A Stochastic Gravitational Wave Background

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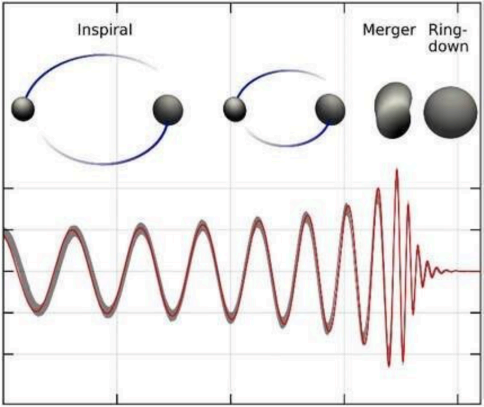
Outline

Introduction

Since about a decade we are observing GWs from inspiraling binaries. Besides presenting a novel test of GR, these GW signals have already taught us a lot about compact binary systems.



LIGO 2015

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- **Phase transitions in the early Universe**
- Reheating
- **Inflation**
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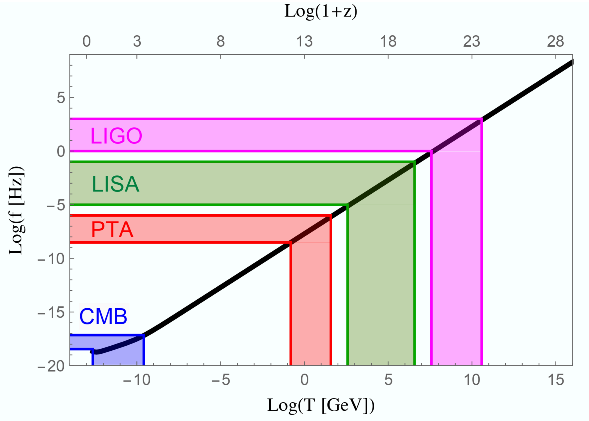
In this talk I shall mainly show that both, generation of gravitational waves after inflation (causal generation) and generation during inflation, lead to two (different) **Universal spectra** of GWs at large scales.



(from Caprini & Figueroa 2018)

Phase transitions and other 'causal' stochastic generation mechanisms of GWs

(Caprini, RD, 2006, see also Pi & Sasaki, 2020)
To generate gravitational waves we need significant tensor anisotropic stress.

$$ds^2 = a^2(t)\left(-dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right) \qquad h^i_i = \partial_j h^j_i = 0$$

$$\ddot{h}_{ij} + 2\mathcal{H}\dot{h}_{ij} + k^2 h_{ij} = 16\pi G a^2 \Pi_{ij}\,.$$

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The anisotropic stress Π_{ij} can be generated by violent processes like phase transition or at second order by large scalar fluctuations on small scales. However, these resulting anisotropic stresses are uncorrelated on super horizon scales,

$$\langle \Pi_{ij}(t,\mathbf{x})\Pi_{lm}(t,\mathbf{y})\rangle = 0 \qquad \text{for} \;\; |\mathbf{x}-\mathbf{y}| > t\,.$$

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Therefore, the power spectrum of Π is analytic at $\mathbf{k}=0$.

$$\langle \Pi_{ij}(t,\mathbf{k})\Pi_{\ell m}(t,\mathbf{k}')\rangle = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')\sum_{\lambda=\pm} e^{\lambda}_{ij}e^{\lambda}_{\ell m}P_{\Pi,\lambda}(k)$$

Causality requires that $P_{\Pi,\lambda}(k)$ starts generically as white noise, $\propto k^0$.
On large scales, $kt \ll 1$, we may neglect the k^2 term in the GW equation and solve it with a purely time dependent Green's function,

$$h(t,\mathbf{k}) = 16\pi G\int_{t_{\rm in}}^{t} dt'\,G(t,t')a^2(t')\Pi(t',\mathbf{k}) \qquad \text{for} \quad |\mathbf{k}t| \ll 1\,.$$

