OpenLoops+LTD: developments at one- and two-loops

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Based on ongoing work
in collaboration with **G. Bertolotti, F. Herren, J. Lindert, S. Pozzorini**

Progress in algorithms and numerical tools for QCD, Orsay (France), 10th June 2026

OpenLoops and Loop Tree Duality (LTD): motivation

See also talks from Hirschi, Huber and Sborlini!

NNLO Bottleneck: analytic calculations plagued by complexity and high computational cost

NNLO-Automation still unavailable

Numerical Alternative: LTD approach

Method: analytic integration over loop energy with numerical 3-dimensional integration

$$\int dq^0 \int d\mathbf{q}^{D-1} \mathcal{I}_V\left(q^0, \mathbf{q}\right) = \int d\mathbf{q}^{D-1} I_V\left(\mathbf{q}\right)$$

Advantages: IR cancellations without subtraction terms, multi-scale integrals

History of LTD

Seminal work: [Soper]

General formulation: [Catani, Rodrigo, Krauss, Gleisberg, Winter, Sborlini, Buchta, Chachamis, Draggiotis, Malamos Driencourt-Mangin, Hernandez-Pinto, Kilian, Kleinschmidt, . . .]

Recent Developments: [Capatti, Hirschi, Kermanschah, Pelloni, Ruijl, Vicini, Tramontano, Weinzierl, Runkel, Szor, Vesga, Kromin, Schwanemann, Ramirez-Uribe, Renteria-Olivo, Renteria-Estrada, Martinez, de Lejarza, Torres Bobadilla, Dhani, Cieri, Aguilera-Verdugo, LTD collaboration, Artico, . . .]

Open question: promising framework, can it be developed for automated MC simulations for collider phenomenology?

OpenLoops+LTD project

[Bertolotti, **NG**, Herren, Lindert, Pozzorini]

Loop-Tree Duality at NLO

Automated LTD event generator

- NLO events with real and virtual unresolved particles
- Local UV+Threshold+IR cancellations
- Predictions for **arbitrary fully-differential observables** (e.g. via Rivet analysis)

Focus of this talk

- Cancellations of UV+Threshold+IR singularities
- Numerical validation of the 1-loop LTD integrand
- Applications to multi-leg QCD FSR processes

Loop-Loop Duality at NNLO

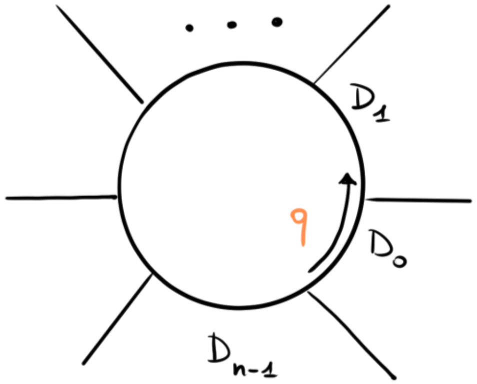
New 2-loop approach

- 1-loop subdiagram handled semi-analytically with OL2
  [Buccioni, Maierhöfer, Lang, Lindert, Pozzorini, Zhang, Zoller ’18]
- Remaining 1-loop integral numerically with OL+LTD@1L

Focus of this talk

- Introduction to the **Loop-Loop Duality** method
- Validation of proof-of-concept applications
- Multi-leg topologies

Loop Tree Duality at NLO [Catani, Rodrigo]



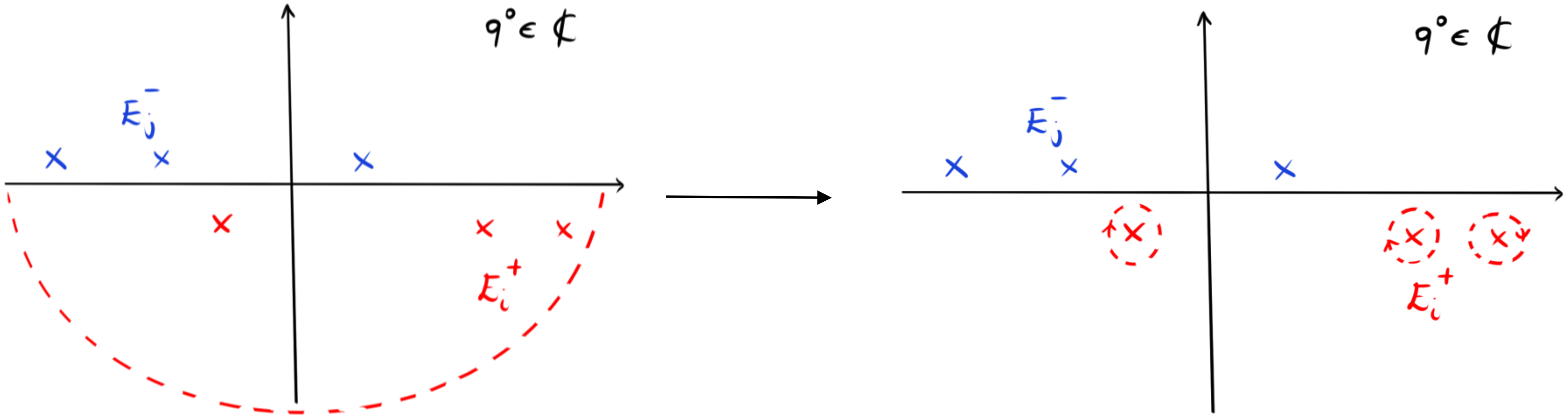
$$D_i = \left(q_0 - E_i^+\right)\left(q_0 - E_i^-\right) \longrightarrow$$ Dependence on q_0 manifest

$$E_i^\pm = -p_i^0 \pm \sqrt{|\vec{q}+\vec{p}_i|^2 + m_i^2 - i\epsilon}$$

$$\mathrm{D}^D q = \mu^{2\epsilon} \frac{\mathrm{d}^D q}{(2\pi)^D}$$

Residue Theorem: **analytic q_0-integration**

$$\mathcal{W}_{\mathrm{V}}^{\mathrm{bare}} = -i \int D^D q \frac{N(q)}{\prod_i D_i(q)} = -\sum_{i=0}^{n-1} \int \mathrm{D}^{D-1}\vec{q} \frac{1}{2\epsilon_i} \frac{N(E_i^+, \vec{q})}{\prod_{j\neq i} D_j(E_i^+, \vec{q})}$$



General and fully differential **NLO LTD-based algorithm** implemented in **OpenLoops**

Automation & Singularities of LTD integrand

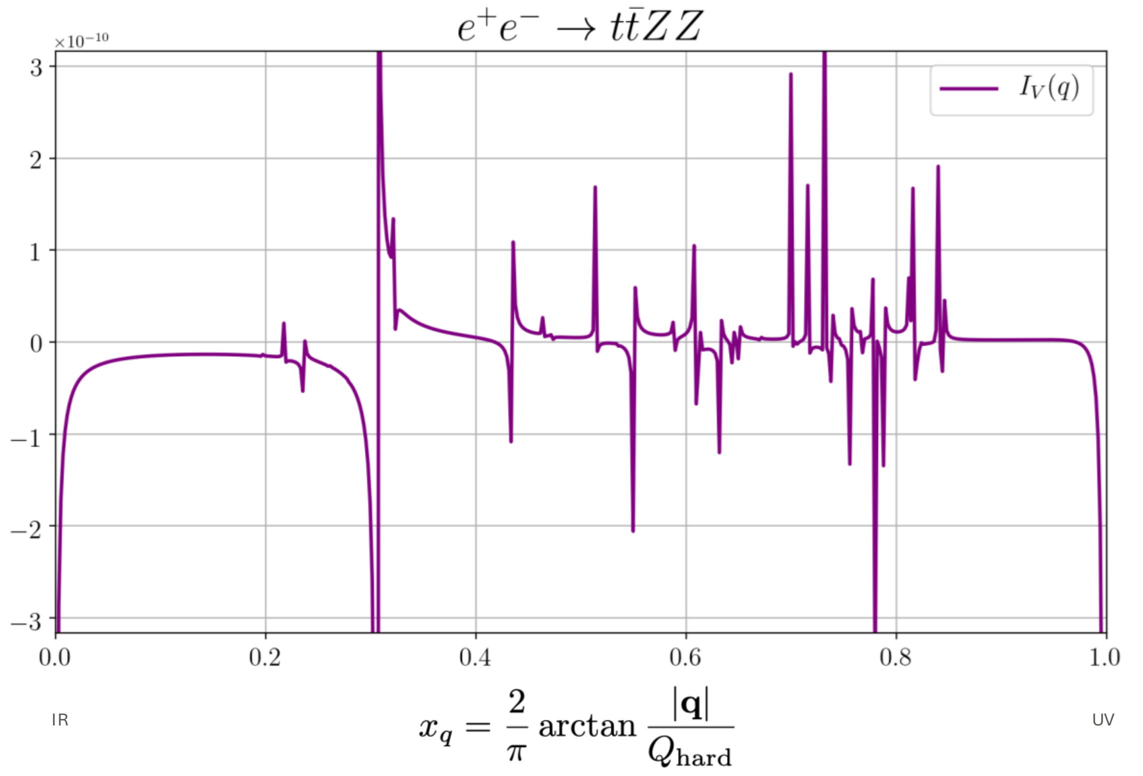
Numerator handled numerically in OL

Ingredients of OpenLoops+LTD generator

$$I_{\rm V}(\mathbf{q}) = -\sum_{i=0}^{n-1} \frac{1}{2\epsilon_i} \frac{N(q)}{\prod_{j\neq i} D_j(q)}\Bigg|_{q^0=E_i^+} = -\sum_{i=0}^{n-1} \frac{1}{2\epsilon_i} \frac{N(E_i^+,\mathbf{q})}{\prod_{j\neq i}(E_i^+ - E_j^+)(E_i^+ - E_j^-)}$$

Local pole cancellation & minimisation of $I_V(\mathbf{q})$ variance

UV singularities	$\lvert\mathbf{q}\rvert \to \infty$	**Local subtraction** based on UV tadpole expansion
Spurious singularities	$E_i^+(\mathbf{q}) \to E_j^+(\mathbf{q})$	**"Dual cancellation" + UV asymptotic expansion**
Threshold singularities	$E_i^+(\mathbf{q}) \to E_j^-(\mathbf{q})$	Local subtraction
Infrared singularities	$q \to 0 \quad q \parallel p_k$	Automated algorithm for **local V+R cancellation**

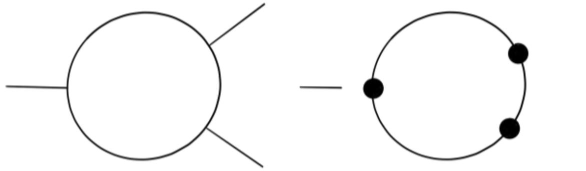


Thanks to Gloria for the inspiration for this slide!

UV Singularities at 1-Loop

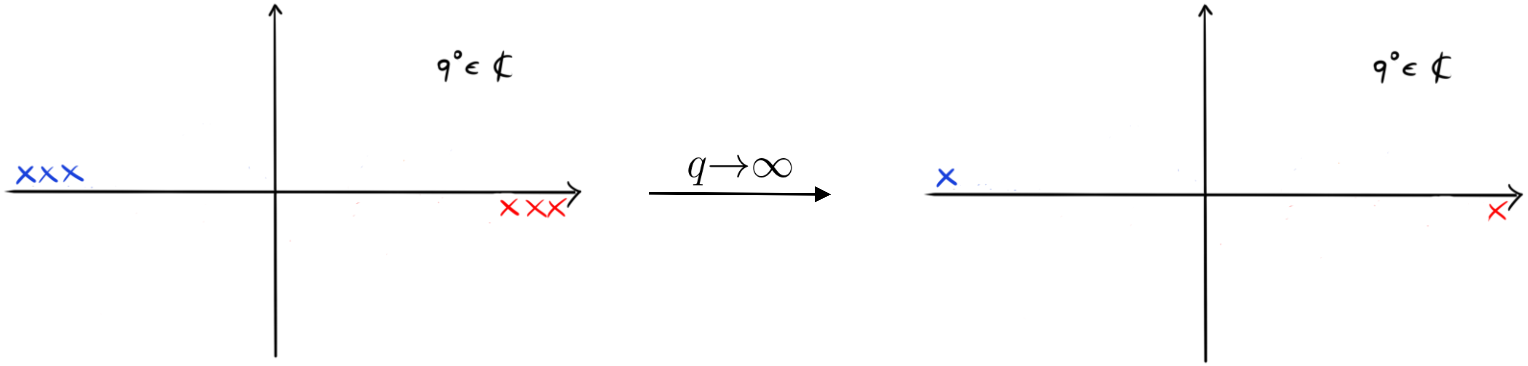
Local subtraction based on tadpole expansion in the limit $q^2 \to \infty$ [Chetyrkin, Misiak, Münz]

$$(q+p_i)^2 - m_i^2 \overset{q\to\infty}{=} q^2 - \mu_{\text{tad}}^2 + \mathcal{O}(q)$$



Expansion + analytic q_0–integration $\longrightarrow$ single residue of higher order pole

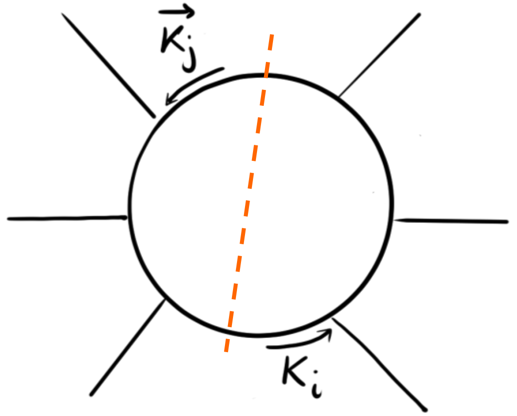
$$I(q) = \frac{N(q)}{\prod_{i=0}^{n-1} D_i(q)} \xrightarrow{q\to\infty} I_{\text{tad}}(q) = \sum_{k=n}^{n+p} \frac{U_k(q)}{(q^2-\mu_{\text{tad}}^2)^k} + \mathcal{O}(q^{-X})$$



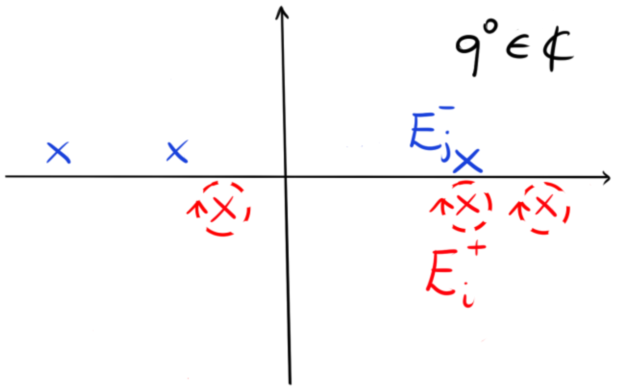
Renormalisation: analytically integrated subtraction terms added back and matched to UV renormalisation constants

Fully automated in OL, including higher-order expansions to avoid huge LTD residue cancellations at high $|\vec{q}|$

Threshold Singularities at 1-Loop



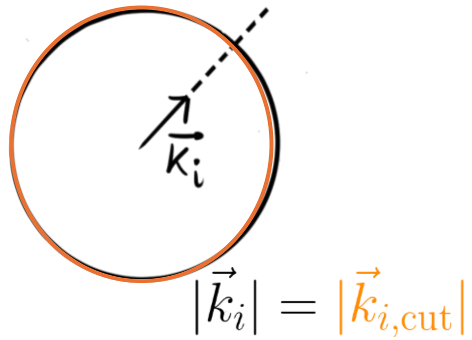
Singularity: q_0–contour **pinched** by poles at $E_i^+ \to E_j^-$



$$\mathcal{W}_{ij}^{\mathrm{thr}} = \int \frac{\mathrm{D}^{D-1}\vec{k}_i}{2\epsilon_i} \frac{g_{ij}(k_i)}{(E_i^+ - E_j^+)(E_i^+ - E_j^- - i0)}$$

