Neutrino Physics – Exercises for the Ecole doctorale PHENIICS

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1 Sterile neutrinos and non-unitarity

Without resorting to any particular model, one may assume in full generality that n extra fermions mix with the light neutrinos. The leptonic mixing matrix $\mathcal{U}$ that diagonalizes the full neutrino mass matrix and connects flavor and mass eigenstates is then a unitary $(3+n)\times(3+n)$ matrix:

$$\mathcal{U} = \begin{pmatrix} N & \Theta \\ R & S \end{pmatrix}. \tag{1}$$

The 3×3 block N plays the role of an effective PMNS matrix and is in general non-unitary:

$$NN^\dagger \neq \mathbb{I}. \tag{2}$$

It can be parametrized as:

$$N = (I - T)U, \tag{3}$$

where $(I - T)$ is a lower triangular matrix:

$$T = \begin{pmatrix} \alpha_{ee} & 0 & 0 \\ \alpha_{\mu e} & \alpha_{\mu\mu} & 0 \\ \alpha_{\tau e} & \alpha_{\tau\mu} & \alpha_{\tau\tau} \end{pmatrix}, \tag{4}$$

and U is unitary.

Exercise 1

1.1 Why can N be written in the form $N = (I - T)U$ without loss of generality?

1.2 Prove that:

$$\alpha_{\beta\beta} = \frac{1}{2}\left(\Theta\Theta^\dagger\right)_{\beta\beta} = \frac{1}{2}\sum_{i=4}^{3+n} |\mathcal{U}_{\beta i}|^2, \tag{5}$$

$$\alpha_{\gamma\beta} = \left(\Theta\Theta^\dagger\right)_{\gamma\beta} = \sum_{i=4}^{3+n} \mathcal{U}_{\gamma i}\mathcal{U}^*_{\beta i}. \tag{6}$$

Exercise 2

The oscillation probability in vacuum is:

$$P_{\gamma\beta} = \left|\left(\mathcal{U}\mathcal{S}\mathcal{U}^\dagger\right)_{\beta\gamma}\right|^2, \tag{7}$$

with

$$\mathcal{S} = \text{diag}\left(\exp\left(-i\frac{\Delta m^2_{j1} L}{2E}\right)\right). \tag{8}$$

2.1 Assuming a distance $L = 574\,\text{m}$ and neutrino energy $E = 1\,\text{GeV}$, what is the expected standard oscillation result?

2.2 Assuming one sterile neutrino with $\Delta m^2_{41} = 10\,\text{eV}^2$, compute the probability $P_{\gamma\beta}$ for:

- appearance,
- disappearance.

2.3 What happens to this probability in the limit $\Delta m^2_{41} \to \infty$?

2.4 Taking into account that detectors have a finite energy resolution, can the previous issue be resolved?

2.5 Express this oscillation probability in terms of the α parametrization.

Exercise 3

3.1 An experiment observes a signal in $P_{\mu e}$ but no signal in P_{ee} or $P_{\mu\mu}$.

Is this consistent with a scenario containing only one sterile neutrino?

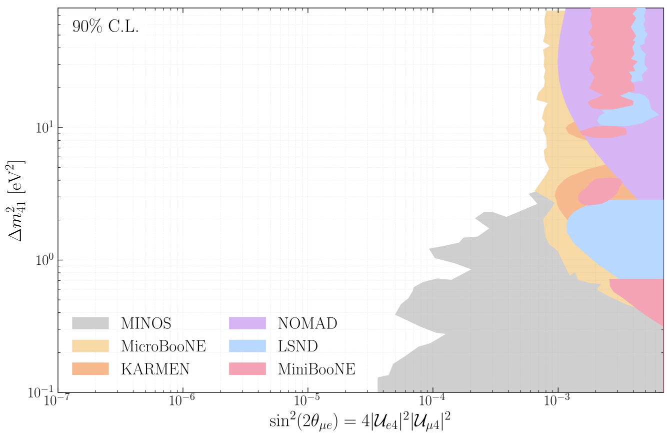

Figure 1: The shaded pink, orange, yellow, and gray areas show current constraints from other experiments, while the shaded purple regions are favored at 99% CL by the LSND and MiniBooNE anomalies.

2 Simple neutrino mass models

The Linear-Inverse Seesaw (LISS) combines elements from both the inverse and linear seesaw mechanisms, providing a framework that generates the required neutrino mass spectrum and mixing patterns.

A minimal realization is obtained by extending the SM with two right-handed neutrinos, N_R^1 and N_R^2, at a new physics scale Λ, carrying opposite lepton numbers:

$$L(N_R^1) = +1, \qquad L(N_R^2) = -1.$$

The Lagrangian is:

$$\mathcal{L} = \mathcal{L}_{\rm SM} + i\bar{N}_R^1 \partial\!\!\!/ N_R^1 + i\bar{N}_R^2 \partial\!\!\!/ N_R^2 - \left(Y_\alpha\, \bar{\ell}_\alpha \tilde{\phi} N_R^1 + \epsilon\, Y'_\alpha\, \bar{\ell}_\alpha \tilde{\phi} N_R^2 + \frac{\Lambda}{2}\, \bar{N}_R^{1c} N_R^2 + \frac{\mu}{2}\, \bar{N}_R^{2c} N_R^2 \right) + \text{h.c.} \quad (9)$$

After electroweak symmetry breaking, the neutrino mass terms can be written as:

$$-\mathcal{L}_{m_\nu} = \frac{1}{2}\, n_L^T\, C\, \mathcal{M}_{\rm LISS}\, n_L + \text{h.c.}, \qquad (10)$$

where

$$n_L = (\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}, N_1^c, N_2^c)^T, \qquad C = i\gamma^2\gamma^0.$$

Exercise 5

5.1 Write explicitly the 5×5 neutrino mass matrix $\mathcal{M}_{\rm LISS}$.

5.2 How many light neutrinos can be massive? Is it compatible with neutrino oscillation data?

Defining $m_D = \mathbf{Y}v$, $m_L = \epsilon\mathbf{Y}'v$, and in the following limit

$$|m_D = \mathbf{Y}v|,\ |m_L = \epsilon\mathbf{Y}'v|,\ |\mu| \ll \Lambda,$$

we can perform a block diagonalization:

$$U_B^T \mathcal{M}_{\rm LISS} U_B = \begin{pmatrix} m_{\rm light}^{3\times3} & 0 \\ 0 & M_{\rm heavy}^{2\times2} \end{pmatrix}, \tag{11}$$

where U_B is a unitary matrix and, at leading order:

$$m_{\rm light} = \frac{\mu}{\Lambda^2} m_D m_D^T - \frac{1}{\Lambda}\left(m_D m_L^T + m_L m_D^T\right).$$

$$M_{\rm heavy}^{2\times2} \simeq \begin{pmatrix} 0 & \Lambda \\ \Lambda & \mu \end{pmatrix}$$

Exercise 7

7.1 Identify the contributions to $m_{\rm light}$ proportional to μ and to ϵ. Explain why they are referred to as the *inverse seesaw* and *linear seesaw* terms.

Exercise 8

8.1 Obtain the masses of the heavy neutrinos in the model.

8.2 What happens in the limit $\mu \to 0$?

8.3 Do you understand why they would be called a pseudo-Dirac pair?