We assume that the gravitational wave source is active from $t_{\rm in}$ up to some time $t_{\rm end}$, after which we can set $\Pi_{ij}(t,\mathbf{k}) \sim 0$. For $t > t_{\rm end}$ and $kt_{\rm end} \ll 1$ the GW solution is therefore of the form

$$h(t,\mathbf{k}) = A_1(\mathbf{k})h_1(t,k) + A_2(\mathbf{k})h_2(t,k)$$

where $h_{1,2}$ are the homogenous solutions of the GW equation and

$$A_{1,2} \simeq k^{-1}\int_{t_{\rm in}}^{t_{\rm end}} dt\, f_{1,2}(t_{\rm end},t)\Pi(t,\mathbf{k})\,.$$

Form this we draw the first important conclusion:

The gravitational wave energy spectrum from causal stochastic sources in the early Universe is always white noise on large scales

$$\langle \dot h(t,\mathbf{k})\dot h(t,\mathbf{k}')\rangle = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')P_{\dot h}(k,t)\,, \qquad k^3P_{\dot h}(k,t) \propto k^3$$

so that

$$\frac{d\Omega_{GW}}{d\log k} = \frac{k^3P_{\dot h}(k,t)}{(2\pi)^3G\rho_c a^2} \propto k^3\,, \qquad \Omega_{GW} = \int_0^\infty \frac{dk}{k}\frac{d\Omega_{GW}}{d\log k}$$

Phase transitions and other 'causal' stochastic generation mechanisms

Another important conclusion is: The frequency of GWs from a stochastic, statistically homogeneous source is given by the **wave number** of the source and not by its (usually much slower) time dependence.
Comparing to a localised oscillating source:

$$h(t,\mathbf{x}) = 4G\int d^3x'\,\frac{\Pi(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,.$$

If the source has a fixed frequency ω, i.e., $\Pi(t,\mathbf{x}) = e^{i\omega t}\Pi(\mathbf{x})$ and it is small, then this is the frequency of the GW.

$$\begin{aligned} h(t,\mathbf{x}) &= 4Ge^{i\omega t}\int d^3x'\,e^{i\omega|\mathbf{x}-\mathbf{x}'|}\,\frac{\Pi(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \\ &\simeq 4G\frac{e^{i\omega(t-r)}}{r}\int d^3x'\,e^{i\mathbf{k}\mathbf{x}'}\,\Pi(\mathbf{x}') = 4G\frac{e^{i\omega(t-r)}}{r}\Pi(\mathbf{k})\,, \\ \text{with} \quad & \mathbf{k} = \omega\widehat{\mathbf{x}}\,. \end{aligned}$$

The frequency of the GW equals the frequency of the source.

On the other hand, if the source is stochastic and extends over all of space, it is better to work directly in Fourier space. Neglecting for simplicity the expansion of the Universe, we have

$$h(t,\mathbf{k}) = 16\pi G \int_{t_{\rm in}}^{t} dt'\,\Pi(t',\mathbf{k})\frac{\sin(k(t-t'))}{k}$$

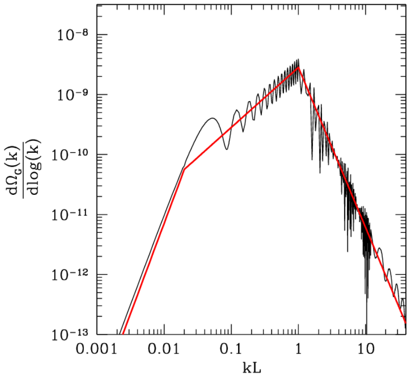
After $t_{\rm fin}$,

$$h(t,\mathbf{k}) = A\sin(kt) + B\cos(kt), \qquad \begin{array}{l} A = 16\pi G \int_{t_{\rm in}}^{t_{\rm fin}} dt'\,\Pi(t',\mathbf{k})\cos(k(t-t')) \\ B = 16\pi G \int_{t_{\rm in}}^{t_{\rm fin}} dt'\,\Pi(t',\mathbf{k})\frac{\sin(k(t-t'))}{k} \end{array}$$

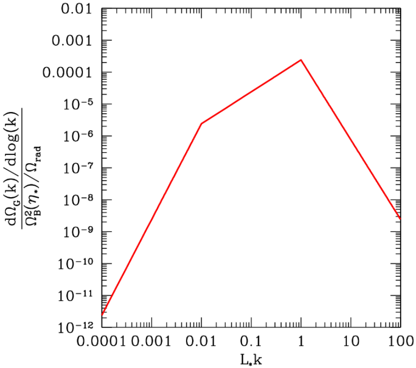
The frequency of the GW is determined by the wave number of the source.