Local Subtraction: 1-dimensional subtraction with simple integrated subtraction terms [see also Kermanschah]

In the CM-frame: $$W_{ij}^{(thr)} = \int d\Omega_i \int_{m_i}^{\infty} d\epsilon_i \underbrace{\frac{1}{\epsilon_i - \epsilon_i^{cut} - i0}}_{\mathrm{PV}\left(\frac{1}{\epsilon_i - \epsilon_i^{cut}}\right) + i\pi\delta(\epsilon_i - \epsilon_i^{cut})} \left[f_{ij}(\epsilon_i, \Omega_i) - \rho(\epsilon_i) f_{ij}(\epsilon_i^{cut}, \Omega_i)\right]$$

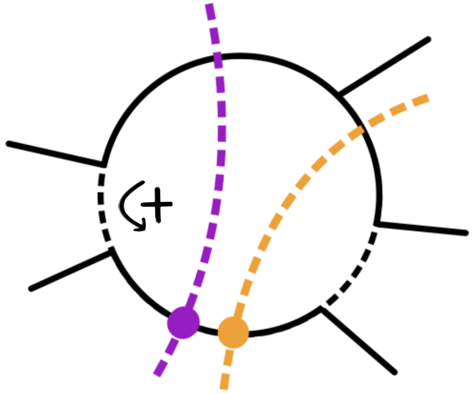


Fully **automated in QCD** and applicable to overlapping cuts

Threshold singularities: triple-cut local subtraction

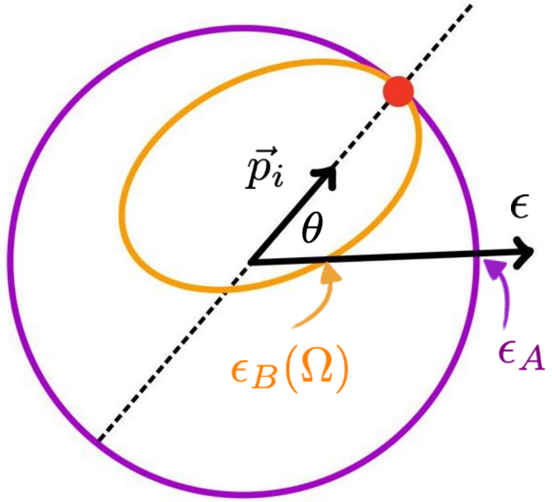
Strategy for triple-cut local cancellation in massless FS QCD

Choose the energy-flow such that both cuts lie on the same residue, i.e. they are parametrised by a common variable ε



$$\mathcal{W}|_{Rj} = \int \mathrm{D}^{D-1}\mathbf{k}_i \, \frac{1}{2\epsilon_j} \frac{g(\mathbf{k}_i)}{(E_j^+ - E_i^-)(E_j^+ - E_k^-)}$$

$$\longrightarrow \int \mathrm{d}\Omega \int \mathrm{d}\epsilon \; \frac{f(\epsilon)}{(\epsilon - \epsilon_A)(\epsilon - \epsilon_B(\Omega))}$$



Double and triple cuts are cancelled via standard double-cut subtractions at fixed Ω

$$\frac{f(\epsilon)}{(\epsilon-\epsilon_A)(\epsilon-\epsilon_B)} - \frac{f(\epsilon_A)}{\epsilon-\epsilon_A} - \frac{f(\epsilon_B)}{\epsilon-\epsilon_B} = \frac{1}{\epsilon_B-\epsilon_A}\left[\frac{f(\epsilon)-f(\epsilon_A)}{\epsilon-\epsilon_A} - \frac{f(\epsilon)-f(\epsilon_B)}{\epsilon-\epsilon_B}\right]$$

Partial fractioning

Regular at triple cut!

$\epsilon = \epsilon_A = \epsilon_B$

Free from ji-cut sing.

Free from jk-cut sing.

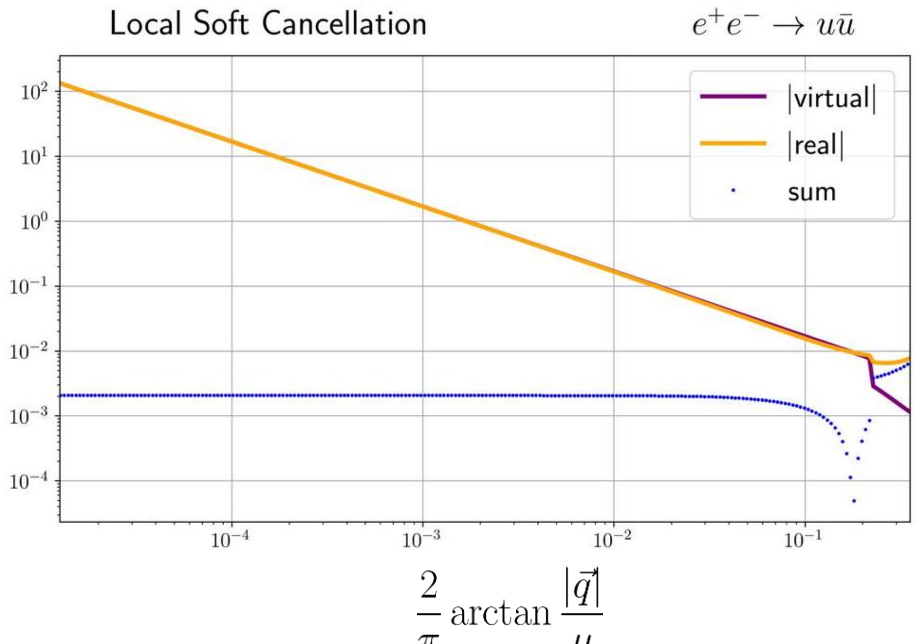
Local IR cancellations at NLO

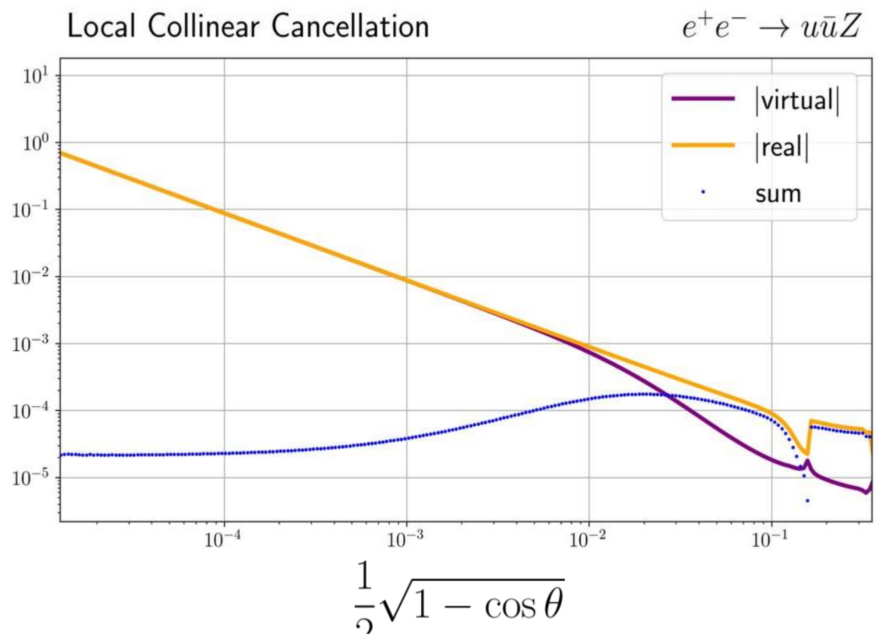
Local Finiteness: Virtual+Real corrections are combined in a single 3-dim integral

$$\sigma_{\mathrm{NLO}}^{(\mathrm{V+R})} = \int d\Phi_B \int d^3\vec{q} \sum_{\alpha} [I_{\mathrm{V},\alpha}(\Phi_{\mathrm{B}},\vec{q}) + I_{\mathrm{R}}(\Phi_{\mathrm{R}}^{\alpha}(\Phi_{\mathrm{B}},\vec{q}))]$$

Local IR cancellation **without auxiliary subtraction terms** obtained by:

- Dynamic loop-momentum shifts such that all virtual IR singularities lie at $\vec{q} \to 0$ and $\vec{q} \parallel \vec{p}_i$
- Collinear sectors where virtual and real IR singularities aligned through appropriate mappings
- Cancellation of fake soft-threshold singularities via $\vec{q} \to -\vec{q}$ symmetrisation





Combined MC integration over Born$\otimes\vec{q}$-space

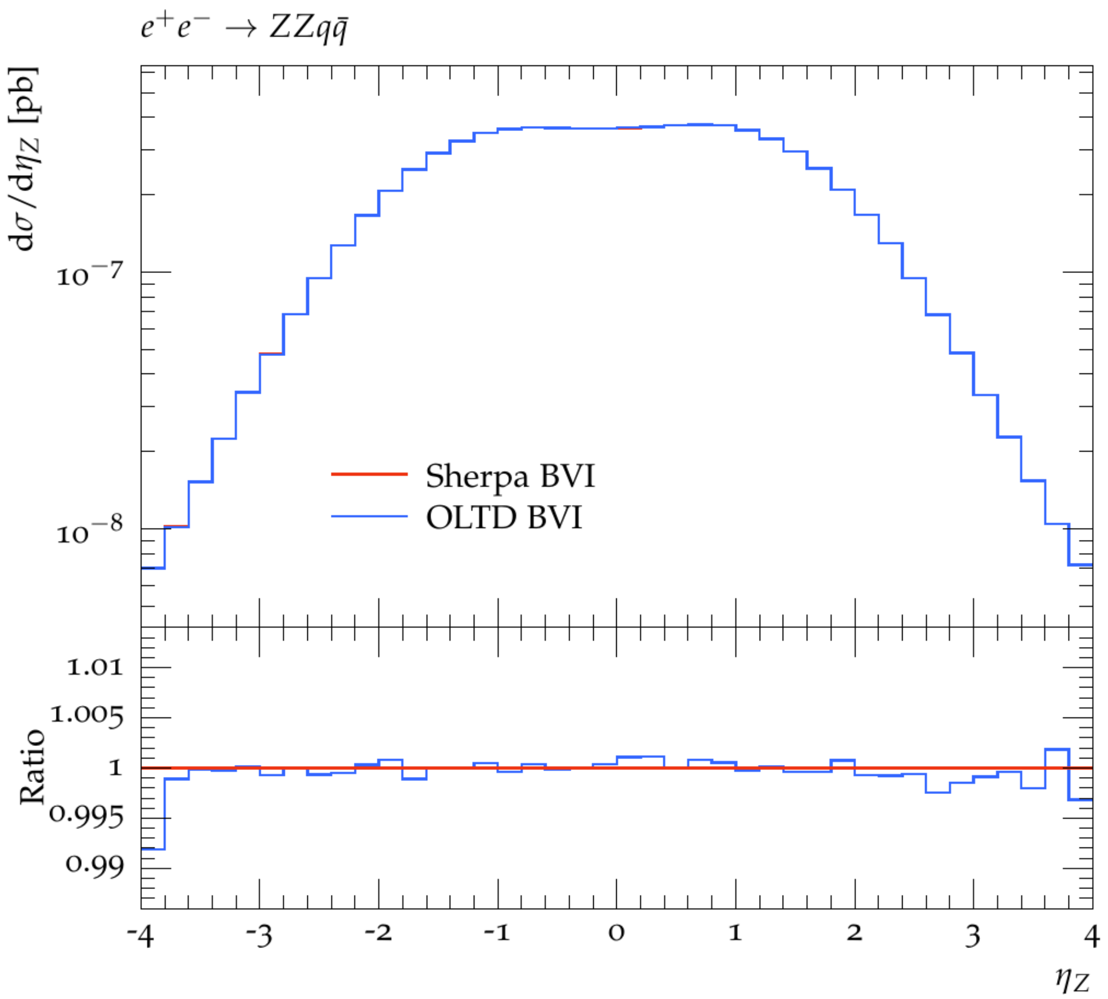
Strategy for optimal convergence

- We first **generate an unweighted Born sample**
- For each Born event we generate a few virtual momenta and mapping-related real emissions

Relative MC uncertainty of $\kappa = \frac{\mathrm{NLO}}{\mathrm{LO}} - 1$ is

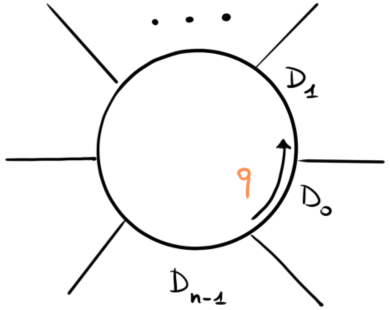
$$\frac{\delta\kappa}{\kappa} = \sqrt{\frac{1+\Delta_\kappa^2}{N}} \quad \text{with} \quad \Delta_\kappa = \frac{\sqrt{\mathrm{Var}(\kappa)}}{\kappa}$$

We typically achieve $\Delta_\kappa = \mathcal{O}(1)$ $\longrightarrow$ optimal MC convergence



Applicable to arbitrary differential observables

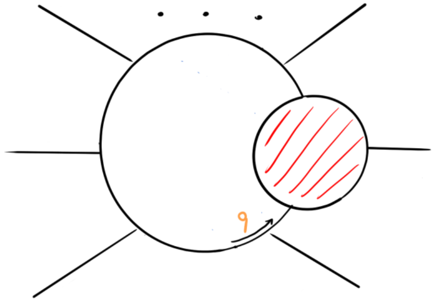
From One Loop to Two Loops



$$= -i \int \mathrm{D}^D q \frac{N(q)}{\prod_i D_i(q)}$$

1-loop: **LTD+OpenLoops** automated algorithm

- **Numerical** integral with **tree-like** integrand

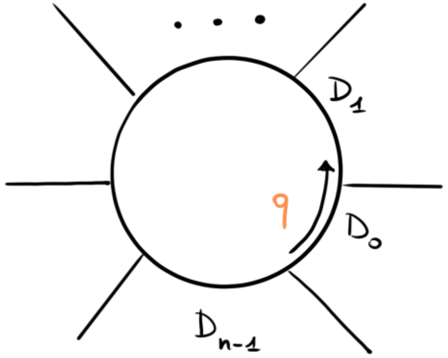


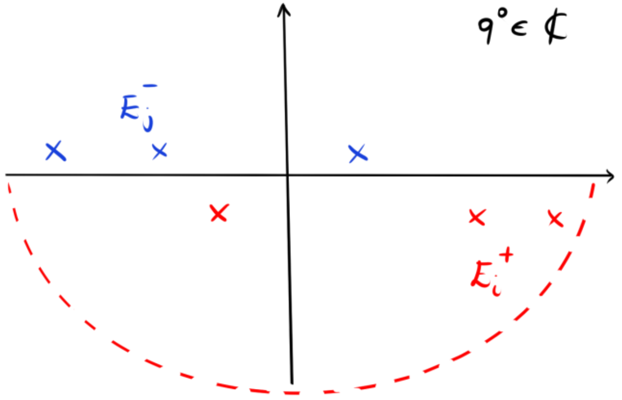
$$= -i \int \mathrm{D}^D q \frac{\mathcal{A}(q)}{\prod_i D_i(q)}$$

