Non-compact QCD axion and its gravitational origin

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References

Georgios Karananas, MS, and Sebastian Zell:

- Weyl-invariant Einstein-Cartan gravity: unifying the strong CP and hierarchy puzzles, 2406.11956
- Gravitational Origin of the QCD Axion, 2506.11836
- A non-compact QCD axion, 2512.20290

Georgios Karananas, MS

- QCD axion from broken scale symmetry 2605.14684
- A scaling non-compact QCD axion, 2606.17019

Outline

- Short reminder:
  - Strong CP problem
  - PQ symmetry and the compact axion
  - What is actually needed to solve the strong CP problem dynamically?
- Origin of the non-compact axion
  - Dilaton of broken scale symmetry
  - Weyl-invariant Einstein–Cartan gravity
- Cosmology of the non-compact axion
  - Compact axion (reminder) and generic features of the non-compact one
  - Mass-term bias (EC gravity): inflation, domain walls, DM, GWs
  - Exponential bias (scale symmetry): the scaling attractor
  - Two scenarios compared
- Experimental consequences
- Conclusions

The strong CP problem: a reminder

The QCD Lagrangian admits a CP-odd term:

$$\mathscr{L}_\theta = \theta \frac{\alpha_s}{8\pi} G^b_{\mu\nu} \tilde{G}^{b\mu\nu} \equiv \theta Q$$

This term is a total derivative, but it is physical because of non-trivial topology of gauge fields.

Observable consequence: neutron electric dipole moment

$$d_n \simeq 10^{-16} \theta e \cdot \mathrm{cm} \Rightarrow |\theta| \lesssim 10^{-10}$$

Strong CP problem = Why θ << 1 in the Standard Model

PQ symmetry and the compact axion

Peccei–Quinn '77, Weinberg '78, Wilczek '78: a global U(1) $_{PQ}$ symmetry, spontaneously broken at the scale F_{PQ} and explicitly broken only by the QCD anomaly. This promotes θ to a dynamical (pseudo)scalar field coupled to the topological charge density Q:

$$\theta \;\rightarrow\; \theta(x) = \frac{a(x)}{f_a}, \qquad \mathscr{L} = -\frac{1}{2}\left(\partial_\mu a\right)^2 + \frac{a}{f_a} Q$$

The QCD vacuum energy is minimised at θ = 0:

$$V_{\mathrm{QCD}}(\theta) = -\, m_\pi^2 f_\pi^2 \sqrt{1 - \frac{4 m_u m_d}{\left(m_u + m_d\right)^2} \sin^2 \frac{\theta}{2}} \;\Rightarrow\; \left\langle \theta \right\rangle = 0$$

Prediction: existence of a scalar particle - the axion, with the mass

$$M_a \simeq 5.7 \times 10^{-15}\,\mathrm{GeV} \left(\frac{10^{12}\,\mathrm{GeV}}{f_a} \right)$$

PQ symmetry and the compact axion

The axion is the phase of a complex scalar: an angular variable — the field space is a circle

$$a \equiv a + 2\pi f_a$$

Two celebrated models incorporating axion:

- KSVZ (new heavy quark + complex scalar singlet),
- DFSZ (one extra Higgs doublet + complex scalar singlet):

Both models have several new degrees of freedom to kill one "unnatural" number (10^{-10}) and contain two more "unnatural" parameters:

Hierarchy of scales, to make axion invisible

$$v_{EW}/F_{PQ} \lesssim 10^{-7}$$

Quality of PQ symmetry, to ensure the solution to strong CP-problem

$$m_{\cancel{PQ}}/F_{PQ} \lesssim 10^{-25}$$

What is the need for the PQ symmetry, besides the solution of the strong CP problem?

What is actually needed to solve strong CP-problem?

Requirements for the mechanism to work:

(i) a scalar field a with (almost) exact shift symmetry
(ii) the coupling $a\, Q/f_a$
(iii) any non-QCD contribution to the potential of a must be small enough, so that $|\theta| \lesssim 10^{-10}$

No need to require that the field a is a phase:

non-compact axion

Origin of the non-compact axion: spontaneously broken scale invariance

What is the source of masses?

The Higgs boson gives mass to W, Z, quarks and leptons.

Questions:

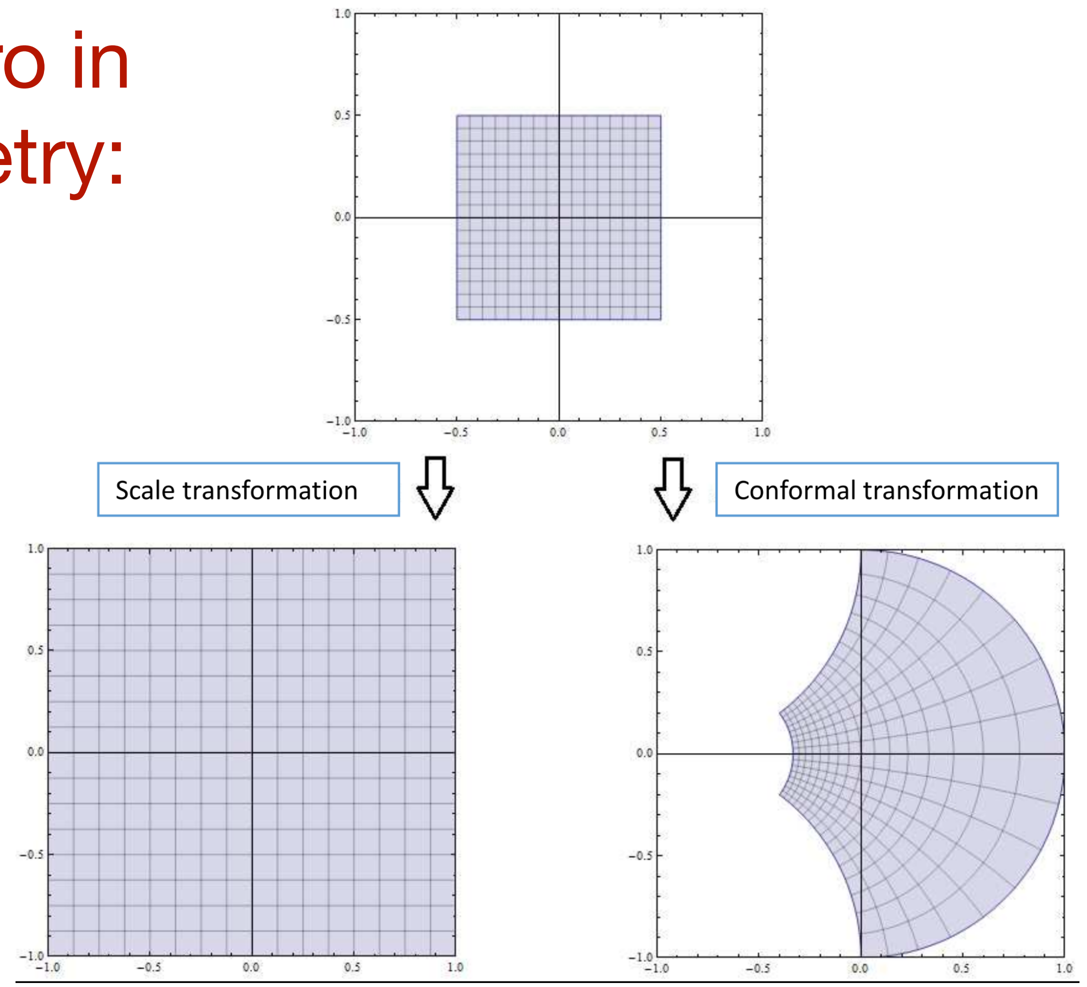
- Who gives mass to the Higgs boson itself?
- Who gives mass to the proton?
- Who determines the Newton gravity constant?

Broken scale symmetry: motivations

If the mass of the Higgs boson is put to zero in the SM, the Lagrangian has a wider symmetry: it is scale and conformally invariant.

Dilatations - global scale transformations (σ = const): $\Psi(x) \rightarrow \sigma^n \Psi(\sigma x)$, n = 1 for scalars and vectors and n = 3/2 for fermions

Attractive features of scale invariant/ conformal theories: no mass parameters, only dimensionless couplings. A hint for solution of the hierarchy problem: why the Higgs boson mass is so small in comparison with the Planck scale?



Scale symmetry can be made anomaly free at the quantum level: replace the renormalisation scale by a field - dilaton

Englert, Truffin, Gastmans, ’76; Wetterich ’88; MS, Zenhäusern ’09

The minimal model - scale invariant νMSM

Particles of the SM

+

graviton

+

dilaton

+

3 Majorana leptons

Roles of different particles

The roles of dilaton:

- determine the Planck mass
- give mass to the Higgs boson
- give masses to 3 Majorana leptons (leading to neutrino masses and oscillations, baryogenesis, and dark matter)
- solve the strong CP-problem: dilaton=axion

Roles of the Higgs boson:

- give masses to fermions and vector bosons of the SM
- provide inflation

Scale-invariant Lagrangian

$$\mathscr{L}_{\nu\mathrm{MSM}} = \mathscr{L}_{\mathrm{SM[M\to 0]}} + \mathscr{L}_G + \frac{1}{2}(\partial_\mu\chi)^2 - V(\varphi,\chi) + \left(\overline{N_I} i\gamma^\mu\partial_\mu N_I - h_{\alpha I}\overline{L}_\alpha N_I \widetilde{\varphi} - f_I \overline{N_I}^c N_I \chi + \mathrm{h.c.}\right)$$

$$+\mathscr{L}_{\mathrm{topology}}$$

Potential (χ - dilaton, φ - Higgs boson, $\varphi^\dagger\varphi = 2h^2$):

$$V(\varphi,\chi) = \lambda\left(\varphi^\dagger\varphi - \frac{\alpha}{2\lambda}\chi^2\right)^2 + \beta\chi^4$$

Gravity part:

$$\mathscr{L}_G = -\left(\xi_\chi\chi^2 + 2\xi_h\varphi^\dagger\varphi\right)\frac{R}{2}$$

Topological part:

$$\mathscr{L}_{\mathrm{topology}} = \log\left(\frac{\chi}{M}\right)\left[\alpha_Q Q + \alpha_E E_4 + \alpha_P P_4\right]$$

Scale invariant since Q, E_4, and P_4 are total derivatives

$$Q = \frac{\alpha_s}{8\pi} G^b_{\mu\nu}\widetilde{G}^{b\mu\nu},\ E_4 = \frac{1}{4}\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}R^{\alpha\beta}_{\mu\nu}R^{\gamma\delta}_{\rho\sigma},\ P_4 = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta}_{\mu\nu}R^{\gamma\delta\mu\nu}$$

SM, QCD | Euler/Gauss-Bonnet | Pontryagin

Properties of the dilaton/axion

- Go to the Einstein frame, and introduce a canonically normalised dilaton/axion field, $\chi \propto M_P e^{a/\Lambda}$.

- QCD and gravity non-perturbative effects (instantons) generate non-trivial potential for a:

"i" in Euclidean path integral

No "i" in Euclidean path integral

$$V_Q(a) \sim \Lambda_Q^4 \cos\left(\frac{a}{f_a} - \theta_{\rm QCD}\right), V_{P_4}(a) \sim \Lambda_P^4 \cos\left(\frac{a}{\tilde{F}} - \theta_{\rm grav}\right), V_{E_4}(a) \sim \Lambda_E^4 e^{-a/F},$$

with unknown parameters $\Lambda_Q, f_a, \ldots$. If the QCD part dominates, the strong CP problem is solved. In general, $\theta_{\rm phys} \neq 0$.

Remark: the dilaton/axion can have a gravitational origin: replace Diff invariance by TDiff invariance (transverse diffeomorphisms). Then the dilaton is a propagating determinant of the metric.

