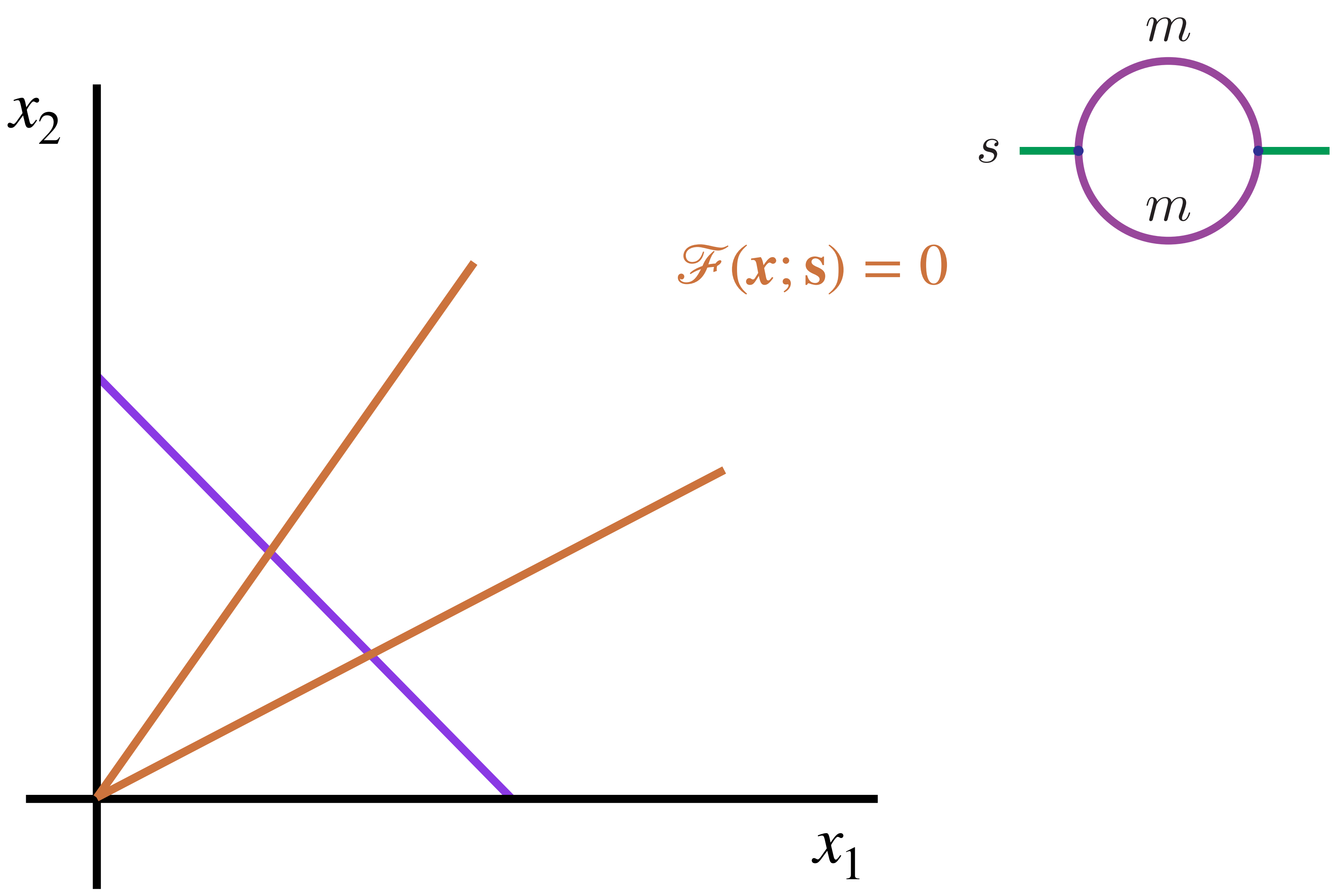


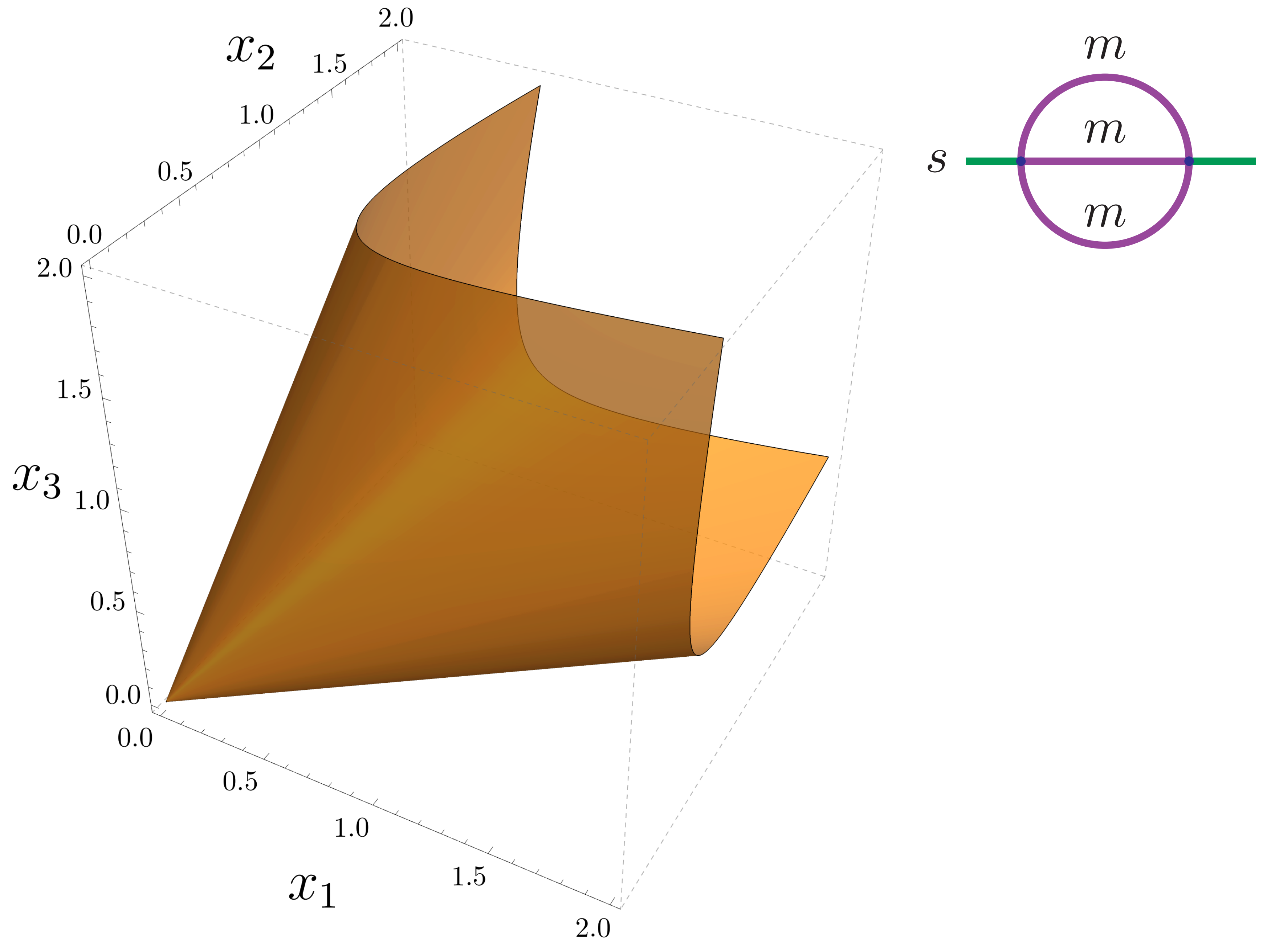
Decomposition

w/ Bennett, Chargeishvili, Magerya, Olsson, Stone

Solutions to $\mathscr{F}(\mathbf{x};\mathbf{s}) = 0$ are rays through the origin (due to homogeneity)

But, rays can trace out non-trivial surfaces on the intersection with $\alpha(\mathbf{x}) = 1$ surface of the δ-functional





Bounded by kinematic thresholds

For a specified kinematic region $\mathbf{s}_R$:

Seek decomposition of $\mathbf{x} \geq 0$ into subspaces in which $\mathscr{F}(\mathbf{x};\mathbf{s})$ has fixed sign

Want parametrisation of the boundaries of this decomposition which define the regions of codim 0

Decomposition

w/ Bennett, Chargeishvili, Magerya, Olsson, Stone

How do we find this in practice?

Want a sign-invariant decomposition of
$\{\mathscr{F}(\mathbf{x};\mathbf{s}) < 0\} \wedge \{0 < \mathbf{x}\} \wedge \mathbf{s}_R,$

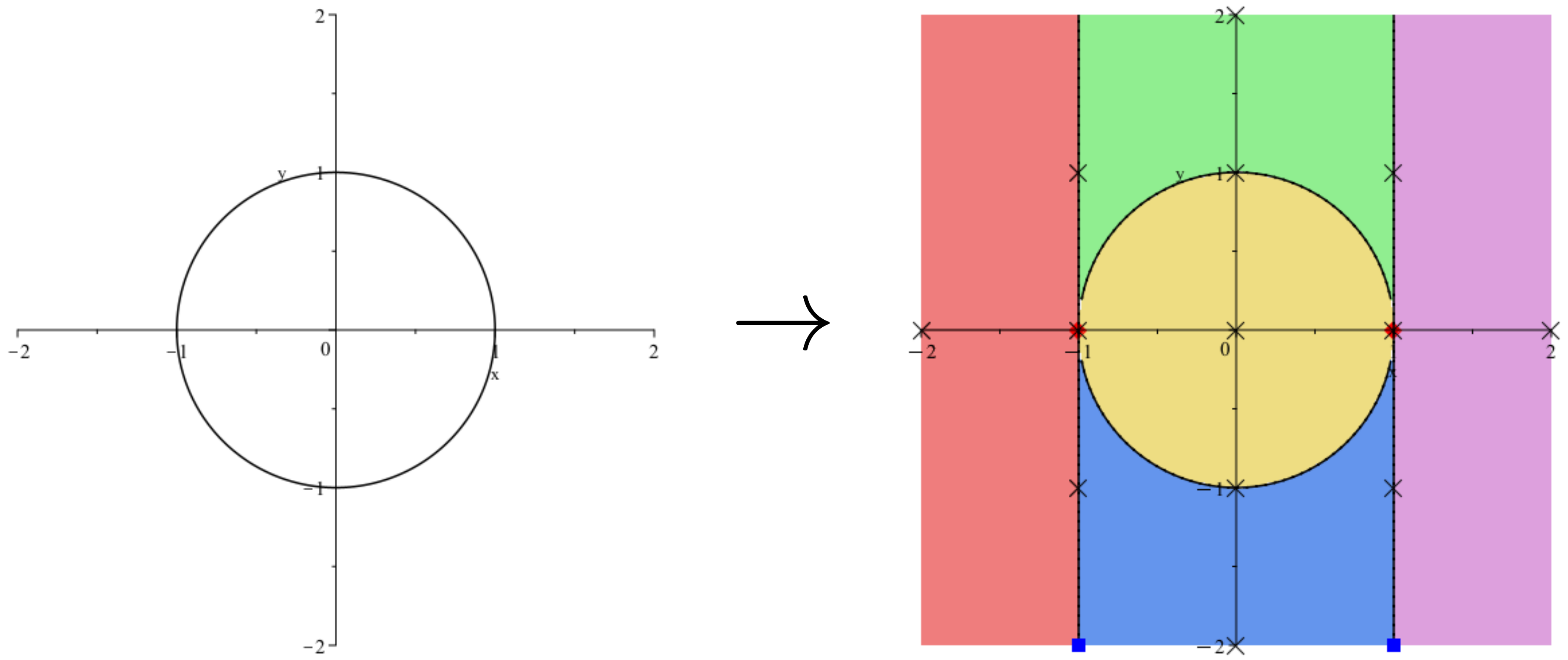
Borrow technique from real algebraic geometry
Generic **C**ylindrical **A**lgebraic **D**ecomposition

codim 0	projection of cells either disjoint or identical	cell boundaries are roots of polynomials	union of disjoint cells is $\mathbb{R}^N$

(CAD) Collins 75; Davenport, Heintz 88; Lazard 94; McCallum 19; …
(GCAD) Strzeboński 00; …

Solves arbitrary systems of strict polynomial inequalities

Downside can be *very* slow!



Lee, del Río, Rahkooy 25

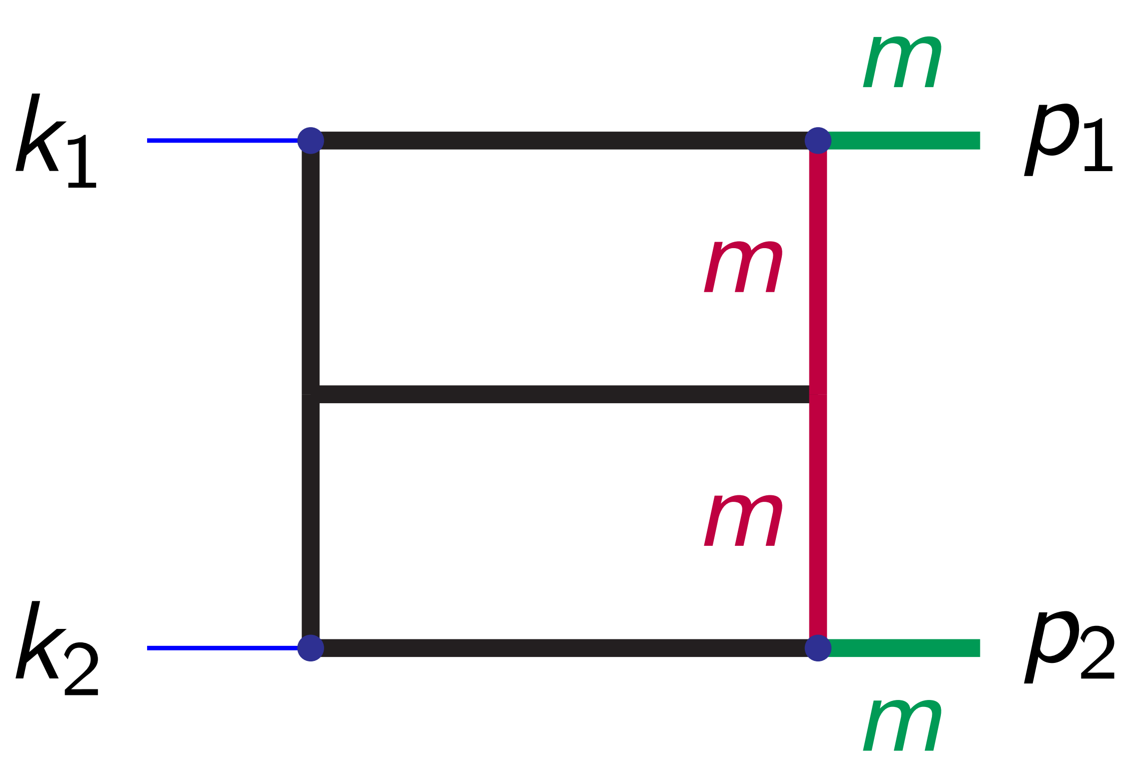
Projection operators: {Leading Coefficient, Discriminant, Resultant}