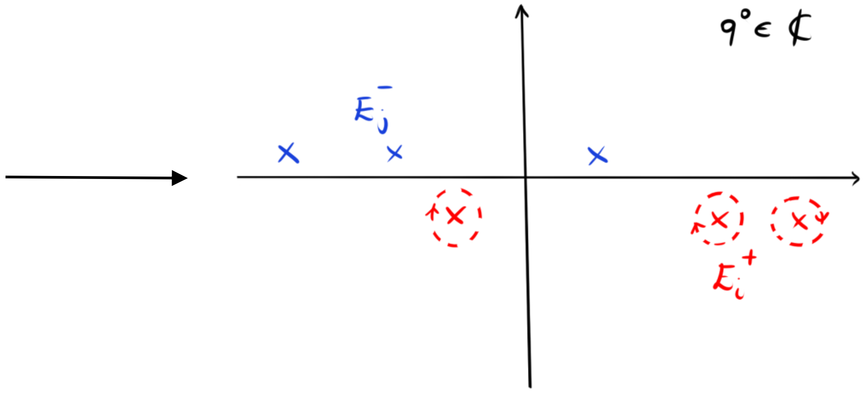
New Loop Loop Duality (LLD) method

- **Numerical** integral with **1-loop-like** integrand
- **"Semi-analytical"** 1-loop subdiagram with OpenLoops2
- Numerical integration with upgraded LTD+OpenLoops 1-loop algorithm

Comparison between LTD and LLD: 1-Loop

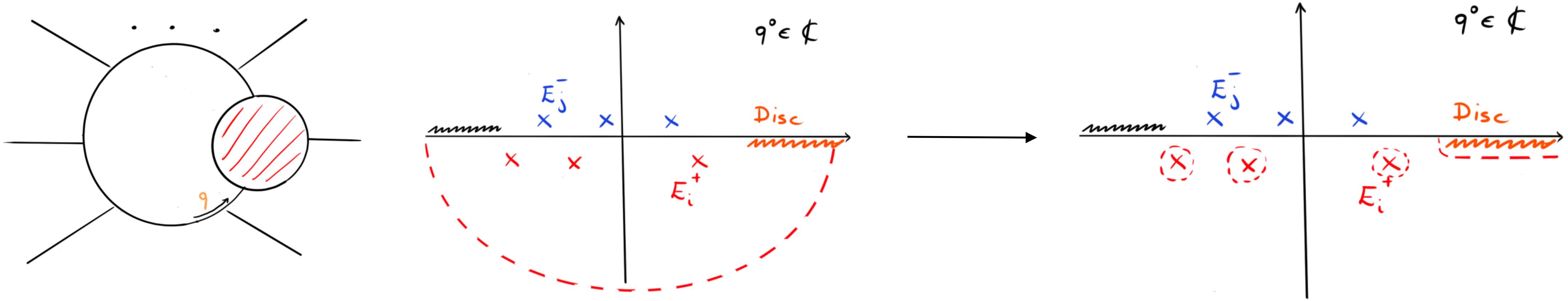






$$\mathcal{W}_{\mathrm{V}}^{\mathrm{bare}} = -i \int \mathrm{D}^D q \frac{N(q)}{\Pi_i D_i(q)} = -\sum_{i=0}^{n-1} \int \mathrm{D}^{D-1}\vec{q} \frac{1}{2\epsilon_i} \frac{N(E_i^+, \vec{q})}{\prod_{j\neq i} D_j(E_i^+, \vec{q})}$$

Comparison between LTD and LLD: 2-Loops

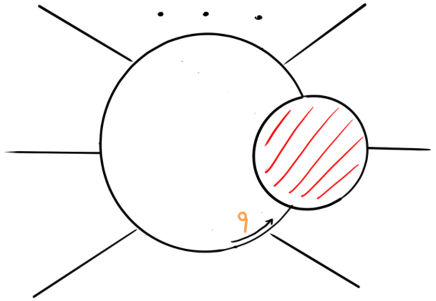


$$\mathcal{W} = i \int \mathrm{D}^{D-1}\vec{q} \int \frac{dq_0}{2\pi} \frac{\mathcal{A}(q)}{\prod_i D_i(q)} = \int \mathrm{D}^{D-1}\vec{q} \left[\sum_i \mathcal{R}_i(\vec{q}) + \frac{i}{2\pi} \int_{q_0^{\mathrm{BP}}}^{\infty} dq_0 \mathcal{C}(q_0, \vec{q}) \right]$$

Main features of the **Loop Loop Duality** method:

- **Residues** can be handled with extension of automated 1-loop LTD+OL algorithm
- Key novelty: **Continuum** q0-integral over Discontinuity

Treatment of semi-analytic 1-loop subdiagram



$$\mathcal{W} = \frac{i}{2\pi} \int \mathrm{D}^D q \frac{\mathcal{A}(q)}{\Pi_i D_i(q)}$$

Numerical integration

OL+LTD algorithm

$$\mathcal{A}(q) = \int \mathrm{D}^D \ell \frac{N(q,\ell)}{\Pi_k D_k(q,\ell)}$$

Analytic integration

OL2 algorithm

Numerator: 2-loop numerator automated in OpenLoops [Pozzorini, Schär, Zoller]

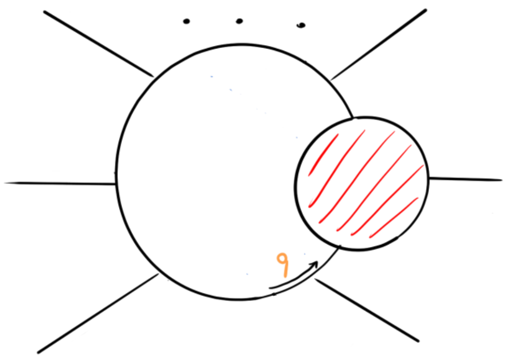
$$N(q,\ell) = \sum_{r=0}^{R} \sum_{s=0}^{S} N_{\mu_1,\ldots,\mu_r;\nu_1,\ldots,\nu_s} q^{\mu_1} \ldots q^{\mu_r} \ell^{\nu_1} \ldots \ell^{\nu_s}$$

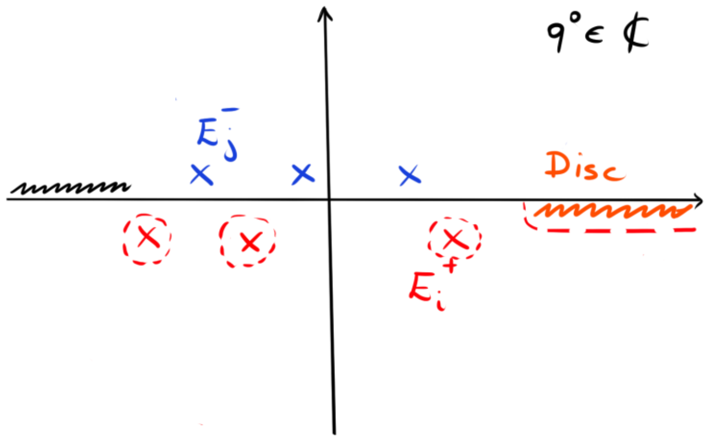
Key: separation of analytic tensorial 1-loop ℓ -integral from numerical LTD q-integral

Discontinuity of 1-loop subdiagram

$$\mathrm{Disc}\mathcal{A}(q^0,\vec{q}) = \mathcal{A}(q^0,\vec{q}) - \mathcal{A}(q^0 - i0,\vec{q})$$

For $\vec{q} \in \mathbb{R}^3$ one can prove that $\mathrm{Disc}\mathcal{A} \neq 0$ only for $q_0 \in \mathbb{R}$

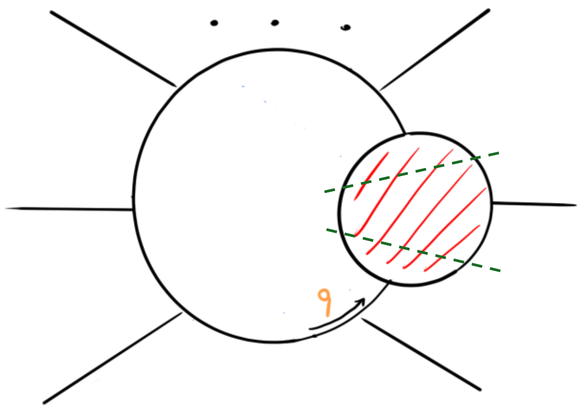


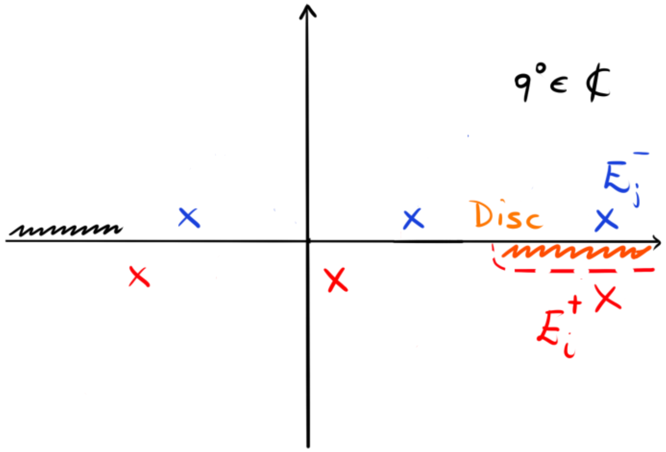


Discontinuity of 1-loop **tensor integrals**

- can be obtained via OpenLoops 2 tensor reduction from scalar-integral discontinuities
- reduction coefficients are rational $\longrightarrow$ free from discontinuities

q_0-continuum integral and LTD pole subtraction





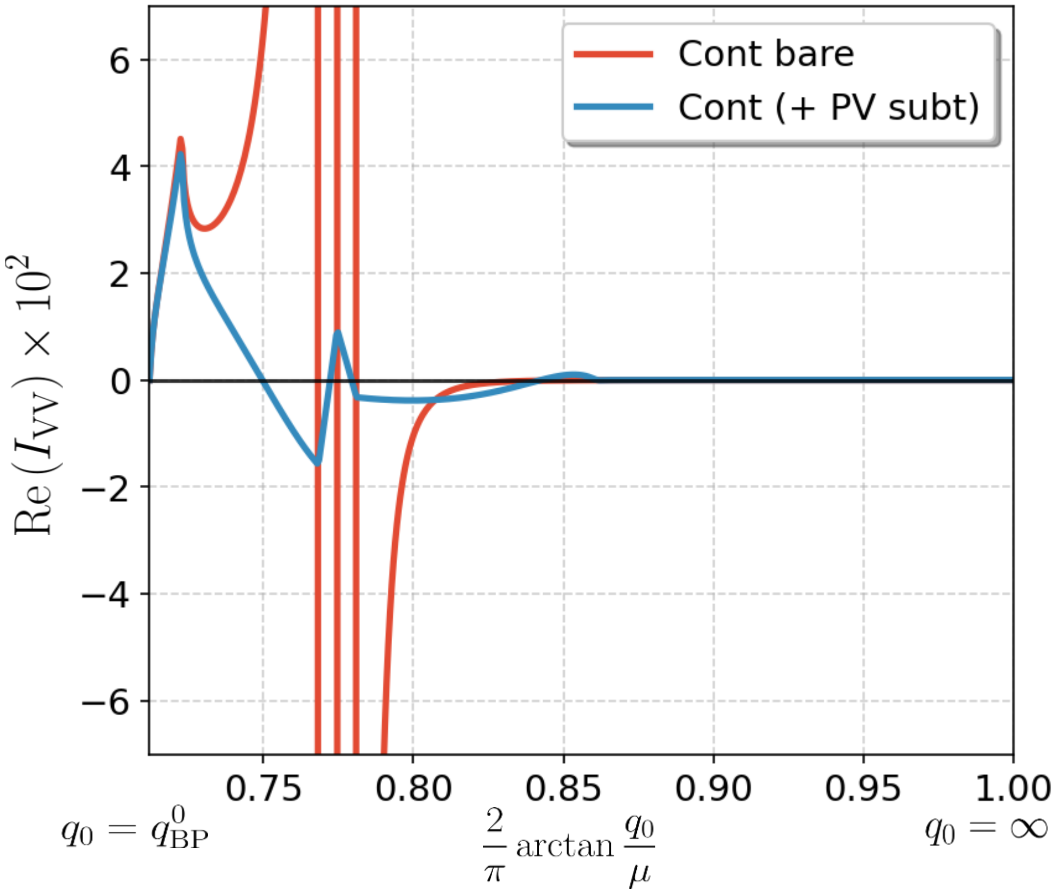
$$\mathcal{W}_{\text{cont}} = -\int \mathrm{D}^{D-1}\vec{q} \int_{q^0_{\text{BP}}}^{\infty} \frac{dq_0}{2\pi i} \, \text{Disc}\mathcal{A}(q_0, \vec{q}) \prod_i \frac{1}{(q_0 - E_i^+)(q_0 - E_i^-)}$$

Subtraction of positive and negative-energy poles along branch cut

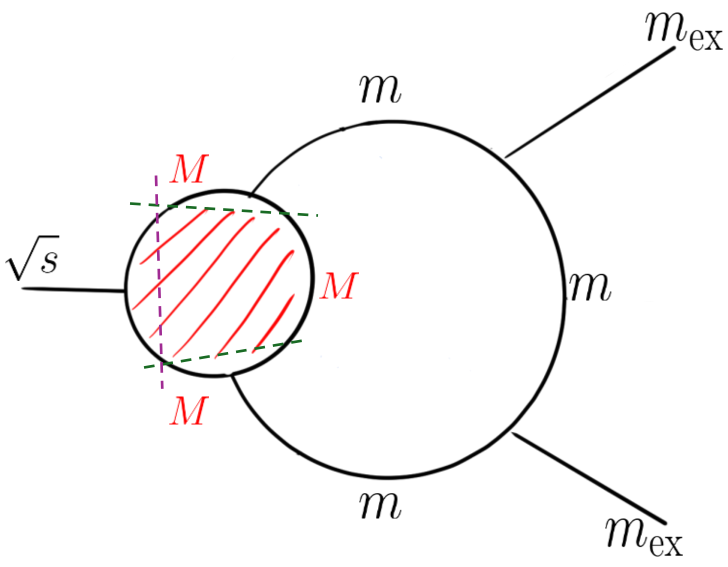
$$\frac{f(q^0)}{q^0 - E_i^\pm} \to \frac{f(q^0)}{q^0 - E_i^\pm} - f(E_i^\pm)\left[\text{PV}\left(\frac{1}{q^0 - E_i^\pm}\right) \pm i\pi\delta(q^0 - E_i^\pm)\right]$$

