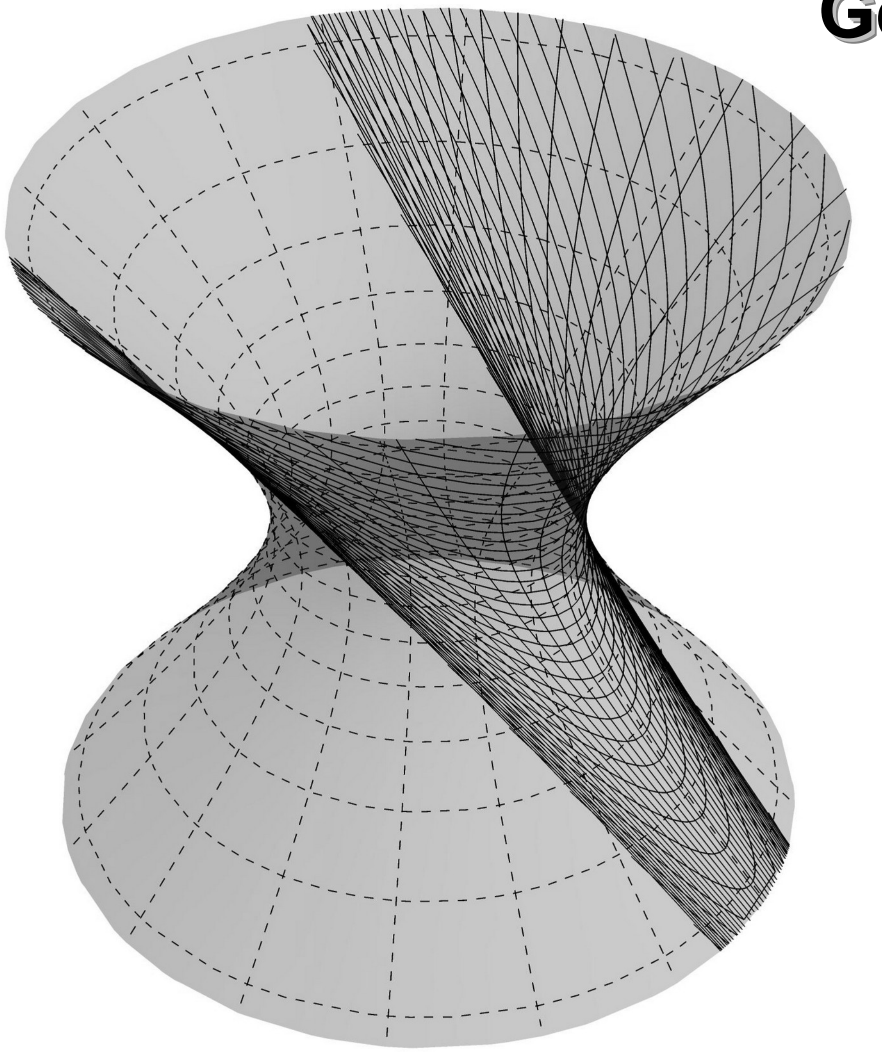
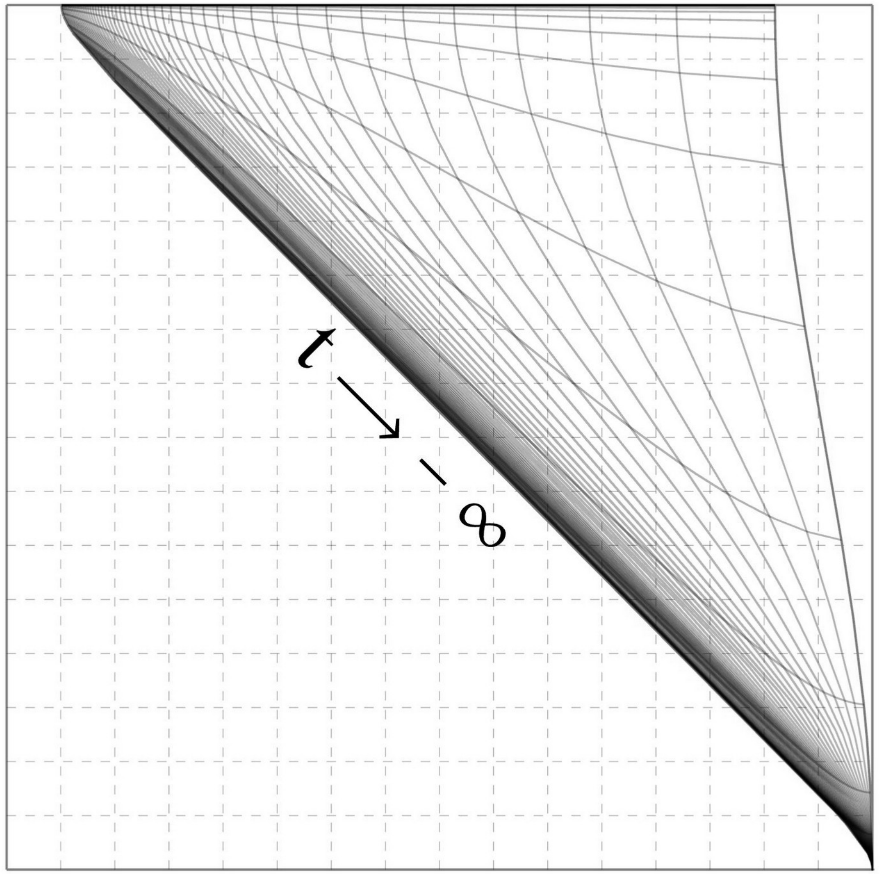
Geodesic Completeness in General Cosmological Scenarios