Similar to projections used in Landau literature

Correia, Giroux, Mizera 25

$$\begin{tikzcd}
\mathcal{F}_n \arrow[d, "\mathrm{proj}"] & \mathrm{CAD}(\mathcal{F}_n) \\
\mathcal{F}_{n-1} \arrow[d, "\mathrm{proj}"] & \mathrm{CAD}(\mathcal{F}_{n-1}) \arrow[u, "\mathrm{Lift}"] \\
\vdots \arrow[d, "\mathrm{proj}"] & \vdots \arrow[u, "\mathrm{Lift}"] \\
\mathcal{F}_2 \arrow[d, "\mathrm{proj}"] & \mathrm{CAD}(\mathcal{F}_2) \arrow[u, "\mathrm{Lift}"] \\
\mathcal{F}_1 \arrow[r, "\text{find roots}", "\text{split into cells}"'] & \mathrm{CAD}(\mathcal{F}_1) \arrow[u, "\mathrm{Lift}"]
\end{tikzcd}$$

Example 1: 2-Loop Box



Abreu, Becchetti, Duhr, Ozcelik 25

$\mathbf{s}^*$: $s = 4,\ t = -1,\ u = -1$ (with $m = 1$)

$$\mathcal{U}(\mathbf{x}) = x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + x_1x_5 + x_2x_5 + x_3x_5 + x_4x_5 + x_1x_6 + x_2x_6 + x_5x_6 + x_3x_7 + x_4x_7 + x_5x_7 + x_6x_7$$

$$\mathcal{F}(\mathbf{x};\mathbf{s}) = -4x_1x_4x_5 + 2\left[x_1x_3x_6 + x_2x_3x_6 + x_2x_5x_6 + x_3x_5x_6 + x_2x_3x_7 + x_2x_4x_7 + x_2x_5x_7 + x_3x_5x_7 + x_2x_6x_7 + x_3x_6x_7 + x_5x_6x_7\right] + (x_1 + x_2 + x_5 + x_7)\,x_6^2 + (x_3 + x_4 + x_5 + x_6)\,x_7^2$$

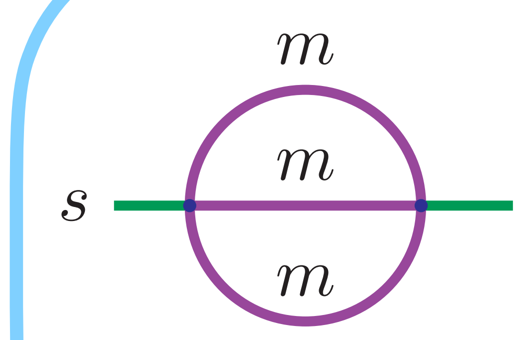
$$\{\mathcal{F}(\mathbf{x}) < 0\} \wedge \{0 < \mathbf{x}\}$$

$$\Big\downarrow \text{ GCAD with projection ordering } x_6, x_7, x_2, x_3, x_5, x_1, x_4$$

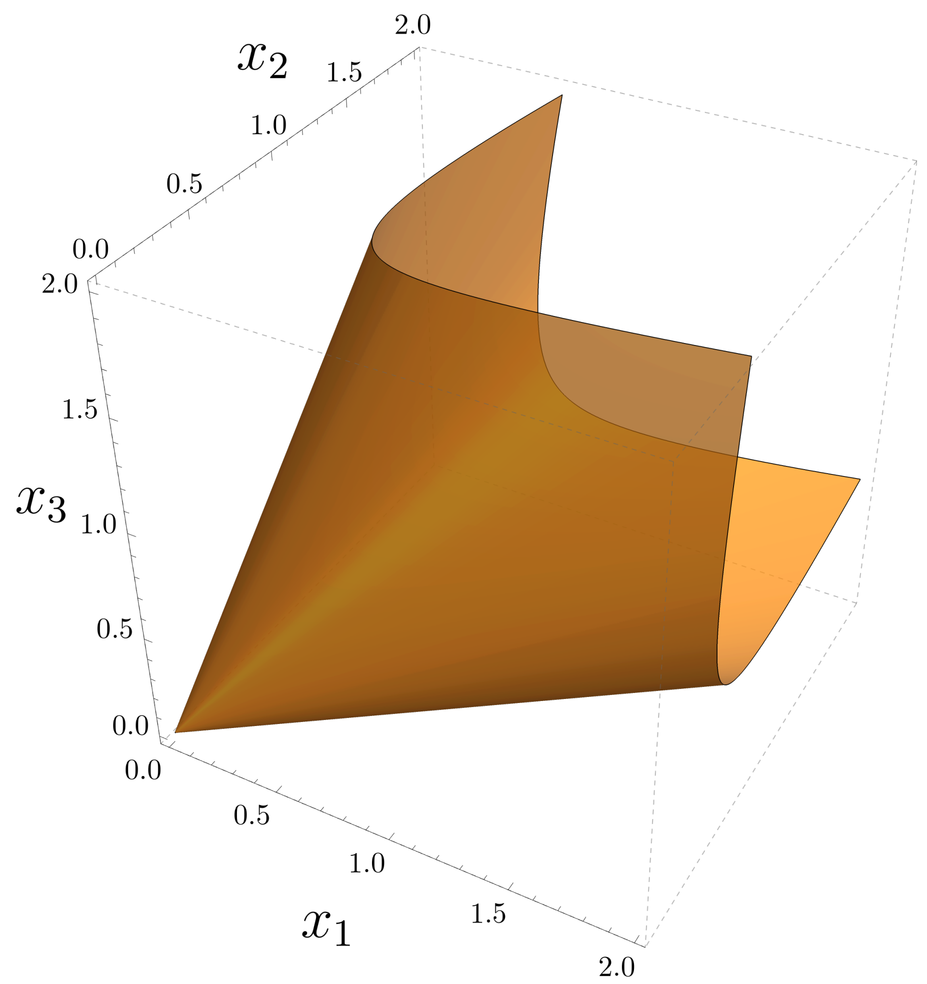
$$0 < x_6 \wedge 0 < x_7 \wedge 0 < x_2 \wedge 0 < x_3 \wedge x_5 < 0 \wedge \frac{(2x_2 + x_7)\,x_7}{4x_5} < x_1 \wedge \frac{\mathcal{N}(\mathbf{x}_{\neq 4})}{4x_1x_5 - (2x_2 + x_7)\,x_7} < x_4$$

$$x_4 \to x_4 + \frac{\mathcal{N}(\mathbf{x}_{\neq 4})}{4x_1x_5 - (2x_2 + x_7)\,x_7} \quad \textbf{THEN} \quad x_1 \to x_1 + \frac{(2x_2 + x_7)x_7}{4x_5}$$

Example 2: 2-loop Sunrise



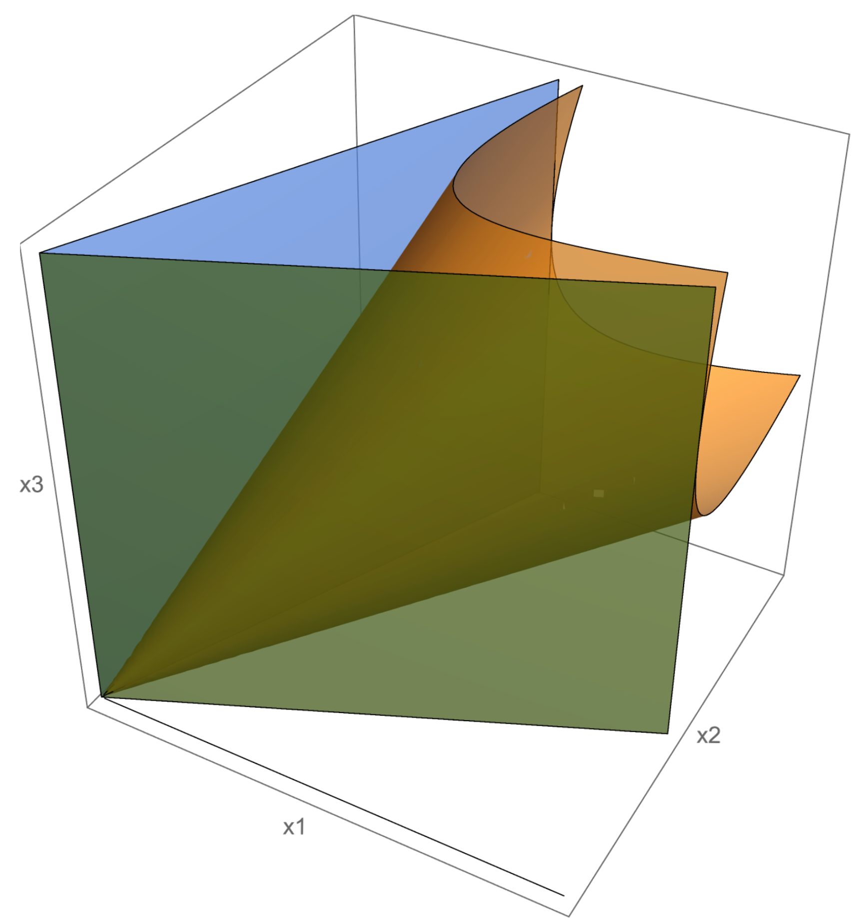
$$\mathcal{U}(\mathbf{x}) = x_1x_2 + x_2x_3 + x_1x_3,$$
$$\mathcal{F}(\mathbf{x};\tilde{s}) = -\tilde{s}x_1x_2x_3 + \mathcal{U}(\mathbf{x})(x_1+x_2+x_3)$$
$$s_R = \{\tilde{s} > 9\}$$



GCAD

$\tilde{s} \prec x_1 \prec x_2 \prec x_3$

4 positive cells
1 negative cell



Negative Cell

$$0 < x_1 \ \wedge\ x_2 < \gamma x_1 \ \wedge\ f_1(x_1,x_2;\tilde{s}) - \sqrt{f_2(x_1,x_2;\tilde{s})} < x_3 < f_1(x_1,x_2;\tilde{s}) + \sqrt{f_2(x_1,x_2;\tilde{s})}$$

$$I^{-,1} = 2^{3-3\epsilon}c_1{}^{3-4\epsilon}c_7\ R_0(x_3)^{1-2\epsilon}\,R_1(x_1,x_2)^{\frac{3}{2}-2\epsilon}\,R_2(x_1,x_2,x_3)^{-1+\epsilon}\left(\sqrt{R_1(x_1,x_2)}\,R_3(x_1,x_2,x_3) - R_2(x_1,x_2,x_3)\,R_4(x_1,x_2)\right)^{-3+3\epsilon}$$

$R_0(x_3) = x_3,$ $\quad R_1(x_1,x_2,x_3) = c_4x_3(x_1+x_2),$

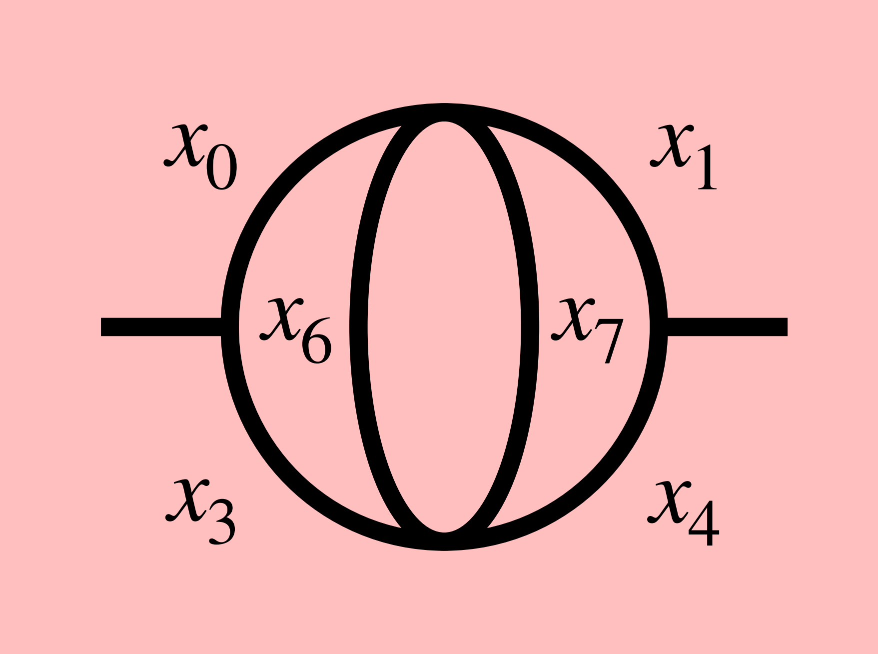
$R_2(x_1,x_2,x_3) = -c_6x_1x_2 - c_5(x_1^2+x_2^2) + 2c_4(x_1+x_2)x_3,$ $\quad R_3(x_1,x_2) = c_1{}^2x_1x_2(c_3x_1x_2 + 4c_2(x_1^2+x_2^2))$

$\gamma, c_1, \ldots, c_7$ are constants depending algebraically on $\tilde{s}$ only

Can still sector decompose algebraic integrand via $x_1 \to y_0^{-1}y_1^0y_2^1y_3^1$, $x_2 \to y_0^{-1}y_1^1y_2^0y_3^1$ and evaluate

In general, we need a method for decomposing sign-invariant algebraic functions

Example 3: Cat's Eye



Lellouch, Lupo, Sjö, Vanhove 26

$$E_5(t) = M^6\, I^{(d)}_{1,1,0,1,1,0,1,1,0}(p^2, M^2) \qquad t = p^2/M^2$$

$$\mathscr{F}(\mathbf{x}; t) = (4-t)\, x_0x_4x_6x_7 + x_0^2x_1x_6 + \ldots \ \text{(50 terms)}$$

$$s_R = t > 4 \wedge t < 16$$ ← Choose kinematic region above 2-particle cut and below 4-particle cut

For simplicity, let us fix $t = 5$ as an example point in s_R ← Ideally want to keep t in GCAD

Ordering $x_4 \prec x_1 \prec x_7 \prec x_6 \prec x_3 \prec x_0$ yields GCAD in ~13 seconds (1 core, Mathematica)

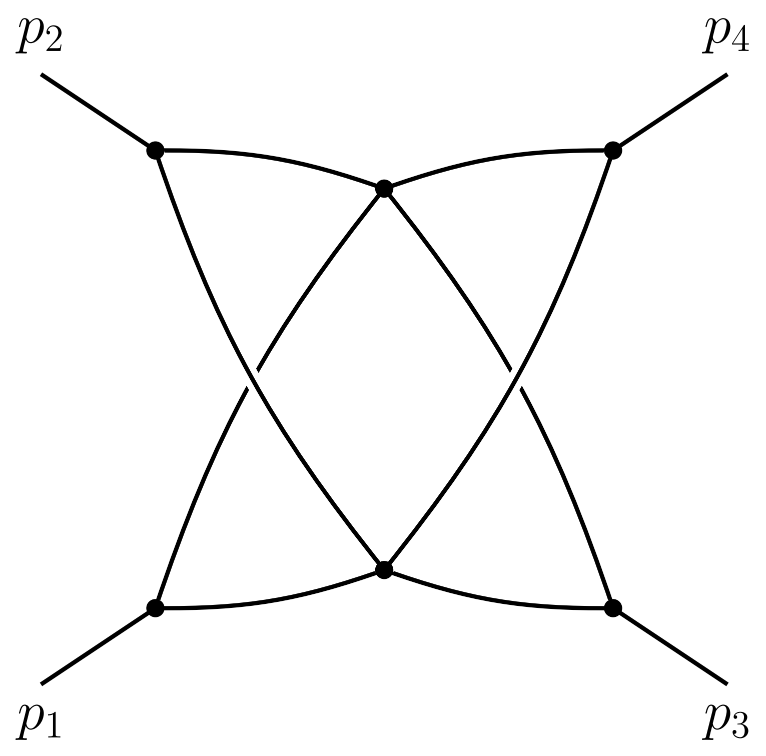
Negative Cells $\mathscr{F}(\mathbf{x}; t) < 0$: 34 **Positive Cells** $\mathscr{F}(\mathbf{x}; t) > 0$: 72

Cell boundaries are algebraic $\Longrightarrow$ need sector decomposition of algebraic functions

$$I(\mathbf{s}) = \sum_{n_+=1}^{72} I^{+,n_+}(\mathbf{s}) + \lim_{\delta\to 0^+} (-1 - i\delta)^{-(N-LD/2)} \sum_{n_-=1}^{34} I^{-,n_-}(\mathbf{s})$$

Did not attempt to obtain this and evaluate the integral yet!