$i\pi$ contributions combined with "residues" of $E_i^\pm$ energy poles

q_0-continuum integrand: example



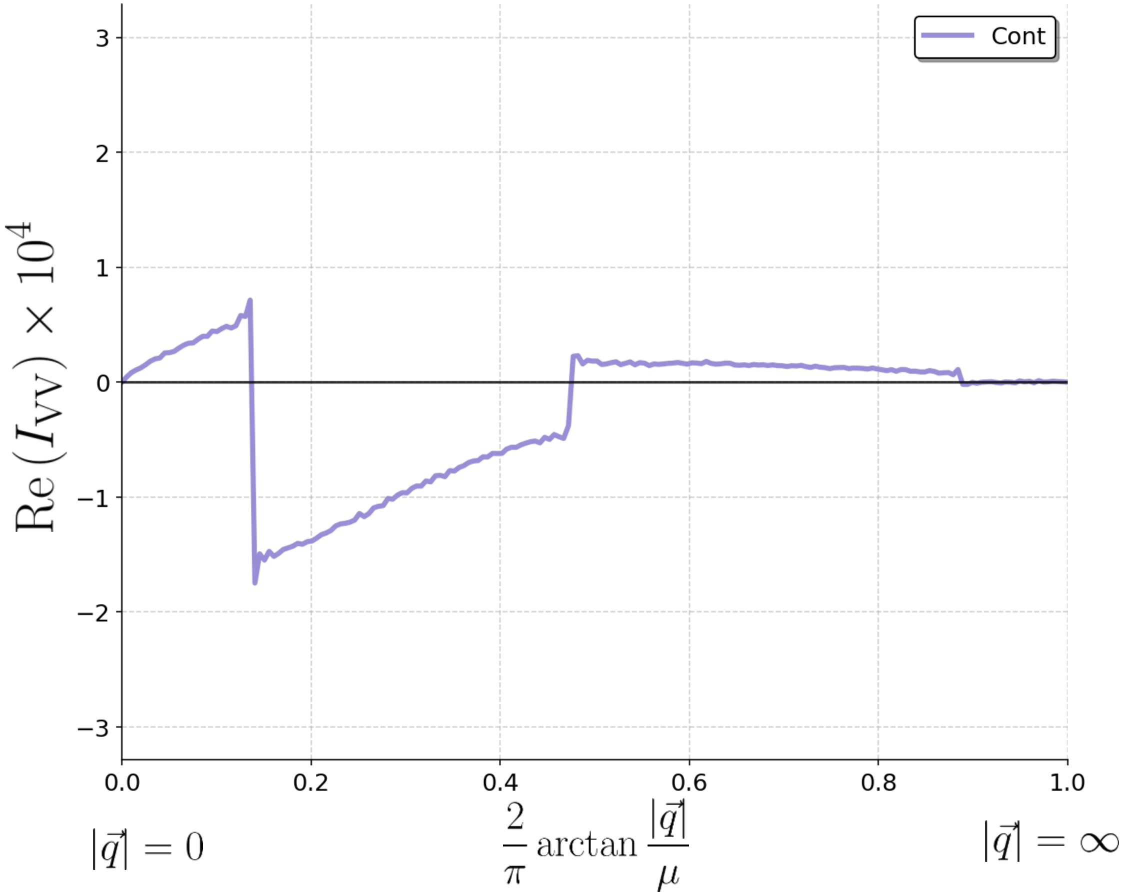
Scalar triangle into triangle



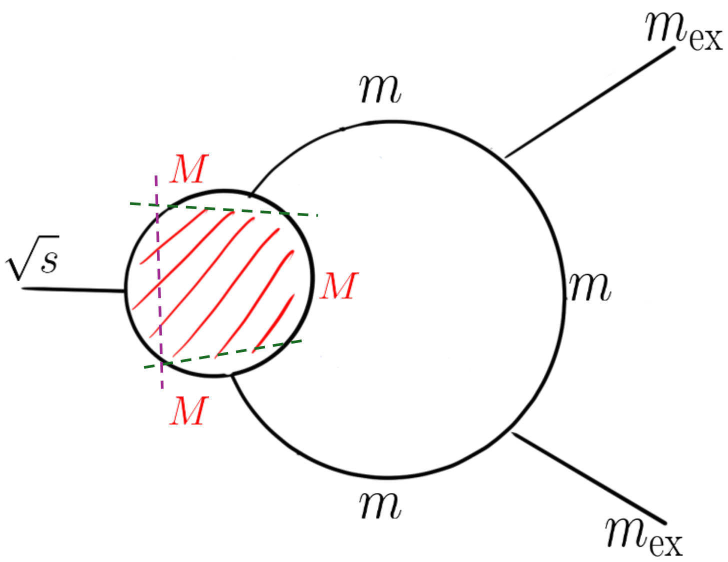
$$(\sqrt{s}, M, m, m_{\mathrm{ex}}) = (12, 3, 10, 2)$$

PV singularities subtracted (room for optimisations) $\longrightarrow$ Integrand ready for MC q_0-integration

Integrand in $\vec{q}$-space: Continuum contribution



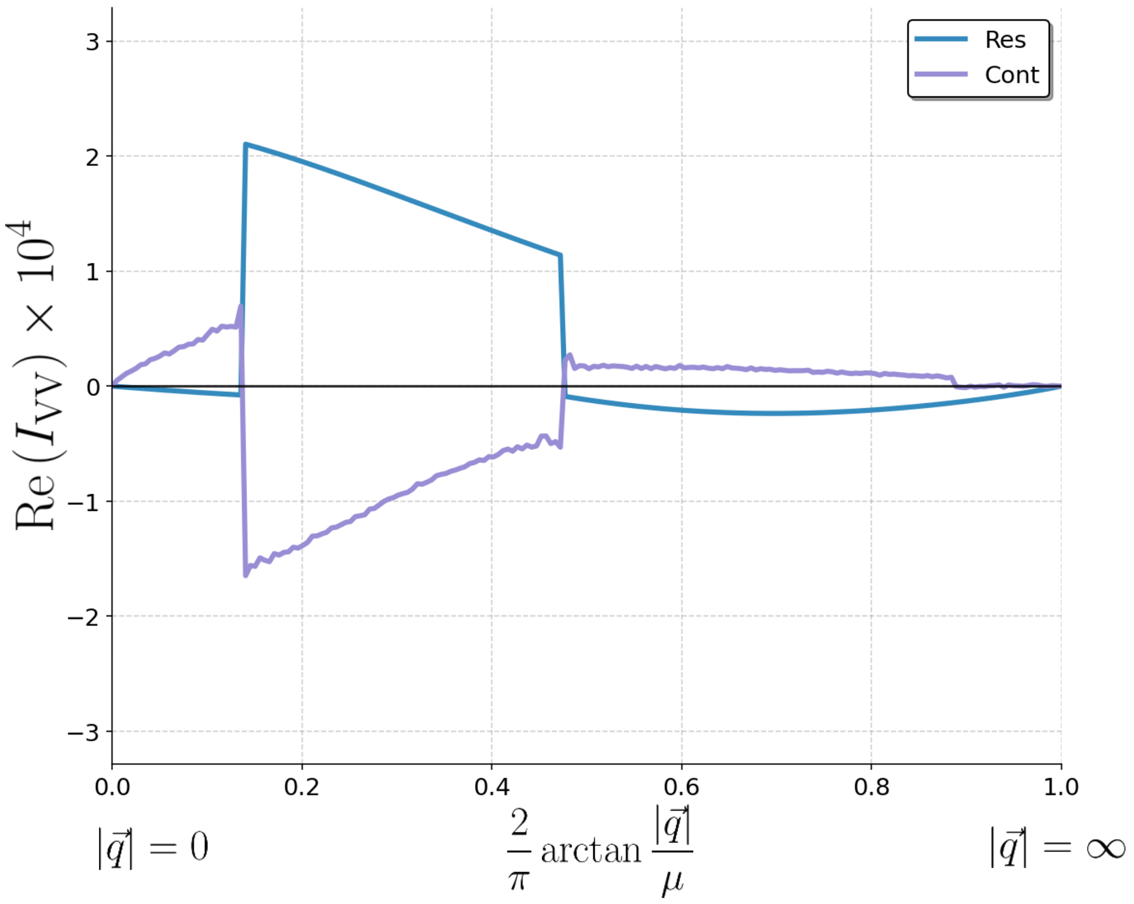
Scalar triangle into triangle



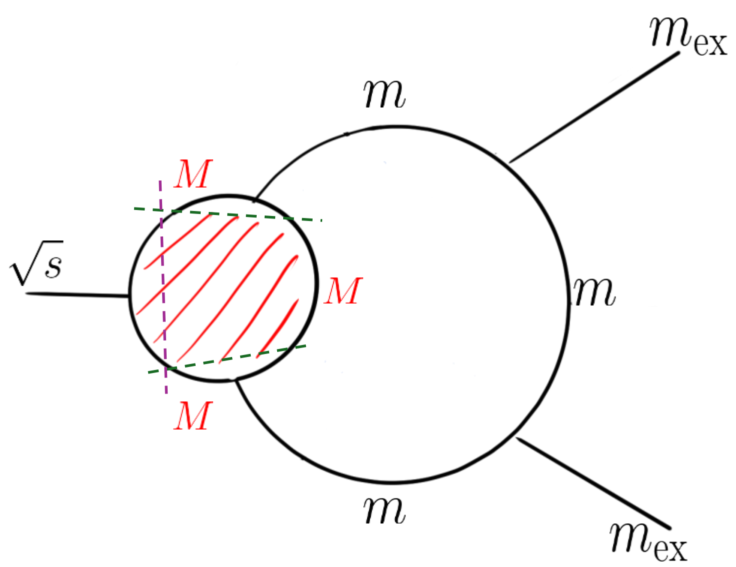
$$(\sqrt{s}, M, m, m_{\mathrm{ex}}) = (12, 3, 10, 2)$$

Finite integrand with "jumps" due to $(i\pi)^2$ contributions from triple cut in $\mathrm{Disc}\mathcal{A}$

Integrand in $\vec{q}$-space: Residue contributions



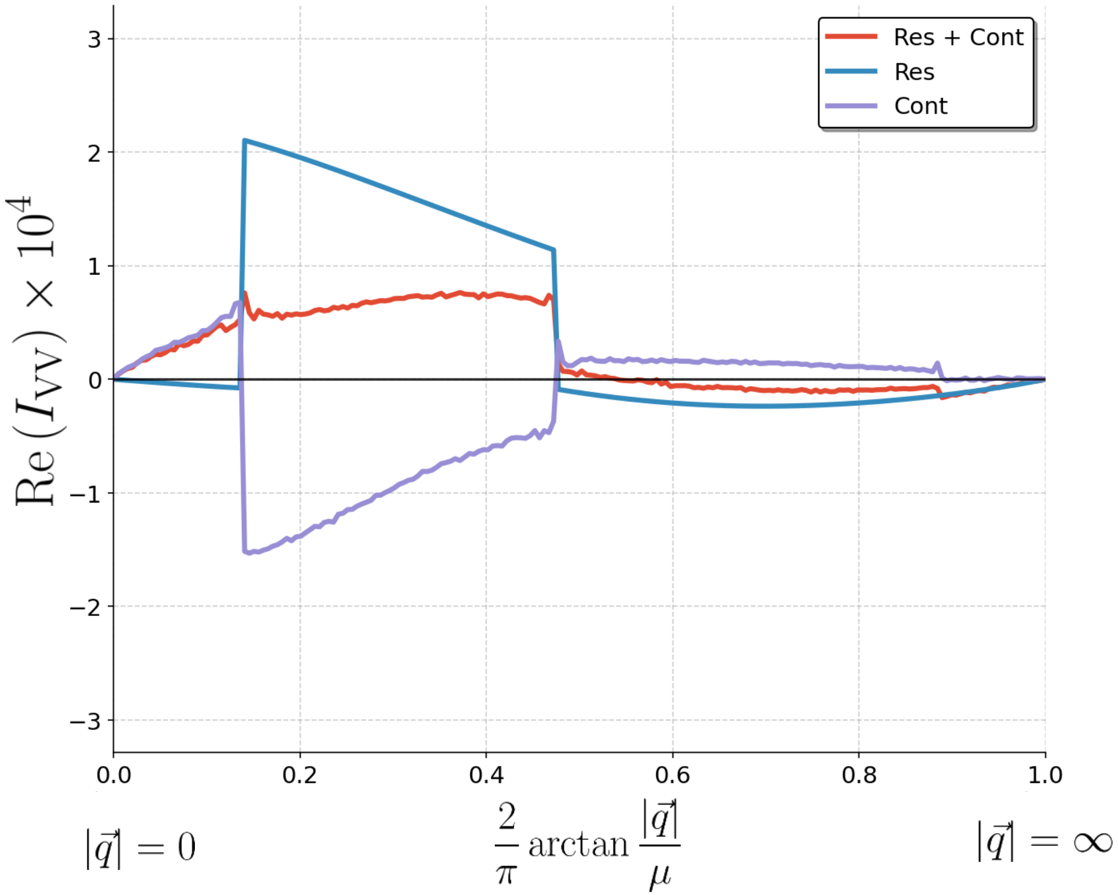
Scalar triangle into triangle



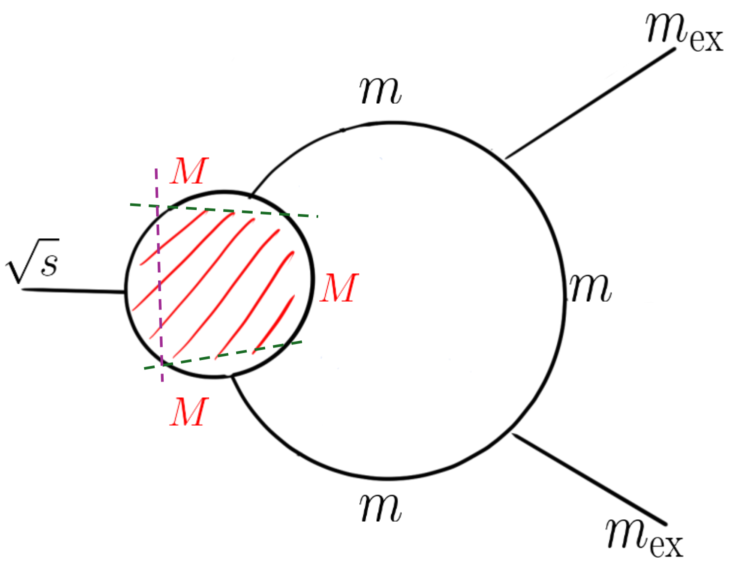
$$(\sqrt{s}, M, m, m_{\mathrm{ex}}) = (12, 3, 10, 2)$$

Opposite "jumps" in Res and Cont due to $(i\pi)^2$ contributions from triple cut in $\mathrm{Disc}\mathcal{A}$

Integrand in $\vec{q}$-space: Continuum + Residues



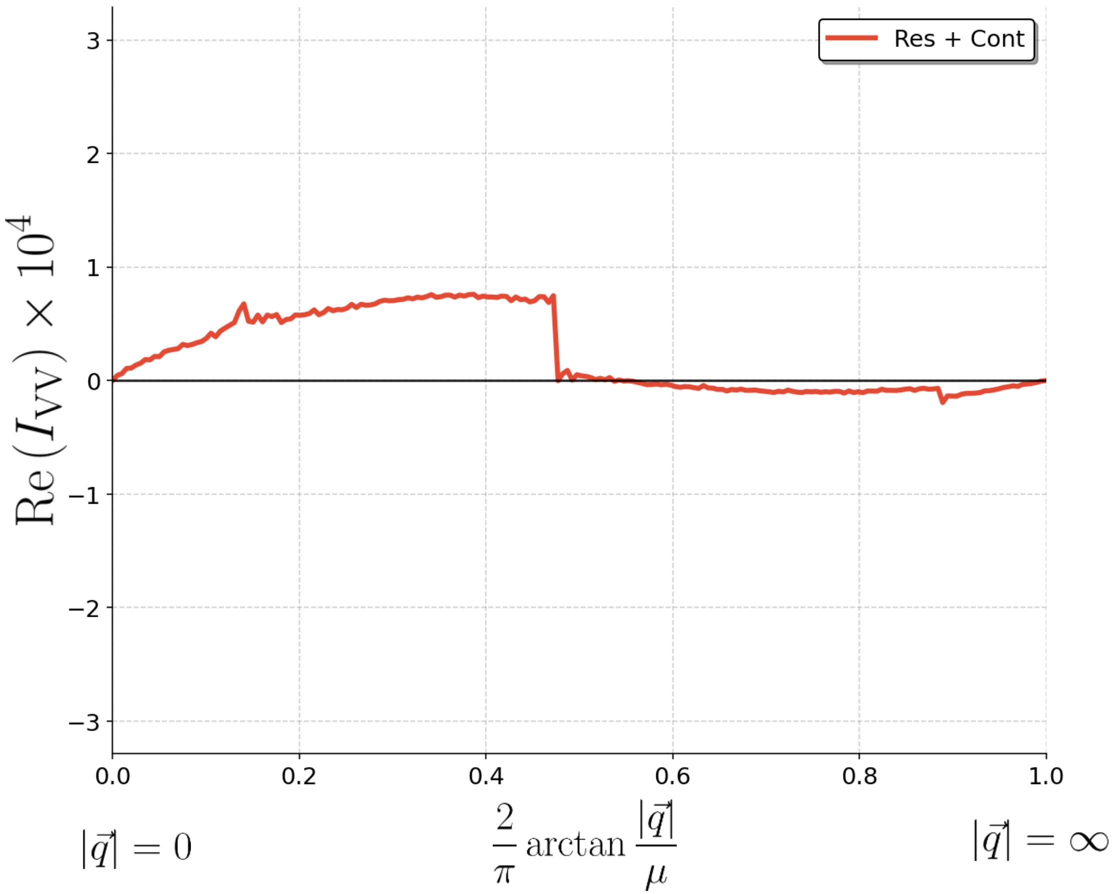
Scalar triangle into triangle



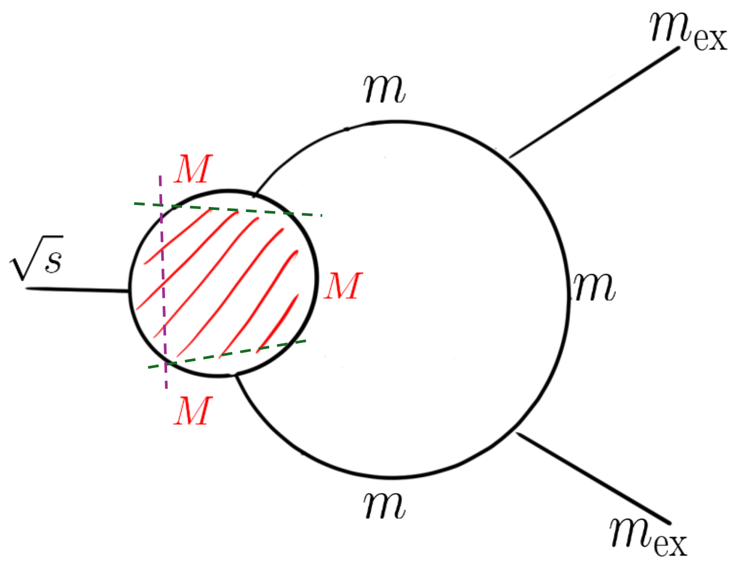
$$(\sqrt{s}, M, m, m_{\mathrm{ex}}) = (12, 3, 10, 2)$$