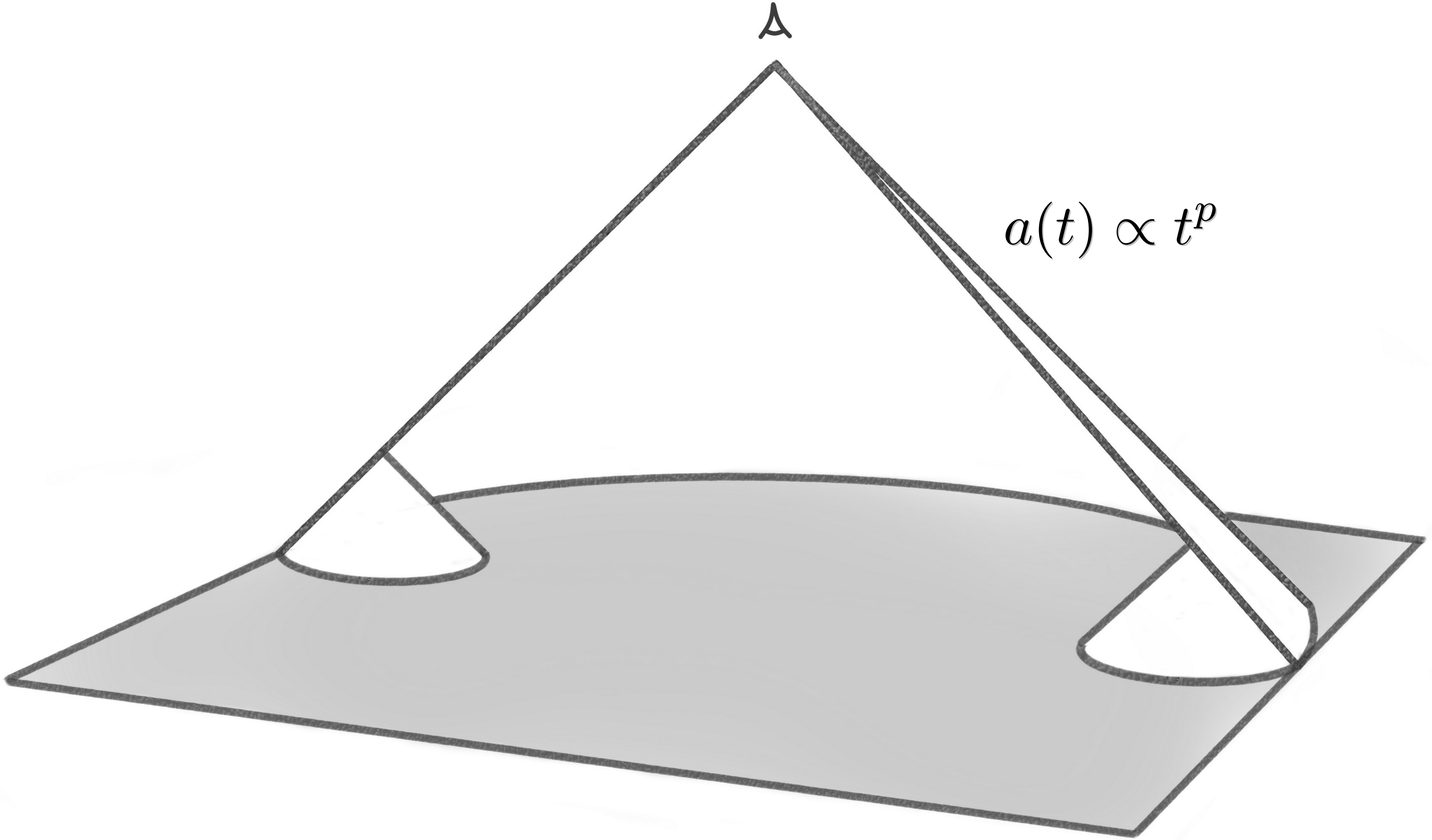




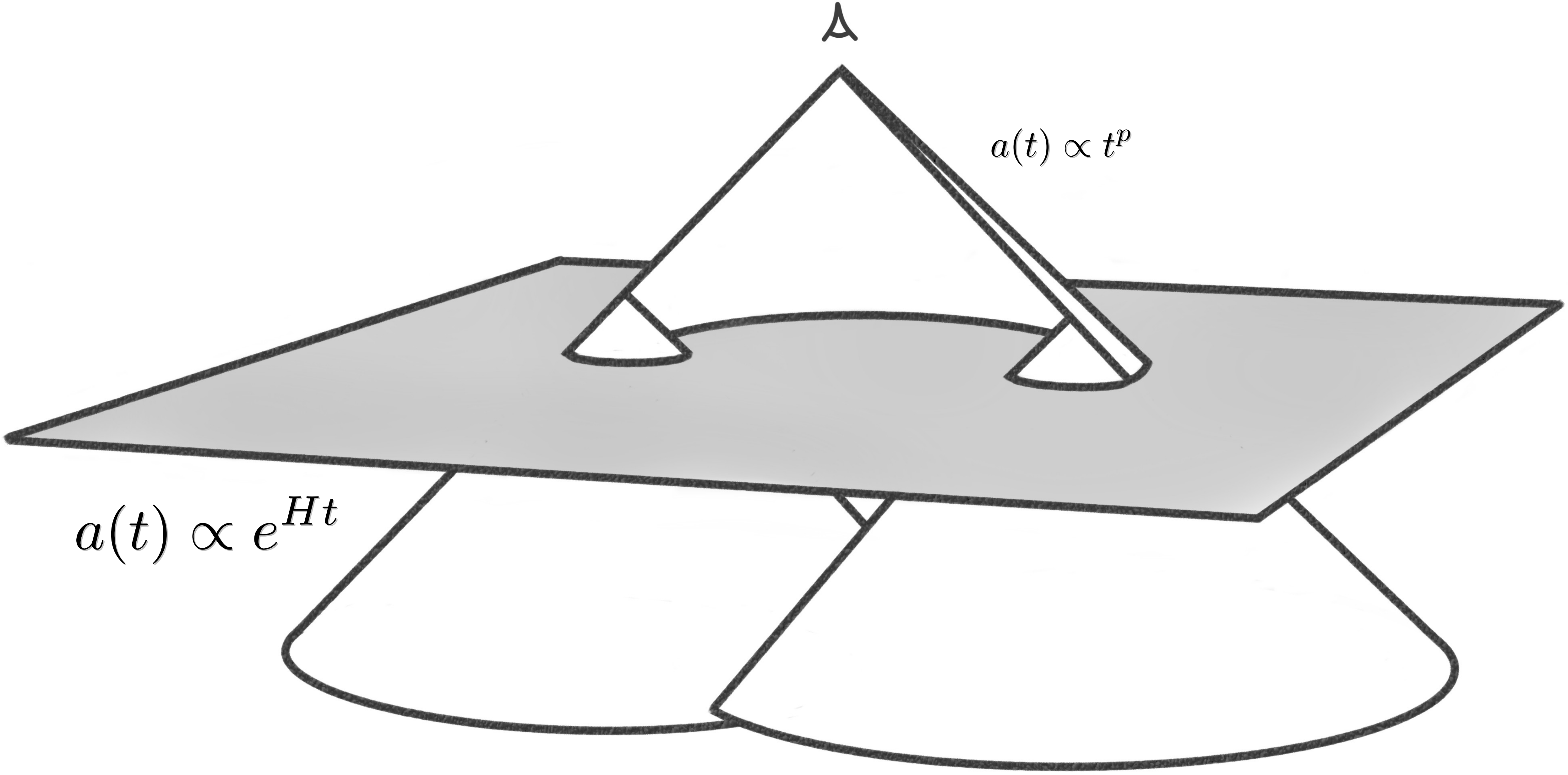
Will Kinney

Particle Physics in the Early Universe
Paris/Orsay
September 2026

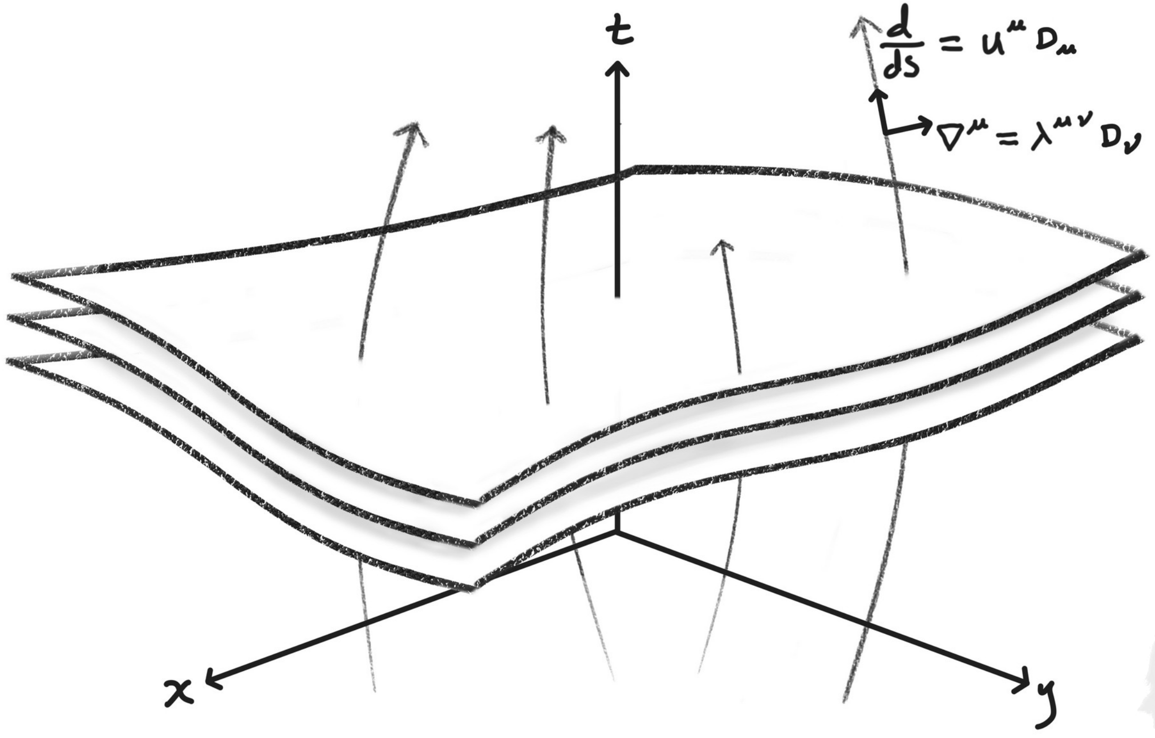
Standard Big Bang: Initial Singularity



Can inflation be past eternal?