Pinched Contours & Hidden Regions



Landau Analysis

To characterise singularities, simultaneously solve 1st and 2nd Landau Equations

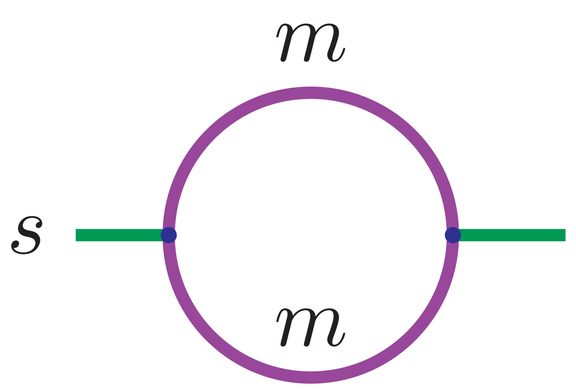
Focus on leading Landau singularities ($\Longrightarrow$ ignore solutions with some $x_j = 0$)

1 $$\mathscr{F}(\mathbf{x};\mathbf{s}) = 0$$ ← **At most quadratic in x_i**

2 $$\frac{\partial \mathscr{F}(\mathbf{x};\mathbf{s})}{\partial x_j} = 0 \quad \forall j$$ ← **At most linear in x_i**

Can we use Generic Cylindrical Algebraic Decomposition (GCAD) to solve this problem?

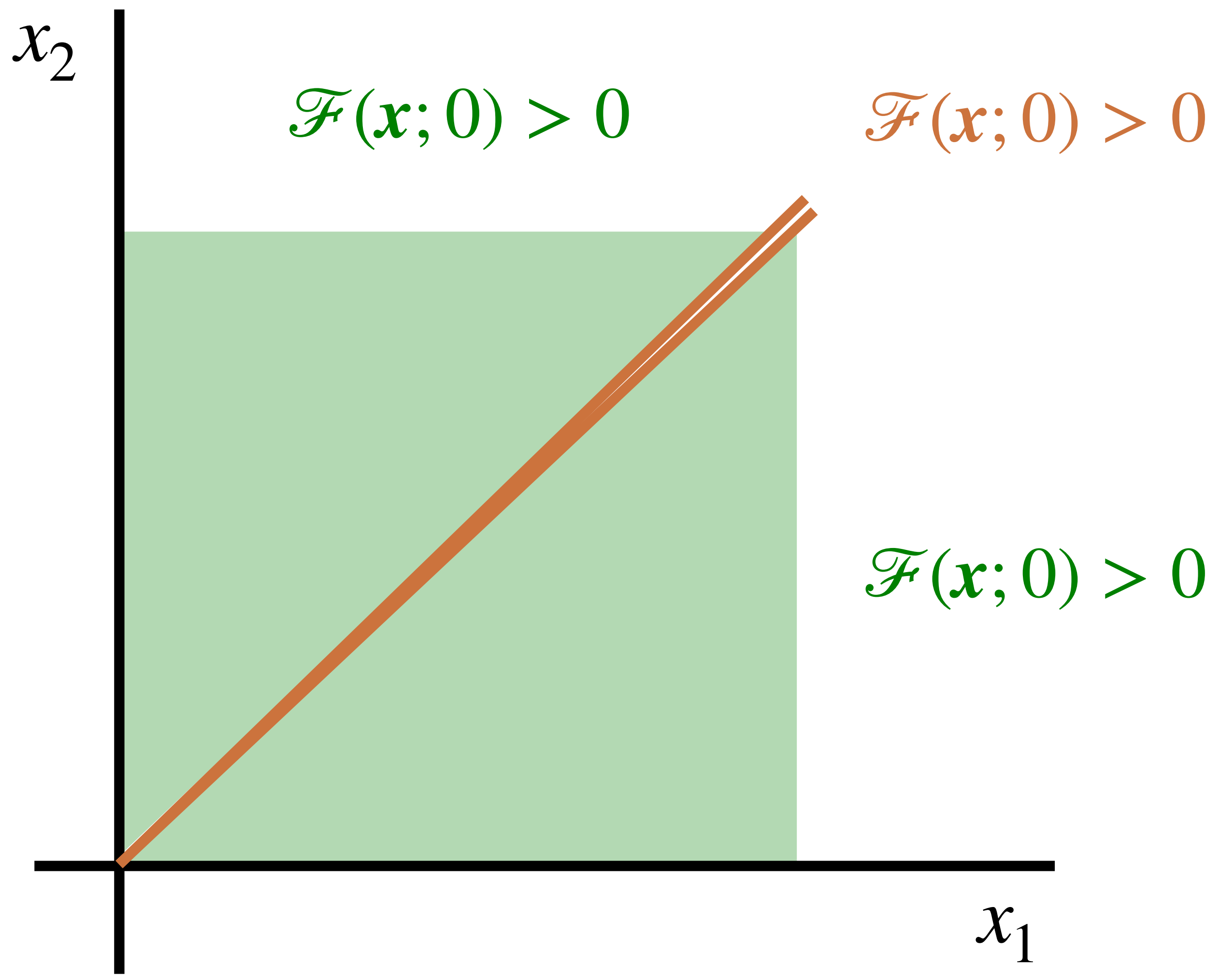
Bubble

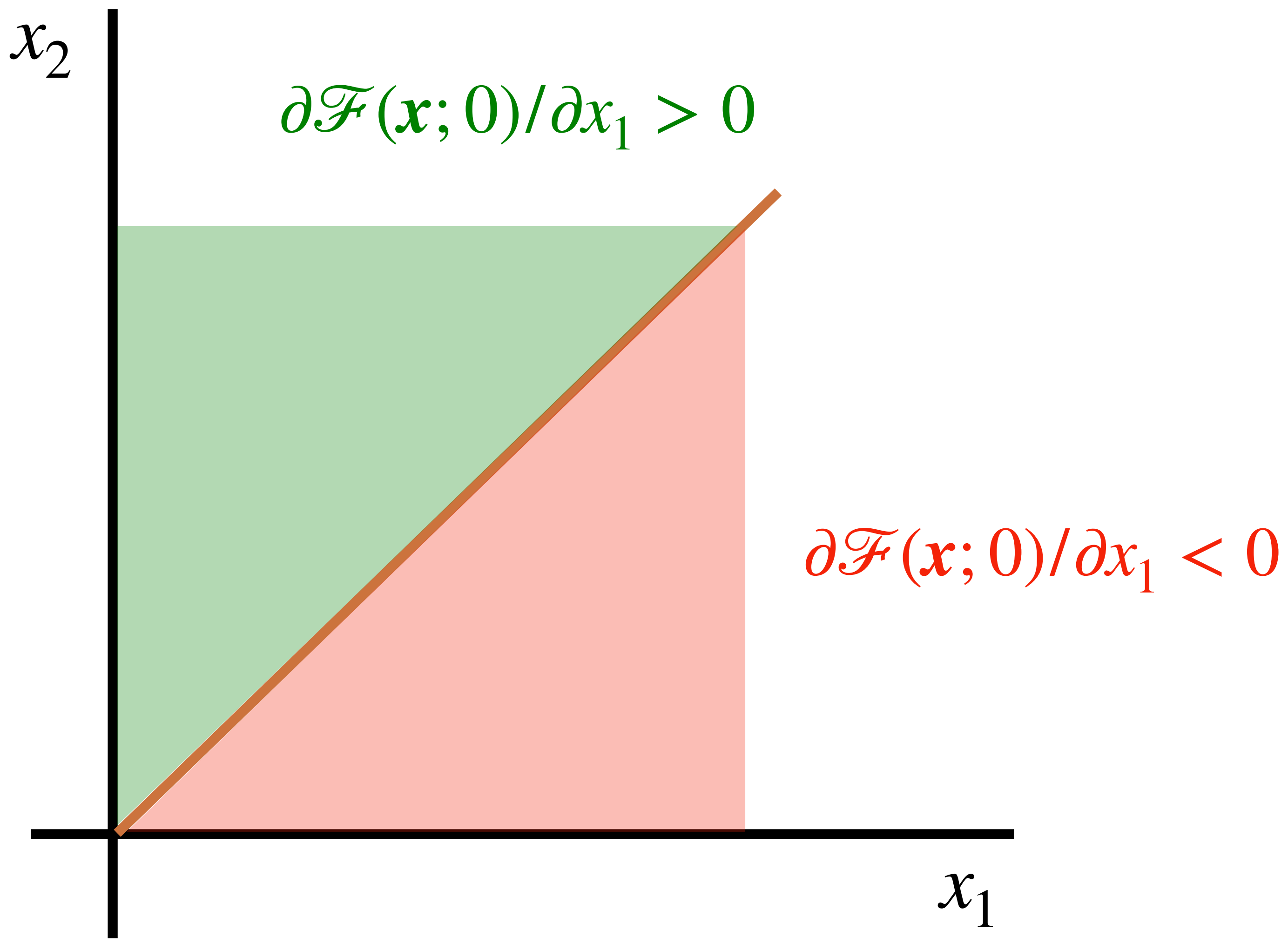


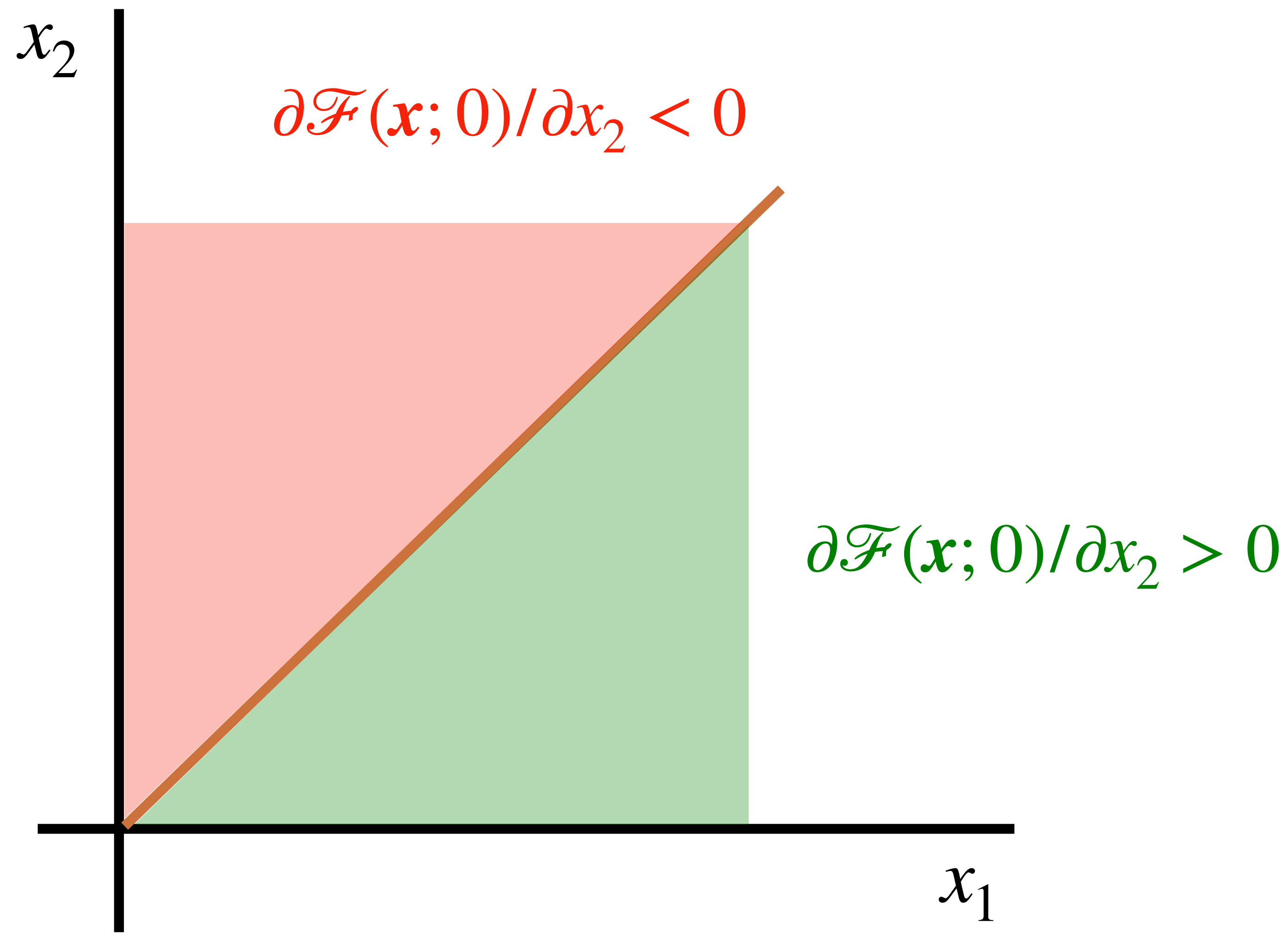
$$\mathscr{F}(\mathbf{x};\mathbf{s}) = -s x_1 x_2 + (m^2 x_1 + m^2 x_2)(x_1 + x_2)$$

Suppose we are interested in evaluating/expanding at $y = s - 4m^2 \sim 0$, consider $y \to \lambda y$ ($m^2 = 1$ and $s = 4$)

$$\mathscr{F}(\mathbf{x};0) = (x_1 - x_2)^2, \qquad \frac{\partial \mathscr{F}(\mathbf{x};0)}{\partial x_1} = 2(x_1 - x_2), \qquad \frac{\partial \mathscr{F}(\mathbf{x};0)}{\partial x_2} = -2(x_1 - x_2)$$







Example 1: Bubble

Let's compute a GCAD of the following systems

$$\mathscr{F}(\mathbf{x};0) > 0 \;\wedge\; \frac{\partial \mathscr{F}(\mathbf{x};0)}{\partial x_1} > 0 \;\wedge\; \frac{\partial \mathscr{F}(\mathbf{x};0)}{\partial x_2} < 0 \;\wedge\; x_1 > 0 \;\wedge\; x_2 > 0 \qquad \Longrightarrow \; x_2 < x_1$$

$$\mathscr{F}(\mathbf{x};0) > 0 \;\wedge\; \frac{\partial \mathscr{F}(\mathbf{x};0)}{\partial x_1} < 0 \;\wedge\; \frac{\partial \mathscr{F}(\mathbf{x};0)}{\partial x_2} > 0 \;\wedge\; x_1 > 0 \;\wedge\; x_2 > 0 \qquad \Longrightarrow \; x_1 < x_2$$

Can dissect integral into two integrals that have solutions of the Landau Equations only on the boundary