Cancellation in the sum between residue and continuum

Integrand in $\vec{q}$-space: Continuum + Residues



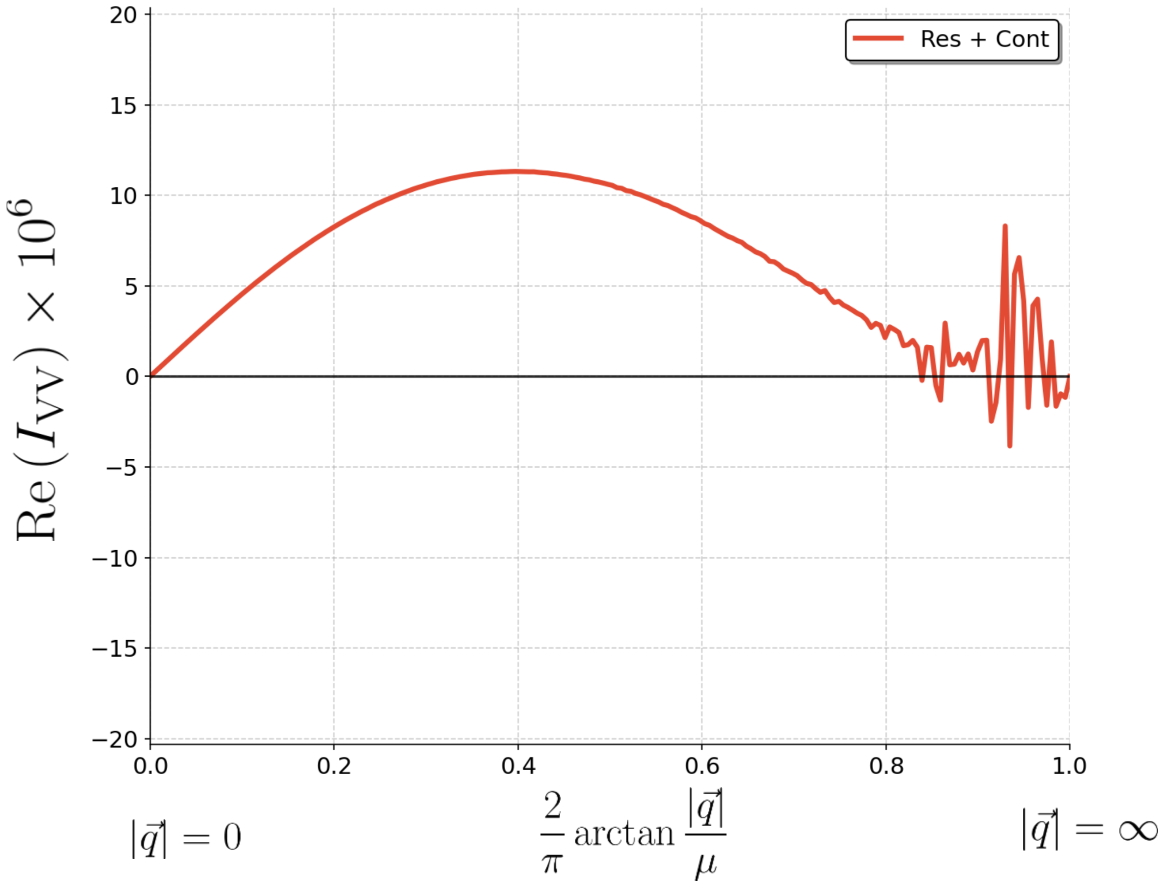
Scalar triangle into triangle



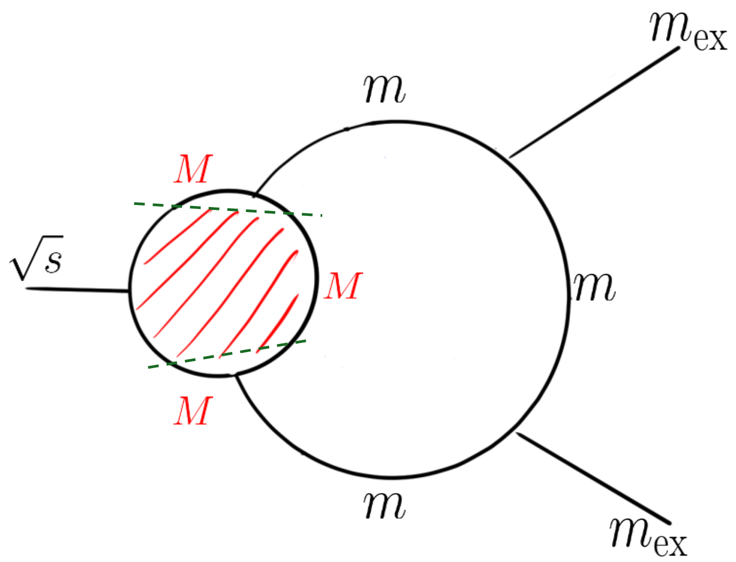
$$\left(\sqrt{s}, M, m, m_{\mathrm{ex}}\right) = (12, 3, 10, 2)$$

Well behaved integrand, **ready for MC integration**

Statistical fluctuations at large $|\vec{q}|$



Scalar triangle into triangle

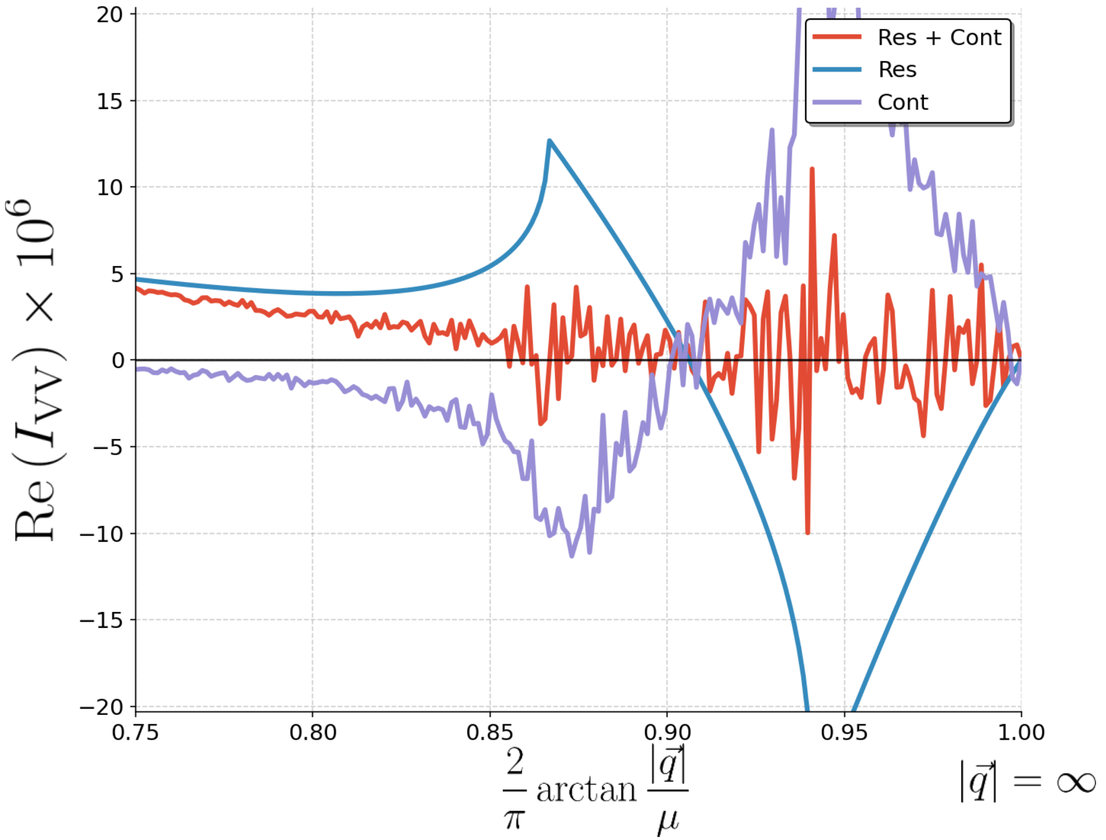


$$(\sqrt{s}, M, m, m_{\mathrm{ex}}) = (8, 10, 5, 2)$$

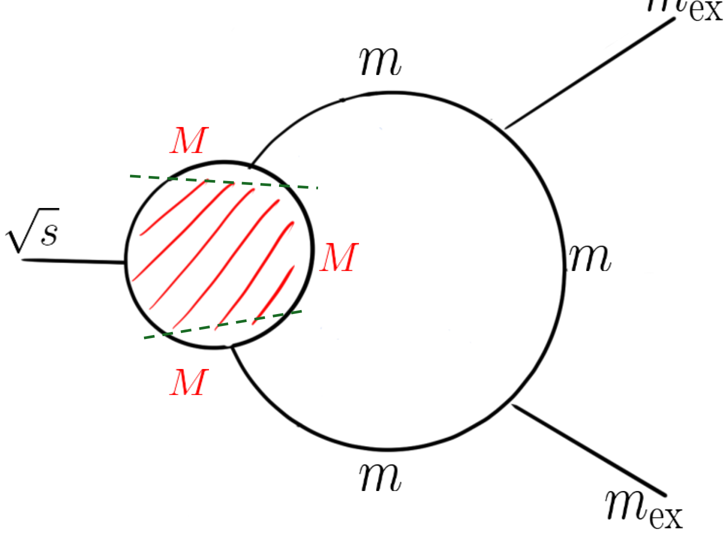
Stability: smooth integrand in most of the $\vec{q}$-space

Issue: severe fluctuations in the UV region $\longrightarrow$ reduced convergence of $\vec{q}$ -integration

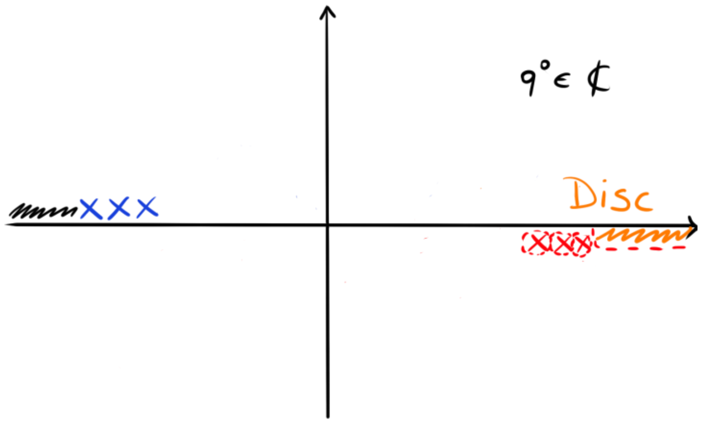
Origin of fluctuations: large res-cont cancellations



Scalar triangle into triangle



$$(\sqrt{s}, M, m, m_{\mathrm{ex}}) = (8, 10, 5, 2)$$

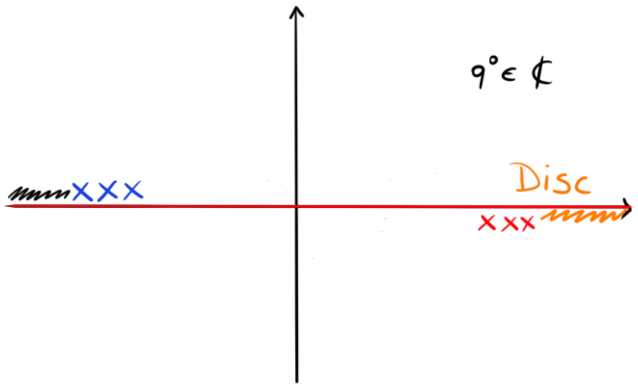


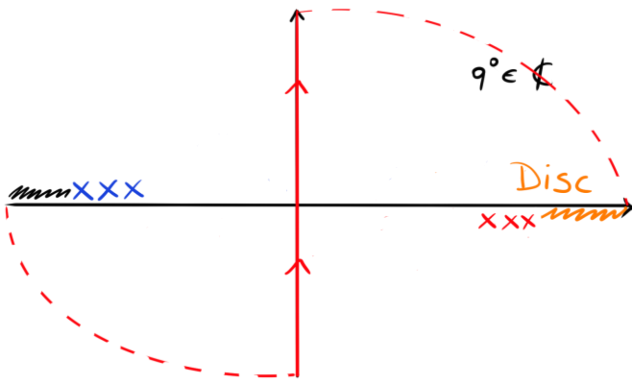
Origin: at $|\vec{q}| \gg \sqrt{s}$ all E_i^+ poles approach the branch point

$$E_i^+ \sim E_{\mathrm{BP}} = |\vec{q}|\left[1 + \mathcal{O}\left(\frac{\sqrt{s}}{|\vec{q}|}\right)\right]$$

Solution: Wick rotation

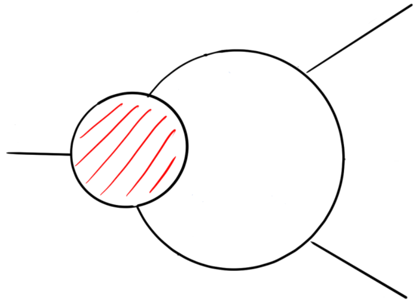
At $|\vec{q}| \gg \sqrt{s}$ all poles and cuts configuration enables Wick rotation