Geodesic Completeness



A space is *geodesically complete* iff all geodesic paths are both future- and past-infinite.

$$\int_{-\infty}^{t} ds \to \infty, \qquad \int_{t}^{\infty} ds \to \infty$$

Simple example: de Sitter space

Geodesic equation: $\frac{d}{ds}\left[a^2(t)\,\frac{d\mathbf{x}}{ds}\right] = 0 \qquad ds^2 = dt^2 - e^{2Ht}d\mathbf{x}^2$

$$\left(\frac{dt}{ds}\right)^2 = 1 + v_0^2 a^{-2}(t)\,, \qquad v_0 = \text{const.}$$

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Comoving Observer: $v_0 = 0 \Rightarrow ds = dt$

$$\Delta s = \int ds = \int_{-\infty}^{0} dt \to \infty$$

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$$\Delta s=\int_{-\infty}^{0}\frac{dt}{\sqrt{1+v_0^2e^{-2Ht}}}=\frac{1}{2H}\ln\left(\frac{\sqrt{1+v_0^2}+1}{\sqrt{1+v_0^2}-1}\right)$$

finite for $v_0\neq 0$

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Lorentz boost

BGV Theorem (Arbitrary FRW Space)

Now consider arbitrary $H(s)$:

$$\int_{s_i}^{s_f} H ds = \int_{t_i}^{t_f} H \frac{ds}{dt} dt$$

$$= \int_{t_i}^{t_f} \frac{1}{a} \left(\frac{da}{dt} \right) \frac{1}{\sqrt{1 + v_0^2/a^2}} dt$$

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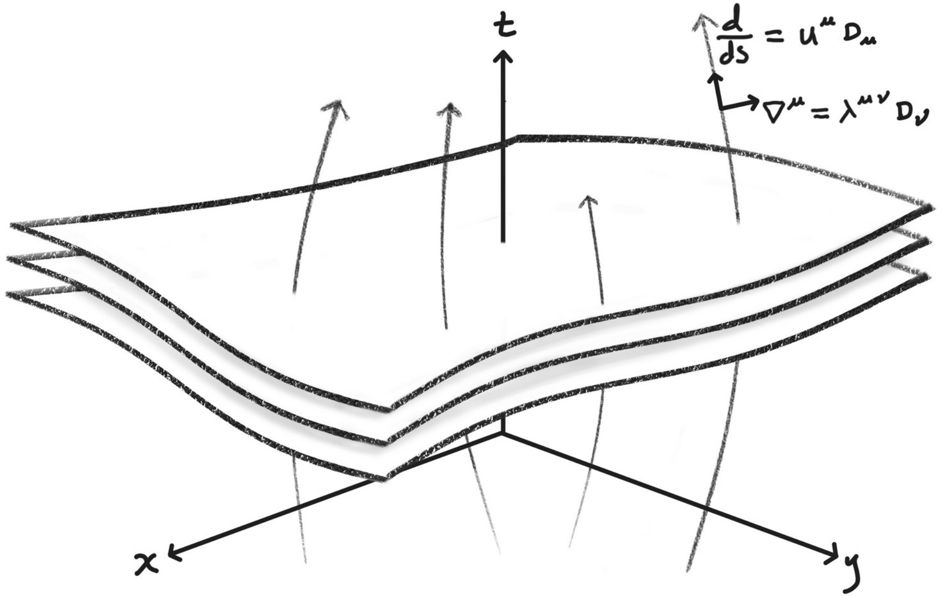
$$= \int_{t_i}^{t_f} \frac{1}{a} \left(\frac{da}{dt} \right) \frac{1}{\sqrt{1 + v_0^2/a^2}} dt$$

$$= \int_{a_i}^{a_f} \frac{da}{\sqrt{a^2 + v_0^2}}$$

$$= \frac{1}{2} \ln \left[\frac{\sqrt{v_0^2 + a^2} + a}{\sqrt{v_0^2 + a^2} - a} \right] \Bigg|_{a_i}^{a_f} = \frac{1}{2} \ln \left(\frac{\gamma_0 + 1}{\gamma_0 - 1} \right)$$

Finite!

The BGV Theorem



Average Expansion:

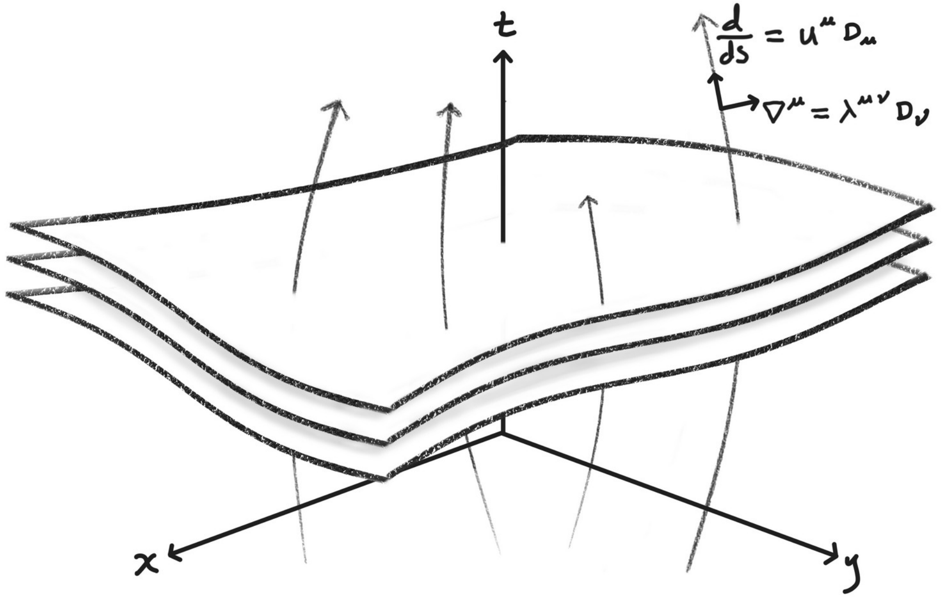
$$\mathcal{H}_{\rm av} \equiv \frac{1}{\Delta s}\int H ds$$

If $\mathcal{H}_{\rm av}$ is *finite* and *positive*, then:

Finite! ⟶ $$\Delta s = \int ds = \frac{1}{\mathcal{H}_{\rm av}}\int H ds$$

(Borde, Guth, & Vilenkin arXiv:gr-qc/0110012)

A Counterexample



Average Expansion:

$$\mathcal{H}_{\rm av} \equiv \frac{1}{\Delta s} \int H ds$$

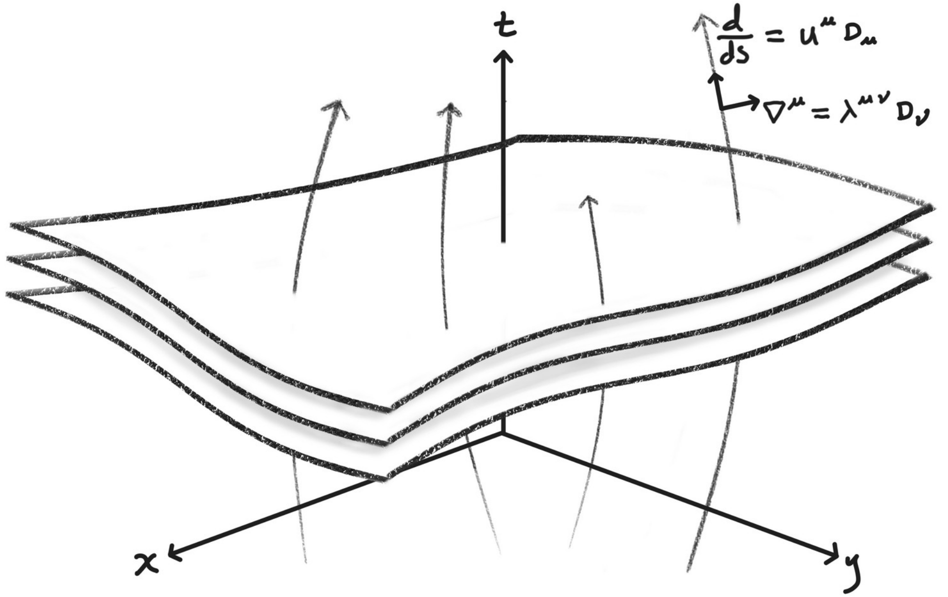
Inflation after an initial loitering phase:

$$a(t) = C + e^{Ht} \qquad \mathcal{H}_{\rm av} > 0$$

Geodesically past-complete!