(See Caprini, RD, Sturani, 2006 for more details.)

Examples (from Caprini, RD, 2006)



From turbulence



From magnetic fields

The coherence scale L or L_* is typically a large fraction of $t_{\rm in}$ often $L \sim t_{\rm in}/10$, so that the typical frequency of the generated GWs is $f/(2\pi) = k \sim 10/t_{\rm in}$.

During radiation domination $t \simeq 10^{10}$sec $(1\text{MeV}/T) = 10^5$ sec $(100\text{GeV}/T)$.

From the electroweak transition we expect GWs dominating the the mHz range (LISA),

while for the QCD transition we expect GWs in the range of 10^{-7}Hz (PTAs).

Particle production during inflation

The quadratic action of a bosonic field with a light mass, $M \ll H$ during inflation can usually be written in the form

$$S = -\frac{1}{2}\int d^4x(\partial_\mu v \partial^\mu v + m^2(t)v^2)\,.$$

Here the index μ is raised with the Minkowski metric and $m^2(t)$ depends on the coupling of the field to the expansion of the Universe.

Up to normalisation, the variable v is usually related to the original field via the scale factor. For example for tensor perturbations, h, of the metric, GWs

$$v = \frac{m_P a}{\sqrt{8\pi}}h \qquad \text{and} \qquad m^2(t) = -\ddot{a}/a \simeq -(2+3\epsilon)/t^2\,.$$

The last $\simeq$ sign is valid for slow roll inflation. This tachyonic mass leads to particle creation of the mode k when $|t| \searrow 1/k$. For gravitational waves we obtain the well know GW spectrum from slow roll inflation (Starobinskii, 1979)

$$\mathcal{P}_h(k) = \frac{1}{2\pi^2}k^3 P_h = \frac{H^2}{m_p^2}\left(\frac{k}{\mathcal{H}}\right)^{-2\epsilon}\,.$$

Secondary GW production during inflation

(Following Teuscher, RD, Martineau & Barrau, 2512.14670)

But not only tensor perturbations of the metric or inflaton perturbation, but also all other (bosonic) fields that are not conformally coupled are generated during inflation. Generically such a fields ϕ leads to an anisotropic stress that is quadratic in the field,

$$\Pi_{ij} \propto \Psi_i \Psi_j$$

Where Ψ_i is typically a spatial or temporal derivative of a field component.

For simplicity we concentrate on power law or slow roll inflation such that $a \propto |t|^q$, $q = 2/(1+3w) \simeq -1-\epsilon$.

Setting $x = kt$ and denoting a derivative w.r.t. x by a $'$ we can write the GW evolution eqn.

$$(ah)'' + \left(1 - \frac{a''}{a}\right)(ah) = \frac{16\pi}{m_p^2 k^2} a^3 \Pi(k,t)$$

with solution

$$\mathcal{P}_h(k,t) = \frac{128}{m_p^4}\frac{1}{ka^2}\int_{x_i}^{x}\int_{x_i}^{x} G(x,y)G(x,z)a^3(y)a^3(z)P_\Pi\left(k,\frac{y}{k},\frac{z}{k}\right)dydz$$

The anisotropic stress of a field generated during inflation

$$\Pi^i_j(\mathbf{k},t) = \Lambda^{i\,m}_{j\,\ell}(\hat{\mathbf{k}})T^\ell_m(\mathbf{k},t)\,, \qquad \Lambda^{i\,m}_{j\,\ell} \text{ is the TT projection operator.}$$

$$T^i_j(\mathbf{k},t) = \int d^3p\,\Psi^i(\mathbf{p},t)\Psi_j(\mathbf{p}-\mathbf{k},t) + \ldots$$

In order to have the right dimension and to depend only on physical quantities the power spectrum of Ψ_i must be of the form

$$k^3 P_\Psi(k,t,t') = \frac{k^n}{a(t)^{n/2}a(t')^{n/2}} M^{4-n}$$

Using Wick's theorem we then obtain for the anisotropic stress power spectrum

$$P_\Pi(k,t,t') = \int \frac{d^3p}{(2\pi)^3} P_\Psi(p,t,t') P_\Psi(|\mathbf{k}-\mathbf{p}|,t,t') f(\theta,\varphi)$$