Origin of the non-compact axion: Weyl invariant Einstein-Cartan gravity

Gravity as a gauge theory: Einstein-Cartan

- Gauge fields of the SM = local SU(3)×SU(2)×U(1). Gravity = local Poincaré (Lorentz) symmetry? Utiyama '56, Kibble '61, Sciama '62
- Basic gauge fields:

$$e_\mu^{\;A}\ \text{(tetrad, 16 fields)}, \qquad \omega_\mu^{\;AB}\ \text{(spin connection, 24 fields)}$$

- Field strengths — curvature and torsion:

$$F_{\mu\nu}^{AB} = \partial_\mu \omega_\nu^{AB} - \partial_\nu \omega_\mu^{AB} + \omega_{\mu C}^{A}\omega_\nu^{CB} - \omega_{\nu C}^{A}\omega_\mu^{CB}$$

$$T_{\mu\nu}^{A} = \partial_\mu e_\nu^{A} - \partial_\nu e_\mu^{A} + \omega_{\mu B}^{A} e_\nu^{B} - \omega_{\nu B}^{A} e_\mu^{B}$$

- Lowest-order invariants and irreducible components of the torsion $T_{\mu\nu\rho} = e_{\mu A} T_{\nu\rho}^{A}$,

$$F \equiv \frac{1}{8\sqrt{g}} \epsilon_{ABCD} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^{AB} e_\rho^{C} e_\sigma^{D} \qquad \widetilde{F} \equiv \frac{1}{\sqrt{g}} \epsilon^{\mu\nu\rho\sigma} e_{\rho C} e_{\sigma D} F_{\mu\nu}^{CD}$$

$$v_\mu = T^{\nu}_{\;\mu\nu},\ a_\mu = \epsilon_{\mu\nu\rho\sigma} T^{\nu\rho\sigma},\ \tau_{\mu\nu\rho} = \frac{2}{3}\left(T_{\mu\nu\rho} - v_{[\nu} g_{\rho]\mu} - T_{[\nu\rho]\mu}\right)$$

Torsion is non-dynamical: only 2 of the 40 components of (e, ω) propagate. Metric gravity is recovered when torsion is integrated out — plus new contact interactions.

Weyl-invariant Einstein-Cartan gravity

- Weyl symmetry: local change of the rulers. The theory has no intrinsic mass parameters

$$e^A_\mu \to \Omega(x)\, e^A_\mu, \qquad \omega^{AB}_\mu \to \omega^{AB}_\mu$$

- The only ghost-free Weyl-invariant, dimension-4 structures (f, f̃ = gauge couplings of the Lorentz group):

$$S = \int d^4x \sqrt{g} \left[\frac{1}{f^2} F^2 + \frac{1}{\tilde{f}^2} \tilde{F}^2 \right] + S_{\rm SM} + \left(h^2 F,\ h^2 \tilde{F},\ a_\mu^2 h^2, \ldots \right)$$

- Introduce auxiliary fields χ (dilaton) and φ (ALP); use Weyl invariance to fix χ = M_P/√2 — spontaneous breaking generates all scales; then integrate out torsion:

$$S_{\rm gr} = \int d^4x \sqrt{g} \left[\chi^2 F + M_P^2 \phi\, \tilde{F} - \frac{f^2 \chi^4}{4} - \frac{\tilde{f}^2 M_P^4 \phi^2}{4} \right], \qquad \chi \to \frac{M_P}{\sqrt{2}}$$

Result: metric gravity + SM + exactly one new scalar a (φ ∝ a²) with canonical kinetic term, plus non-polynomial Higgs kinetic term and 4-fermion interactions.

No quantum Weyl anomaly: Englert, Truffin, Gastmans '76; MS, Zenhäusern '09.

One theory, three hierarchy puzzles

- The gauge couplings of the Lorentz group set the vacuum energy, the tree-level Higgs mass and the mass of the new scalar:

$$\epsilon_{\rm vac} = \frac{f^2 M_P^4}{16}, \qquad m_H^2 = -\frac{f^2 \xi_h M_P^2}{4}, \qquad m_a^2 \propto \frac{\tilde{f}^2 (1+6\xi_h) M_P^2}{48}$$

- Very small Lorentz gauge couplings ("gravity is the weakest gauge force") would explain simultaneously the three puzzles:

$$\frac{m_H}{M_P} \simeq 10^{-16}, \qquad \frac{\epsilon_{\rm vac}^{1/4}}{M_P} \simeq 10^{-31}, \qquad \frac{m_{\cancel{PQ}}}{F_{PQ}} \lesssim 10^{-25}$$

$$\text{tree level:} \quad \epsilon_{\rm vac} \Rightarrow f \sim 10^{-62}, \qquad m_H \Rightarrow f\sqrt{\xi_h} \sim 10^{-16} \quad \text{— cannot both hold}$$

Weyl-invariant perturbation theory: f = f̃ = 0 are fixed points of the RG evolution — small values are radiatively stable. But the tree-level values of the vacuum energy and of the Higgs mass do not match the observations.

Non-perturbative effects (gravity–Higgs instantons) may generate $m_H \propto e^{-S} M_P$ with $S \gg 1$ (Shkerin, MS '18; Shkerin, MS, Zell '21). If they also contribute $\propto m_H^4$ to the vacuum energy, f must be tuned to $f \sim m_H^2/M_P^2 \sim 10^{-32}$ to keep the cosmological constant small; if not, $f \sim 10^{-62}$.

The new scalar a: non-compact, gravity-induced mass $m_a \propto \tilde{f} M_P$ — a candidate non-compact axion.

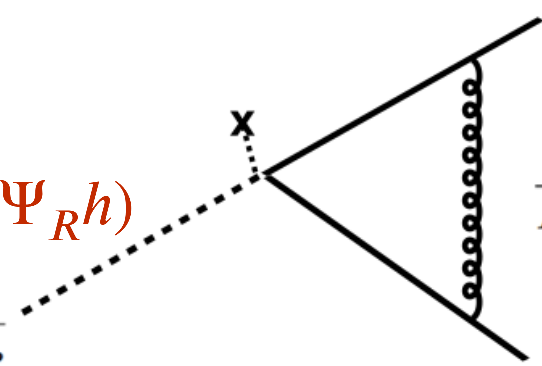
Does the gravitational scalar talk to QCD?