(Ellis & Maartens arXiv:gr-qc/0211082
Lesnefsky, Easson, & Davies arXiv:2207.00955)

A Counterexample



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Inflation after an initial loitering phase:

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Geodesically past-complete!

$\dot{H} > 0$

Violates NEC!

(Ellis & Maartens arXiv:gr-qc/0211082
Lesnefsky, Easson, & Davies arXiv:2207.00955)

BGV Theorem Redux

If there is a $\Delta > 0$ such that

$$\mathcal{H}_{\mathrm{av}} \equiv \frac{1}{s_f - s_0} \int_{s_0}^{s_f} H ds \geq \Delta$$

for all $s_0 \in [s_i, s_f]$ then the space is geodesically incomplete on $[s_i, s_f]$.

(Garcia-Saenz, Hua, Zhao, arXiv:2405.04062)

Null Geodesic Completeness

BGV Theorem: Null Geodesics

Consider a null geodesic congruence, $v^\mu v_\mu = -\gamma^2 + g_{ij} v^i v^j = 0$,

So we can write an effective "Lorentz boost" as $\lambda_{ij} v^i v^j = \gamma^2 = (v^\mu u_\mu)^2$

Then for affine parameter λ:

$$\begin{aligned}
\frac{d\gamma}{d\lambda} &= -\frac{d}{d\lambda}(v^\mu u_\mu) = -v^\mu \frac{du_\mu}{d\lambda} \\
&= -v^\mu v^\nu u_{\mu;\nu} = -\frac{1}{3}\Theta \lambda_{\mu\nu} v^\mu v^\nu \\
&= -\frac{1}{3}\gamma^2 \Theta.
\end{aligned}$$

BGV Theorem: Null Geodesics

Integrating the local expansion rate along the geodesic gives:

$$\frac{1}{3}\int \Theta d\lambda = -\int_{\lambda_i}^{\lambda_f} \frac{d\gamma}{\gamma^2} = \frac{1}{\gamma}\bigg|_{\lambda_i}^{\lambda_f}$$

Since $\gamma > 0$ for any null geodesic, this is automatically finite!

Example: FRW Space

For $ds^2 = dt^2 - a^2\left(t\right)d\mathbf{x}^2$, the affine length along a null geodesic is:

$$d\lambda \propto a\left(t\right)dt$$

and

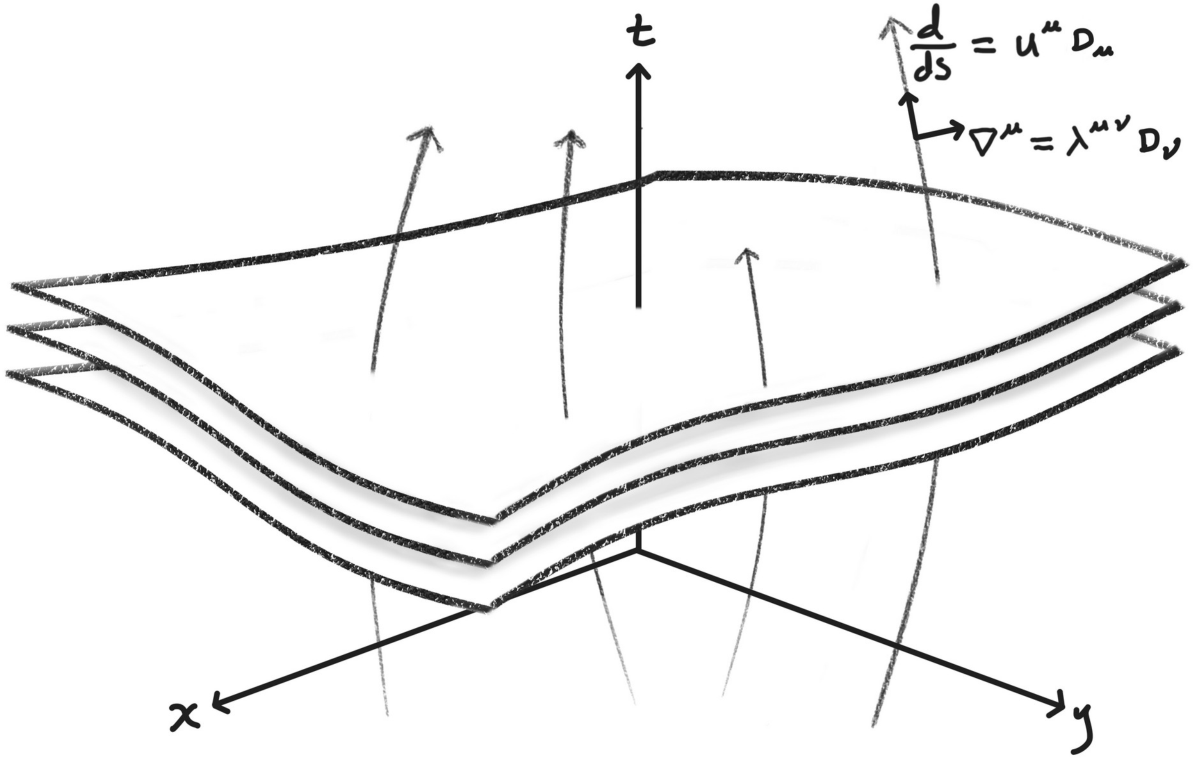
$$\gamma = v^{\mu}u_{\mu} = \frac{dt}{d\lambda} = a\left(t\right)$$

Integrating the expansion rate along the geodesic then gives:

$$\int H d\lambda = a\left(\lambda_f\right) - a\left(\lambda_i\right) = a\left(\lambda_f\right)$$