The anisotropic stress of a field generated during inflation

The resulting anisotropic stress power spectrum depends on the value of n :

(i) $n \geq 3/2$: P_Π is dominated by the UV cutoff and has a white noise spectrum,

$$P_\Pi(k) \propto \Lambda_{UV}^{2n-3} \,.$$

(ii) $3/2 > n \geqslant 0$: we approximate

$$P_\Pi(k) \simeq \int_0^k dp\, p^2 P_\Psi(p) P_\Psi(k) + \int_k^{\Lambda_{\rm UV}} dp\, p^2 P_\Psi(p) P_\Psi(p) \propto k^{2n-3} \,.$$

(iii) $n < 0$: an infrared cutoff Λ_{IR} must be introduced for the integral to converge. This could e.g. be the largest scale that exited the Hubble horizon during inflation. For $k \gg \Lambda_{IR}$,

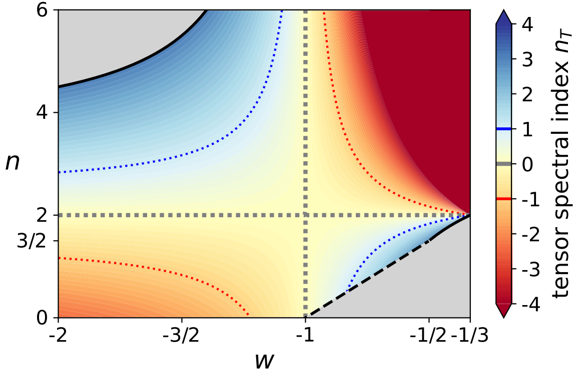
$$P_\Pi(k) \propto k^{n-3} \Lambda_{IR}^n \,.$$

We naturally expect $n \geq 0$ as otherwise additional physics (like the beginning of inflation) will be relevant.

For $n \geq 0$ we can perform the required integrals and obtain

$$\mathcal{P}_h(k,t) \propto \frac{M^{4-2(n-2)} H_{\rm end}^{2q(2-n)}}{M_p^4} k^{2(n-2)(q+1)}\,, \quad q+1 = -\epsilon$$

$$\text{if } n \geq \max\left\{3(1+w)\,, \frac{5+7w}{2(1+w)}\right\}.$$



From Teuscher, RD, Martineau & Barrau, 2512.14670

Secondary GWs induced by a field generated during inflation

For standard slow roll inflation ($w \gtrsim -1$) the above conditions are satisfied and we obtain always obtain a nearly scale invariant spectrum.

$$\mathcal{P}_h(k,t) \propto k^{-2(n-2)\epsilon}$$

Note that only for $n = 3$ this spectrum is equal to the one from direct inflationary production.

For $n = 2$ (e.g. GWs induced from scale invariant scalar perturbations at second order) $\mathcal{P}_h(k,t)$ is always scale invariant.

Contrary to primary GWs generated during inflation, these secondary GWs and not Gaussian. They have an un-suppressed bi-spectrum, $B \sim P^{3/2}$.
This is an important property to distinguish them.

Example: Axion inflation with helical and kinetic coupling to gauge fields

$$\mathcal{L} \supset -\frac{1}{4}(1 + i_1(\phi))F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}i_2(\phi)\widetilde{F}_{\mu\nu}F^{\mu\nu}$$

(see RD, Hollenstein & Jain, 2010; Barnaby, Namba & Peloso, 2011; Caprini & Sorbo, 2014; Fujita & RD, 2019; Teuscher, RD, Martineau & Barrau, 2025 etc.)