Examining the facet regions of the new integrals we obtain

This dissection can be found with Asy2

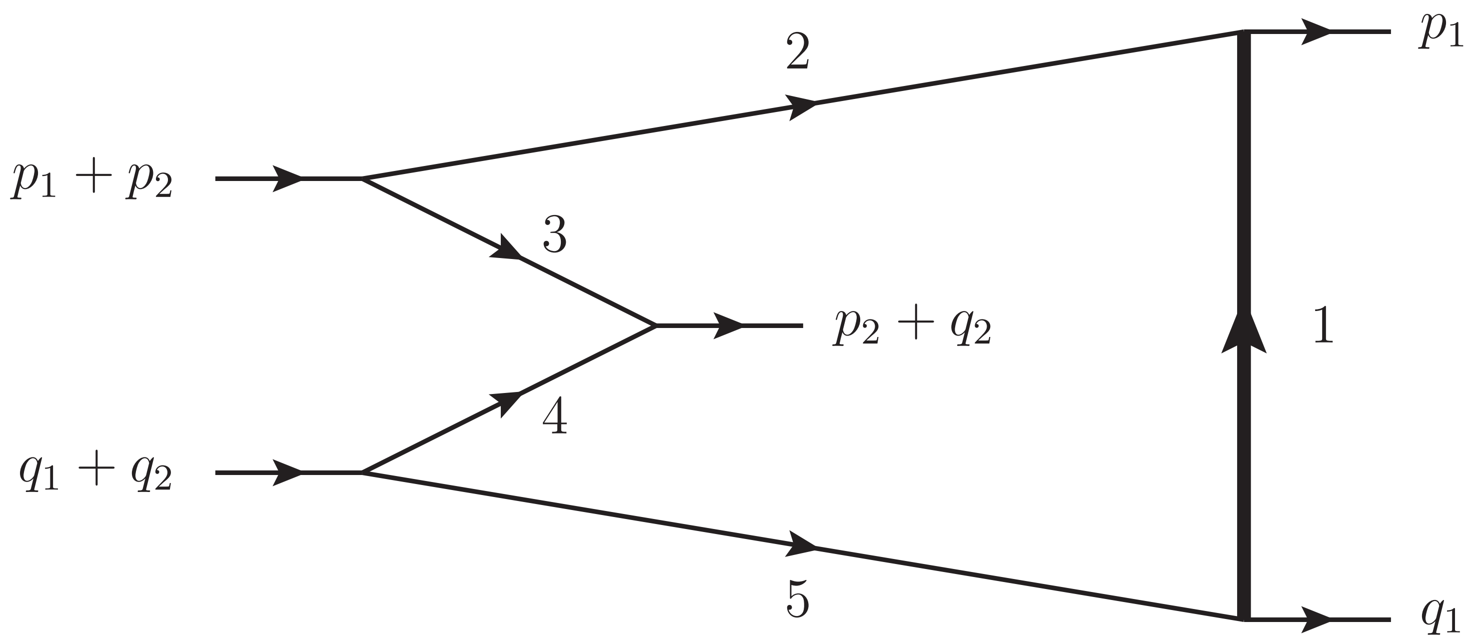
Jantzen, A. Smirnov, V. Smirnov 12

$$\{\lambda^0, \lambda^0\} \;\&\; \{\lambda^0, \lambda^{\frac{1}{2}}\}$$

Hard region:
Facet of original integral

Potential region:
``Hidden" region, facet of dissected integral

Example 2: Pentagon



Jantzen, A. Smirnov, V. Smirnov 12

$$\mathcal{F}(\mathbf{x};\mathbf{s}) = x_1(x_1 + x_2 + x_3 + x_4 + x_5)\ m^2 + (x_2 - x_3)(x_4 - x_5)\ q^2$$

Expansion $m^2 \to \lambda m^2$

$$\mathcal{F}(\mathbf{x};0) = (x_2 - x_3)(x_4 - x_5)\ q^2$$

GCAD system

$$\mathcal{F}(\mathbf{x};0) > 0 \ \wedge\ \frac{\partial \mathcal{F}(\mathbf{x};0)}{\partial x_2} > 0 \ \wedge\ \frac{\partial \mathcal{F}(\mathbf{x};0)}{\partial x_3} < 0 \ \wedge\ \frac{\partial \mathcal{F}(\mathbf{x};0)}{\partial x_4} > 0 \ \wedge\ \frac{\partial \mathcal{F}(\mathbf{x};0)}{\partial x_5} < 0 \ \wedge\ x_i > 0$$

$$x_3 < x_2 \ \wedge\ x_5 < x_4$$

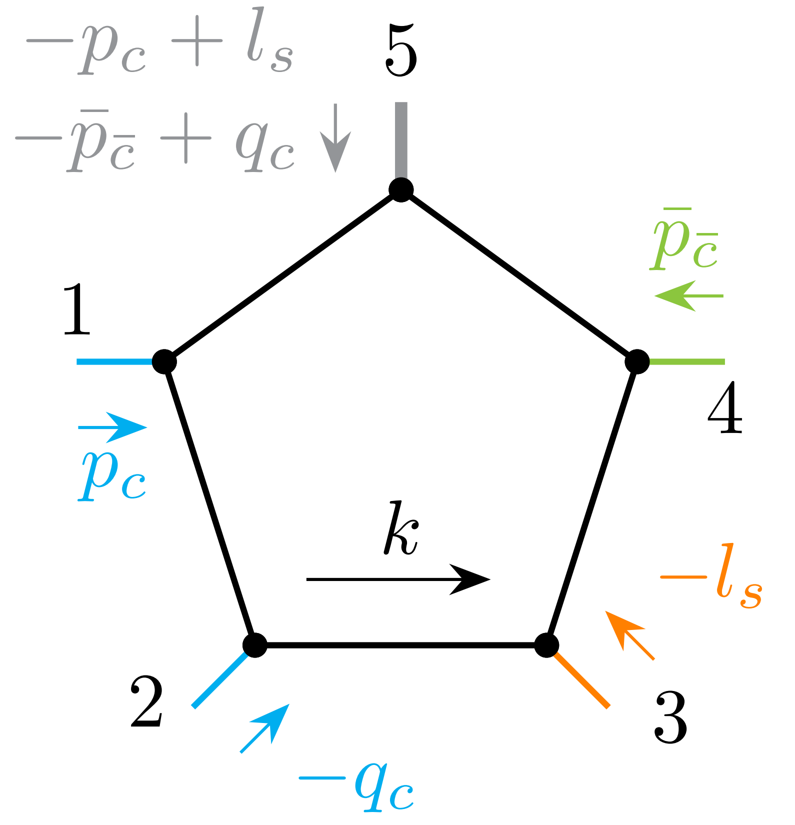
The new facet introduced by this dissection corresponds to a Glauber region

Jantzen, A. Smirnov, V. Smirnov 12

This dissection can be found with Asy2

Jantzen, A. Smirnov, V. Smirnov 12

Example 3: Pentagon II



Becher, Hager, Jaskiewicz, Neubert, Schwienbacher 24

$$\mathcal{F}(\mathbf{x};\mathbf{s}) = -x_1x_3\ s_{23} - x_1x_4\ s_{51} - x_3x_5\ s_{45} - x_4x_5\ m^2 - x_2x_4\ s_{34} - x_2x_5\ s_{12}$$

Expansion $s_{34} \to \lambda s_{34}$, $s_{12} \to \lambda s_{12}$,

$$\mathcal{F}(\mathbf{x};0) = -x_1x_3\ s_{23} - x_1x_4\ s_{51} - x_3x_5\ s_{45} - x_4x_5\ m^2$$

Does not factor in these variables

One of the GCAD systems

(Apologies to the authors if this is not the relevant kinematic region...)

$$\mathcal{F}(\mathbf{x};0) > 0 \ \wedge\ \frac{\partial \mathcal{F}(\mathbf{x};0)}{\partial x_1} < 0 \ \wedge\ \frac{\partial \mathcal{F}(\mathbf{x};0)}{\partial x_3} > 0 \ \wedge\ \frac{\partial \mathcal{F}(\mathbf{x};0)}{\partial x_4} < 0 \ \wedge\ \frac{\partial \mathcal{F}(\mathbf{x};0)}{\partial x_5} < 0 \ \wedge\ x_i > 0 \ \wedge\ s_{23} > 0 \ \wedge\ s_{45} < 0 \ \wedge\ s_{51} < 0$$

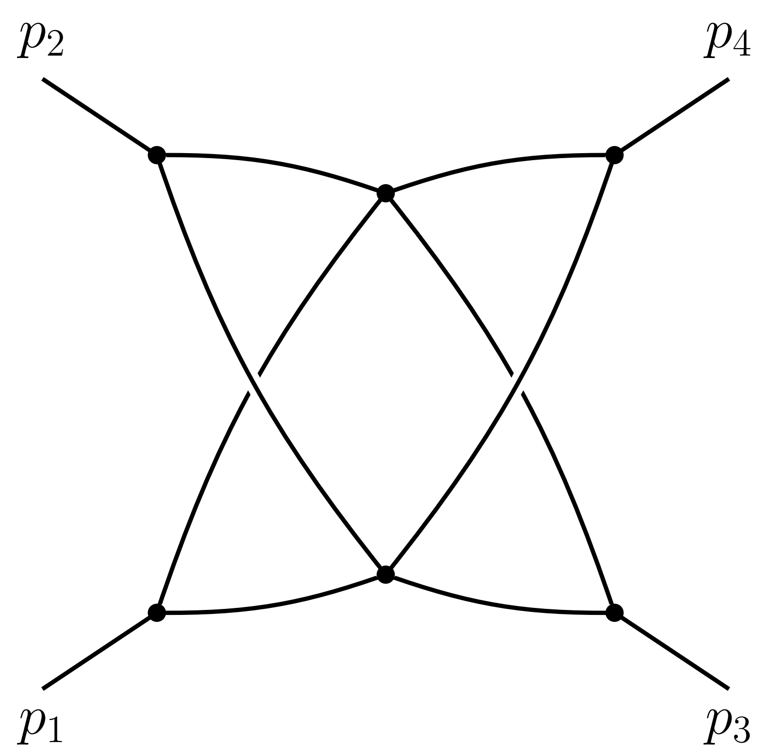
$$\frac{m^2 s_{23}}{s_{45}} < s_{51} \ \wedge\ \frac{s_{23}x_1}{-s_{45}} < x_5 \ \wedge\ \frac{-s_{45}x_3}{m^2} < x_4 < \frac{s_{23}x_3}{-s_{51}}$$

The new facet introduced by this dissection corresponds to a Glauber region

Not found with Asy2

Becher, Hager, Jaskiewicz, Neubert, Schwienbacher 24

Example 4: Crown



Gardi, Herzog, Jones, Ma 24

$$\mathcal{F}(\mathbf{x};\mathbf{s}) = -s_{12}\,(x_1x_4 - x_0x_5)\,(x_3x_6 - x_2x_7) - s_{13}\,(x_1x_2 - x_0x_3)\,(x_5x_6 - x_4x_7) + \;p_i^2\;(\ldots)$$

Consider $p_i^2 \to \lambda p_i^2$ (and insert $s_{12} = 1,\ s_{13} = -1$ as an inessential simplification)

$$\mathcal{F}(\mathbf{x};0) = -\,(x_3x_4 - x_2x_5)(x_1x_6 - x_0x_7)$$

One of the GCAD systems

$$\mathcal{F}(\mathbf{x};0) < 0 \;\wedge\; \frac{\partial\mathcal{F}(\mathbf{x};0)}{\partial x_0} < 0 \;\wedge\; \frac{\partial\mathcal{F}(\mathbf{x};0)}{\partial x_1} > 0 \wedge \frac{\partial\mathcal{F}(\mathbf{x};0)}{\partial x_2} < 0 \wedge \frac{\partial\mathcal{F}(\mathbf{x};0)}{\partial x_3} > 0 \wedge \frac{\partial\mathcal{F}(\mathbf{x};0)}{\partial x_4} > 0 \wedge \frac{\partial\mathcal{F}(\mathbf{x};0)}{\partial x_5} < 0 \wedge \frac{\partial\mathcal{F}(\mathbf{x};0)}{\partial x_6} > 0 \wedge \frac{\partial\mathcal{F}(\mathbf{x};0)}{\partial x_7} < 0 \wedge x_i > 0$$

$$x_5 > \frac{x_3x_4}{x_2} \wedge x_7 > \frac{x_1x_6}{x_0}$$

The new facet introduced by this dissection corresponds to a Landshoff scattering region

Non-linear
Not found with Asy2

Outlook

Method for evaluating integrals in parameter space above threshold *without* contour deformation

Exposes the (potentially integrable) singular structure of Feynman Integrals

Directions to explore

Efficient implementation in numerical computations

Finding hidden regions in the Method of Regions

Utility for analytic calculations and understanding of sequential discontinuities

Open questions

Sector decomposition of algebraic functions

Simpler way to obtain GCAD output

Thank You For Listening!

Existing Tools

Various tools attempt to find re-mappings using **linear** changes of variables

ASY/FIESTA Jantzen, A. Smirnov, V. Smirnov 12

Check all pairs of variables (α_1, α_2) which are part of monomials of opposite sign

For each pair, try to build linear combination $x_1 \to bx'_1, x_2 \to x'_2 + bx'_1$ s.t negative monomial vanishes

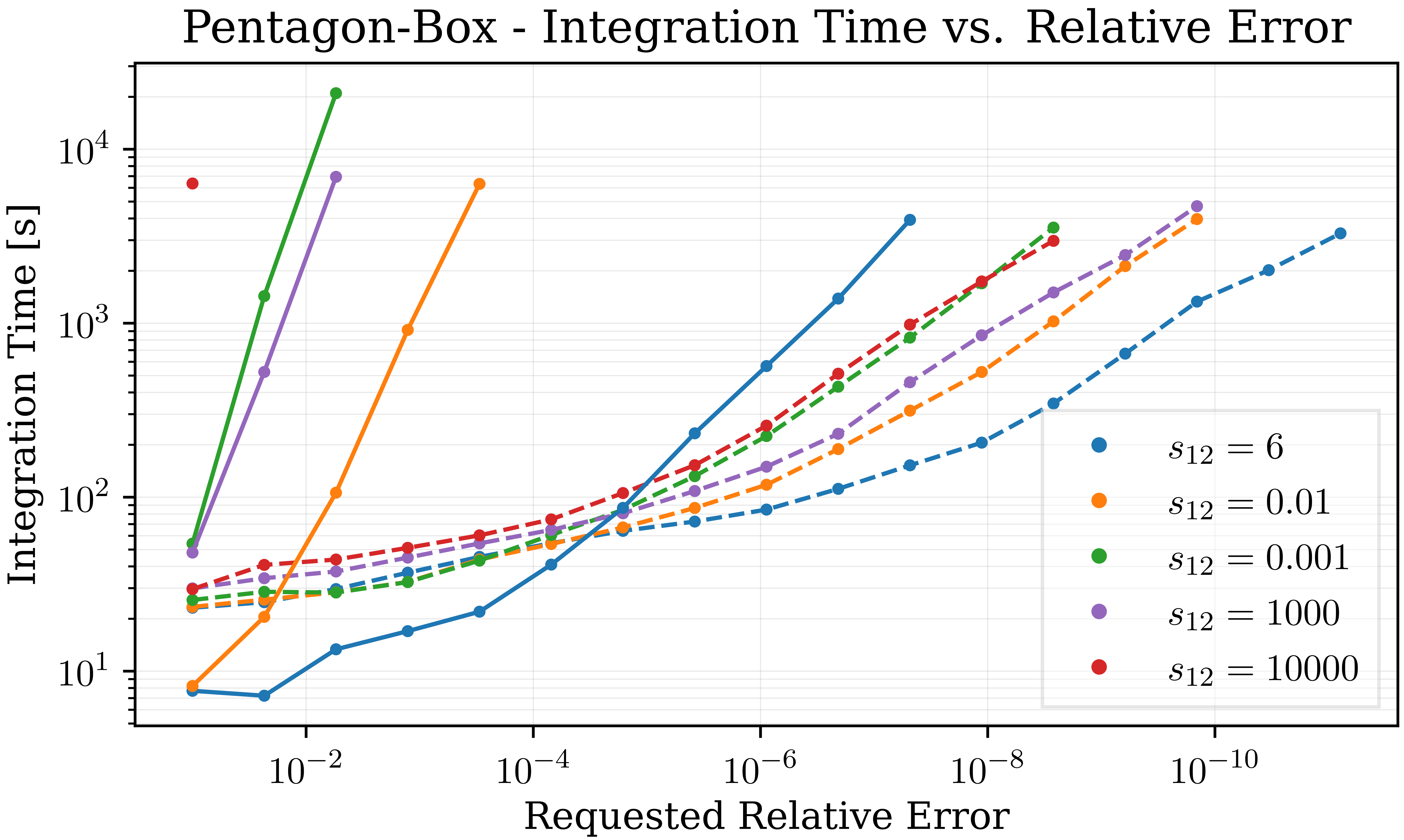
Repeat until all negative monomials vanish **or** warn user