A Coordinate-Independent BGV Theorem

Geodesic Congruences

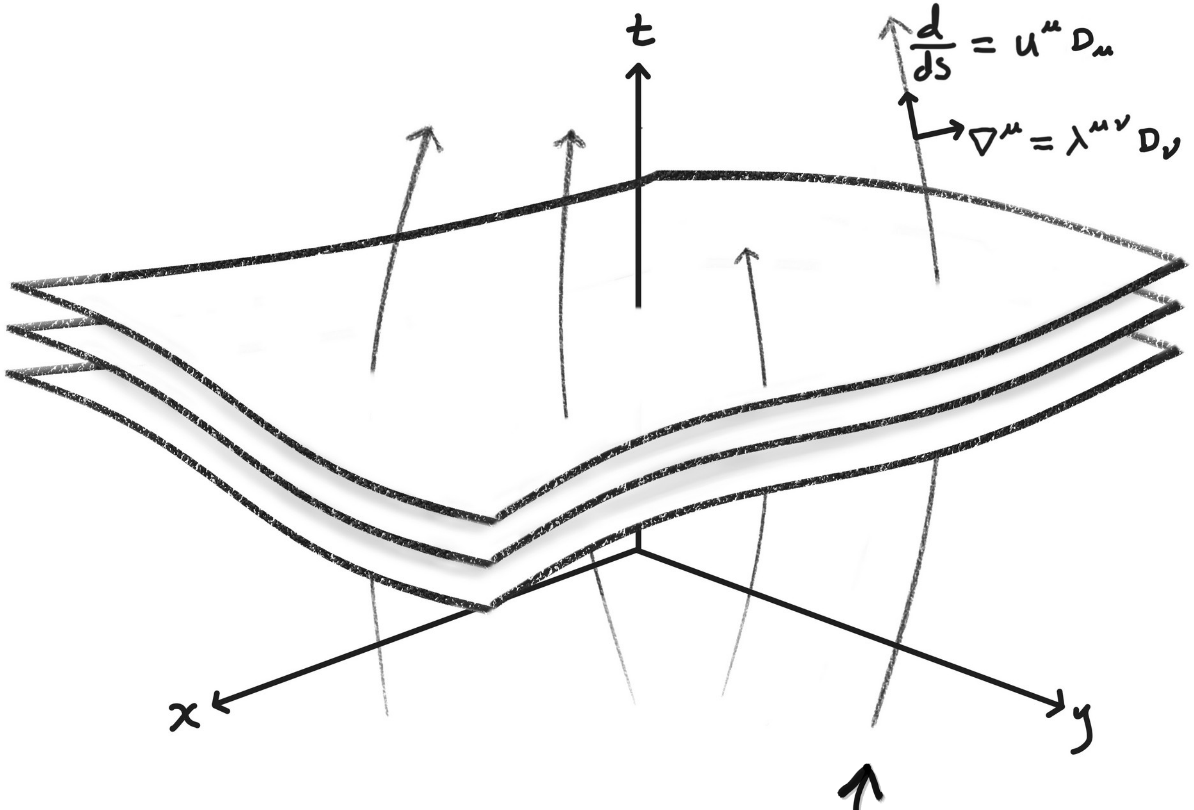


$$\dot{T} \equiv \frac{dT}{ds} = u^\mu T_{;\mu}$$

$$\nabla^\mu T = \lambda^{\mu\nu} T_{;\nu}$$

$$\lambda^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu \Rightarrow \lambda^{\mu\nu} u_\nu = 0$$

Geodesic Congruences



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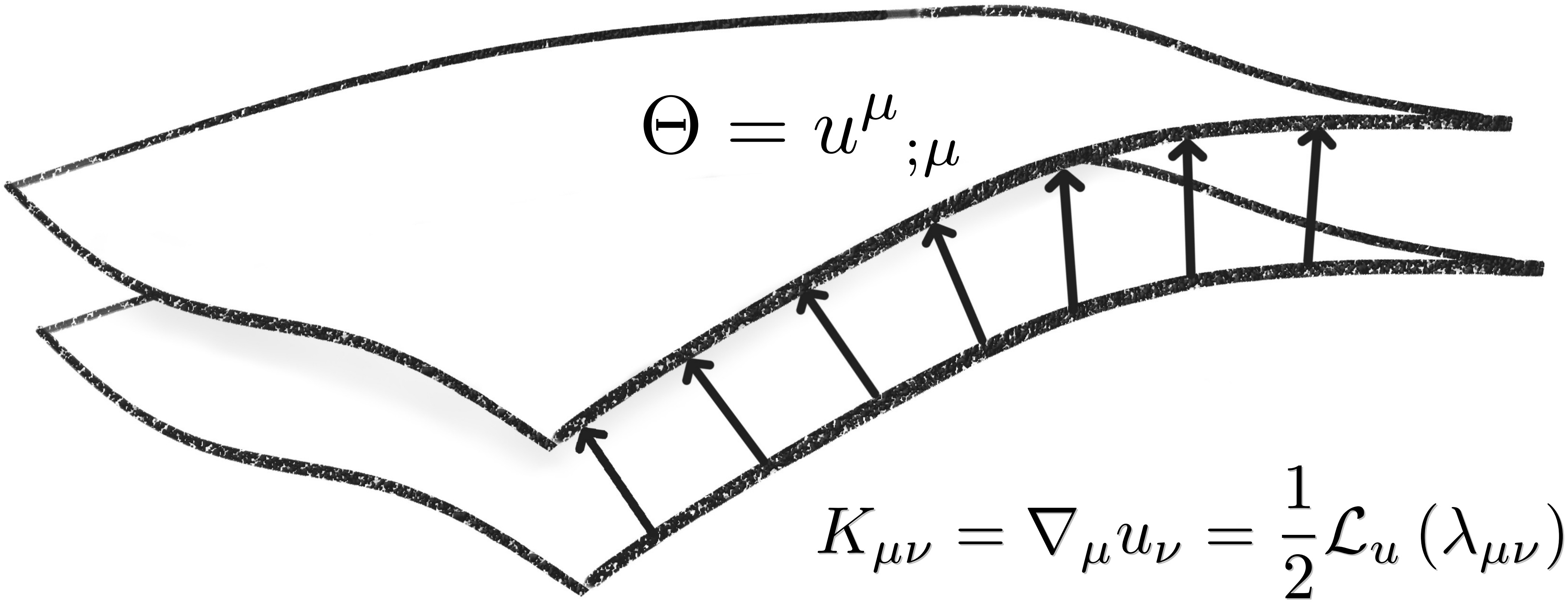
$$\nabla^\mu T = \lambda^{\mu\nu} T_{;\nu}$$

$$\lambda^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu \Rightarrow \lambda^{\mu\nu} u_\nu = 0$$

Geodesic:

$$\frac{du^\mu}{ds} = u^\nu u^\mu{}_{;\nu} = 0$$

Expansion and Extrinsic Curvature



(WHK, Maity, Sriramkumar arXiv: 2307.10958)

BGV Theorem (General)

Now take a *non-comoving* geodesic $v^\mu \neq u^\mu$.

Lorentz boost: $\gamma \equiv -v^\mu u_\mu$ $\qquad \dfrac{d\gamma}{ds} = \dfrac{1}{3}\Theta\left(1-\gamma^2\right).$

Zero shear and vorticity

$$\sigma_{\mu\nu} = \omega_{\mu\nu} = 0$$

The total expansion along the geodesic v^μ is then:

$$\int \Theta ds = \int \frac{3}{1-\gamma^2}\frac{d\gamma}{ds}ds = \int \frac{3d\gamma}{1-\gamma^2}$$

The result is again finite:

$$\int \Theta ds = \frac{3}{2}\ln\left(\frac{\gamma+1}{\gamma-1}\right)$$

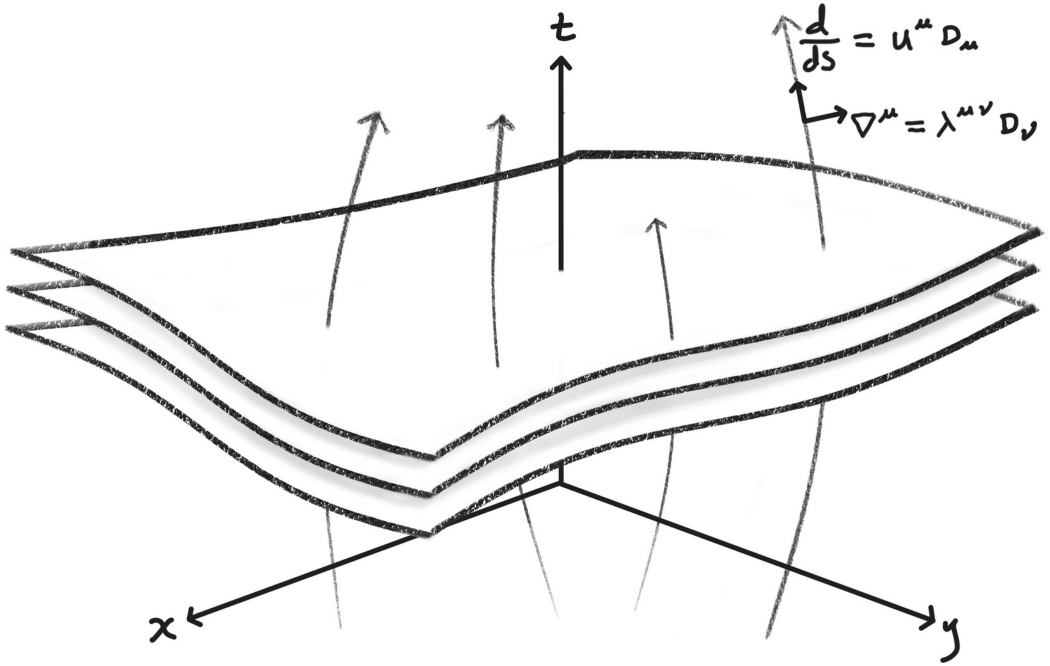
(WHK, Maity, Sriramkumar arXiv: 2307.10958)

BGV and the Null Energy Condition

Geometry vs Dynamics

The BGV theorem is **purely geometric**.