- Interactions of a induced by the Einstein-Cartan nature of gravity (A, V axial and vector fermionic currents):

$$F_1(a,h)\partial_\mu a A^\mu,\; F_2(a,h)\partial_\mu a V^\mu$$

- Not enough: exact shift symmetry a → a + const, and the correlators of A and V have no pole at zero momentum ⇒ no communication with QCD, no solution of the strong CP problem (the point missed in earlier attempts to solve it with gravity)
- The way out: the requirement of quantum Weyl symmetry. In dimensional regularisation Weyl invariance is maintained by dilaton-dependent evanescent operators,

$$(M_P^2 + aM_P)^{D-4}(G\widetilde{G} + \overline{\Psi}_L\Psi_R h)$$



$$\frac{\alpha}{D-4} \quad \propto \frac{\zeta}{M_P}[aG\widetilde{G} + am_q\bar{q}\gamma_5 q]$$

D is the space-time dimension in DimReg $(D-4)\frac{a}{M_P}$

- "High-quality" axion thanks to Weyl symmetry. Strong CP is solved if the gravity-induced mass is smaller than the QCD one:

$$\frac{\tilde{f}}{\zeta} \lesssim 10^{-5}\frac{m_\pi f_\pi}{M_P^2}\frac{\sqrt{m_u m_d}}{m_u + m_d} \sim \mathcal{O}\left(10^{-43}\right)$$

Cosmology of the non-compact axion

Cosmology of the compact axion: a reminder

$U(1)_{PQ}$ broken before/during inflation

- The axion is massless during inflation: fluctuations $\delta a \simeq H_I/2\pi$ are frozen in and become isocurvature perturbations
- CMB (Planck) bound on isocurvature perturbations:

$$H_I \lesssim 10^7\,\mathrm{GeV}\left(\frac{f_a}{10^{11}\,\mathrm{GeV}}\right)^{0.4}$$

incompatible with high-scale inflation (Higgs, Starobinsky, α-attractors: $H_I \sim 10^{13}$ GeV)

$U(1)_{PQ}$ broken after inflation

- Network of axion strings; at T ~ 1 GeV domain walls attached to the strings
- $N_{DW} > 1$: stable walls overclose the Universe unless an ad hoc bias is added;
- $N_{DW} = 1$: walls decay, but f_a is pinned to a narrow window (lower bound from astrophysics, upper bound from axion abundance)

These results are deeply rooted in the compactness of the axion: $a \equiv a + 2\pi f_a$, i.e. from the existence of $U(1)_{PQ}$.

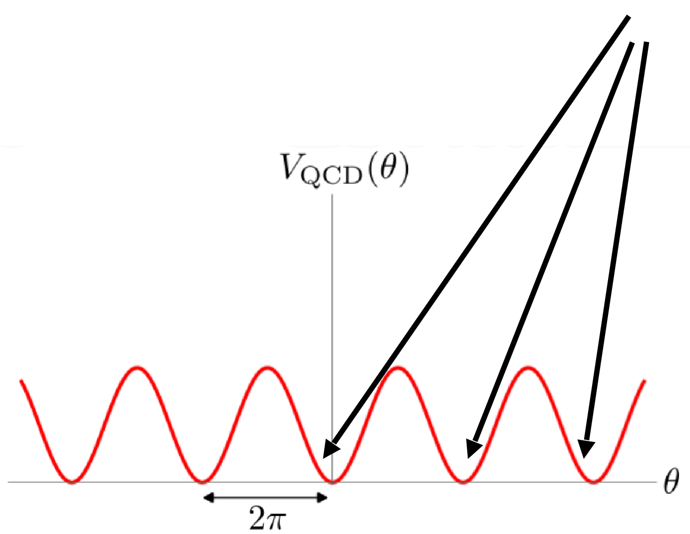
Non-compact axion: effective action

Drop the U(1) symmetry altogether: a real scalar field θ on the real line, not on a circle, coupled to QCD through the topological density, with an approximate shift symmetry:

$$\mathscr{L} = -\frac{f^2(\Phi)}{2}\partial_\mu\theta\partial^\mu\theta + \frac{\theta}{F(\Phi)}Q - \delta V$$

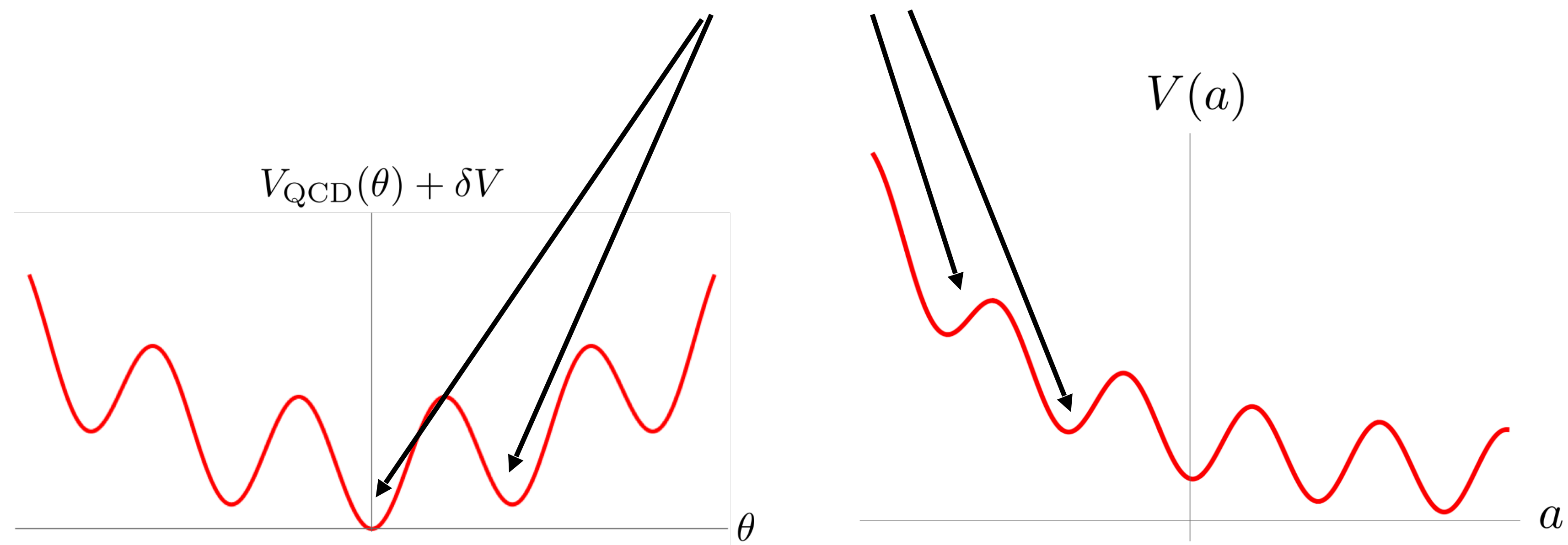
$f(\Phi)$, $F(\Phi)$ are functions of other scalar fields Φ (Higgs, dilaton, ...); δV is a small non-QCD potential.