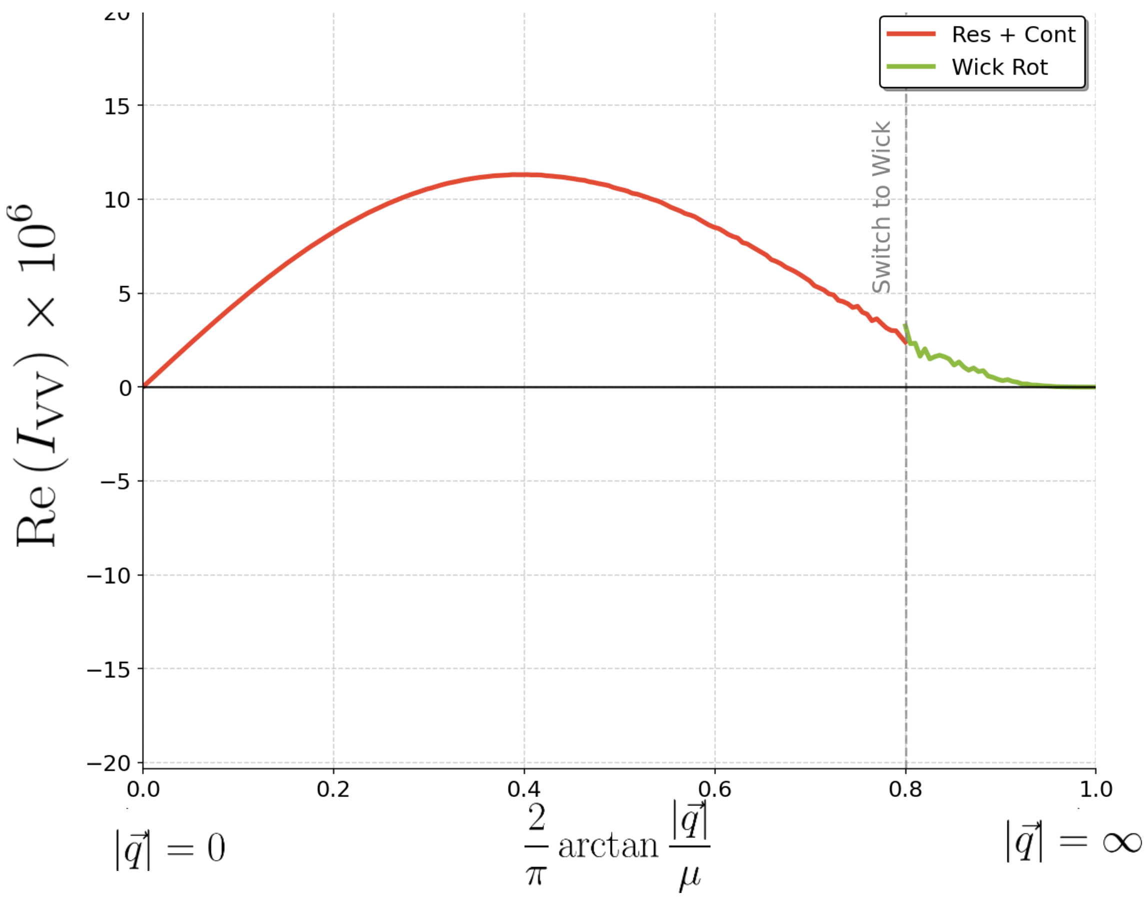
Res+Cont replaced by single (numerical) q^0-integral

$$\mathcal{W} = -\int \mathrm{D}^{D-1}\vec{q} \int_{-\infty}^{\infty} \frac{dq_{\mathrm{E}}^0}{2\pi} \frac{\mathcal{A}(q)}{\prod_i D_i(q)}\bigg|_{q_0 = iq_{\mathrm{E}}^0}$$

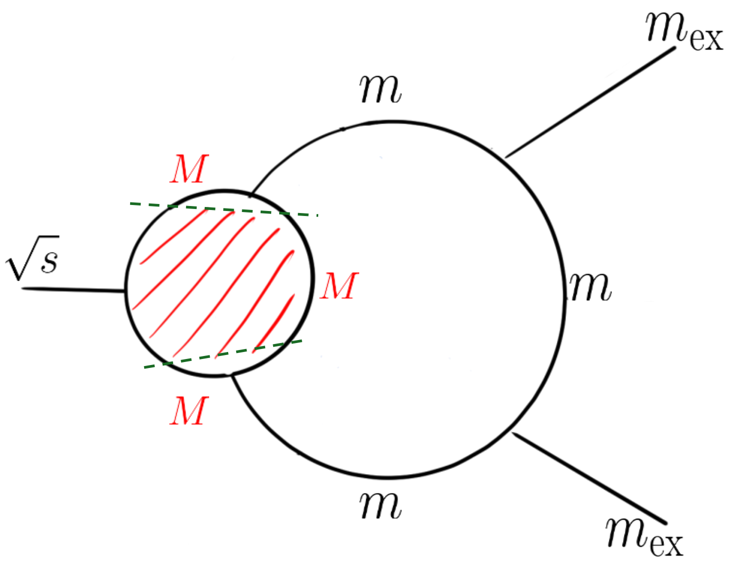


Required extra ingredient: continuation of analytic 1-loop integrals into $q_0 \in \mathbb{C}$ plane [see e.g. Fleischer, Jegerlehner, Tarasov]

Wick Rotation: $\vec{q}$-integrand after q^0-integration



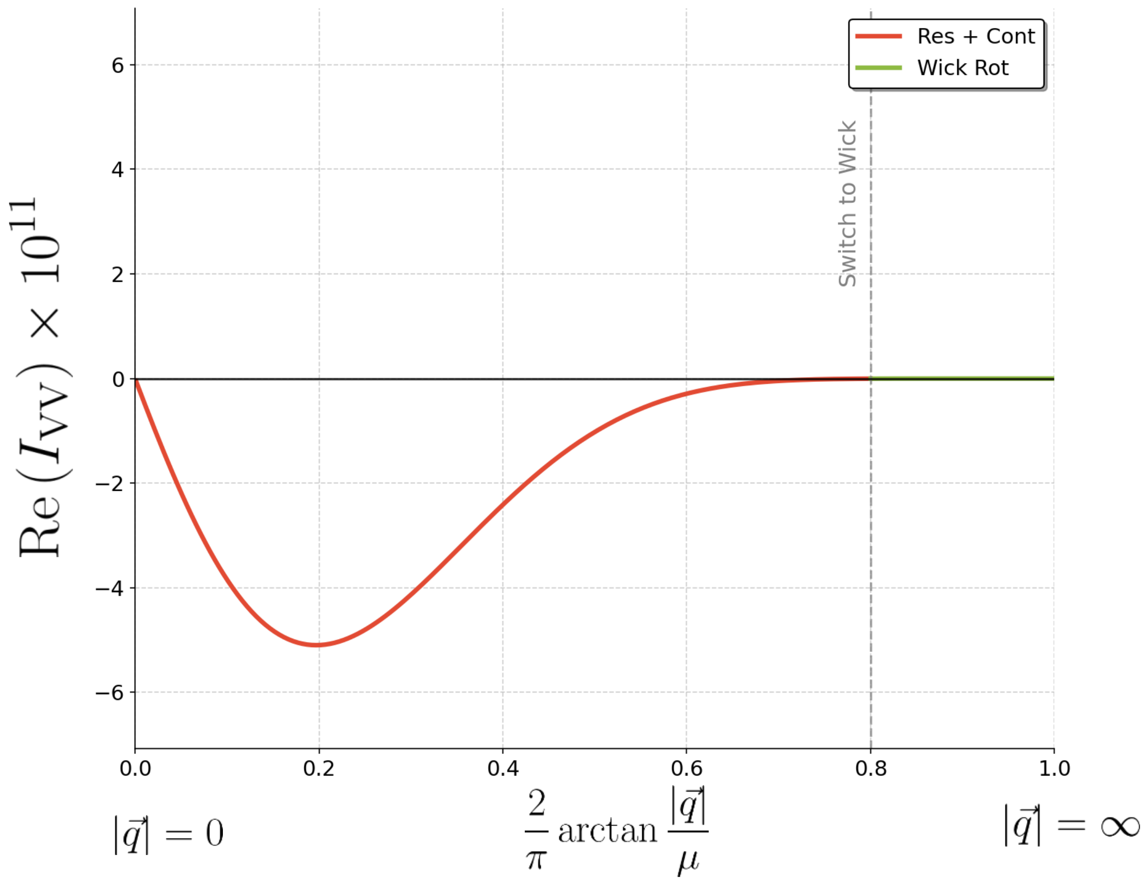
Scalar triangle into triangle



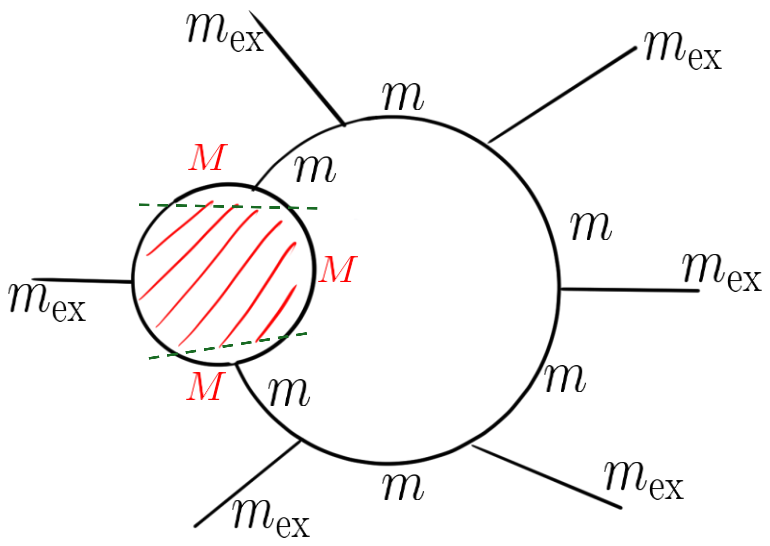
$$\left(\sqrt{s}, M, m, m_{\mathrm{ex}}\right) = (8, 10, 5, 2)$$

Statistical fluctuations in the UV region are **cured** by the Wick rotation

Multi-leg example: $\vec{q}$-integrand



Scalar triangle into hexagon

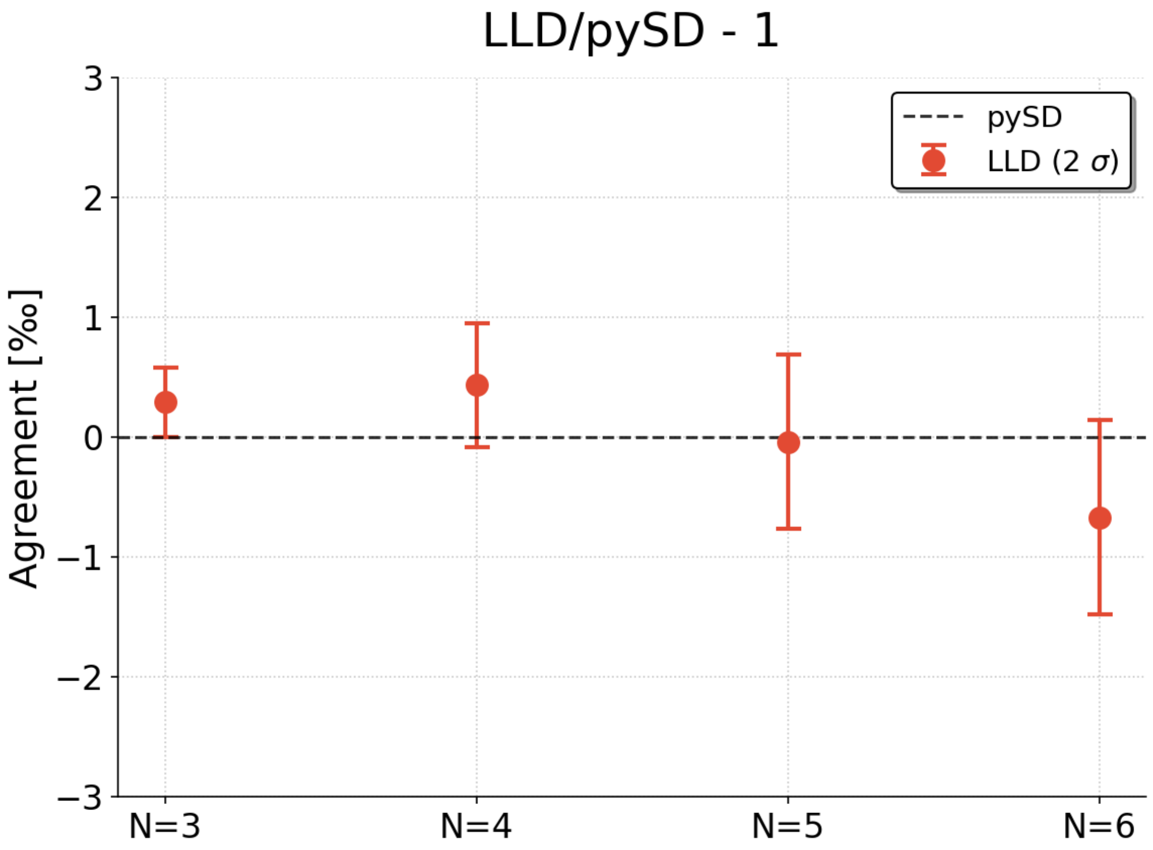


$$\left(\sqrt{s}, M, m, m_{\mathrm{ex}}\right) = (9, 10, 5, 2)$$

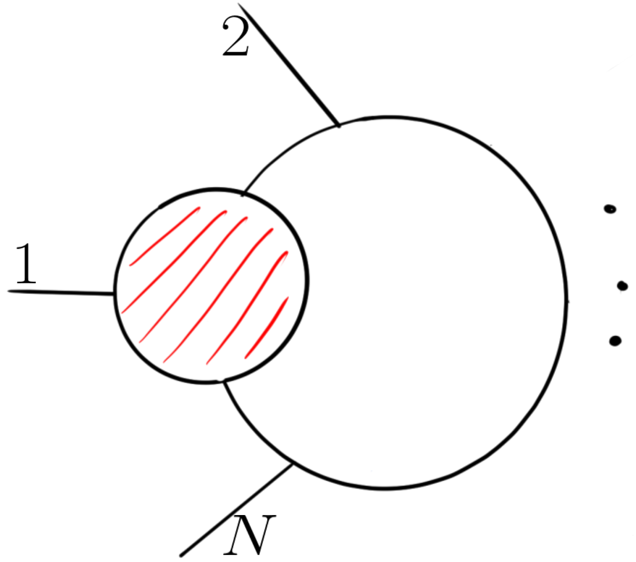
Scalability: the algorithm is applicable to an **arbitrary number** N of legs in the numerical loop

Numerical $\vec{q}$-integration-Comparison against pySecDec

[Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke]



Scalar triangle into N-leg diagram



Sub-per-mille Agreement with O(10^7) $\vec{q}$-evaluations

Relative q-variance $\sqrt{\frac{<I^2>}{<I>^2} - 1} = \mathcal{O}(1)$ ⟶ Efficient MC integration in Born$\otimes\vec{q}$-space

Summary and Outlook

Automated LTD+OpenLoops **NLO** Algorithm

- Numerical loop integrations
- Local UV+threshold+IR cancellations
- Status: automated for e^+e^- processes at NLO QCD

First steps towards NNLO: new **Loop Loop Duality** method

- Combination of numerical and semi-analytical loop integration
- LTD-like algorithm + continuum integrals
- Status: partial automation in OpenLoops with focus on IR-finite 2-loop diagrams