Can we relate it to dynamics via energy conditions?



(WHK, arXiv:2607.28412)

Example: FRW Space

Friedmann Equation

$$\frac{1}{3}\Theta^2 = 3H^2 = 8\pi G\rho - \frac{k}{a^2}$$

Raychaudhuri Equation

$$\dot{\Theta} = 3\dot{H} \leq -3H^2 - 4\pi G\left(\rho + 3p\right)$$
$$\leq -12\pi G\left(\rho + p\right) + \frac{k}{a^2}$$

(WHK, arXiv:2607.28412)

Example: FRW Space

Friedmann Equation

$$\frac{1}{3}\Theta^2 = 3H^2 = 8\pi G\rho - \frac{k}{a^2}$$

Strong Energy Condition

$$R_{\mu\nu}u^\mu u^\nu = 4\pi G\left(\rho + 3p\right) \geq 0$$

Raychaudhuri Equation

$$\dot{\Theta} = 3\dot{H} \leq -3H^2 - 4\pi G\left(\rho + 3p\right)$$

$$\leq -12\pi G\left(\rho + p\right) + \frac{k}{a^2}$$

Null Energy Condition

$$R_{\mu\nu}k^\mu k^\nu = 4\pi G\left(\rho + p\right) \geq 0$$

(WHK, arXiv:2607.28412)

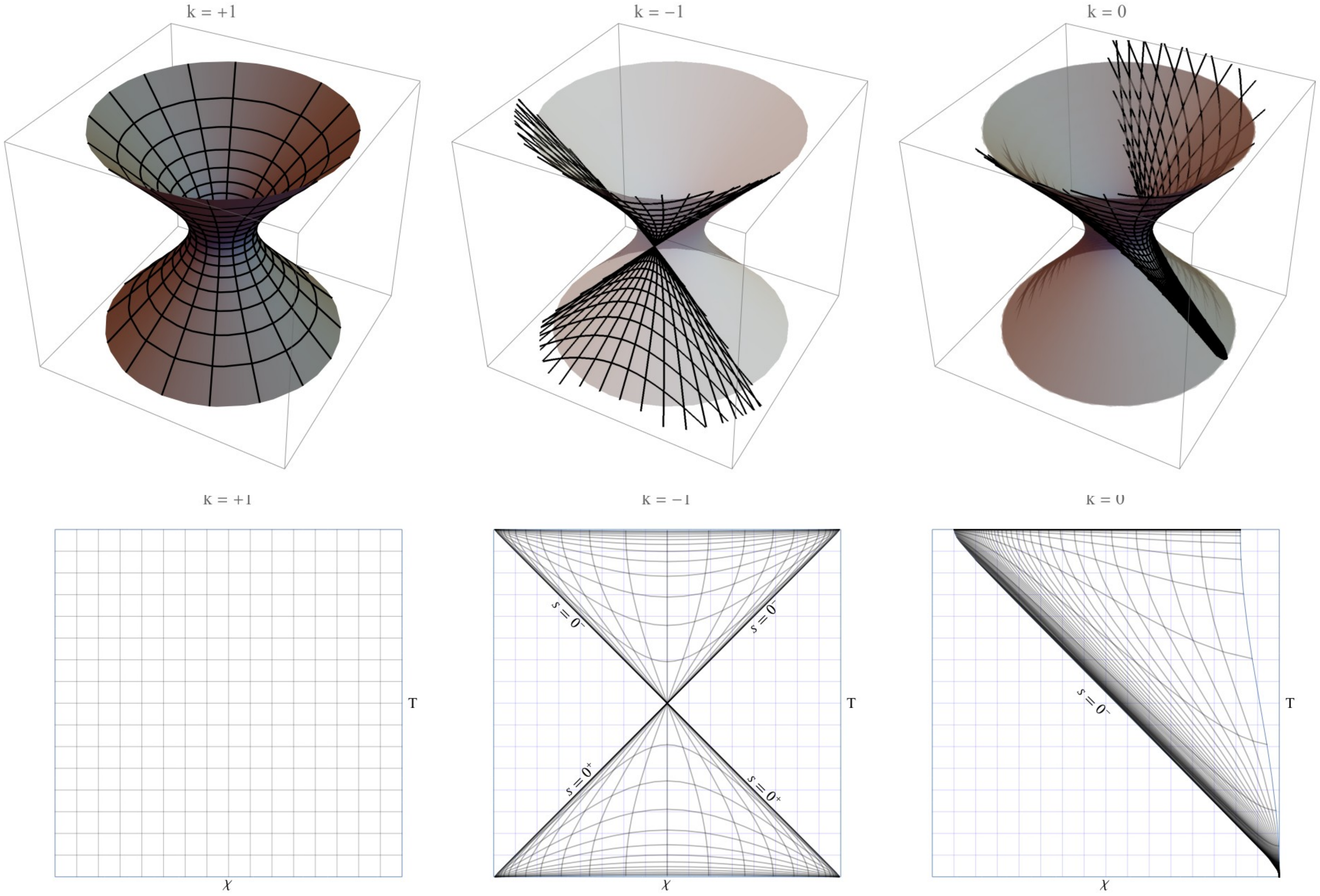
FRW Space

$$\dot{\Theta} \leq -3R_{\mu\nu}k^{\mu}k^{\nu} + \frac{k}{a^2}$$

Null Convergence Condition
plus non-positive curvature ensures $\dot{\Theta} < 0$

(WHK, arXiv:2607.28412)

Coordinate Systems on de Sitter Space



Θ_+ Θ_- Θ_0

General Perfect Fluid

Raychaudhuri Equation

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \left(\sigma^2 - \omega^2\right) = -R_{\mu\nu}u^\mu u^\nu = -4\pi G\ (\rho + 3p)$$

Hamiltonian Constraint (Friedmann Equation)

$$^3\mathcal{R} + \frac{2}{3}\Theta^2 - \sigma^2 = 16\pi G\rho$$

$$\dot{\Theta} = -\frac{3}{2}R_{\mu\nu}k^\mu k^\nu + \frac{1}{2}{}^3\mathcal{R} - \frac{3}{2}\sigma^2$$

(WHK, arXiv:2607.28412)

General Perfect Fluid

$$\dot{\Theta} = -\frac{3}{2} R_{\mu\nu} k^\mu k^\nu + \frac{1}{2}\,{}^3\mathcal{R} - \frac{3}{2}\sigma^2$$

Positive-definite
$\sigma^2 \geq 0$

Null Convergence
$R_{\mu\nu} k^\mu k^\nu \geq 0$

Three-curvature of spatial hypersurfaces

Null Convergence Condition
plus non-positive curvature ensures $\dot{\Theta} < 0$

(WHK, arXiv:2607.28412)

Key Point: Momentum Constraint

Momentum constraint:

$$\nabla_\nu \sigma^{\mu\nu} - \frac{2}{3}\nabla^\mu \Theta = 0$$

Shear-free $\Rightarrow \nabla^\mu \Theta = 0$: expansion constant on spatial hypersurfaces.

$$\dot{\Theta} = -\frac{3}{2} R_{\mu\nu} k^\mu k^\nu + \frac{1}{2}\, {}^3\mathcal{R}$$

Application to BGV Theorem

The expansion Θ is *no longer* the relevant quantity in the BGV integral!

$$\frac{d\gamma}{ds} = -\frac{1}{3}\gamma^2\Theta - \sigma_{\mu\nu}v^\mu v^\nu$$

Timelike

$$\Theta_v \equiv \Theta + \frac{3\sigma_{\mu\nu}v^\mu v^\nu}{\gamma^2 - 1} = \Theta + 3\sigma_{\mu\nu}e^\mu_v e^\nu_v$$

Null

Unit basis vectors:
Directional!