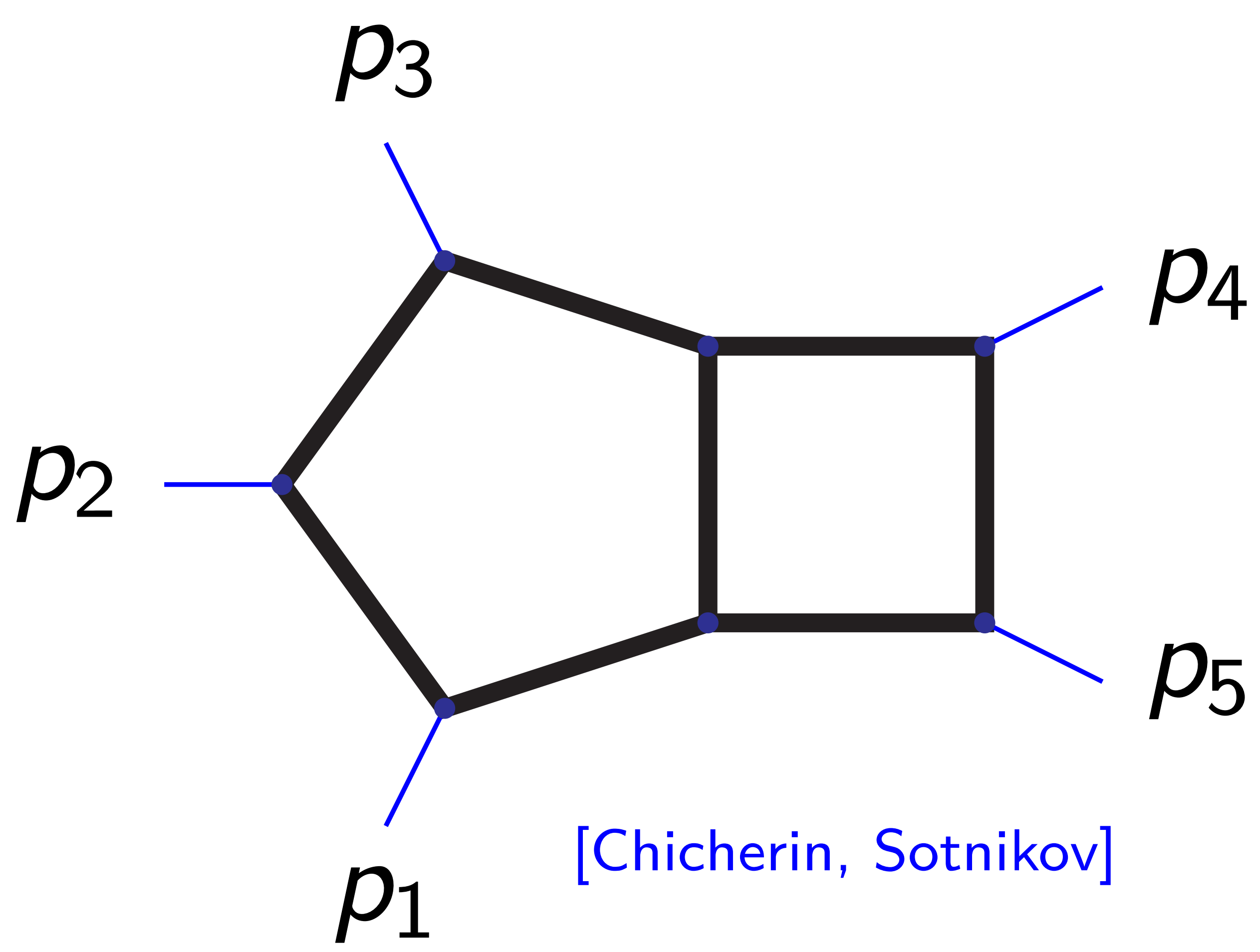
ASPIRE Ananthanarayan, Pal, Ramanan, Sarkar 18; B. Ananthanarayan, Das, Sarkar 20

Consider Gröbner basis of $\left\{\mathscr{F}, \partial\mathscr{F}/\alpha_1, \partial\mathscr{F}/\alpha_2, \ldots\right\}$ (i.e. $\mathscr{F}$ and Landau equations)

Eliminate negative monomials with linear transformations $x_1 \to bx'_1, x_2 \to x'_2 + bx'_1$

This is not enough to expose all regions and/or singularities in parameter space





pySecDec

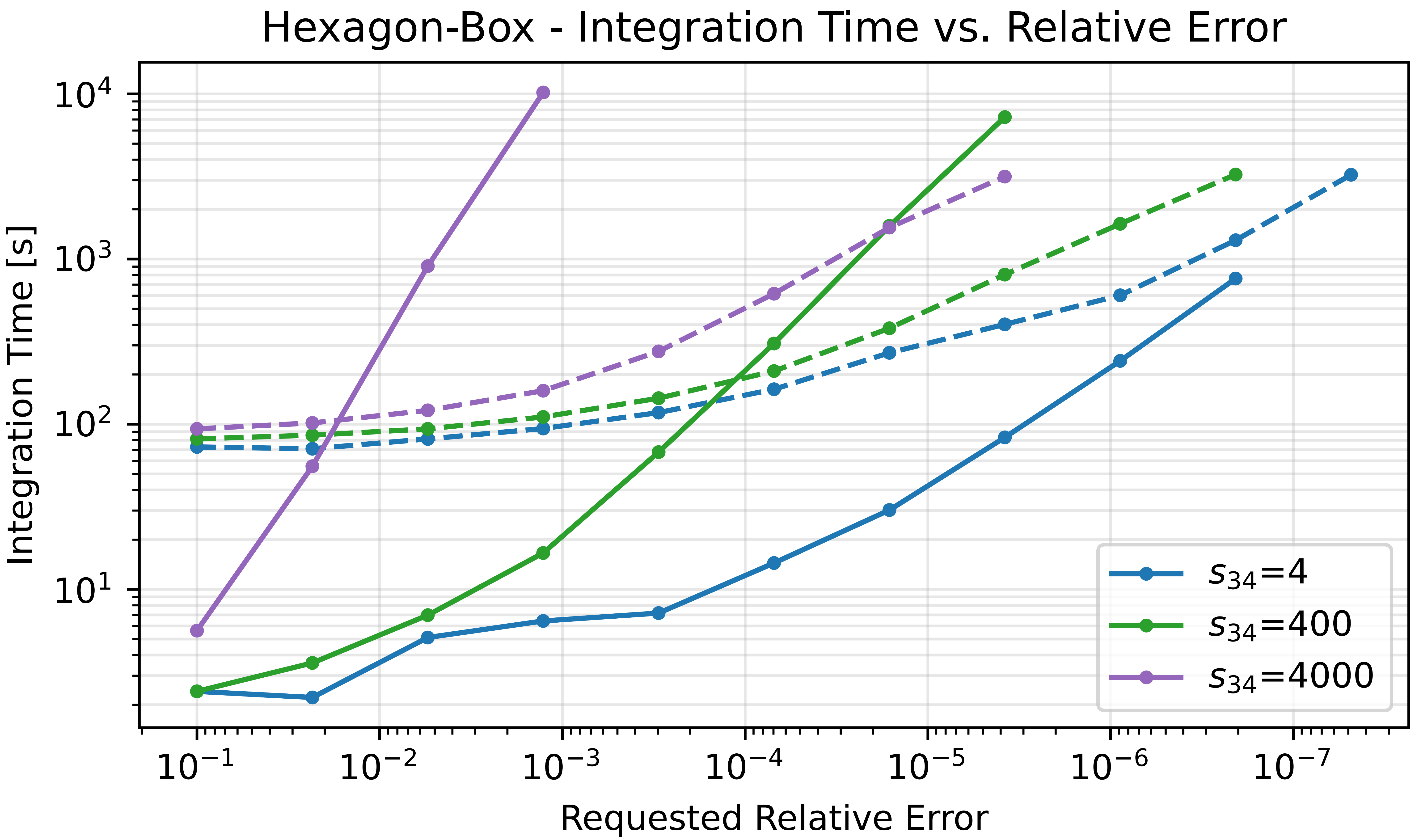
CD $(6-2\epsilon)$

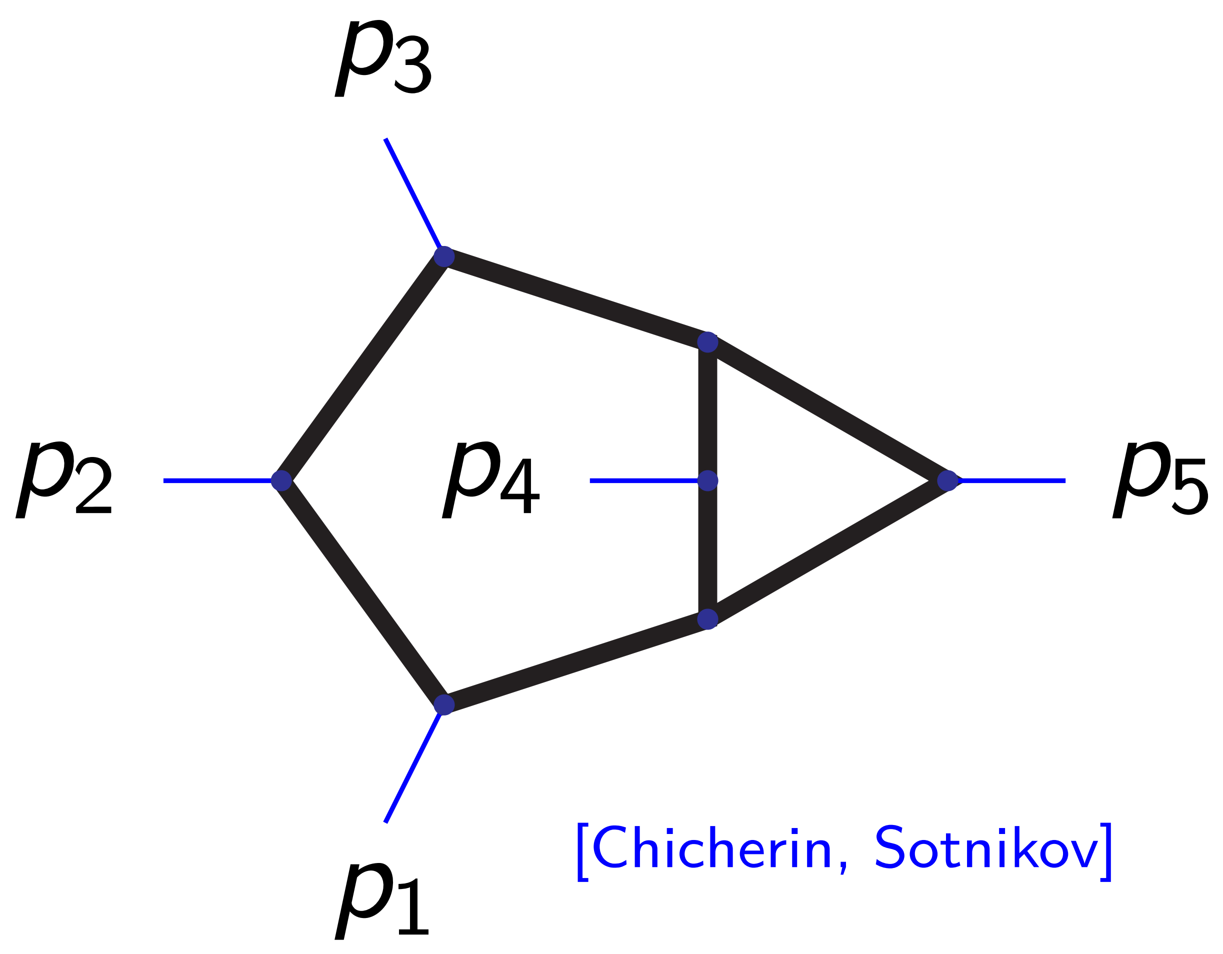
NOCD $(6-2\epsilon)$

$N_+ = 3$ $N_- = 4$

$s_{23} = -5$ $s_{34} = +4$

$s_{45} = +3$ $s_{15} = -1$





[Chicherin, Sotnikov]