identified
vacua



PQ axion

different
vacua



Non-compact axion

QCD generates a periodic contribution to the potential

Generic features

- No U(1) symmetry $\longrightarrow$ no axion strings, only domain walls are possible

- For low scale inflation with $H_I \lesssim 10^7 \mathrm{GeV} \left(\dfrac{f_a}{10^{11} \mathrm{GeV}} \right)^{0.4}$ cosmology of non-compact axion is the same as for compact axion

- Two contributions to the axion mass: QCD + gravity $\longrightarrow$ biased domain walls (if any)

Mass term bias:
inspired by EC gravity

Residual CP-violation

Even a tiny gravitationally-induced non-QCD mass for the axion, necessarily translates into observable CP-violation: the ground state is shifted away from the parity-preserving point.

$$V_{\mathrm{QCD}}(\theta) + \delta V \approx \frac{f_a^2 M_a^2}{2}\left((\theta - \bar{\theta})^2 + \mu^2\theta^2\right)$$

QCD induced mass

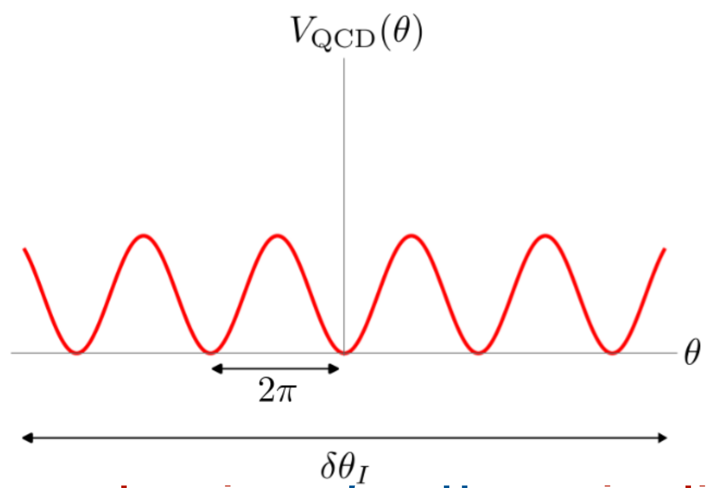
$$M_a \simeq 5.7 \times 10^{-15}\mathrm{GeV}\left(\frac{10^{12}\mathrm{GeV}}{f_a}\right)$$

Gravity induced mass

$$\theta_{\mathrm{phys}}^{\mathrm{exp}} \lesssim 10^{-10} \quad \Longrightarrow \quad \mu^2 \lesssim 10^{-10}$$

High scale inflation

- EC Weyl invariant gravity naturally accommodates the metric Higgs inflation with $n_s = 0.97$ and r=0.0033. For high scale (Higgs) inflation, $H_I \sim 10^{13}$ GeV the amplitude of primordial axion fluctuations is of order $\delta\theta_I \simeq H_I/(2\pi f_a)$.



- Contrary to the compact axion, for the misalignment contribution to the axion abundance the isocurvature bounds from CMB are completely evaded: the primordial axion perturbations feel many vacua, meaning that they imprint on the spectrum as white noise, and their contribution is therefore exponentially suppressed (Kofman, Linde, Lyth)

$$\frac{\delta\rho}{\rho}\Big|_{\rm mis.} \sim \frac{H_I}{2\pi f_a}\left(\frac{l}{l_c}\right)^{-\frac{H_I^2}{8\pi^2 f_a^2}} \Omega_\theta$$

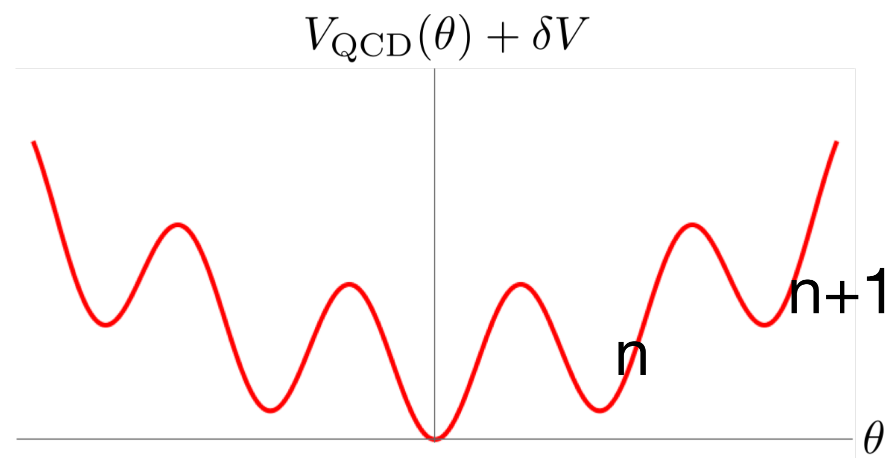
Here Ω_θ is the axion abundance, $l_c \sim 10^{17}$cm is the present-day size of the modes created around the QCD scale

Domain walls and their fate

- At temperatures $T_{\rm osc} \sim 1$ GeV, the QCD-induced axion mass is of the order of the Hubble rate and the non-compact field creates a complicated network of domain walls (there are no axion strings, contrary to PQ case).