$$\Theta_k \equiv \Theta + \frac{3\sigma_{\mu\nu}k^\mu k^\nu}{\gamma^2} = \Theta + 3\sigma_{\mu\nu}e^\mu_k e^\nu_k$$

(WHK, arXiv:2607.28412)

Non-Geodesic Threading

Add a non-geodesic threading $\dot{u}^\mu \neq 0$:

$$\frac{d\gamma}{ds} = -\frac{1}{3}\gamma^2\Theta - \sigma_{\mu\nu}v^\mu v^\nu - \gamma\dot{u}_\mu v^\mu$$

Timelike

$$\tilde{\Theta}_v = \Theta + 3\sigma_{\mu\nu}e_v^\mu e_v^\nu + \frac{3}{\beta}\dot{u}_\mu e_v^\mu$$

Null

Unit basis vectors:
Directional!

$$\Theta_k = \Theta + 3\sigma_{\mu\nu}e_k^\mu e_k^\nu + 3\dot{u}_\mu e_k^\mu$$

(WHK, arXiv:2607.28412)

General BGV Condition

$$\frac{1}{s_f - s_0}\int_{s_0}^{s_f}\left(\Theta + 3\sigma_{\mu\nu}e^\mu_v e^\nu_v + \frac{3}{\beta}\dot{u}_\mu e^\mu_v\right) ds \geq \Delta > 0 \qquad \forall s_0 \in (s_i, s_f)$$

Need only hold along one test geodesic v^μ !

$$\dot{\Theta} = -\frac{3}{2}R_{\mu\nu}k^\mu k^\nu + \frac{1}{2}{}^3\mathcal{R} - \frac{3}{2}\sigma^2 + \dot{u}^\mu{}_{;\mu}$$

Either sign!!

(WHK, arXiv:2607.28412)

Application: The Comoving Curvature Perturbation

Comoving World Lines are Non-Geodesic

At linear order:

$$\dot{u}_0 = 0 \qquad \dot{u}_i = \partial_i \left[\zeta - (aH)\, B\right]$$

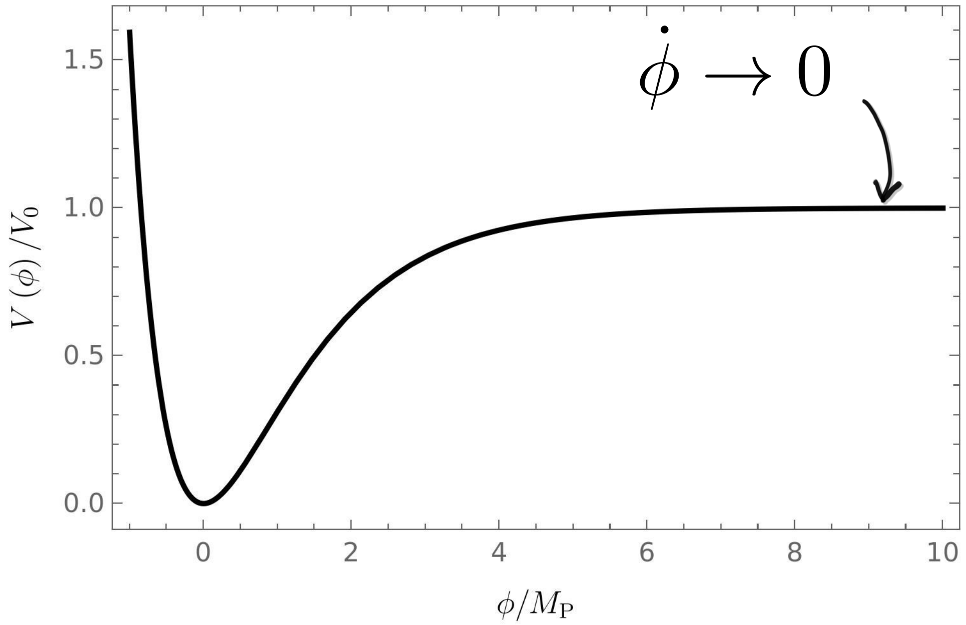
Curvature perturbation

Shift perturbation

Gauge-invariant Mukhanov variable

$$u = a\left(\delta\phi + \frac{\dot{\phi}}{H}\zeta\right)$$

Eternal Inflation on the Plateau



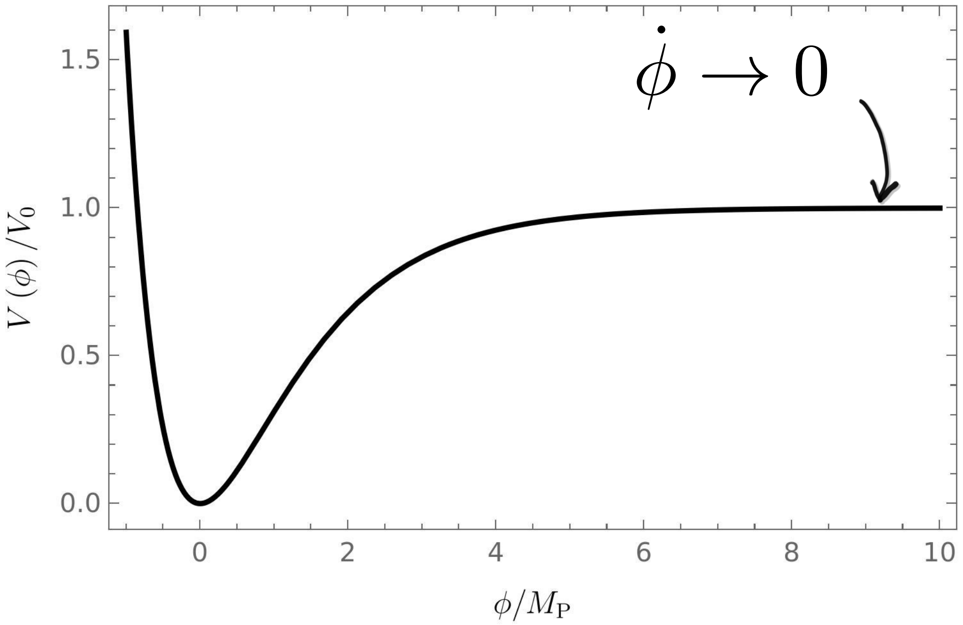
Comoving curvature perturbation diverges!

$$\zeta \sim \frac{H^2}{\dot{\phi}} \to \infty$$

Mukhanov variable is well-behaved:

$$u = a\left(\delta\phi + \frac{\dot{\phi}}{H}\zeta\right) \to a\delta\phi$$

Eternal Inflation on the Plateau



Perturbations in flat gauge:

$$\sqrt{\langle \delta\phi^2 \rangle_{k=aH}} = \frac{H}{2\pi} = \frac{\Theta}{6\pi}$$

$$\delta T_{\mu\nu} \sim \partial_\mu (\delta\phi)\, \partial_\nu (\delta\phi)$$

$$\frac{\delta\tilde{\Theta}}{\tilde{\Theta}} \sim \mathcal{O}\left[\left(\frac{\Theta}{M_{\mathrm{P}}}\right)^2\right] \qquad \frac{\delta\dot{\Theta}}{\Theta^2} \sim \mathcal{O}\left[\left(\frac{\Theta}{M_{\mathrm{P}}}\right)^2\right]$$