pySecDec

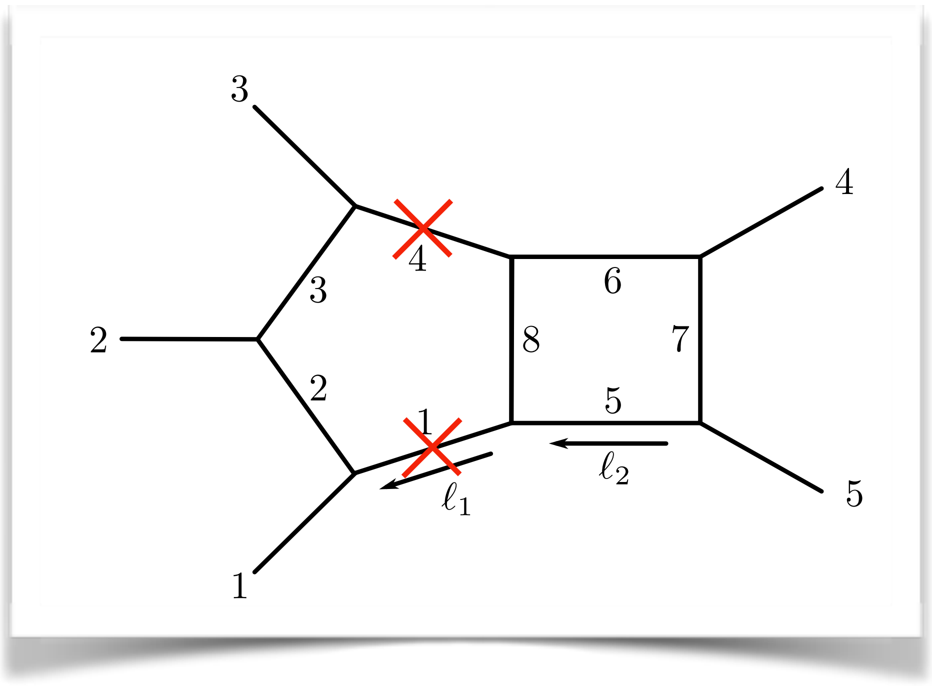
CD $(6-2\epsilon)$

NOCD $(6-2\epsilon)$

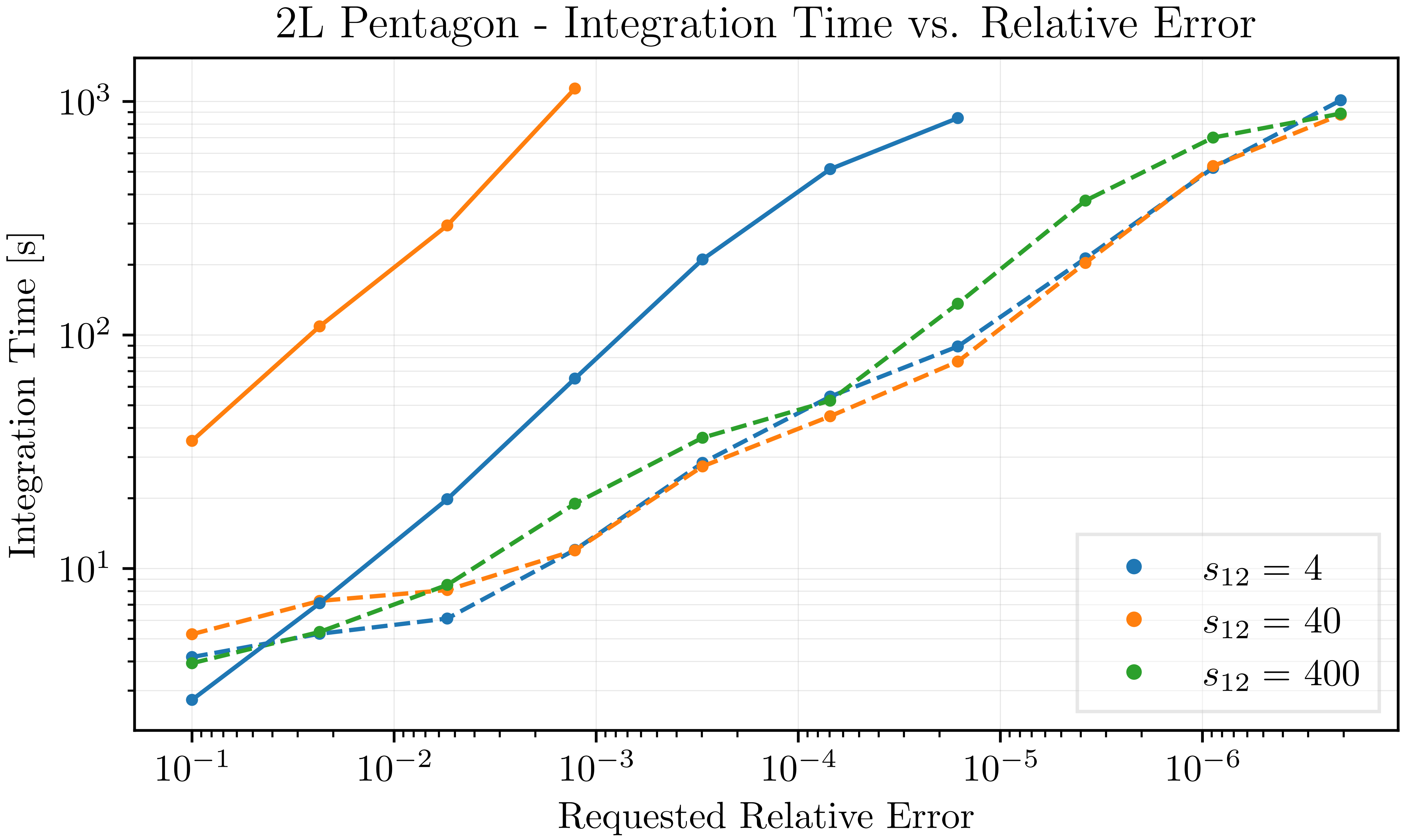
$N_+ = 4 \qquad N_- = 4$

$s_{12} = +6 \qquad s_{23} = -5$

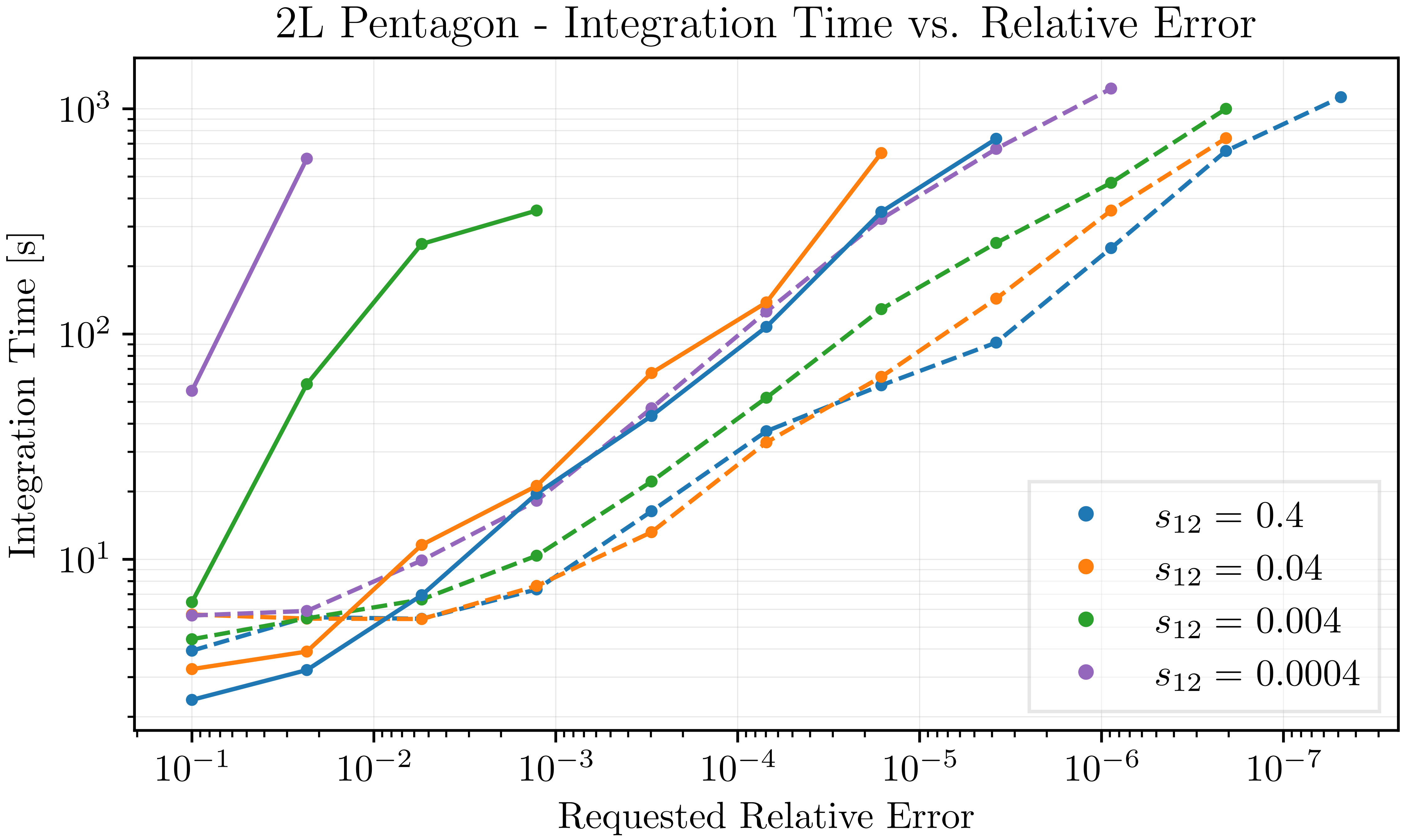
$s_{45} = +3 \qquad s_{15} = -1$



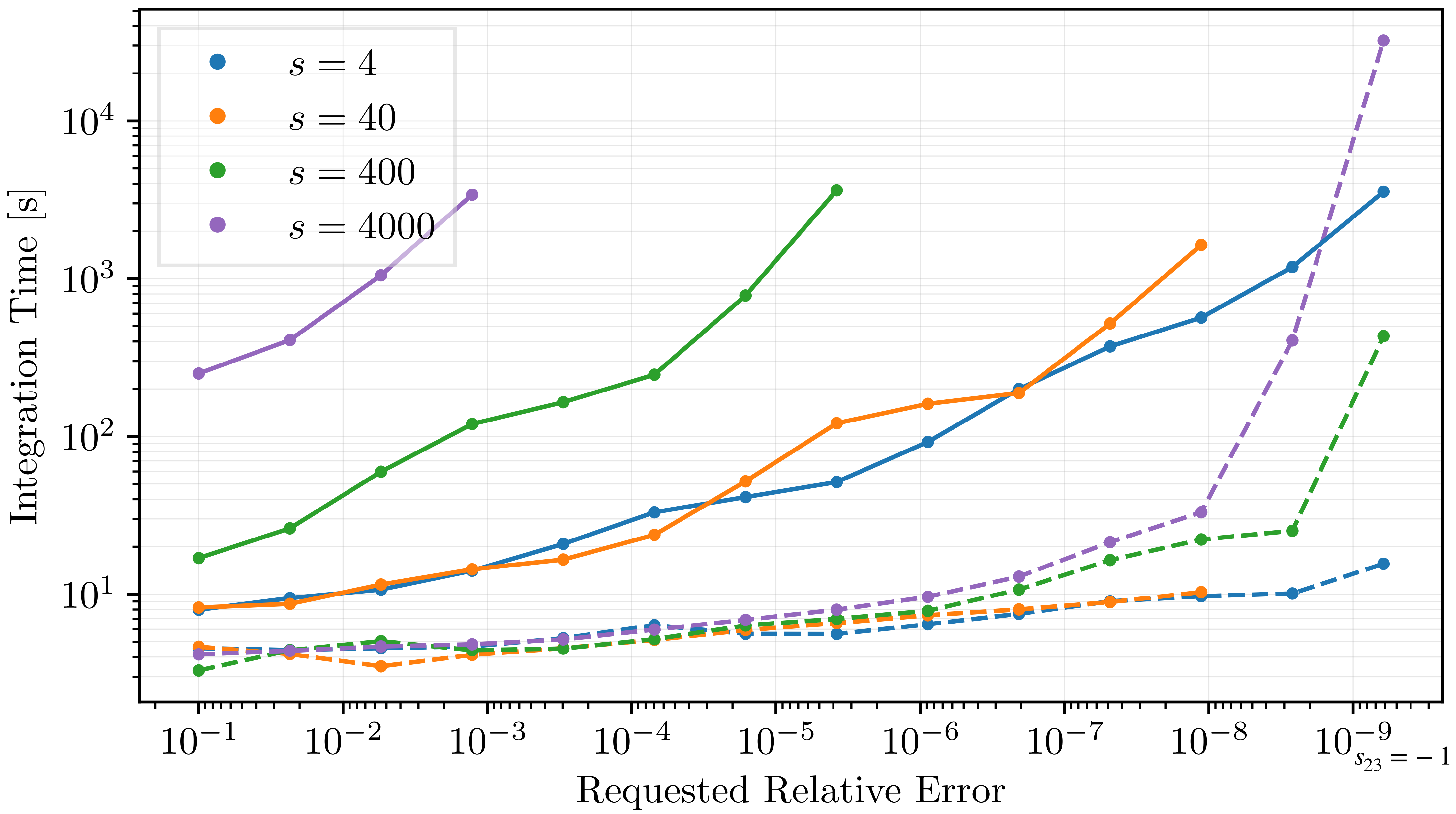
3
4
2
1
5
4
3
2
1
6
8
7
5
ℓ_1
ℓ_2

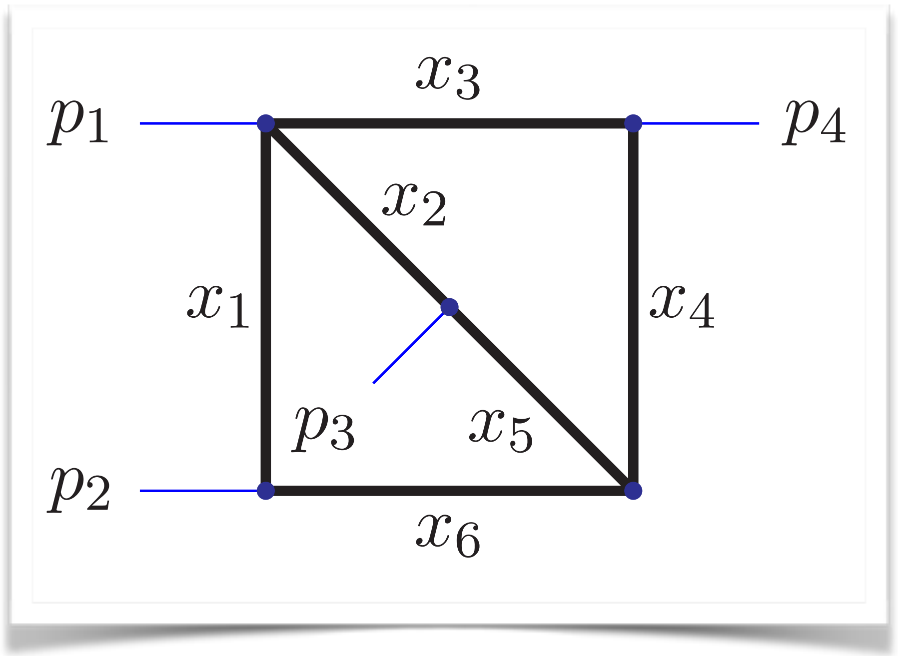


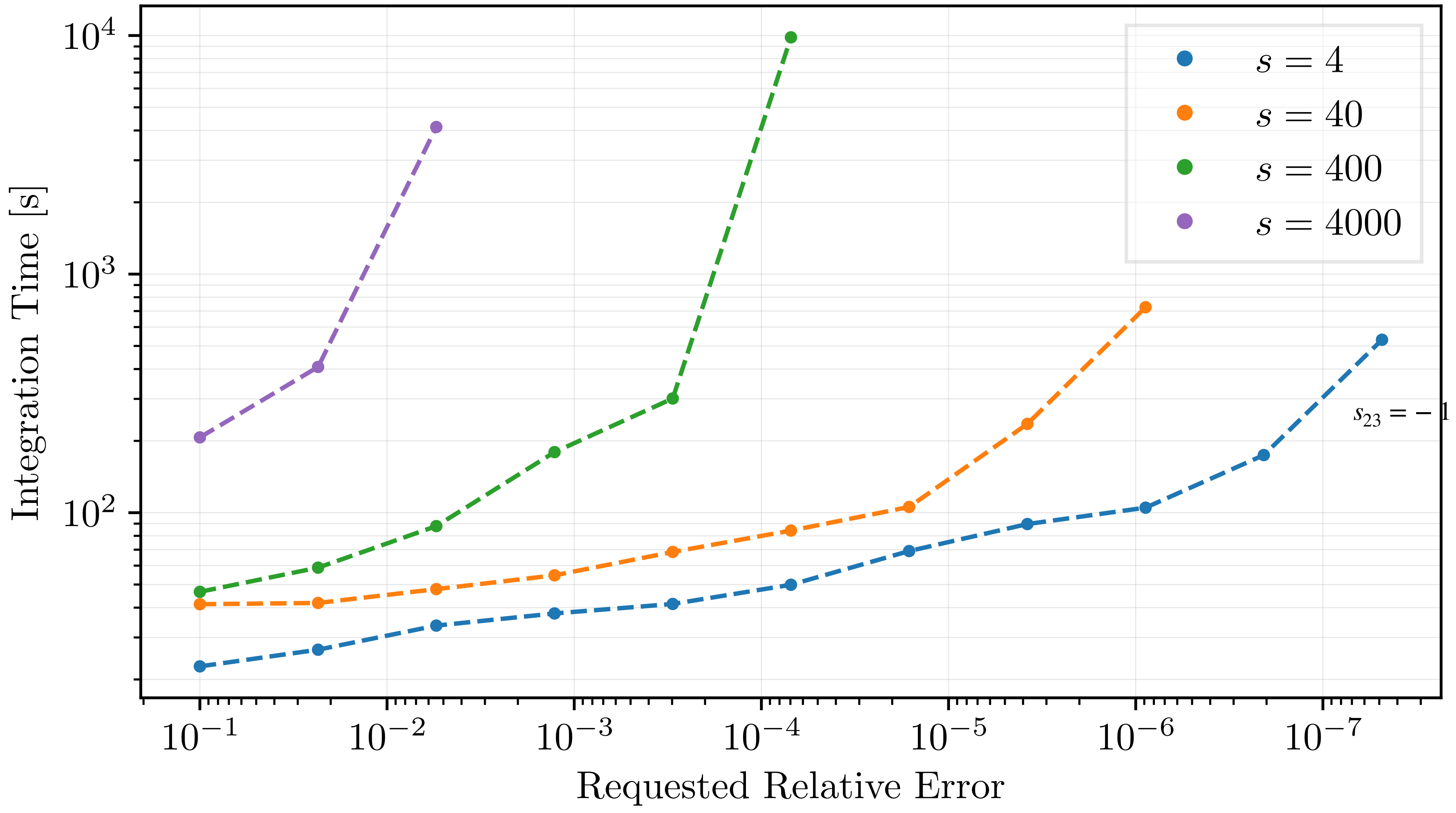
2L Pentagon - Integration Time vs. Relative Error
Integration Time [s]
Requested Relative Error
$s_{12} = 4$
$s_{12} = 40$
$s_{12} = 400$

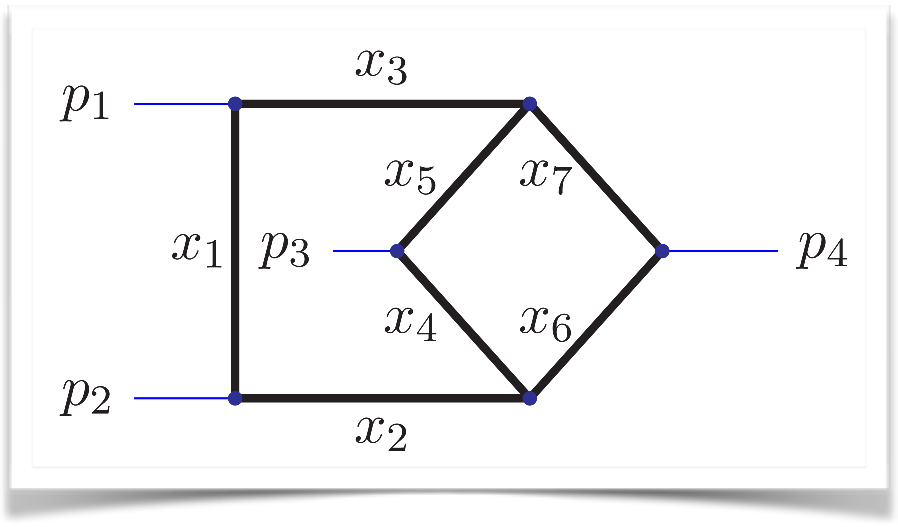


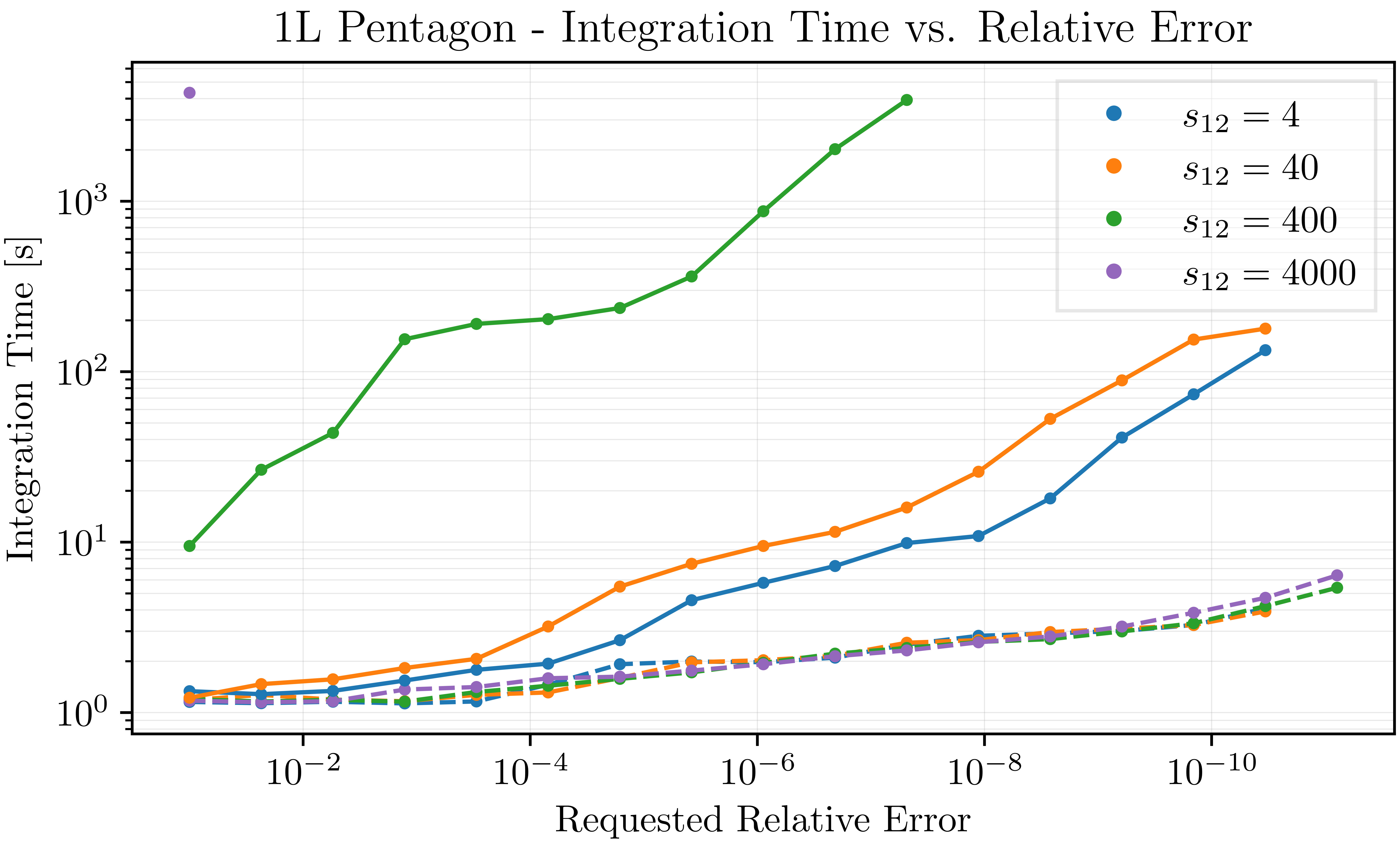