- $N \sim H_I/f_a \gg 1$different types of domain walls.

- Energetically favourable domains expand at the expense of those with higher energy density.



- Another source of axion production: collisions of the domain walls,
$$\frac{\Omega_{aDW}}{\Omega_{DM}} \sim \kappa\left(\frac{10^{-6}}{\mu}\right)\left(\frac{f_a}{2\times 10^9\mathrm{GeV}}\right)^{1.5}$$
, $\kappa < 1$ is the efficiency of the energy transfer of domain wall energy into non-relativistic axions. Domain walls keep a memory about superhorizon isocurvature perturbations (Kitajima, Lee, Takahashi, Yin '26)

    - If $\kappa \sim 1$, axions may constitute only 10^{-5} of DM because of the isocurvature CMB bounds

    - If $\kappa \ll 1$, axions can be all the DM

Domain walls and their fate

The energy density of the wall network redshifts more slowly than radiation and would soon dominate the Universe. Because of the bias δV the domains with higher energy shrink, and the network collapses after a time

$$t_* \sim \frac{\sigma_{\mathrm{DW}}}{\Delta V} \sim \frac{1}{\mu^2 M_a}, \qquad \sigma_{\mathrm{DW}} \sim 8 M_a f_a^2, \quad \Delta V \sim \mu^2 M_a^2 f_a^2$$

This must happen before Big Bang nucleosynthesis, $t_* \lesssim 1$ s, which gives a lower bound on the gravitationally induced axion mass — that is, a lower bound on the residual CP violation:

$$\mu^2 \gtrsim \left(\frac{10^{-12}\,\mathrm{GeV}}{M_a}\right) \times 10^{-13} = 1.8 \times 10^{-11} \left(\frac{f_a}{10^{12}\,\mathrm{GeV}}\right)$$

Together with the nEDM bound $\mu^2 \lesssim 10^{-10}$ this leaves a window which closes for $f_a \gtrsim 5.6{\times}10^{12}$ GeV (see the parameter space below).

Axion dark matter

The ratio between Ω_θ and the DM abundance due to misalignment is of the order

$$\frac{\Omega_\theta}{\Omega_{\rm DM}} \simeq \left\langle \delta\theta_I^2 \right\rangle \left(\frac{f_a}{9 \times 10^{11}\mathrm{GeV}} \right)^{1.165} \quad \text{with}$$

$$\left\langle \delta\theta_I^2 \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta\theta^2 = \frac{\pi^2}{3} \,.$$

If the gravitational axion is all DM, then there must be strong CP-violation not far from the present constraints on it:

$$\theta_{\rm phys} \gtrsim 4.3 \times 10^{-12} \,.$$

This is roughly 30 times smaller than the current bound on the neutron electric dipole moment. The use of the astrophysical constraint $f_a > 10^9$ weakens the bound by two orders of magnitude, $\mu^2 > 2 \times 10^{-14}$.

Gravitational waves due to collapse of domain walls

The characteristic frequency of gravitational waves is $\omega_* \sim 1/t_*$, where t_* is the time when domain walls disappear, $t_* \sim \sigma_{\rm DW}/\Delta V$ ($\sigma_{\rm DW}$ is the surface tension of DW),

$$\omega_* \sim \frac{1}{t_*} \sim \mu^2 \left(\frac{M_a}{10^{-12}\rm GeV} \right) \times 10^{13} \rm s^{-1} \,.$$

Estimate of the energy density $\Omega_{\rm DW}$ of the produced gravitational waves:

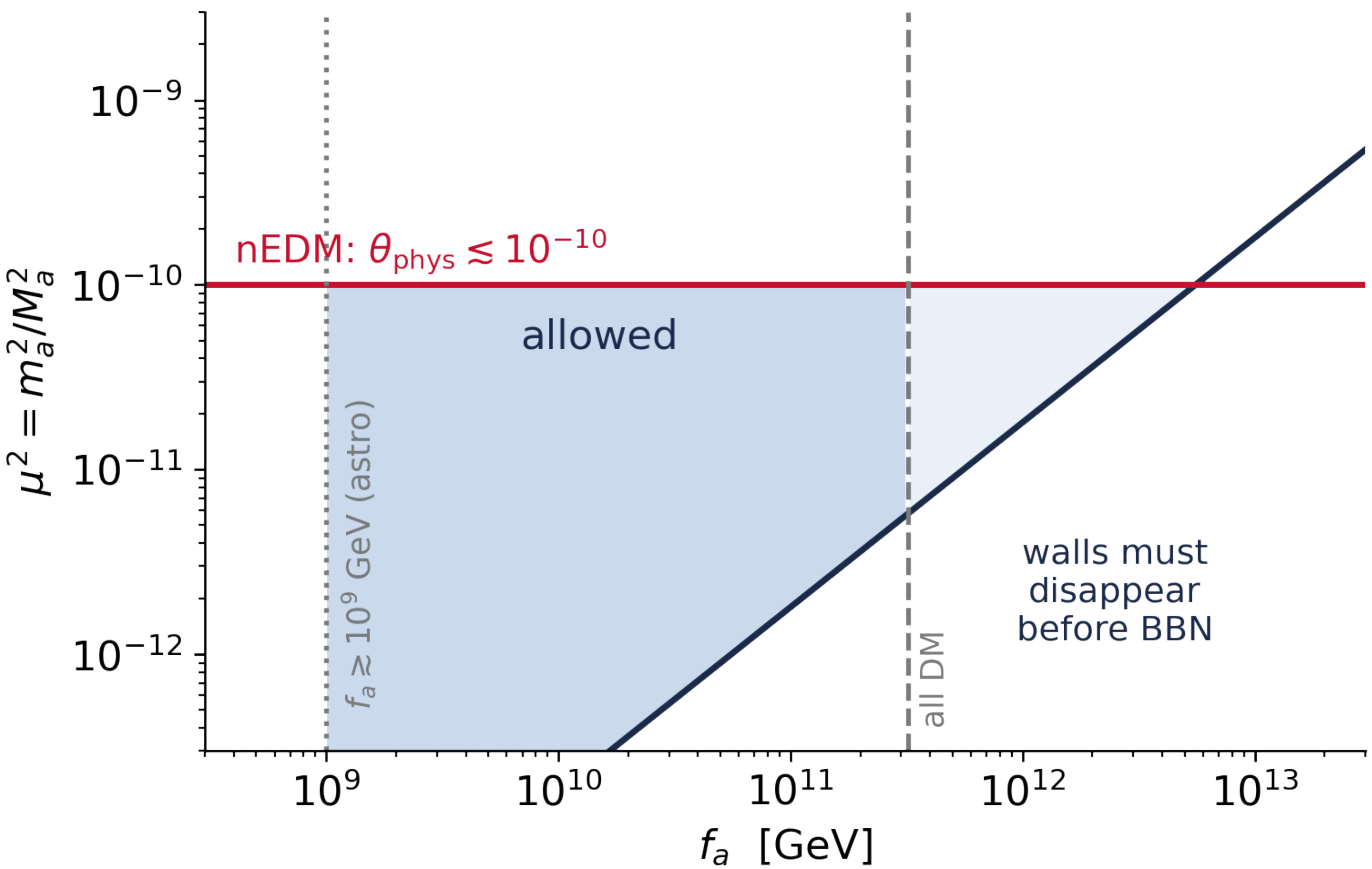
$$\Omega_{\rm GW,*} \simeq 2 \times 10^{-2} \mu^{-4} \left(\frac{f_a}{M_{\rm Pl}} \right)^4 \,.$$

Redshifting to the present day, and using, e.g. $\mu = 10^{-6}$ and $f_a = 3.2 \times 10^{11}$ GeV yields the following peak frequency and amplitude:

$$\omega_0 \sim \mathcal{O}(1) \rm nHz, \Omega_{\rm GW,0} \sim \mathcal{O}(10^{-10})$$

within the frequency band of pulsar timing arrays, and may be within reach of future observations.

Parameter space of the non-compact axion



Viable window:

$$10^9 \text{ GeV} \lesssim f_a \lesssim 5.6 \times 10^{12} \text{ GeV}$$

- upper bound: walls must decay before BBN with $\mu^2 \lesssim 10^{-10}$
- lower bound: astrophysics (SN 1987A, ...)
- all of dark matter: f_a ≲ 3.2×10^{11} GeV
- EFT validity: f_a ≳ 10^{-3} H_I

Assumption: the misalignment contribution dominates

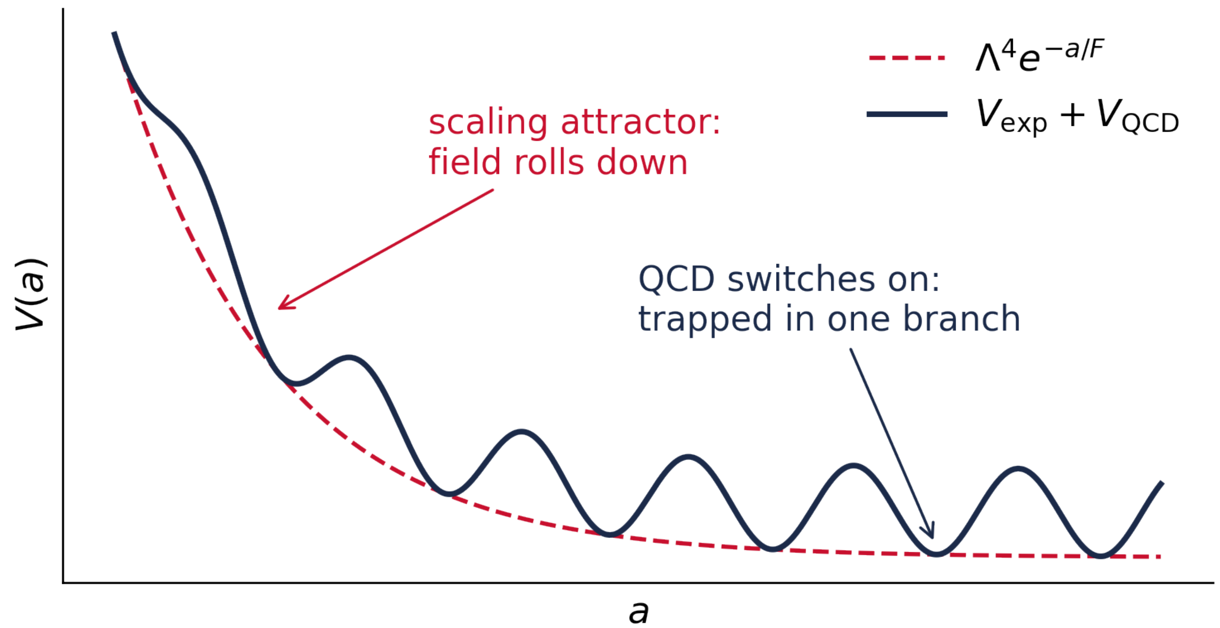
Three correlated signals:
strong CP violation θ_phys ~ μ^2
nHz gravitational waves
axion dark matter

Exponential potential bias: inspired by scale symmetry

A scaling non-compact QCD axion

- If the exponential tilt is not small, the axion rolls down during radiation domination:

$$\mathscr{L} = -\frac{1}{2}\left(\partial_\mu a\right)^2 - \Lambda^4 e^{-a/F} + \frac{a}{f_a} Q$$



- The field traverses ~ $(2F/\pi f_a) \log(T_1/T_2) \gg 1$ QCD periods: a genuinely non-compact field range is essential

Scaling attractor (H = 1/2t):

$$\bar{a} = 2F \log\frac{t}{t_0} + \text{const}, \qquad \dot{\bar{a}} = 4FH, \qquad \Omega_a = 4\left(\frac{F}{M_P}\right)^2$$

A global late-time attractor: every patch, whatever its initial value from inflation, converges to the same trajectory before QCD switches on.