Cyclic Cosmology

Cyclic universes and entropy

NOVEMBER 1, 1931 PHYSICAL REVIEW VOLUME 38

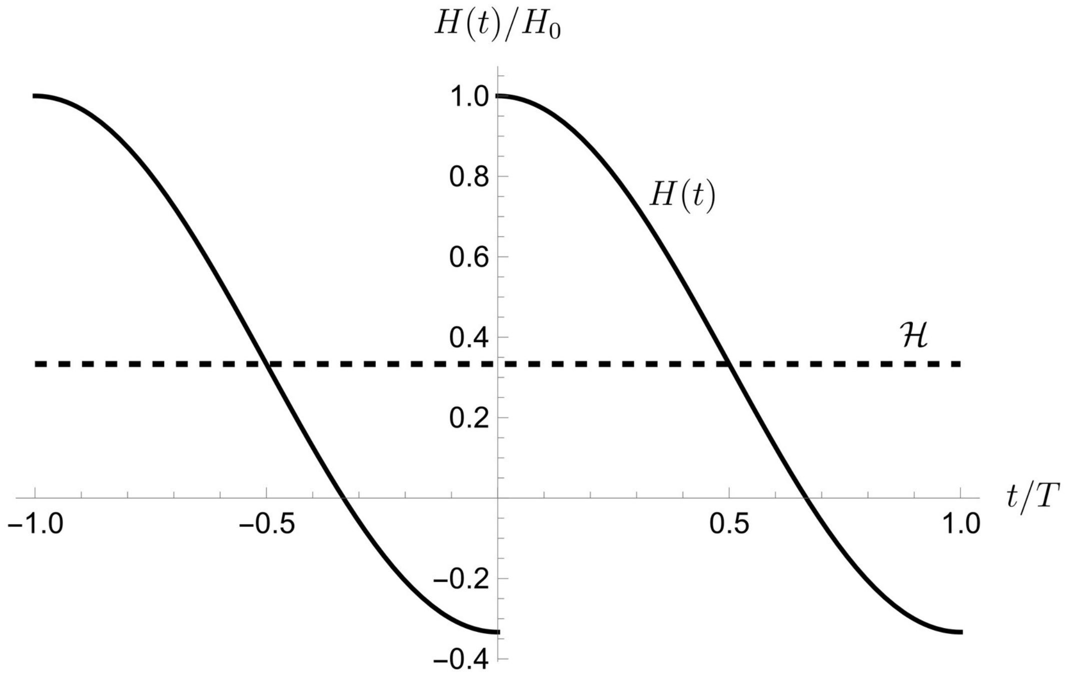
ON THE THEORETICAL REQUIREMENTS FOR A PERIODIC BEHAVIOUR OF THE UNIVERSE

BY RICHARD C. TOLMAN
CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA
(Received September 18, 1931)

ABSTRACT

This article investigates, on the basis of relativistic mechanics and relativistic thermodynamics, the theoretical requirements for nonstatic models of the universe to exhibit a behaviour which is periodic in time. The discussion is limited to models which can be regarded from a large-scale point of view as filled with a uniform distribution of fluid, and the time coordinate is purposely chosen so as to agree with the proper time that would be used by local observers at rest in this fluid. It is first shown that the analytical requirements which would correspond to the continuous expansion and contraction of such models between definite maximum and minimum limits, together with the thermodynamic requirement that the expansion and contraction must be reversible, could not both be simultaneously satisfied with any fluid for filling the model which has reasonable properties. Attention is then turned to models

Ijjas / Steinhardt Cyclic Model



Hubble parameter is cyclic:

$$H(t+T) = H(t)$$

Scale factor grows:

$$a(t+T) > a(t)$$

Exponential expansion / slow contraction dilutes entropy!

(Ijjas & Steinhardt, arXiv:1904.08022)

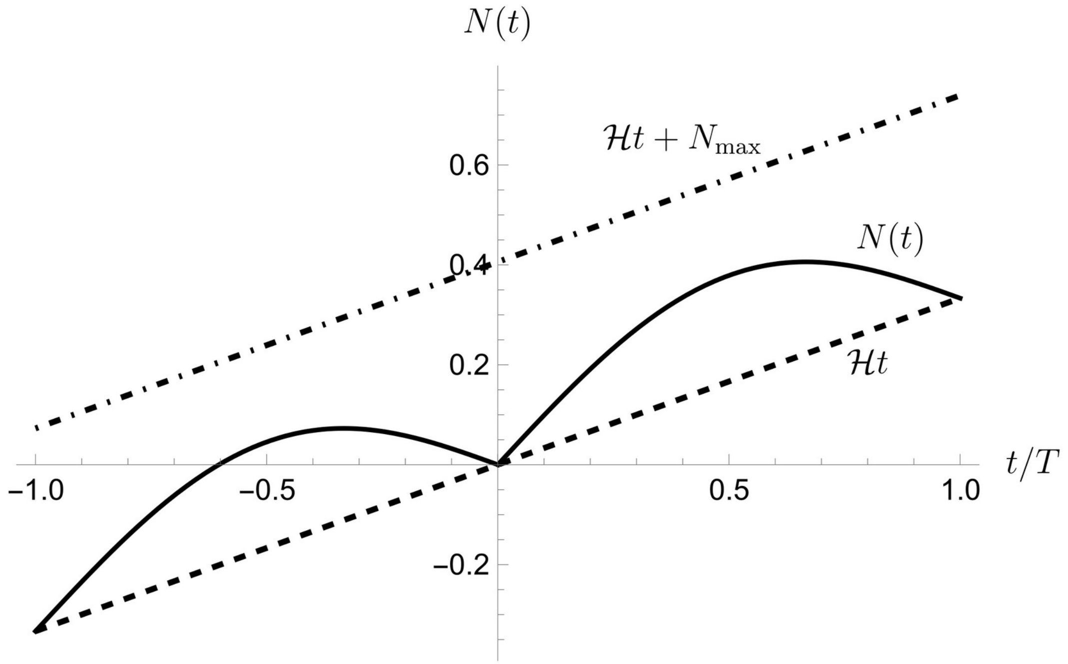
Ijjas / Steinhardt Cyclic Model

Construct bounding de Sitter spaces:

$$a(t) = e^{\mathcal{H}t}, \quad e^{\mathcal{H}t + N_{\max}}$$

$$\mathcal{H} \equiv \frac{1}{T}\int_0^T H dt$$

average expansion rate over one cycle



(WHK, Stein arXiv:2110.15380)

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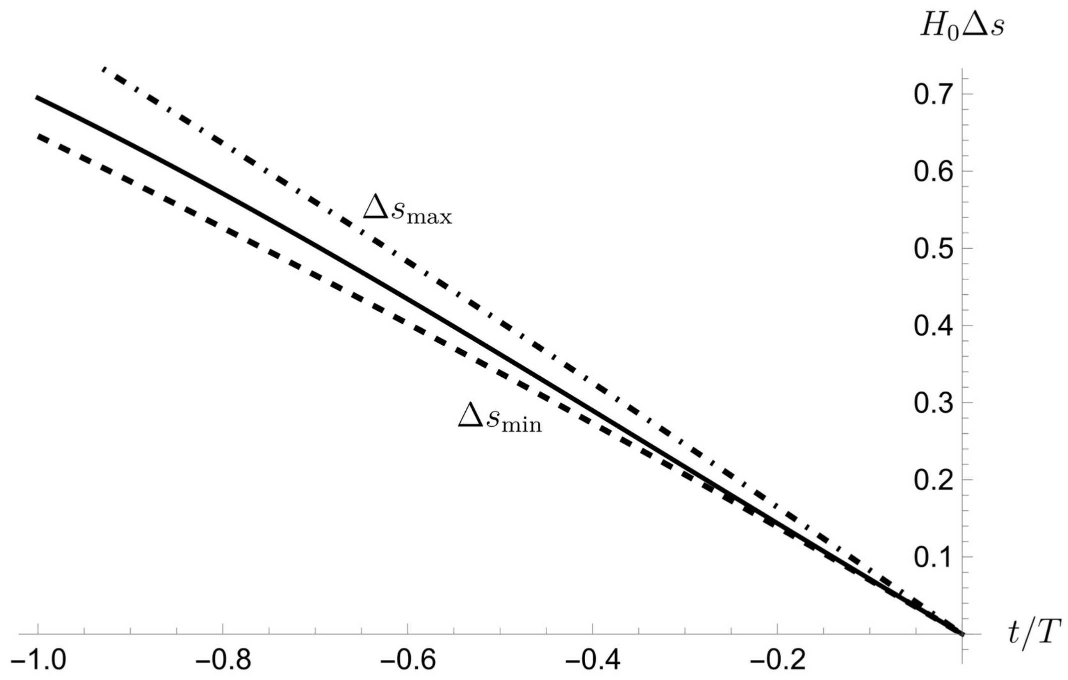

Proper time bounded from above and below by finite values!

$$\Delta s_{\min} < \Delta s < \Delta s_{\max}$$