2L Pentagon - Integration Time vs. Relative Error
Integration Time [s]
Requested Relative Error
$s_{12} = 0.4$
$s_{12} = 0.04$
$s_{12} = 0.004$
$s_{12} = 0.0004$

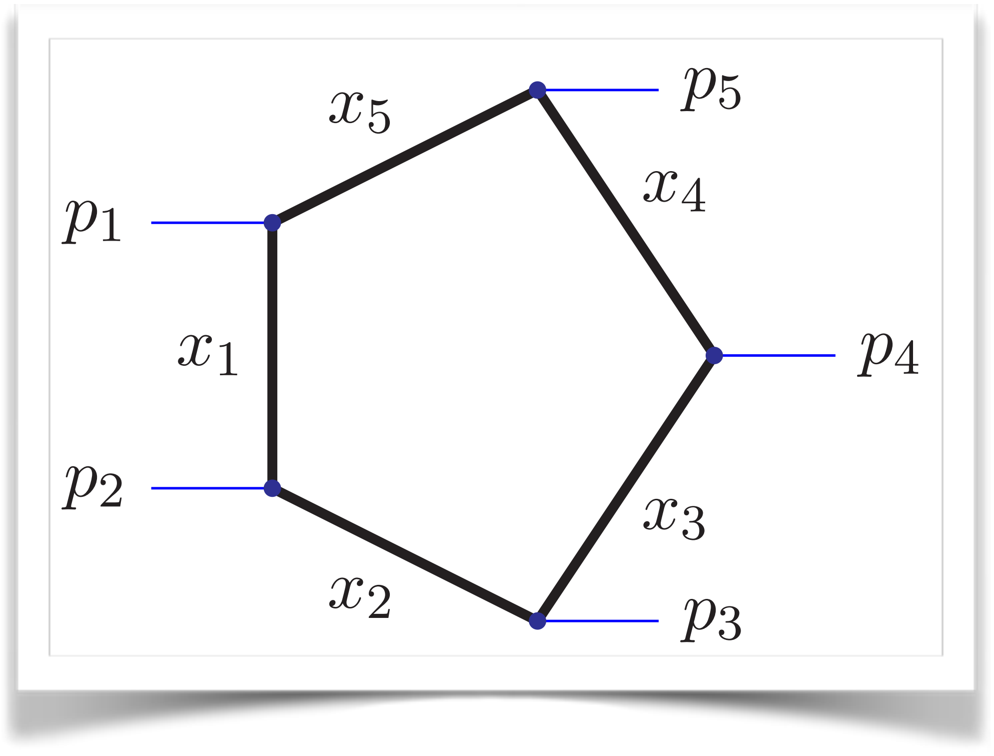

$s = 4$
$s = 40$
$s = 400$
$s = 4000$
Integration Time [s]
10^1 10^2 10^3 10^4
10^{-1} 10^{-2} 10^{-3} 10^{-4} 10^{-5} 10^{-6} 10^{-7} 10^{-8} 10^{-9}
$s_{23} = -1$
Requested Relative Error


p_1
x_3
p_4
x_2
x_1
x_4
p_3
x_5
p_2
x_6

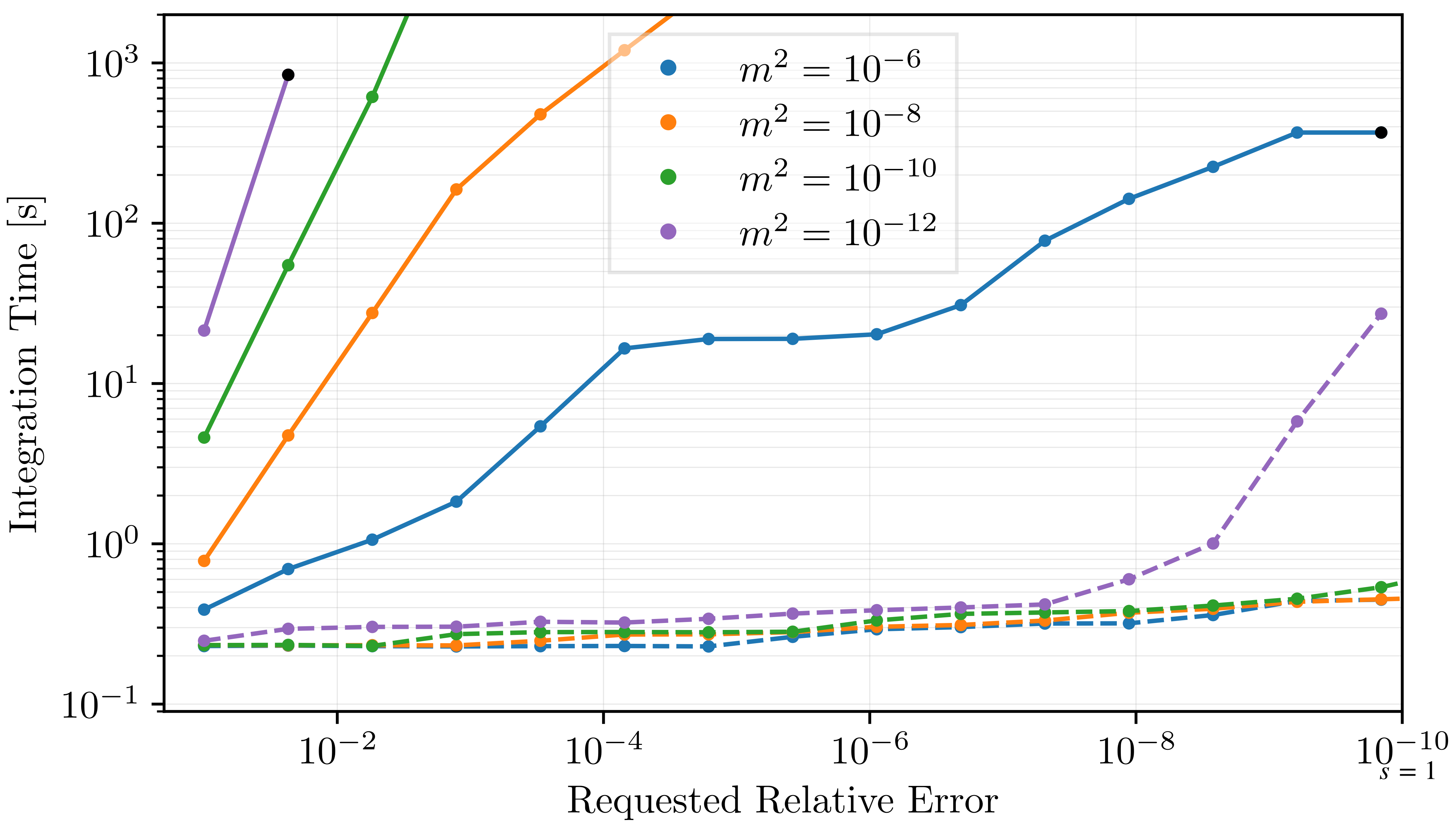

$s = 4$
$s = 40$
$s = 400$
$s = 4000$
Integration Time [s]
10^2 10^3 10^4
10^{-1} 10^{-2} 10^{-3} 10^{-4} 10^{-5} 10^{-6} 10^{-7}
$s_{23} = -1$
Requested Relative Error