Superhorizon perturbations decay:

$$\delta a_k \propto \sqrt{\frac{T_2}{T_1}}, \qquad \frac{H_I}{2\pi F}\sqrt{\frac{T_2}{T_1}} \ll 10^{-5}$$

Isocurvature is erased dynamically, not washed out statistically. $T_1/T_2 \gtrsim 10^{12}$ is enough.

Scaling axion: dark matter and CP violation

- QCD potential takes over when the two forces become comparable, at

$$T_2 \simeq 2\ \mathrm{GeV}\left(\frac{4\times10^{11}\ \mathrm{GeV}}{F}\right)^{1/12}\left(\frac{10^9\ \mathrm{GeV}}{f_a}\right)^{1/12}$$

Over-barrier, $F > f_a$: kinetic misalignment

The field overshoots many minima before being trapped at $T^* \approx 1.3$ GeV; the abundance is set by F alone:

$$\Omega_a h^2 \sim 0.12\left(\frac{F}{4\times10^{11}\ \mathrm{GeV}}\right)^{7/6}$$

All of DM for $F \sim$ few $\times 10^{11}$ GeV, independently of f_a

Under-barrier, $F < f_a$: standard misalignment

The field is captured at once in one branch with

$\theta_i \sim O(1)$;

$\Omega_a h^2 \approx 3.2 \times 10^{-5}$ θ_i² ($f_a/10^9$ GeV)^{7/6}

θ_{phys} ~

few $\times 10^{-12}$ ($F/10^8$ GeV)^{2/3} ($f_a/10^9$GeV)^{2/3}

- In the kinetic regime dark matter fixes F, and the residual CP violation is predicted:

$$\theta_{\mathrm{phys}} \sim \mathcal{O}(\mathrm{few}) \times 10^{-11}\left(\frac{F}{4\times10^{11}\ \mathrm{GeV}}\right)^{1/3}\left(\frac{f_a}{10^9\ \mathrm{GeV}}\right)$$

Single branch everywhere ⇒ no domain walls, no strings, no nHz gravitational waves. Vacuum tunnelling to lower branches: $S_4 \sim 10^{73}$, utterly negligible.

Two non-compact scenarios

	Statistical (mass-term bias)	Dynamical (scaling attractor)
Non-QCD potential	$\delta V = \frac{1}{2} m_a^2 f_a^2 \theta^2$, gravity-induced (Weyl-EC) or small exponential tilt (dilaton)	Exponential $\Lambda^4 e^{-a/F}$, large tilt
Isocurvature suppression	Averaging over many populated QCD vacua (white noise)	Attractor homogenises the field before QCD switches on
Domain walls / strings	N types of walls, no strings; collapse before BBN	None (single branch); no strings
Gravitational waves	nHz background, $\Omega_{GW} \sim 10^{-10}$–$10^{-12}$, pulsar timing arrays	None from the axion sector
Residual CP violation	$\theta_{phys} \sim \mu^2$; window 10^{-12}–10^{-10} if axion is DM	$\theta_{phys} \propto F^{1/3} f_a$; few × 10^{-11} for DM benchmark
Dark matter	Misalignment + walls, $f_a \lesssim 3 \times 10^{11}$ GeV	Kinetic misalignment, set by $F \sim$ few × 10^{11} GeV
Inflation	$H_I \sim 10^{12}$–10^{13} GeV needed	$H_I \lesssim 10^{12}$ GeV $(f_a/10^9$ GeV$)$ from unitarity
Decisive tests	EDMs + PTA	EDMs, direct detection

Experimental consequences

Electric dipole moments

Residual strong CP violation in the window

$$10^{-12} \lesssim \theta_{\rm phys} \lesssim 10^{-10}$$

Current nEDM bound: $\theta \lesssim 10^{-10}$

Storage-ring proton EDM (Anastassopoulos et al. 2016; Alexander et al. 2022): 3 orders of magnitude improvement, sensitive to $\theta \gtrsim 10^{-12}$

Gravitational waves

Stochastic background from the collapse of the wall network, peaked at ω_0 ~ nHz with Ω_GW ~ 10^{-10}–10^{-12}

Pulsar timing arrays (NANOGrav 15 yr, EPTA, IPTA, SKA); spectral shape differs from that of supermassive black-hole binaries and of N_DW = 1 PQ walls

Observation of either signal would provide non-trivial information about the microscopic nature of the axion.

Simultaneous detection of strong CP violation and of a nHz background would strongly point towards non-compact axion dynamics.

Direct axion searches (haloscopes) in the M_a ≈ 10^{-5} eV range remain fully relevant.

Conclusions

- A solution of the strong CP problem does not need a Peccei–Quinn symmetry: a non-compact scalar coupled to Q, with a tiny non-QCD potential, does the job
- Non-compactness changes the cosmology: inflationary fluctuations populate many QCD vacua for high scale inflation, isocurvature bounds are evaded (statistically or dynamically), there are no strings, and the walls collapse thanks to the bias
- Generic prediction: CP violation in QCD, $\theta_{phys} \sim 10^{-12} - 10^{-10}$ for an axion that is the dark matter, within reach of proton EDM experiments; in the mass biased scenario, a nHz gravitational-wave background for pulsar timing arrays
- Two gravitational origins:
    - Weyl-invariant Einstein-Cartan gravity (one extra scalar; the smallness of the cosmological constant, the Higgs boson mass, and axion gravitational mass are coming from weak Lorentz gauge couplings)
    - the dilaton of broken scale symmetry