The equation of motion for the gauge potential is

$$\ddot{A}^{\pm} + \left(k^2 \mp k\frac{\frac{d}{dt}i_2}{1+i_1} - \frac{\frac{d^2}{dt^2}\sqrt{i_1+1}}{\sqrt{i_1+1}}\right)A^{\pm} = 0$$

Setting $(1 + i_1) \propto |t|^{\gamma_1}$, $0 < \gamma_1 < 4$, and $d/dt(i_2)/(1 + i_1) = \gamma_2/t$ we obtain

$$k^3|B|^2 \propto k^{4+\gamma_1}\,. \qquad k^3|E|^2 \propto k^{4-\gamma_1}\,. \qquad |B^+|^2 - |B^-|^2 \propto \sinh(\pi\gamma_2)\,, \qquad |B|^2 \propto e^{\pi\gamma_2}\,.$$

γ_1 determines the spectral index and γ_2 determines amplitude & helicity.

Example: Axion inflation with helical and kinetic coupling to gauge fields

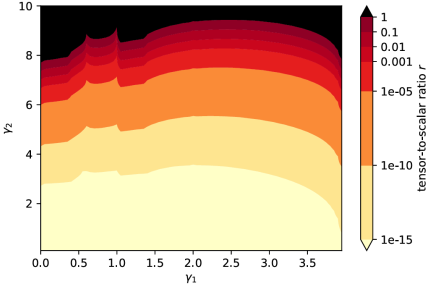
Primordial electromagnetic fields, amplified out of the vacuum by non-conformal couplings like in the Ratra model (Ratra, 1988). In this case their stress tensor reads

$$T_{ij} = E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}(E^2 + B^2)$$

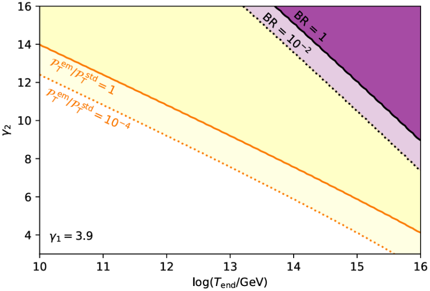
The exact spectral index n of the anisotropic stress depends on the model. For axion inflation with helical and small kinetic coupling it is $n \sim 4$ leading to a helical GW background with power spectrum

$$\mathcal{P}_h \propto k^{-4\epsilon}$$

Example: Axion inflation with helical and kinetic coupling to gauge fields



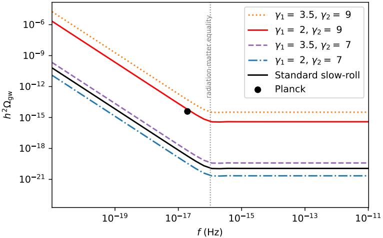
tensor-to-scalar ratio
Exponential growth with γ_2

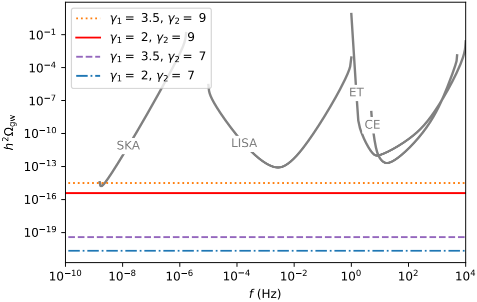


Backreaction
mainly from the axion eom.

(From Teuscher, RD, Martineau & Barrau, 2510.00869)

Standard inflation: $P_h \propto \left(\frac{H_i}{M_P}\right)^2$ EM induced: $P_h \propto \exp(\pi\gamma_2)\left(\frac{H_i}{M_P}\right)^4$





(From Teuscher, RD, Martineau & Barrau, 2510.00869)

Conclusions

- We have seen that GWs induced by stochastic sources of anisotropic stress after inflation, typically have a white noise spectrum, $d\Omega_{GW}/d\log k \propto k^3$.
- Also, it is the *spatial* Fourier components of the source that determine the frequency spectrum after the disappearence of the source (via $k = \omega$), not the temporal spectrum of the source.
- Arbitrary bosonic fields amplified during inflation generate a spectrum of secondary GWs. For slow roll inflation this spectrum is nearly scale invariant. For general power law inflation $\mathcal{P}_k \propto k^{2(n-2)(q+1)} \propto k^{-2(n-2)\epsilon}$ where n is the spectral index of the generating anisotropic stress and $a \propto |t|^q \propto |t|^{-1-\epsilon}$.
- These secondary GWs are typically not Gaussian and have large bispectra.