Preliminary numerical investigations indicate quite encouraging convergence

Still a lot of work ahead of us

Summary and Outlook

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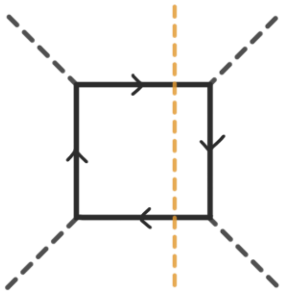
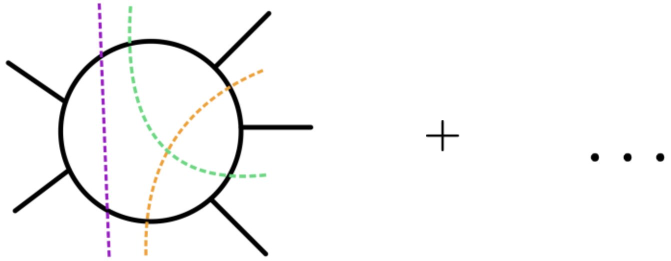
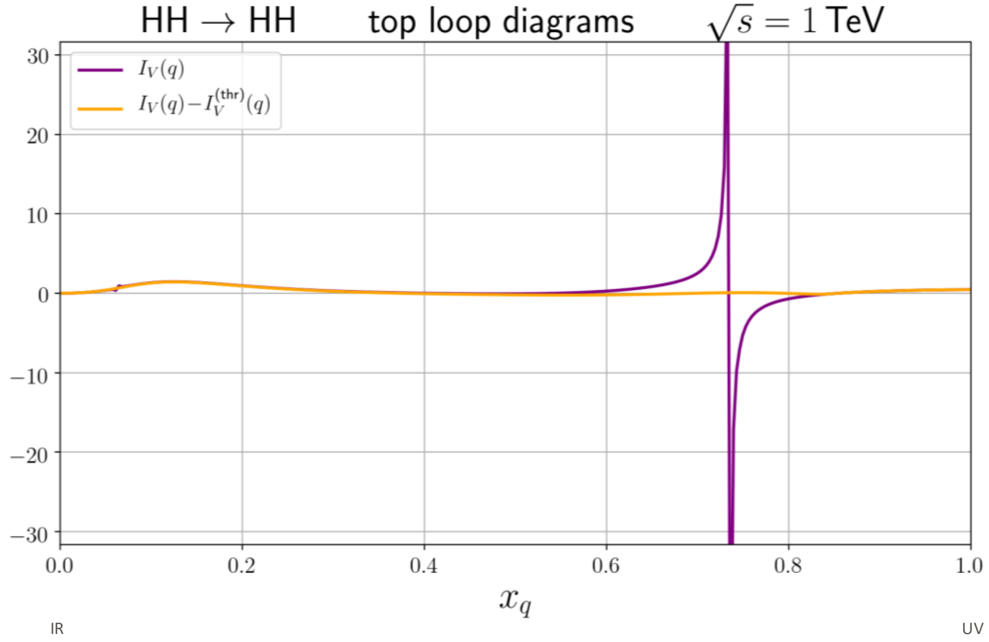
Thank you for your attention

Backup

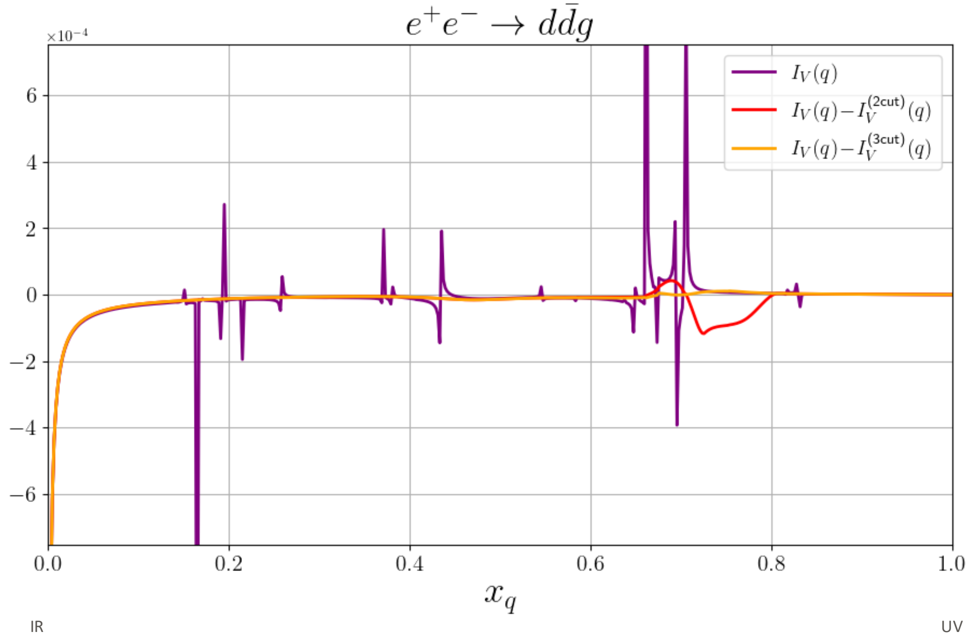
Threshold subtraction at work







$[\, x_\theta = 0.5\,,\ x_\phi = 0.5 \,]$



$[\, x_\theta = 0.5\,,\ x_\phi = 0.1 \,]$

Local cancellation of V+R singularities

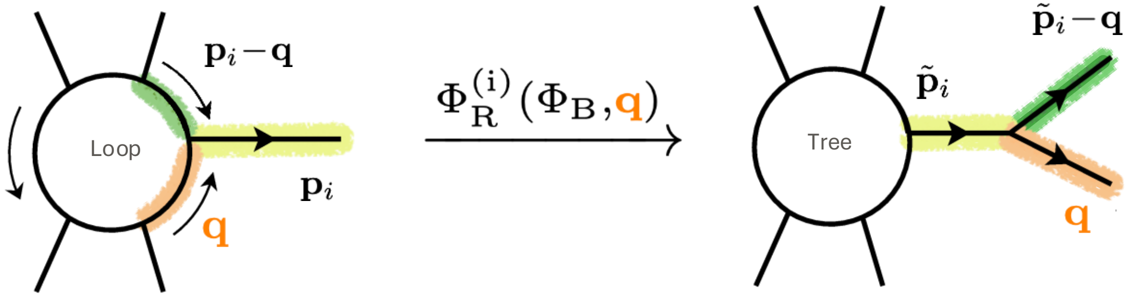
Challenge: each collinear region requires a dedicated mapping $(\Phi_{\rm B}, \mathbf{q}) \to \Phi_{\rm R}(\Phi_{\rm B}, \mathbf{q})$

Global Sectors

Split the virtual and real phase spaces into different collinear secors (à la FKS)

Align phase spaces with Catani-Seymour **mappings**

Identify the loop momentum **q** with real unresolved momentum



Master formula for V+R integration

$$\sigma_{\rm NLO}^{(\rm V+R)} = \int {\rm d}\Phi_{\rm B} \sum_{\alpha} \int {\rm D}^3\mathbf{q} \left[\epsilon_\alpha^{(\rm V)}(\Phi_{\rm V})\, I_{\rm V}(\Phi_{\rm V}) + \epsilon_\alpha^{(\rm R)}(\Phi_{\rm R}^{(\alpha)})\, I_{\rm R}(\Phi_{\rm R}^{(\alpha)}) \right]_{\Phi_{\rm R}^{(\alpha)} = \Phi_{\rm R}^{(\alpha)}(\Phi_{\rm V})}$$

- ☑ **Local IR cancellations** achieved sector-by-sector
- ☑ Enables **MC event generation** & **finite predictions** for any IR-safe observables

MC integration: generation of "virtual" events

Efficient MC integration over the combined virtual $\Phi_\mathrm{V} = (\Phi_\mathrm{B}, \mathbf{q})$ space

$$I_\mathrm{V} \equiv \sum_{\mathrm{diagr}} \left[I_\mathrm{V}^{(\mathrm{bare})} - I_\mathrm{V}^{(\mathrm{UV})} - I_\mathrm{V}^{(\mathrm{thr})} + I_\mathrm{V}^{(\mathrm{frc})} \right]$$

$$\sigma_\mathrm{NLO}^{(\mathrm{V})} = \int \mathrm{d}\Phi_\mathrm{V}\, I_\mathrm{V}(\Phi_\mathrm{V}) = \int \mathrm{d}\Phi_\mathrm{B} \int \mathrm{D}^3\mathbf{q}\, I_\mathrm{V}(\Phi_\mathrm{B}, \mathbf{q})$$

$$= \int \mathrm{d}\Phi_\mathrm{B}\, I_\mathrm{B}(\Phi_\mathrm{B}) \int \mathrm{D}^3\mathbf{q}\, \frac{I_\mathrm{V}(\Phi_\mathrm{B}, \mathbf{q})}{I_\mathrm{B}(\Phi_\mathrm{B})}$$

Much smoother!

$$= \int \mathrm{d}\Phi_\mathrm{B}\, I_\mathrm{B}(\Phi_\mathrm{B}) \int_{[0,1]^3} \mathrm{d}^3\mathbf{x}\, w_\mathrm{MC}(\mathbf{q}(\mathbf{x}))\, R_{\frac{\mathrm{V}}{\mathrm{B}}}(\Phi_\mathrm{B}, \mathbf{q}(\mathbf{x}))$$

Key aspects:

- **Factorisation** of Born contribution
- Generation of **unweighted Born sample**

- "*Default*" MC integration: **Born sampling** + **single** random **q-point per Born event**

$$\sigma_\mathrm{NLO}^{(\mathrm{V})} = \sigma_\mathrm{B} \frac{1}{N_\mathrm{B}} \sum_{i=1}^{N_\mathrm{B}} w_\mathrm{MC}(\mathbf{q}^{(i)})\, R_{\frac{\mathrm{V}}{\mathrm{B}}}(\Phi_\mathrm{B}^{(i)}, \mathbf{q}^{(i)})$$

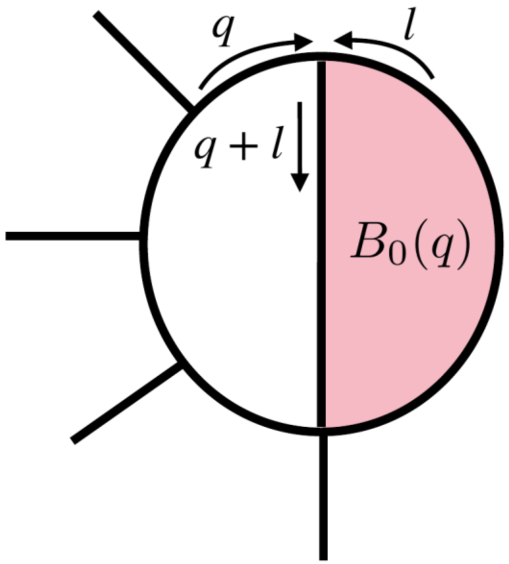
- "*Improved*" MC integration: **Born sampling** + **Loop sampling** at fixed Born point

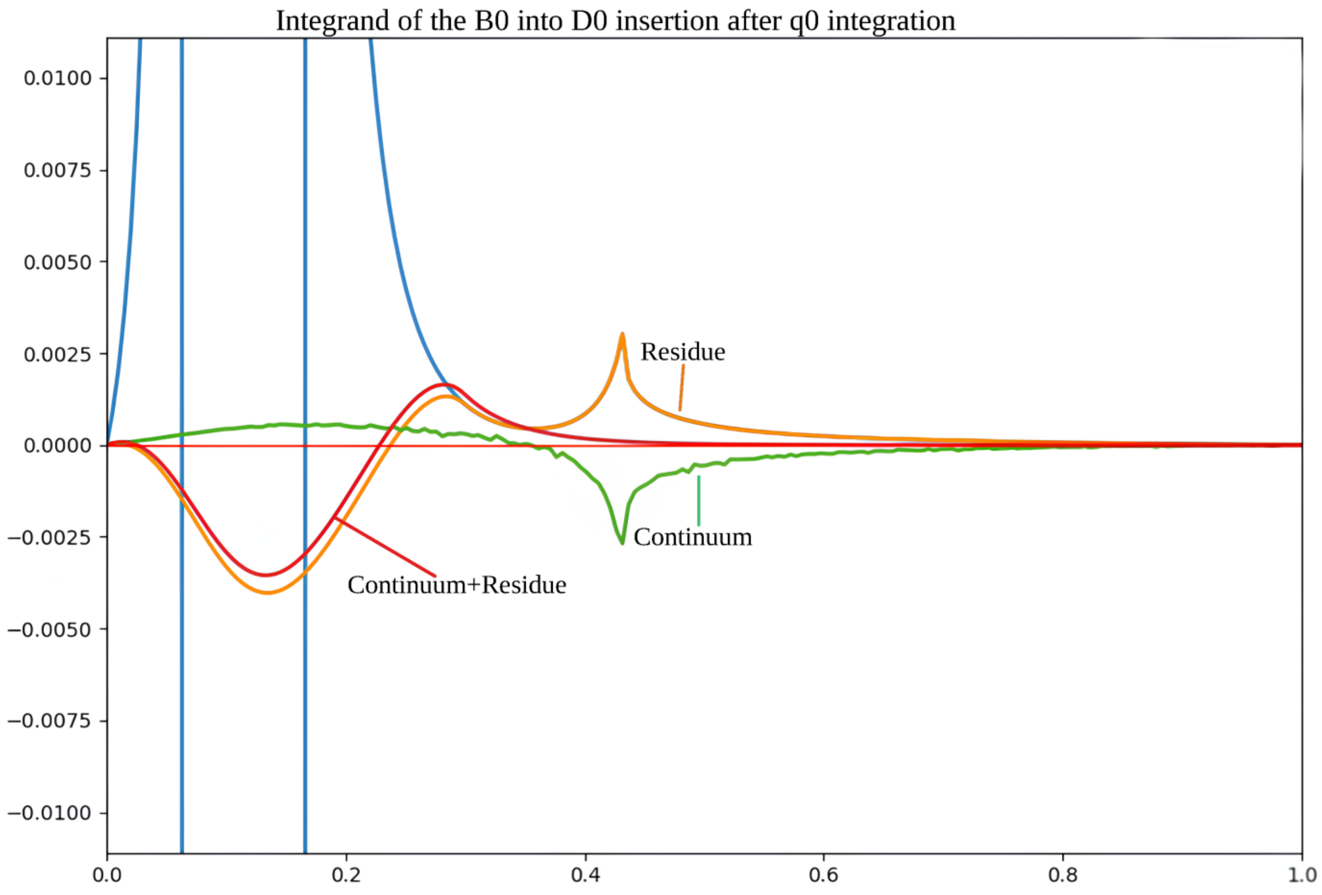
$$\sigma_\mathrm{NLO}^{(\mathrm{V})} = \sigma_\mathrm{B} \frac{1}{N_\mathrm{B}} \sum_{i=1}^{N_\mathrm{B}} \frac{1}{N_\mathbf{q}} \sum_{j=1}^{N_\mathbf{q}} w_\mathrm{MC}(\mathbf{q}^{(ij)})\, R_{\frac{\mathrm{V}}{\mathrm{B}}}(\Phi_\mathrm{B}^{(i)}, \mathbf{q}^{(ij)})$$

Improved MC convergence
with multiple q evaluations!