p_1
x_3
x_5
x_7
x_1
p_3
p_4
x_4
x_6
p_2
x_2

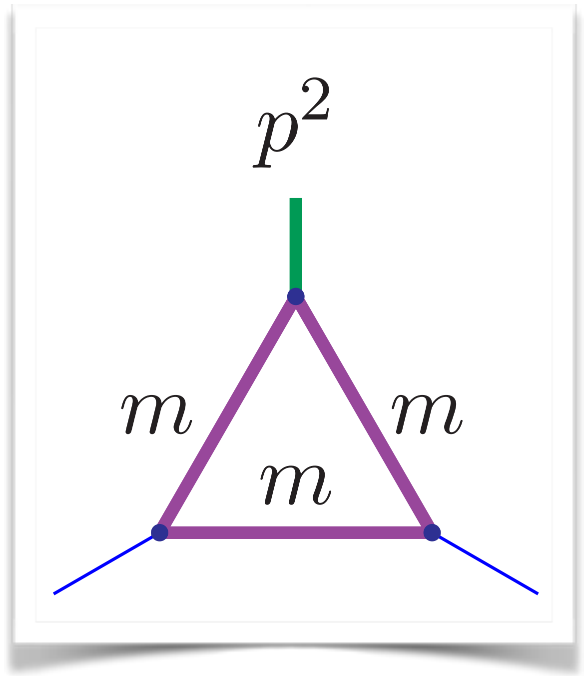




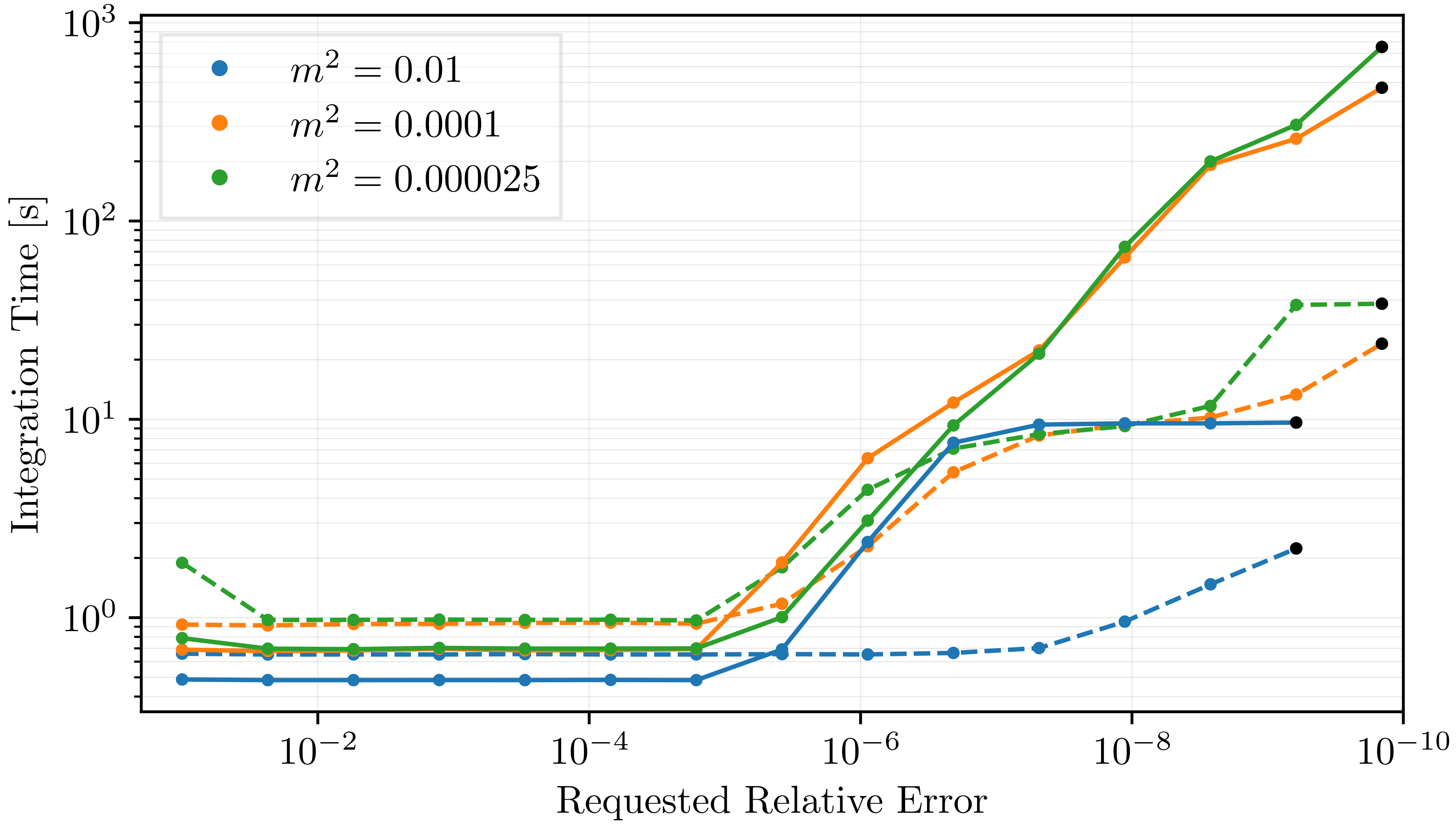
$(s_{23}, s_{34}, s_{45}, s_{51}) = (-3, 2.5, -3, 5)$



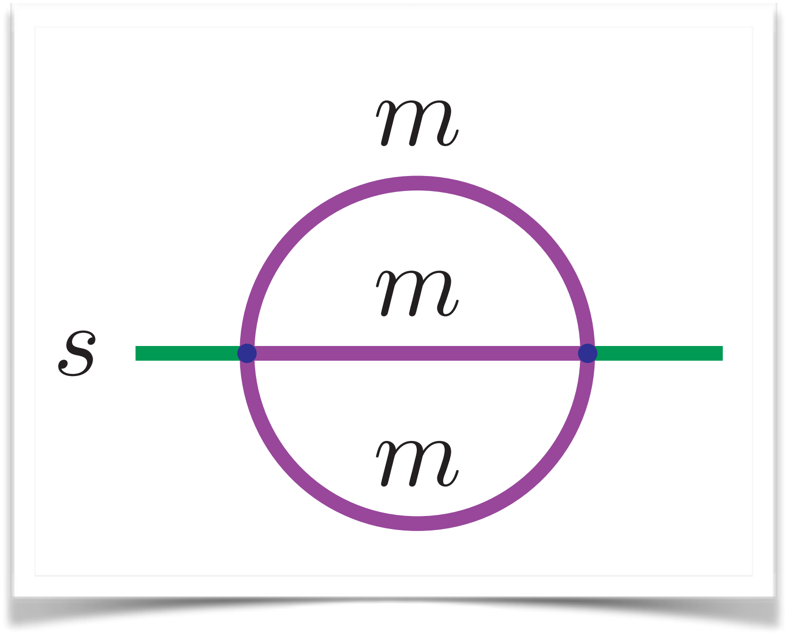
$m^2 = 10^{-6}$
$m^2 = 10^{-8}$
$m^2 = 10^{-10}$
$m^2 = 10^{-12}$
Integration Time [s]
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
$s = 1$
Requested Relative Error



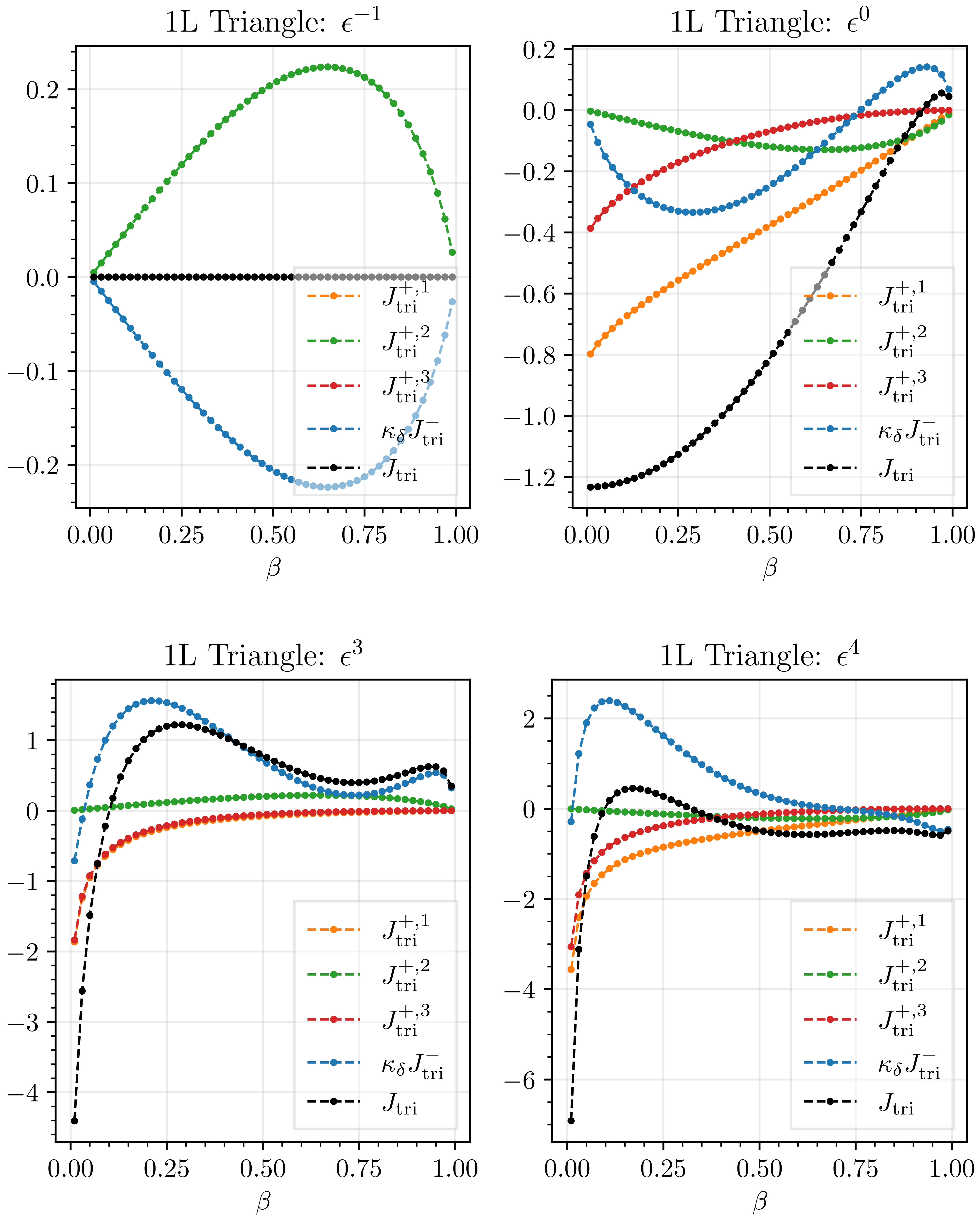
p^2
m
m
m



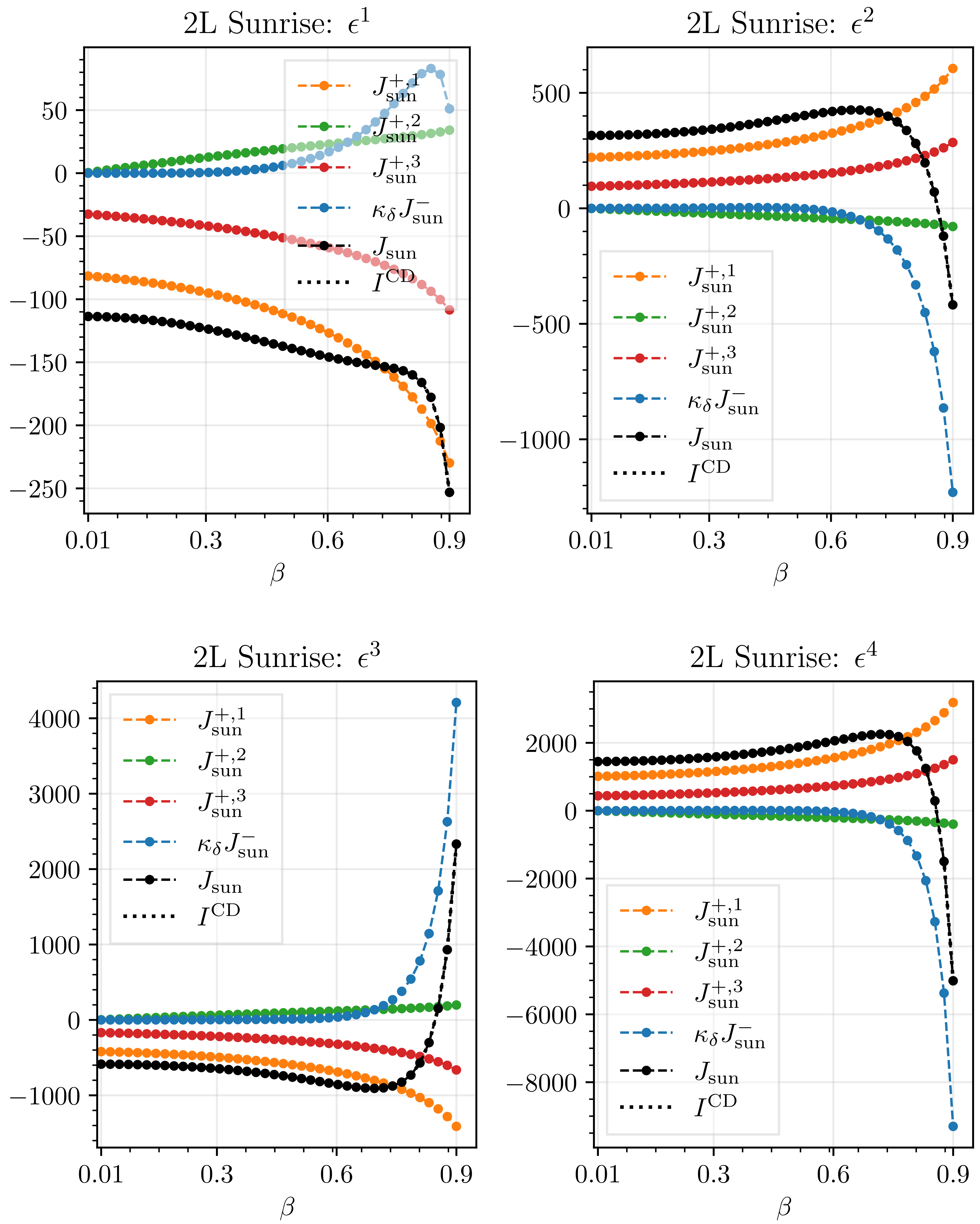
$m^2 = 0.01$
$m^2 = 0.0001$
$m^2 = 0.000025$
Integration Time [s]
10^{0}
10^{1}
10^{2}
10^{3}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
Requested Relative Error



m
m
m
s



1L Triangle: ϵ^{-1}
1L Triangle: ϵ^{0}
1L Triangle: ϵ^{3}
1L Triangle: ϵ^{4}
β
$J^{+,1}_{\mathrm{tri}}$
$J^{+,2}_{\mathrm{tri}}$
$J^{+,3}_{\mathrm{tri}}$
$\kappa_\delta J^{-}_{\mathrm{tri}}$
J_{tri}



2L Sunrise: ϵ^1
2L Sunrise: ϵ^2
2L Sunrise: ϵ^3
2L Sunrise: ϵ^4
$J_{\rm sun}^{+,1}$
$J_{\rm sun}^{+,2}$
$J_{\rm sun}^{+,3}$
$\kappa_\delta J_{\rm sun}^{-}$
$J_{\rm sun}$
$I^{\rm CD}$
β
0.01
0.3
0.6
0.9