- Very fast for $N_\mathbf{q} \leq 10-100$ in OL
- Stat. Uncertainties recduce by up to a factor $\sqrt{N_\mathrm{q}}$

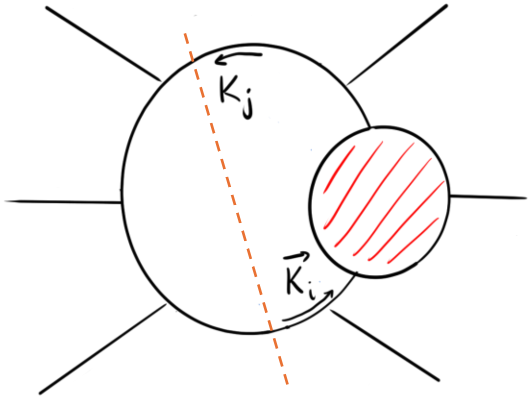
Integrand in $\vec{q}$-space: B0 insertion



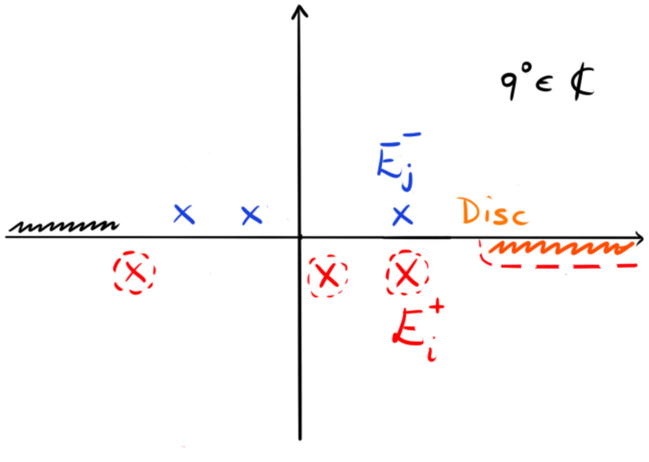


$|\vec{q}| = 0$ $\qquad \frac{2}{\pi} \arctan \frac{|\vec{q}|}{\mu}$ $\qquad |\vec{q}| = \infty$

Numerical Threshold Singularities



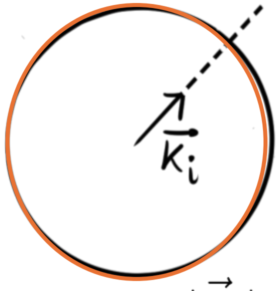
Similar $E_i^+ \to E_j^-$ singularity structure as in one loop case, with 1-loop numerator $\mathcal{A}_{ij}(k_i)$



$$W_{ij}^{\text{thr}} = \int \frac{\mathrm{D}^{D-1}\vec{k}_i}{2\epsilon_i} \frac{\mathcal{A}_{ij}(k_i)}{(E_i^+ - E_j^+)(E_i^+ - E_j^- - i0)}$$

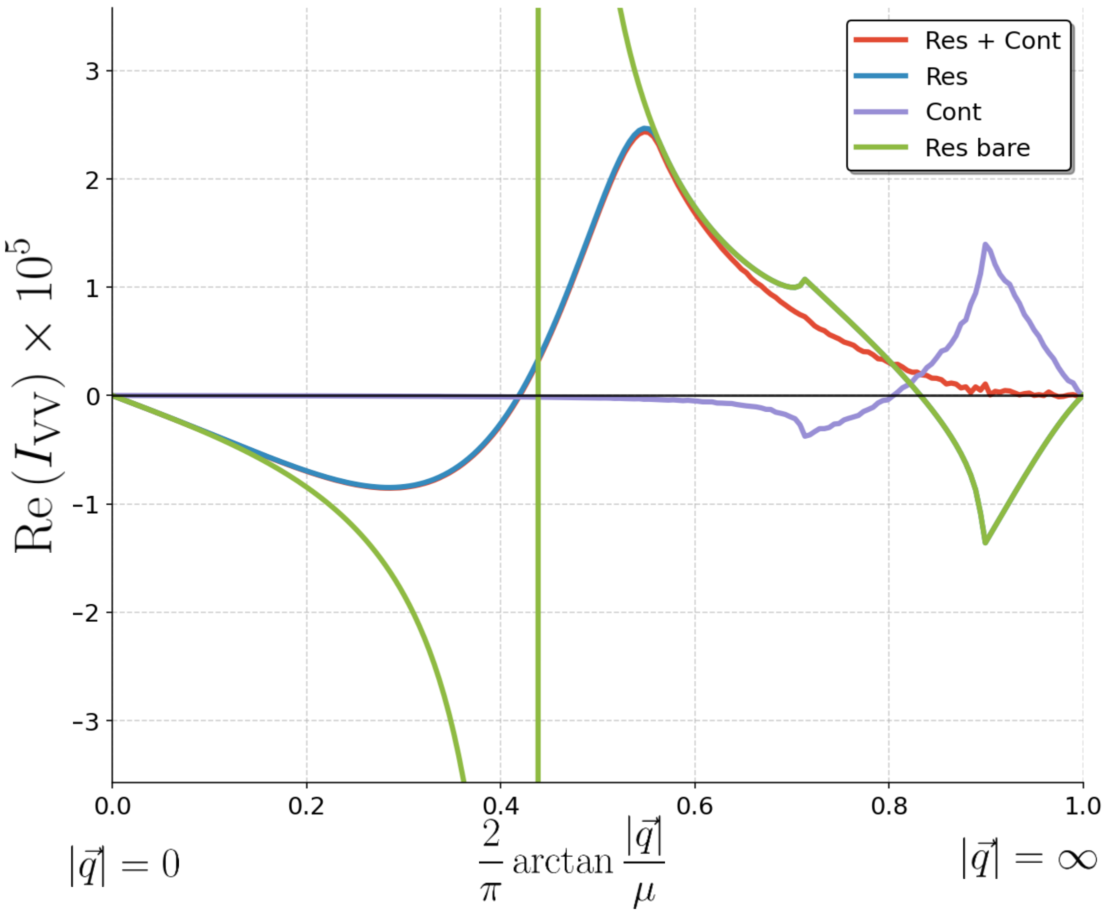
Local Subtraction: using same 1-dim subtraction with simple integrated subtraction terms as in 1-loop case

In the CM-frame: $$W_{ij}^{(thr)} = \int d\Omega_i \int_{m_i}^{\infty} d\epsilon_i \underbrace{\frac{1}{\epsilon_i - \epsilon_i^{cut} - i0}}_{\mathrm{PV}\left(\frac{1}{\epsilon_i - \epsilon_i^{cut}}\right) + i\pi\delta(\epsilon_i - \epsilon_i^{cut})} \left[f_{ij}(\epsilon_i, \Omega_i) - \rho(\epsilon_i) f_{ij}(\epsilon_i^{cut}, \Omega_i)\right]$$

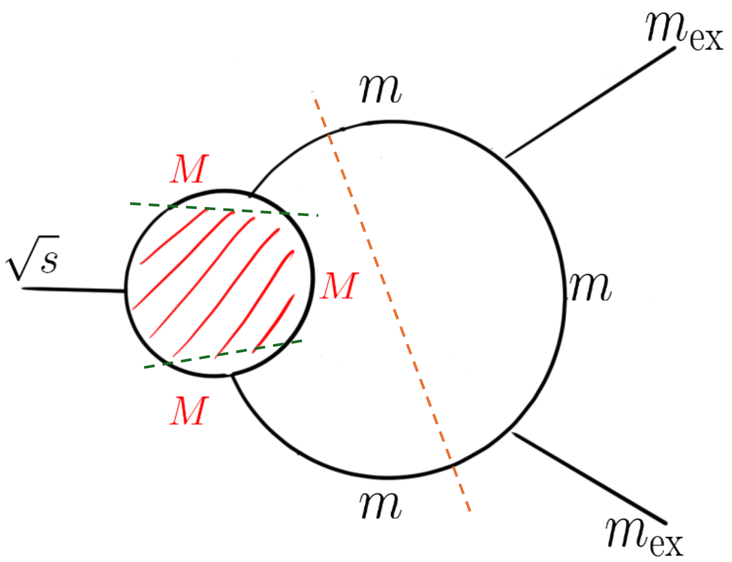


$|\vec{k}_i| = |\vec{k}_{i,\text{cut}}|$

Numerical threshold singularities: $\vec{q}$-space integrand



Scalar triangle into triangle



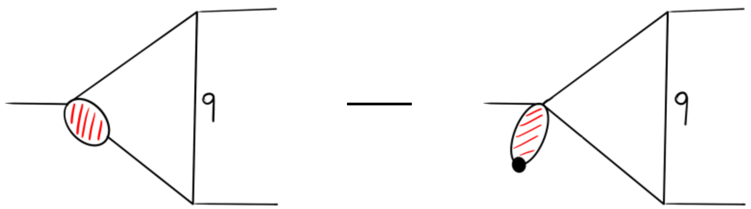
$$(\sqrt{s}, M, m, m_{\mathrm{ex}}) = (12, 10, 5, 2)$$

Threshold subtraction: removes singularity enabling numerical $\vec{q}$-integration (room for optimisation)

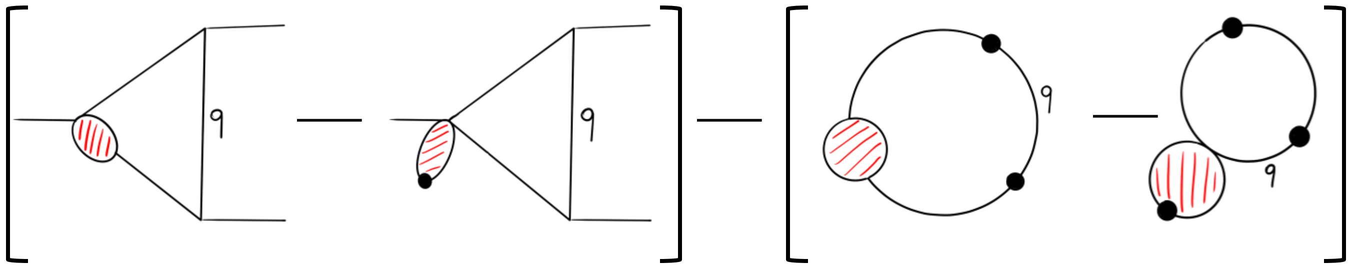
Local Subtraction of UV Singularities at 2-Loops

Local subtraction of UV singularities via R-operation and tadpole expansion [Chetyrkin, Misiak, Münz]

(1) Subtraction of 1-loop sub-divergencies ⟶ $$I^{(1)}_{\mathrm{tad}} = \int \mathrm{D}^D q \frac{\mathcal{U}_1(q)}{\prod_i D_i(q)} \int \mathrm{D}^D \ell \frac{\mathcal{T}(\ell)}{(\ell^2 - \mu^2_{\mathrm{tad}})^b}$$



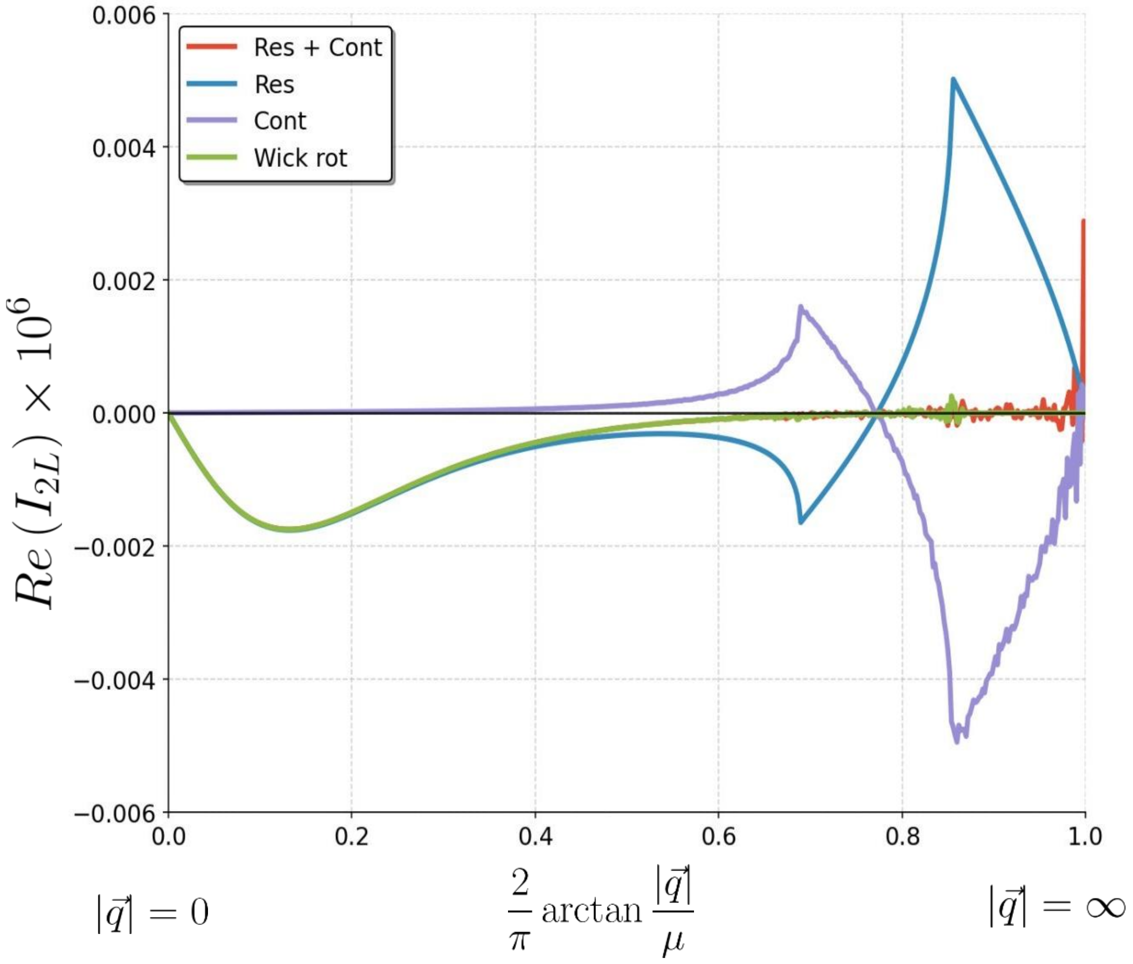
(2) Subtraction of two-loop global divergencies ⟶ $$I^{(2)}_{\mathrm{tad}} = \int \mathrm{D}^D q \int \mathrm{D}^D \ell \frac{\mathcal{U}_2(q,\ell)}{(q^2 - \mu^2_{\mathrm{tad}})^a (\ell^2 - \mu^2_{\mathrm{tad}})^b ((q+\ell)^2 - \mu^2_{\mathrm{tad}})^c}$$



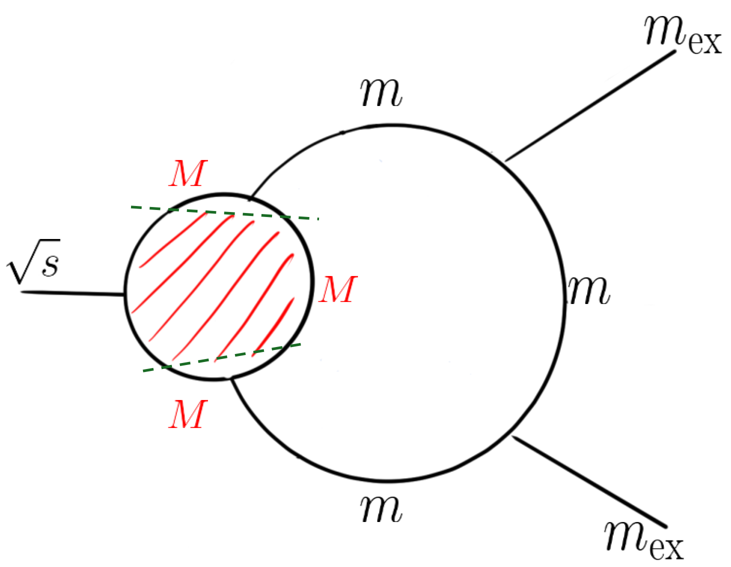
- UV-subtracted diagram **converted into LLD** representation and numerically integrated in 3-dim
- Analytically integrated tadpoles added back and matched to UV renormalisation constants

Tensor integrals: simple example

Can be treated with similar algorithm as for scalar integrals



rank-1 triangle into triangle



$$(\sqrt{s}, M, m, m_{\text{ex}}) = (8, 10, 5, 2)$$

Integration with O(10^7) -evaluations yields **sub-per-mille agreement with pySecDec**